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Abstract

In this paper we argue that differences in the investment projects' features can help to explain the observed differentials in output growth and in output volatility across countries. This result is achieved by studying analytically an endogenous growth model where investments are (generically) distributed over multi-period flexible projects leading to new capital once completed. Recently developed techniques in dynamic programming are adapted and used to fully characterized the balanced growth path and transitional dynamics of this model. Based on this analytical ground, several numerical exercises are performed to show how the key results of our analysis are also quantitatively relevant.

Keywords: investment projects, distributed delays, optimal control, dynamic programming, infinite dimensional problem.

JEL classification: E22, E32, O40

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1 Introduction

The notion of investment project was introduced for the first time in a general equilibrium model by Kydland and Prescott [31] to allow for gestation lags in the production of capital goods. A project, as defined by these authors, has three features. First, it requires several stages before its completion and, once completed, leads to new productive capital; therefore an (exogenously given) lag of several periods exists between the beginning of a project and the formation of new productive capital. Secondly, the amount of resources allocated to a project, as well as its objective, are decided at its beginning and cannot be adjusted afterward. As a consequence, a project initiated one year ago and to be completed in the current year cannot be enhanced, reduced, or even scrapped whatever positive or negative shock may affect the economy. For this reason, we may well refer to this kind of projects as *fixed projects.* [1] Lastly, the investment over the fixed projects is either spread evenly over all the projects independently on their degree of completion (uniform distribution), or concentrated on the project at its earliest stage (i.e. pure investment lag case). Therefore the investment distribution is exogenously given and, furthermore, not generic. [2]

While the first feature is confirmed by several empirical evidences (e.g. Koeva, [29]) the other two features are less convincing. For example, the assumption of fixed projects seems quite restrictive especially when the capital stock is construed broadly “to encompass human capital, knowledge, public infrastructure, and so on" (see Barro and Sala-i-Martin [6]) as it is usually the case with endogenous growth models having linear technology.

In fact, several empirical evidences point to projects with a certain degree of flexibility. Among them, those on public infrastructures are probably the most popular. In the United Kingdom, the government started in 2009, as a consequence of the recession, a public spending review which comprehended 217 projects, totalling £34 billion (The Independent 17 June 2010); following the review, several of these projects were reduced or even axed as the building of new schools for around £5 billion (Guardian, 6 July 2010), or the building of new hospitals for more than £2.5 billion (The Telegraph, 3 March 2009). Evidences of opposite sign can be also found in the literature. Recently Flyvbjerg et al. [23], [22] have estimated that additional resources were required to complete around 90% of a sample of 258 public transportation infrastructure projects in the United States and that the additional resources added over time amount for the 20%-40% of the initially planned investment. Finally modifications to public works are also contemplated and regulated by law in some European Countries as shown, for example, by the Italian Law 109 approved in 1994.

The third feature is also not confirmed by several recent contributions pointing out to alternative distributions over the projects. One of these is the *decreasing exponential distribution* according to which a declining proportion of resources is allocated to the projects closer to completion. In a model with projects lasting 4 quarters, Altug [2] estimates that the 70 per cent of the resources are allocated in the first two quarters and strongly reject the hypothesis of uniform distribution. Similar results are found by Park [38] when the projects take three quarters to be completed. On the other hand, some authors have found evidences in support of an *increasing exponential distribution* according to which a close to zero proportion of resources is allocated in the first stage of the project (planning) and increasingly higher in the other stages (construction); this is indeed referred as the time to plan specification and was originally suggested by Christiano and Todd [14]. This specification has been also empirically confirmed by Del Boca et al. [8] looking at the annual Italian firms investment data on structures. Other distributions identified by the literature are a *U-shaped distribution* (e.g. Zhou [45], and Peeters [40]) and a *hump-shaped distribution* (e.g. Altug [2]). Interestingly, there is also evidence that the heterogeneity in the distributions can be country-specific (e.g. Peeters [40]).

The heterogeneity in the project's characteristics (i.e. investment distribution and project's length) seems even more compelling when we consider not only physical capital but also human capital, public

1. It is worth noting that a project already started is fixed not because the investment is irreversible but because the resources necessary to complete it are predetermined or committed at its beginning. This difference will result plainly clear in Section 2.1 where we will formally define the projects.

2. More precisely, Kydland and Prescott propose a model setup with a generic distribution but the equilibrium path is numerically computed by assuming the two previously described distributions (i.e. uniform distribution or pure investment lag).

infrastructure, etc. For example, human capital formation projects, such as first university degrees, have been quite different even across European countries in the period 1950-1999. [3] Ruegg [42] distinguishes countries with a first university degree of 3 years (such as the U.K. bachelor) from other countries with a longer degree of at least 4 years (e.g. Italy, Spain, Austria, Finland) which is usually prolonged by more than 2 years (e.g. MIUR [35]). Similarly the realization of public infrastructures projects varies significantly across countries with some reporting significant delays in their completion as documented, for example, by the list of the incomplete public projects recently published by the Italian Public Infrastructure and Transport Ministry (see "Elenco Anagrafe Opere Incompiute", Ministero per le Infrastrutture e Transporti). Of course, the heterogeneity in the project's features becomes even more evident when we compare developed with developing countries as emerges from a quite large literature on construction projects pointing out how the actual project's length is, on average, longer in developing countries where it can arrive to be twice the estimated project duration. [4]

Therefore, the aim of this paper is to develop an endogenous growth model characterized by *generically distributed investment over flexible multi-period projects* to account for the empirical evidences just described and to investigate how the growth rate, welfare, and transitional dynamics can be affected both qualitatively and quantitatively by differences in the project's characteristics. In this extent, we depart from Kydland and Prescott original framework by modifying the second and third feature of an investment project and embedding it in an endogenous growth framework.

The engine of growth in our economy is the presence of constant returns to scale in the capital stock which is the only accumulating factor of production; for this reason, capital is indeed defined in a broader sense, and our results can be related to the empirical evidences on investments in public infrastructures, human capital formation, and construction mentioned before. In our economy, the benevolent social planner decides, as usual, how much to consume and save in each period; however the aggregate investment contributes to the development of all the projects not yet completed (*flexibility*), each of them leading to new capital at different dates in the future. Then new capital is obtained as the weighted (Riemann) sum of all the investments undertaken over a given (finite) time interval, and as its limit when we move to continuous time. If we think again to the capital stock as public infrastructure and assuming two years long projects, then shares of the investment today are allocated, for example, to the development/realization of new bridges today, new roads within one year, and new schools within two years. Resources allocated to investments in public infrastructure a year later, will be useful to build up new more roads in the current year, new more schools within one year, and start a project for the realization of new hospitals in two years. The other departure from the existing literature is to allow for a *generic distribution* of the investment over the (flexible) projects by keeping generic, but still exogenous as in Kydland and Prescott, the weights in the previously mentioned (Riemann) sum.

The paper contributes to the existing literature in three ways. First it provides a full analytical characterization of an endogenous growth model with investment generically distributed over flexible multi-period projects; this is done in the core part of the paper where we use a dynamic programming approach to unveil the transitional dynamics and balanced growth path properties of the economy as well as the closed-form optimal path of all the aggregate variables. These results are important since represent the solid ground where the quantitative analysis is built on. Significantly, the dynamic programming approach used in this paper represents a methodological contribution to the existing theoretical literature as the first economic application of this method to a general equilibrium model where the state equation is an integro-differential equation (see the next section for more details).

Secondly, our analysis shows that countries with different projects' characteristics grow at different rates and that the heterogeneity in the projects' characteristics across countries can help to explain a quantitatively relevant part of the growth rate differentials observed in the empirical literature for choices of the parameters values usually considered realistic. For example, our first quantitative exercise (Subsection 8.1) shows that the income gap after 100 years between two countries, which are similar but

3. In the Bologna declaration signed in 1999, the education ministers of many European countries agreed to establish a framework of study programmes similar to the anglo-saxon tradition of the bachelor's and master's qualification (see Ruegg [42], page 366).

4. Koushki et al. [30] shows that the estimated residential construction project duration in Kuwait is on average 8.3 months (planning) plus 9.4 months (construction) while the actual is 8.3 plus 18.2 months. Similar results are found in studies focusing on other developing countries such as Nigeria (Mansfield et al. [34]), Jordan (Al-Momani [1]), etc.

their investmentís distributions, is 9.4% when the two countries has the same length of a project, d, equal to two years but the richest is characterized by time-to-plan (i.e. increasing exponential distribution) while the poorest by a uniform distribution of the investment over the projects. The income gap changes to 37.02% when the poorest country is characterized by pure investment lags in production (i.e. Dirac's Delta in $-d$). Even larger differentials are observed if the project length changes from 2 to 3, 4, and 5 years.[5] In the latter, the income gap after 100 years is 109.6% when the poorest country has pure investment lags in production while there is time-to-plan in the richest.

Moreover we show (Subsection 8.2) that the investment projects are a source of endogenous fluctuations with output, capital, and investment converging by damping fluctuations toward the balanced growth path. Measuring the average and maximum absolute deviation from the GDP as well as the speed of convergence towards the balanced growth path of the optimal path of output reveals that the heterogeneity in the projects' features across countries implies quantitatively relevant differences in the persistence of output deviations from the balanced growth path and can be one of the determinants of i) the negative correlation between mean output growth and output growth volatility and ii) the substantial variation in the output volatility across countries as observed in several empirical studies (e.g. Ramey and Ramey [41]).[6]

Lastly, although we make no attempt to bring the model to the data, our analysis carries several interesting implications for policy and future empirical works as it will be discussed in the concluding section.

The rest of the paper is organized as follows. The next Section 2 is devoted to present the related theoretical literature while Section 3 contains an accurate description of the model setup; in particular, we define the flexible multi-period investment projects and the investment distribution (Subsection 3.1), then we write the social planner problem (Subsection 3.2) and state it formally as an optimal control problem (Subsection 3.3). In Section 4 we first state some important preliminary results (Subsection 4.1) and then we explain the methodology used to solve the problem (Subsection 4.2). In Section 5 we apply the procedure explained in Subsection 4.2 to our problem. This section is divided in 5 subsection for the sake of clarity. In Section 6 and Section 7 we use the results of Section 5 to describe the balanced growth paths (Section 6) and the transitional dynamics (Section 7) of the economy; Section 8 focuses on a numerical analysis of the quantitative implications of our model in term economic growth (Subsection 8.1) and endogenous fluctuations (Subsection 8.2). Appendix A is devoted to present the decentralized economy, Appendix B contain some notations on function spaces that can be useful for the reader, while Appendix C collects all the proofs.

2 Related Theoretical Literature

In our economy, new capital requires time to be built: new capital is added to the existing capital stock once a project, initiated in the past, is completed. Then the accumulation of capital is sluggish, but once added, capital becomes productive immediately. This definition of "time to build" differs from the one commonly proposed in the recent economic growth literature (see, among others, Benhabib and Rustichini [7], example 7, page 332, Asea and Zak [3], Bambi [4], and Bambi et al. [5]) for three reasons. First of all, the resources necessary to complete a project are allocated at different stages and not concentrated in only one stage of the project itself; then the investment distribution used throughout our paper is kept generic, while in the other contributions a Dirac's delta in the last stage of the projects is used. Secondly, the projects are flexible in our paper while fixed, in the sense explained previously, in these other contributions. The third and last difference is more subtle: the contributions, just cited, consider always economies where net investments become immediately new productive capital but final output takes time to be built because there is a delay between the use of the new added capital and the production of new goods. In contrast, we consider a delay between the initial investment and the arrival of new capital with the latter immediately productive once added to the production process. This second

5. Projects longer than two years is reasonable since with a linear technology, capital is broadly defined to include also human capital, knowledge, etc..

6. The deviation from the balanced growth path depends on an initial exogenous shock to the existing amount of capital stock.

description of "time to build" was used for the first time by Kalecki [28], who called it *gestation lags* in production, and later used by Kydland and Prescott [31] and by Lucas [33]. It is indeed the specification which is mostly used in the business cycle literature. With respect to the last cited contribution, our paper shares some similarities. Lucas [33] studies the optimal investment policy for a single firm whose objective is to maximize the discounted flow of profits by choosing the number of projects to initiate taking into account that the limit of the weighted (Riemann) sum of all the initiated projects undertaken over a given (finite) time interval generates new capital stock. Therefore the capital's formation equation is an integro-differential equation, closely resembling ours. [7] However, there are three crucial differences with respect to his contribution. The first is the ownership right of capital; in our model, the firms rent capital at each date from the households and then their profit maximization problem is static while the households' problem is dynamic and their saving/investment is spread over the not yet completed projects as described previously in this introduction. [8] The second difference is the aim of the analysis, since we are interested in understanding how different distributions and lengths of the projects may affect the growth rate, welfare, and transitional dynamics of the economy. The last difference is that the main analytical results in Lucas (see [33], page 43) are obtained by restricting the analysis to those distributions which allow the author to convert the original complicated problem to a classical problem in the calculus of variations. On the other hand, our approach avoids any restriction on the distribution of the investment over the projects. In this extent we contribute also from a methodological viewpoint by providing an approach which can be used to solve a broader class of problems.

Similarly, our paper is related to the stream of literature on optimal dynamic advertising and to the one of vintage capital. Concerning optimal advertising, the original model of Arrow and Nerlove [36] describes the optimal decision of a monopolistic firm which has to decide the stock of advertising goodwill which maximizes the discounted flow of profits taking into account that advertising is a costly activity, and it has a positive, but decreasing over time, effect on the revenue. As extensively documented by the survey of Feichtinger et al. [21], several contributions have generalized the seminal model and one line of literature investigates the optimal investment decision when the instantaneous rate of change of the stock of advertising goodwill follows an integro-differential equation in order to account for two effects: the lag between the investment in advertising and the corresponding increase of goodwill, and the distribution of the forgetting time (for more details, Feichtinger et al. [21], page 200). The resulting law of motion of the stock of advertising goodwill is similar to the capital's formation used in our model. Also in this case, our paper is different in the assumption on the ownership of the stock variable and on the scope of the analysis. Moreover, all these contributions (e.g. Pauwels [39] and Hartl [26]) characterize analytically the optimal investment decision for specific distributions using a modification of the maximum principle, while in our paper we apply dynamic programming techniques and we are able to find the optimal plan of the economy without imposing any restriction on the distribution of the investment over the projects.

Concerning vintage capital models, there is indeed an extensive literature as summarized recently by Boucekkine et al. [9]. For the sake of brevity, we remind here just three papers: Benhabib and Rustichini [7] who were probably the first to show how some depreciation profiles, as for example the one-hoss shay, may lead to endogenous fluctuations once embedded in an otherwise standard Ramsey model; Boucekkine et al. [12] who extended this result further by characterizing the dynamics of the same economy when the scrapping time is endogenous; and Feichtinger et al. [20] who considered a vintage capital framework where agents are allowed to invest in capital goods having different vintages.

Finally, this paper belongs to the class of optimal control problems where the state equation is a functional-differential equation. From a methodological viewpoint, most of the papers treating this kind of problems, and in particular all those just cited, use maximum principle techniques. Recently, starting from Fabbri and Gozzi [17], new techniques based on the dynamic programming approach have been developed and applied to these problems. These techniques allow to solve such problems more explicitly; in particular it is possible to find the closed loop policy function and unveiling economic mechanisms which were otherwise hidden (e.g., see Bambi et al. [5], Boucekkine et al. [10, 11]).

7. Lucas [33] represents a generalization of the results in Lucas [27] to the case of distributed delays.

8. In the main text, we solve a social planner problem but in Appendix A we have also briefly described the decentralized economy.

3 Model Setup

3.1 Description of the Flexible Multi-period Projects

We start with a description of the flexible-multiperiod projects, or investment plans, when time is discrete and then we move to its continuous counterpart. In line with Kydland and Prescott, we indicate with $s_{j,t}$ a project at time t, j stages from completion. [9] Once completed a project generates new capital:

$$s_{1,t} = \Delta k_{t+1},$$

no project requires more than d periods to be completed, $s_{d+1,t} = 0$, and the initial capital stock, k_0, is exogenously given.

On the other hand, the dynamics of a project is different from the specification in Kydland and Prescott to allow for modifications in the already initiated projects; in fact, in our economy, the aggregate investment at time t is allocated over a menu of d projects:

$$s_{1,t},\ s_{2,t},\ \dots\ s_{d,t}$$

so that each project completed after j periods, receives an (exogenous) share, a_j, of the aggregate investments, i_t. Formally, an investment distribution is so defined:

Definition 3.1 (Investment's Distribution). *Given the (maximal) project's length $d \in \mathbb{N}$, an investment distribution over the projects is a vector*

$$(a_1, a_2, ..., a_d) \in \mathbb{R}^d_+ \qquad \textit{with} \ \sum_{j=1}^{d} a_j = 1 \ \ \textit{and} \ \ a_j \geq 0 \ \ \forall j$$

where, for every j and t, $a_j i_t$ is the share of the investment i_t over the projects j periods from completion. [10]

To understand the dynamics of the projects, it may be useful to follow how one of them evolves over time, from its beginning to its completion after, for example, $d = 3$ periods. At time $t = 0$ the resources allocated to it are

$$a_3 i_0 = s_{3,0}$$

For example, we can think to the raw resources allocated initially to build up some single-lane roads. In the next period, new resources are available and a share of them can be again allocated to the project

$$a_3 i_0 + a_2 i_1 = s_{2,1}$$

These new resources are used, for example, to add another lane to the previously planned single-lane roads. Finally in the last stage we have

$$a_3 i_0 + a_2 i_1 + a_1 i_2 = s_{1,2}$$

meaning that further resources are allocated to the project to add, for example, a couple of roundabouts. The double-lane roads and the roundabouts are now completed and can be added to the existing capital at the beginning of the next period:

$$s_{1,2} = \Delta k_3$$

Then we may obtain recursively the dynamics of the projects as:

$$s_{j,t} = s_{j+1,t-1} + a_j i_t$$

9. It is worth noting that there is no relevant change in the analytical derivations and interpretation of the results if $s_{j,t}$ indicates the *group of* projects at time t, j-stages from completion.

10. Observe that the investment distribution can be read as a probabilistic distribution with a_j the probability of investing in a project j stages from completion.

It is indeed worth noting that all the resources added to a project, even those at the very last stage, increase the capital stock generated by completing the project itself: for this reason, the project is said to be flexible. In our example, it is clear that

$$i_2 \uparrow \qquad \Rightarrow \qquad k_3 \uparrow$$

since it implies a change in the project started at $t = 0$. Flexibility implies also that the total resources needed to complete a project are not determined at its beginning but only at the end because a project can be modified at each stage. It is also worth noting that in the continuous-counterpart, which will be adopted later, a completed project leads instantaneously to new capital and therefore the investment decision at that date may modify its magnitude. Flexibility as just described, is fully specified as long as the initial history of the investment, i_t with $t \in [-d, 0)$ is exogenously given; therefore any investment decision taken at $t = 0$ influences, according to equation (3.1), all the projects not yet completed at that date.

Flexibility is indeed the first main difference we introduce to the Kydland and Prescott's specification. In fact, in their framework, the projects dynamics is given by $s_{j,t} = s_{j+1,t-1}$ and aggregate investment is equal to $i_t = \sum_{j=1}^{d} b_j s_{j,t}$ with $\sum_{j=1}^{d} b_j = 1$ and $b_j \geq 0$ for all j. Moreover the exogenously given initial conditions are k_0 and $s_{j,0}$, with $j \in [1, d-1]$ which, together with (3.1), imply that Δk_{d-j} are predetermined. Focusing, as before, on the case $d = 3$ and $t = 0$ we have that

$$i_0 = b_1 s_{1,0} + b_2 s_{2,0} + b_3 s_{3,0} \qquad or \qquad \frac{1}{b_3}\left(i_0 - \underbrace{b_1 \Delta k_1 - b_2 \Delta k_2}_{exog.\ given}\right) = \Delta k_3$$

and, therefore, i_0 determines completely $s_{3,0}$ and then k_3. For the same reason, it follows that

$$i_1 \uparrow \ \text{and/or} \ \ i_2 \uparrow \qquad \Rightarrow \qquad k_3 \ unchanged$$

since these investments do not affect the project started at date $t = 0$. Then, it is clear that, in Kydland and Prescott, the resources to be allocated to the different projects are decided at the very beginning while, in our context, more resources can be added during the works in progress and crucially till the last period before the projects' completion.

Disinvestment from existing project is also possible. In fact, investments are assumed reversible and, therefore, any project can be reduced or scrapped even before its completion. Then the main difference with the no gestation lags case, $d = 1$, is that a disinvestment can affect resources not yet productive. Coming back to our example on public infrastructure, it could be that a reallocation of resources takes place in the last stage of the project and instead of some double-lane roads and a couple of roundabouts, only one double-lane road is built up. This feature is probably more appealing from an empirical point of view than the fixed plan case, as for example testified by the empirical evidences provided in the introduction. [11]

The continuous counterpart of (3.1) is

$$-\frac{\partial s}{\partial j} + \frac{\partial s}{\partial t} = a(j)i(t), \qquad j \in [0, d], \quad t \geq 0$$

with the boundary condition $s(d, t) = 0$ and $s(0, t) = k'(t)$ for every $t \geq 0$ which are the continuous counterpart of the discrete time ones. Integrating this last equation leads to

$$s(0, t) - s(d, t - d) = \int_0^d \left(-\frac{\partial}{\partial j} + \frac{\partial}{\partial t}\right) s(j, t - j)dj = \int_0^d a(j)i(t - j)dj$$

so, thanks to the boundary conditions, we have

$$k'(t) = \int_0^d a(j)i(t - j)dj$$

11. It is also worth noting that the realization of "big" projects is possible in the sense that the capital added at the end of the project can be higher than the total amount of resources available at its very beginning (in our example, and assuming a linear technology, it could be that $a_3 i_0 + a_2 i_1 + a_1 i_2 = \Delta k_3 > A k_0$).

or by replacing j with $-r$

$$k'(t) = \int_{-d}^{0} a(r)i(t+r)dr$$

whose notation is now the one commonly used in optimal control problem where the state equation is a delay differential equation with distributed delays. Consistently with Definition 3.1, we also assume that Before moving to the social planner problem, it is worth noting that (3.1) embeds all the specifications of time-to-build used in the literature. This feature is the other main difference of our framework with respect to the existing contributions where a specific distribution is always assumed. Finally it can be observed that our generic distribution contains the purely gestation lags case

$$i(t-d) = k'(t)$$

when $a(r) = \delta_{-d}$, the standard case with no time-to-build

$$i(t) = k'(t)$$

when $a(r) = \delta_0$, the case of time-to-plan when the distribution is increasing exponential, etc [12].

3.2 The Social Planner Problem

In this section we embed the project structure just described in an AK model. The social planner problem for this economy is presented in the following while the decentralized version can be found in Appendix A. The social planner wants to maximize the functional

$$\int_0^{+\infty} \frac{c(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0, \ \sigma \neq 1,$$

subject to the resource constraint of the economy and the production function, which write respectively

$$c(t) = y(t) - i(t), \qquad \textit{and} \qquad y(t) = Ak(t) \qquad \textit{with } A > 0,$$

the **state equation** describing the capital formation

$$\begin{cases} k'(t) = \int_{-d}^{0} a(r)i(t+r)dr, & t \geq 0, \\ k(0) = k_0, \quad i(s) = i_0(s), \ s \in [-d, 0), \end{cases} \tag{3.1}$$

and the pointwise (inequality) constraints

$$\textbf{(c1)} \ \ k(t) > 0, \quad \forall t \geq 0; \qquad \textbf{(c2)} \ \ i(t) \leq Ak(t), \ \text{ for a.e. } t \geq 0. \tag{3.2}$$

In the problem above, k is the state variable and i is the control variable. The constraint (c1) is a constraint on the state variable imposing the nonnegativity of capital, and immediately implies to consider initial data $k_0 > 0$; the constraint (c2) is a mixed state-control constraint imposing that the current investment cannot exceed the current production, so it is a kind of no-borrowing constraint. We note that in (3.1), $i_0(s)$ must be assigned for a.e. $s \in [-d, 0)$ and is not part of the control: indeed, it is an initial datum together with k_0.

The state equation (3.1) is an example of differential equation with delays (DDE) in the control variable. The fact that the initial datum is a real number k_0 together with a function i_0 illustrates that the nature of the problem is infinite dimensional.

We notice that we do not impose a priori nonnegativity constraints either on the function i_0 or on the function i: in particular, since i may be negative, disinvestment is allowed, i.e. the investment is reversible.

12. It is worth noting that in the above cases a can also be a measure (e.g. in the gestation lag case, a is the Dirac delta δ_{-d}). For sake of simplicity, the theoretical part focuses only on the case where a is a function. However, a straightforward generalization of the arguments presented in this paper can be done to include also the case where a is a measure (e.g. Bambi et al. [5, 17]). For this reason, our quantitative analysis (Section 8) considers also the pure gestation lag case and the one without time-to-build.

3.3 Formal Statement of the Control Problem

As explained before, standard pointwise initial conditions are not enough to determine solution paths of DDEs. Rather, we need an initial function defined on a time-interval whose span depends on the delays' structure; in particular, the initial conditions in our model consist of the initial, exogenously given, capital stock k_0 and the history of investment, $i_0(s)$ with $s \in [-d, 0)$ where the latter is an exogenously given function. Accordingly, we shall work on functional spaces. Therefore it is useful to introduce by the very beginning some notations for functional spaces. This is done in Appendix B which provides a list of all the functional spaces used in our analysis.

Consistently with the definition 3.1 of investment distribution, we also assume, from now on, the following.

Hypothesis 3.2. *The share of investment a is specified such as $a \in L^2([-d, 0]; \mathbb{R}^+)$ and $\int_{-d}^0 a(r)dr = 1$ with $a(r) \geq 0$ for any $r \in (-d, 0)$.*

Then we are ready to rewrite our optimal control problem. First of all, we may write more formally (3.1): given a control strategy $i_0 \in L^2([-d, 0]; \mathbb{R})$ and $i \in L^2_{\text{loc}}([0, +\infty); \mathbb{R})$, we denote by $\tilde{\imath} : [-d, +\infty) \longrightarrow \mathbb{R}$ the function in $L^2_{\text{loc}}([-d, +\infty); \mathbb{R})$ defined as follows.

$$\tilde{\imath}(s) = \begin{cases} i_0(s), & s \in [-d, 0) \\ i(s), & s \in [0, +\infty). \end{cases} \tag{3.3}$$

For every $i_0 \in L^2([-d, 0]; \mathbb{R})$, $k_0 \in \mathbb{R}$ and $i \in L^2_{\text{loc}}([0, +\infty); \mathbb{R})$ there exists a unique continuously differentiable solution to (3.1), i.e. a function of class $C^1([0, +\infty); \mathbb{R})$, which will be denoted by $k_{(k_0,i_0),i}(\cdot)$, verifying pointwise (3.1) for each $t \geq 0$ [13]. Using (3.3), such a solution can be explicitly written in integral form as

$$k_{(k_0,i_0),i}(t) = k_0 + \int_0^t \int_{-d}^0 a(r)\tilde{\imath}(s + r)dr\, ds, \qquad t \geq 0 \, . \tag{3.4}$$

The fact that $k_{(k_0,i_0),i} \in C^1([0, +\infty); \mathbb{R})$ is due to the continuity of the function $s \mapsto \int_{-d}^0 a(r)\tilde{\imath}(s + r)dr$.

The functional to maximize is

$$J((k_0, i_0); i) \stackrel{def}{=} \int_0^{+\infty} \frac{(Ak_{(k_0,i_0),i}(t) - i(t))^{1-\sigma}}{1 - \sigma} e^{-\rho t} dt, \qquad \sigma > 0, \quad \sigma \neq 1,$$

under the admissibility constraints (3.2), i.e. over the set

$$\mathcal{I}_{(k_0,i_0)} \stackrel{def}{=} \{i \in L^2_{\text{loc}}([0, +\infty); \mathbb{R}) \ : \ k_{(k_0,i_0);i}(t) > 0 \ \ \forall t \geq 0, \ \ i(t) \leq Ak_{(k_0,i_0),i}(t) \text{ for a.e. } t \geq 0\}.$$

We call **(P)** the problem of finding an optimal control strategy, i.e. finding $i^* \in \mathcal{I}_{(k_0,i_0)}$ such that

$$J((k_0, i_0); i^*) = V(k_0, i_0) \stackrel{def}{=} \sup_{i \in \mathcal{I}_{(k_0,i_0)}} J((k_0, i_0); i). \tag{3.5}$$

The function V defined above is the so called **value function** of the problem, which will be a key tool for solving our problem and whose properties will be studied in Section 4.1.

4 Methodology and Preliminary Results

In this section we provide some preliminary results and, after that, we will explain the methodology we are going to use to solve the problem **(P)**. The preliminary results include consideration on i) the maximal growth rate of capital; ii) the existence and uniqueness of optimal paths; iii) the properties of the value function V; and iv) the asymptotic behavior of admissible paths.

13. Hereafter, the symbol $\cdot$ in the argument of a function represents the running independent variable of the function.

4.1 Preliminary Results

To find conditions ensuring the finiteness of the value function, we need first to give an upper bound for all the state trajectories. This is done by determining the maximum asymptotic growth rate of the function $k_{(k_0,i_0),i}$.

Equation (3.4), Hypothesis 3.2 (nonnegativity of the function a) and the structure of $\mathcal{I}_{(k_0,i_0)}$ suggest that the capital k is the highest possible when the control i satisfies the feedback relation $i(t) = Ak(t)$ for all $t \geq 0$ (which is the maximum of the range of admissibility). Then, plugging the control defined by the feedback relation $i(t) = Ak(t)$ for $t \geq 0$ into the state equation (3.1), we get the corresponding closed loop DDE

$$\begin{cases} k'(t) = \int_{-d}^{(-d)\vee(-t)} a(r)i_0(t+r)dr + A\int_{(-d)\vee(-t)}^{0} a(r)k(t+r)dr \ , \quad t \geq 0, \\ \\ k(0) = k_0, \qquad i_0(s), \quad s \in [-d, 0). \end{cases} \tag{4.1}$$

We notice that in (4.1) the delay is now in the state variable. [14]

Proposition 4.1.

1. *For every $(k_0, i_0) \in H$, the DDE (4.1) has a unique continuously differentiable solution denoted by $k^M_{(k_0,i_0)}(\cdot)$.*
2. *Let $(k_0, i_0) \in H$, $i \in \mathcal{I}_{(k_0,i_0)}$. Then $k_{(k_0,i_0),i}(t) \leq k^M_{(k_0,i_0)}(t)$ for every $t \geq 0$.*
3. *If $k^M_{(k_0,i_0)}(\cdot) > 0$, then $\mathcal{I}_{(k_0,i_0)} \neq \emptyset$.*

Now we want to study the DDE (4.1) which becomes, for $t \geq d$,

$$k'(t) = A\int_{-d}^{0} a(r)k(t+r)dr. \tag{4.2}$$

For such equation we can apply standard statements on DDEs. Studying the properties of this equation is crucial to fully characterize the solution of our problem. We begin with the *characteristic equation* of (4.2), which is the transcendental equation

$$z = A\int_{-d}^{0} a(r)\, e^{rz} dr \ , \quad z \in \mathbb{C} \ . \tag{4.3}$$

The next proposition describes the properties of its spectrum of infinite roots [15].

Proposition 4.2 (Maximal Growth of Capital).

1. *There exists a unique real root ξ of (4.3). It is simple and strictly positive.*
2. *If $\lambda = \mu + i\nu$ is a complex root of (4.3) (i.e. with $\nu \neq 0$) then also $\bar{\lambda} = \mu - i\nu$ is a root of (4.3). Taking the one with $\nu > 0$ we have*

$$-Ae^{-\mu d} < \mu < \xi, \qquad \frac{\xi}{d} < \nu < A\left(1 \vee e^{-\mu d}\right) \tag{4.4}$$

In particular the real part of all complex roots is strictly smaller than ξ. The real number ξ is then called the ***maximal root*** *associated to (4.3).*

3. *There exists a decreasing real sequence $\{\mu_j\}$ and a positive real sequence $\{\nu_j\}$ such that all the complex and non real roots of (4.3) are given by $\{\lambda_j = \mu_j + i\nu_j, \bar{\lambda}_j = \mu_j - i\nu_j\}_{j\in\mathbb{N}}$.*

14. Recall that given two real numbers a and b, by $a \vee b$ (respectively $a \wedge b$) we mean $\max\{a, b\}$ (respectively $\min\{a, b\}$).
15. See Diekmann et al [15, Ch. 1] for more details about the characteristic equation associated to a general linear DDE.

4. *Let a_1 and a_2 be two functions in $L^2([-d,0];\mathbb{R}^+)$ satisfying Hypothesis 3.2 and let ξ_1 and ξ_2 be the corresponding maximal roots.*

$$\int_{-d}^{s} a_1(r)dr \leq \int_{-d}^{s} a_2(r)dr, \quad \forall s \in [-d,0] \quad \Longrightarrow \quad \xi_1 \geq \xi_2.$$

Since $\int_{-d}^{0} a_1(r)dr = \int_{-d}^{0} a_2(r)dr = 1$, this is true in particular if a_1 is increasing and a_2 is decreasing. □

As one can infer from the discussion above, ξ is the maximal log-run growth rate of capital which can be obtained by setting $c(t) = 0$, and therefore $y(t) = i(t)$ at each date t. This is formalized in the next proposition.

Proposition 4.3. *Let $(k_0, i_0) \in H$. Then for some $\varepsilon > 0$, it holds the following.*

$$k^M_{(k_0,i_0)}(t) = \alpha_0 e^{\xi t} + o(e^{(\xi-\varepsilon)t}),$$

where α_0 is a coefficient depending on (k_0, i_0) and

$$\lim_{t\to\infty} \left| \frac{o(e^{(\xi-\varepsilon)t})}{e^{(\xi-\varepsilon)t}} \right| = 0.$$

We can now study the optimal control. Concerning the existence, it will be proved in the Section 5 through the solution of the HJB equation [16]. In this section, we focus on uniqueness and we prove that the optimal control, whenever it exists, it is unique. This result is proved in the following proposition.

Proposition 4.4 (Uniqueness of Optimal Paths). *Let $(k_0, i_0) \in H$ and assume that $V(k_0, i_0)$ is finite. There exists at most one optimal control for the problem* (**P**) *with the initial datum (k_0, i_0).*

Now we study some properties of the value function which was defined in (3.5) on the points of H^+ with the agreement that $V(k_0, i_0) = -\infty$ if $\mathcal{I}_{(k_0,i_0)} = \emptyset$. However, to work on the Hilbert space H [17], we define V on the whole space, i.e. also for initial data with negative capital, even though this case has no economic interest. For initial data $k_0 \leq 0$ the set of admissible strategies is trivially empty, so $V = -\infty$ over $(H^+)^c$. On the other hand, as we will see, if ρ is large enough, then $V > -\infty$ in H^{++}. Hence, letting

$$dom(V) \stackrel{def}{=} \{(k_0, i_0) \in H \mid V(k_0, i_0) > -\infty\},$$

we have the inclusions

$$H^{++} \subset dom(V) \subset H^+$$

In order to guarantee finiteness to the problem from now on we assume the following restriction on parameters.

Hypothesis 4.5. *The parameters are set such as $\rho > \xi(1 - \sigma)$.*

This assumption is the counterpart of the restriction on parameters $A(1-\sigma) < \rho$ in the standard AK model without time-to-build. It requires that the level of technology (i.e. the maximal growth rate of capital) has to be sufficiently low to avoid unbounded utility. In our model, the maximal growth rate of capital is no more A but rather $\xi < A$. The last inequality reflects the "inefficiency" due to the presence of delays in the formation of new capital stock.

Proposition 4.6 (Finiteness and homogeneity of the value function).

1. *$V(k_0, i_0)$ is finite for all $(k_0, i_0) \in H^{++}$. In particular:*
   a) *if $\sigma \in (0,1)$, then $0 \leq V(k_0, i_0) < +\infty$.*

16. Alternatively we could have used tools of Convex and Functional Analysis to prove it directly.
17. This is important since to solve the problem we pass through another problem whose set of initial data includes also the case of negative capital.

b) If $\sigma > 1$, then $-\infty < V(k_0, i_0) \leq 0$.

2. *The set where V is finite is a cone of H and V is homogeneous of degree $(1-\sigma)$ therein:*

$$V\left(\alpha(k_0, i_0)\right) = \alpha^{1-\sigma} V(k_0, i_0), \qquad \forall \alpha > 0.$$

Finally we conclude this section with some information on the asymptotic behavior of capital paths. Let us recall some known facts from the theory of DDEs. Let us consider the so called fundamental solution to the equation for the maximal growth, i.e. the function γ solving the DDE

$$\begin{cases} \gamma'(t) = A\int_{-d}^{0} a(r)\gamma(t+r)dr, & t \geq 0, \\ \\ \gamma(0) = 1, \quad \gamma(s) = 0, \ s \in [-d, 0). \end{cases} \tag{4.5}$$

Notice that, since $a(\cdot) \geq 0$, we have

$$\gamma(t) > 0, \qquad \forall t \geq 0. \tag{4.6}$$

From Diekmann et al [15, Th. 5.4, p. 34] and using the fact that ξ is the solution to (4.2) with the highest real part (as proved in Proposition 4.2), we have, for some $\alpha_\xi > 0$ and every $\varepsilon > 0$,

$$\gamma(t) = \alpha_\xi e^{\xi t} + o(e^{(\xi-\varepsilon)t}), \tag{4.7}$$

where $\lim_{t\to\infty} \left| \frac{o(e^{(\xi-\varepsilon)t})}{e^{(\xi-\varepsilon)t}} \right| = 0$.

Let $(k_0, i_0) \in H$ and $i \in \mathcal{I}_{(k_0,i_0)}$, and consider the corresponding consumption strategy

$$c(t) = Ak_{(k_0,i_0),i}(t) - i(t) \geq 0. \tag{4.8}$$

Clearly we can rewrite the evolution of $k_{(k_0,i_0),i}$ in terms of c as

$$k'_{(k_0,i_0),i}(t) = \int_{-d}^{(-d)\vee(-t)} a(r) i_0(t+r)dr + \int_{(-d)\vee(-t)}^{0} a(r)(Ak_{(k_0,i_0),i}(t+r) - c(t+r))dr, \quad t \geq 0.$$

Hence

$$k'_{(k_0,i_0),i}(t) = \int_{-d}^{0} a(r)(Ak_{(k_0,i_0),i}(t+r) - c(t+r))dr, \quad t \geq d.$$

Then setting

$$\bar{k}_0 := k_{(k_0,i_0),i}(d), \quad \bar{\imath}_0(s) := i(s+d), \qquad s \in [-d, 0], \tag{4.9}$$

the variation of constants formula (see Hale and Lunel [25], Ch. 6, p. 170) allows to write $k_{(k_0,i_0),i}$ in terms of $k^M_{(\bar{k}_0,\bar{\imath}_0)}$ (see the definition of k^M in Proposition 4.1) and γ as

$$k_{(k_0,i_0),i}(t) = k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d) - \int_d^t \gamma(t-s)ds \int_{-d}^{0} a(r)c(s+r)dr, \qquad t \geq d. \tag{4.10}$$

Lemma 4.7. *Let $(k_0, i_0) \in H$, $i \in \mathcal{I}_{(k_0,i_0)}$ and c as in (4.8). Then*

$$\exists \lim_{t\to+\infty} e^{-\xi t} \int_d^t \gamma(t-s)ds \int_{-d}^{0} a(r)c(s+r)dr \in [0, \bar{\alpha}_0],$$

where $\bar{\alpha}_0$ is the constant of Proposition 4.3 related to the initial data $(\bar{k}_0, \bar{\imath}_0)$ defined in (4.9).

Proposition 4.8. *Let $(k_0, i_0) \in H$, $i \in \mathcal{I}_{(k_0,i_0)}$ and let $\bar{\alpha}_0$ be the constant of Lemma 4.7.*

1. *There exists finite $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i}(t) \in [0, \bar{\alpha}_0]$,*
2. *Let $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i}(t) = \alpha > 0$. Then, for every $\beta \in (0, \alpha)$ there exists $i_\beta \in \mathcal{I}_{(k_0,i_0)}$ such that*
   - $k_{(k_0,i_0),i_\beta}(\cdot) \leq k_{(k_0,i_0),i}(\cdot)$,
   - $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i_\beta}(t) \leq \beta$;

– $J((k_0, i_0); i_\beta) > J((k_0, i_0); i)$.

The result of Proposition 4.8(1) is a nontrivial generalization of a much easier result in the case without delays. In fact, the case without delays has $\gamma(t-s) = e^{\xi(t-s)}$ and so the function $t \mapsto e^{-\xi t} \int_0^t \gamma(t-s)ds$ is clearly monotone, whereas in the delay case this is not true anymore and one has to control the fluctuations of such a function (see the proof of Lemma 4.7).

It is also worth noting that Proposition 4.8(2) has an immediate corollary: if the investment strategy $i^* \in \mathcal{I}_{(k_0,i_0)}$ is optimal, then $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i^*}(t) = 0$. This will be crucial in Section 5.4 to prove the main results in Theorem 5.11 and then find the solution of problem (**P**).

4.2 Methodology

Our problem **(P)** is an optimal control problem with state constraints where the state equation (3.1) is a delay differential equation (DDE). This kind of problems is usually difficult to solve for two reasons. Firstly, these problems are intrinsically infinite dimensional due to the fact that the solution of the state equation (such as equation (3.1)) can be found only specifying an initial condition which is not a point in $\mathbb{R}^n$ but a function (in our case, the initial capital stock and the past history of the investment), which is an element of an infinite dimensional space. Secondly, there are state inequality constraints (in our case **c1** and **c2**).

The dynamic programming approach can be used successfully to solve these problems if a "regular" (i.e. differentiable in a suitable sense) solution of the associated Hamilton-Jacobi-Bellman (HJB) equation can be found and if such solution is indeed the value function V for, at least, a subset of initial data. The first contribution in the economic literature which successfully dealt with an infinite dimensional optimal control problem with state constraint was Fabbri and Gozzi [17], while other more recent contributions are Bambi et al. [5], and Boucekkine et al. [10, 11] [18].

However the presence of the distributed delay term in the state equation, and the absence of the irreversibility constraint make our problem more difficult than those faced in the previous contributions [19]. For this reason we have developed a specific strategy to solve problem **(P)**; this strategy can be summarized in the following steps:

1. We rewrite **(P)** as an equivalent infinite dimensional problem ($\mathbf{P^H}$) (with value function V^H) to which the dynamic programming approach can be applied. This is done in section 5.1.
2. We write the HJB equation associated to ($\mathbf{P^H}$) and we find an explicit solution v. This is done in section 5.2 [20].
3. We show that such solution (as the HJB equation itself) is defined on a larger set than the one of V^H and, through a verification theorem, that it is equal to the value function $\tilde{V}^H$ of another control problem (that we call ($\tilde{\mathbf{P}}^{\mathbf{H}}$)) which is easier to solve. This is addressed in section 5.3.
4. We consider the problem ($\tilde{\mathbf{P}}$) which is the optimal control problem equivalent to ($\tilde{\mathbf{P}}^{\mathbf{H}}$) and we derive its solution through the one found for ($\tilde{\mathbf{P}}^{\mathbf{H}}$). Section 5.4 is dedicated to this, in particular Proposition 5.9.
5. We show, through a delicate analysis of the asymptotic behavior of admissible trajectories that indeed the problems (**P**) and ($\tilde{\mathbf{P}}$) are equivalent on a suitable set of initial data. Section 5.5 contains this results.

We conclude this discussion saying that we are forced to use this quite complicated procedure because the verification theorem for the problem ($\mathbf{P^H}$) cannot be proved directly due to the absence of the irreversibility constraints.

18. Fabbri and Gozzi [18] is an extended version of [17] which provides more details on the techniques and on the literature.
19. In particular, the proof that the solution found for the HJB equation is indeed the value function is much more difficult without an irreversibility constraint.
20. In this section, Remark 5.4 contains additional details on the procedure used to solve the problem.

5 Solution of the Optimal Control Problem

In this section we apply the methodology just described. Each subsection corresponds to one of the steps required by our strategy to solve the problem.

5.1 Rewriting (P) in Infinite Dimension

To rewrite our control problem with delay as a control problem without delay in infinite dimension we define a new state variable (which we call *structural state*) and find the state equation that it satisfies. To accomplish this task we first write the DDE (3.1) in a more formal way, defining suitable functions and operators.

Our infinite dimensional setting is represented by the Hilbert space $H \stackrel{def}{=} \mathbb{R} \times L^2([-d,0];\mathbb{R})$ as defined in Appendix B. In order to write an infinite dimensional differential abstract equation satisfied by the new state variable in the Hilbert space H, we introduce some operators and their adjoints.

Let us define the linear functional $\mathcal{C} \in H^*$ as

$$\psi \mapsto \mathcal{C}(\psi^0, \psi^1) \stackrel{def}{=} \int_{-d}^{0} a(r)\psi^1(r)dr = \langle a, \psi^1 \rangle_{L^2([-d,0];\mathbb{R})}, \qquad \psi \in H.$$

One immediately sees that the adjoint $\mathcal{C}^* : \mathbb{R} \to H$ is the operator defined by

$$\mathcal{C}^*(s) = (0, sa), \quad s \in \mathbb{R},$$

since, for $\psi \in H$,

$$\langle s\,, \mathcal{C}\psi \rangle_{\mathbb{R}} = s\langle a, \psi^1 \rangle_{L^2([-d,0];\mathbb{R})} = s\int_{-d}^{0} a(r)\psi^1(r)dr = \langle \psi, \mathcal{C}^* s \rangle_H.$$

Let us $\mathcal{D}$ denote the derivative operator in the Sobolev space $W^{1,2}([-d,0];\mathbb{R})$. Let $\mathcal{B} : D(\mathcal{B}) \subset H \to H$ be the closed densely defined unbounded linear operator on H defined by

$$(\psi^0, \psi^1) \mapsto \mathcal{B}(\psi^0, \psi^1) \stackrel{def}{=} (0, \mathcal{D}\psi^1),$$

where

$$D(\mathcal{B}) \stackrel{def}{=} \{(\psi^0, \psi^1) \in H \ : \ \psi^1 \in W^{1,2}([-d,0];\mathbb{R}), \psi^0 = \psi^1(0)\}.$$

It is well known that $\mathcal{B}$ generates a strongly continuous semigroup on H, whose explicit expression is

$$S_{\mathcal{B}}(t)\psi = (\psi_0, \psi_0 \mathbf{1}_{[0,+\infty)}(t+\cdot) + \psi_1(t+\cdot)\mathbf{1}_{(-\infty,0)}(t+\cdot)), \quad \psi = (\psi_0, \psi_1) \in H.$$

The adjoint operator of $\mathcal{B}$ is the operator (see e.g. Vinter [43] and Vinter and Kwong [44])

$$\phi = (\phi^0, \phi^1) \mapsto \mathcal{B}^*\phi = \left(\phi^1(0), -\mathcal{D}\phi^1\right), \qquad \phi \in D(\mathcal{B}^*).$$

where

$$D(\mathcal{B}^*) = \{\phi = (\phi^0, \phi^1) \in H \ : \ \phi^1 \in W^{1,2}([-d,0];\mathbb{R}),\ \phi^1(-d) = 0\} \subset H.$$

The operator $\mathcal{B}^*$ generates the strongly continuous semigroup $S_{\mathcal{B}^*}$ on H, whose explicit expression, computable starting from the expression of $\mathcal{B}$, is given by

$$S_{\mathcal{B}^*}(t)\varphi = \left(\varphi^0 + \int_{(-t)\wedge(-d)}^{0} \varphi^1(r)dr,\ \varphi^1(\cdot - t)\mathbf{1}_{[-d,0]}(\cdot - t)\right), \qquad \varphi = (\varphi^0, \varphi^1) \in H.$$

Given $i \in L^2_{loc}([0,+\infty);\mathbb{R})$, $z_0 \in H$, we consider the abstract equation in H

$$\begin{cases} Y'(t) = \mathcal{B}^* Y(t) + \mathcal{C}^* i(t), & t > 0, \\ Y(0) = z_0. \end{cases} \tag{5.1}$$

We will use two concepts of solution to (5.1), that in our case coincide. For details we refer to Li and Yong [32, Ch. 2, Sec. 5].

Definition 5.1. *1. We call mild solution of* (5.1) *the function* $Y \in C^0([0,+\infty);H)$ *defined as*

$$Y(t) \stackrel{def}{=} S_{\mathcal{B}^*}(t)z_0 + \int_0^t S_{\mathcal{B}^*}(t-r)\,\mathcal{C}^* i(r)\,dr\ , \quad t \geq 0,$$

where the integral above is understood as Bochner integral of H-valued functions.

2. We call weak solution of (5.1) *a function* $Y \in C^0([0,+\infty);H)$ *such that*

$$\langle Y(t), h\rangle_H = \langle z_0, h\rangle_H + \int_0^t \langle Y(r), Bh\rangle_H dr + \int_0^t \langle \mathcal{C}^* i(r), h\rangle_H dr, \quad \forall t \geq 0, \quad \forall h \in D(\mathcal{B}). \tag{5.2}$$

We notice that (5.2) can be rewritten as

$$\langle Y(t), h\rangle_H = \langle z_0, h\rangle_H + \int_0^t \langle Y(r), Bh\rangle_H dr + Ch\int_0^t i(r)dr, \quad \forall t \geq 0, \quad \forall h \in D(\mathcal{B}). \tag{5.3}$$

From now on we denote by $Y_{z_0,i}(\cdot)$ the mild solution of (5.1) in H. We notice that the definition of mild solution is the infinite-dimensional version of the variation of constant formula and, by definition the mild solution exists and it is unique. By a well-known result (see Li and Yong [32, Ch. 2, Prop. 5.2]), the mild solution is also the (unique) weak solution.

Now we want to connect the infinite dimensional differential problem defined above with the original problem in DDE form. For that, let us introduce now the bounded linear operator

$$F : L^2([-d,0];\mathbb{R}) \longrightarrow L^2([-d,0];\mathbb{R}),$$

defined on $f \in L^2([-d,0];\mathbb{R})$ as

$$[Ff](s) \stackrel{def}{=} \int_{-d}^s a(r)f(r-s)dr, \qquad s \in [-d,0].$$

By using Hölder's inequality, straightforward computations show that

$$\|f * a\|_{L^2([-d,0];\mathbb{R})} \leq \|a\|_{L^2([-d,0];\mathbb{R})}\|f\|_{L^2([-d,0];\mathbb{R})},$$

which shows that F is bounded. Consider now the bounded linear operator

$$\mathcal{Q} : H \longrightarrow H, \qquad (r,f) \mapsto (r, Ff).$$

Given $t \geq 0$, define

$$\tilde{\imath}_t : [-d,0] \to \mathbb{R}; \qquad \tilde{\imath}_t(s) \stackrel{def}{=} \tilde{\imath}(t+s), \qquad s \in [-d,0].$$

The link between (3.1) and (5.1) is provided by the following.

Theorem 5.2. *Let* $(k_0, i_0) \in H$, $i \in L^2_{\text{loc}}([0,+\infty);\mathbb{R})$. *Set*

$$z_0 \stackrel{def}{=} \mathcal{Q}(k_0,i_0) \in H; \qquad Y(t) \stackrel{def}{=} Y_{z_0,i}(t), \quad k(t) \stackrel{def}{=} k_{(k_0,i_0),i}(t), \quad t \geq 0.$$

Then

$$Y(t) = \mathcal{Q}(Y^0(t), \tilde{\imath}_t), \qquad \forall t \geq 0.$$

and

$$Y^0(t) = k(t), \qquad \forall t \geq 0.$$

We are now ready to reformulate our optimal control problem (**P**) in the space H. For a given $z_0 \in H$, the new set of admissible controls is

$$\mathcal{I}^H_{z_0} \stackrel{def}{=} \{i \in L^2_{\text{loc}}([0,+\infty);\mathbb{R}) \ :\ \ Y^0_{z_0}(t) > 0\ \ \forall t \geq 0, \ \ i(t) \leq AY^0_{z_0,i}(t) \text{ for a.e. } t \geq 0\}.$$

The objective functional over $i \in \mathcal{I}^H_{z_0}$ is

$$J^H(z_0; i) \stackrel{def}{=} \int_0^\infty e^{-\rho t} \frac{(AY^0_{z_0,i}(t) - i(t))^{1-\sigma}}{1-\sigma} dt, \tag{5.4}$$

and the value function in this setting is the function

$$V^H : H \longrightarrow \mathbb{R}, \qquad V^H(z_0) \stackrel{def}{=} \sup_{i \in \mathcal{I}^H_{z_0}} J^H(z_0; i),$$

with the convention $\sup \emptyset = -\infty$. This problem will be called problem $(\mathbf{P}^H)$.
Due to Theorem 5.2 the connection between $(\mathbf{P})$ and $(\mathbf{P}^H)$ is the following: let $(k_0, i_0) \in H$ and set $z_0 = \mathcal{Q}(k_0, i_0)$; then

(i) $\mathcal{I}_{(k_0,i_0)} = \mathcal{I}^H_{z_0}$; (ii) $J^H(z_0; i) = J((k_0, i_0); i)$, $\forall i \in \mathcal{I}_{(k_0,i_0)} = \mathcal{I}^H_{z_0}$; (iii) $V^H(z_0) = V(k_0, i_0)$.

Remark 5.3. *The set $\mathcal{Q}(H) \subset H$ is the subset of initial data in the Hilbert setting corresponding to the initial data in the DDE setting. We notice that it is possible to prove that $\mathcal{Q}(H)$ is dense, but not closed in H. In particular $\mathcal{Q}(H) \neq H$, so the problem we have defined in H contains more initial data with respect to the ones coming from the DDE setting, which are represented as points of $\mathcal{Q}(H)$.*

5.2 Solving the HJB Equation associated to $(\mathbf{P}^H)$

We are going to study the problem by the dynamic programming approach in infinite dimension. The core of the dynamic programming approach to control problems is represented by the so called HJB equation that we are going to define for our problem.

Let $y = (y^0, y^1) \in H$. Define

$$E \stackrel{def}{=} \{(y, P, i) \in H \times D(\mathcal{B}) \times \mathbb{R} : \quad i \leq Ay^0\}.$$

On the set E, we define the function *current value Hamiltonian* $\mathcal{H}_{CV} : E \longrightarrow \overline{\mathbb{R}}$, as

$$\mathcal{H}_{CV}(y, P, i) \stackrel{def}{=} \langle y, \mathcal{B}P \rangle_H + \langle i, \mathcal{C}P \rangle_{\mathbb{R}} + \frac{(Ay^0 - i)^{1-\sigma}}{1-\sigma}.$$

For $\sigma \in (0, 1)$, the map $\mathcal{H}_{CV}$ is well-defined on E. When $\sigma > 1$ the above is not defined in the points in which $Ay^0 = i$. In such points we set then $\mathcal{H}_{CV} = -\infty$. Also we define the *maximum value Hamiltonian* (or simply *Hamiltonian*) of the system as

$$\mathcal{H} : H \times D(\mathcal{B}) \longrightarrow \mathbb{R}, \qquad \mathcal{H} : (y, P) \longmapsto \sup_{i \leq Ay^0} \mathcal{H}_{CV}(y, P, i).$$

The *HJB equation* of our infinite dimensional control problem is then

$$\rho v(y) - \mathcal{H}(y, \nabla v(y)) = 0, \quad y \in H. \tag{5.5}$$

Remark 5.4. *We notice that we have defined the HJB equation in a larger set than the natural set where it should be defined. Indeed, from the state constraint, we know that $V^H(y^0, y^1) = -\infty$ when $y^0 \leq 0$, so it does not make sense to associate an HJB equation to V^H over the set where $y^0 \leq 0$. Nevertheless nothing prevents us to consider the HJB equation to be defined over the whole H, and actually this makes sense: the reason to do that is that in the infinite-dimensional setting the natural constraint for the control problem is not $y^0 > 0$, but a more involved one, which allows also the case $y^0 \leq 0$. What we shall do is to solve the HJB equation above in a set containing also points where $y^0 \leq 0$, and associate to this equation another control problem (with different constraints) with value function $\tilde{V}^H$. Then we do the inverse path: we rephrase this new infinite-dimensional problem in the DDE setting and state its equivalence with a DDE control problem with a new constraint. This problem will have a value function $\tilde{V}$, which is in principle different from the original value function V. But at the end we show that, for some initial data (the ones we are interested in), we have actually the equality $\tilde{V} = V$. This will provide the solution of the original problem for such data.*

Making more explicit the expression of $\mathcal{H}$ in a specific case, we notice that if $(\mathcal{C}P)^{-1/\sigma} > 0$, then the unique maximum point of $\mathcal{H}_{CV}(y, P; \cdot)$ over $(-\infty, Ay^0]$ is

$$i^{MAX} = Ay^0 - (\mathcal{C}P)^{-1/\sigma} > 0. \tag{5.6}$$

It follows that

$$(\mathcal{C}P)^{-1/\sigma} > 0 \implies \mathcal{H}(y, P) = \langle Y, \mathcal{B}P \rangle_H + Ay^0(\mathcal{C}P) + \frac{\sigma}{1-\sigma}(\mathcal{C}P)^{\frac{\sigma-1}{\sigma}}, \tag{5.7}$$

so the HJB equation (5.5) can be rewritten in this case as

$$\rho v(y) - \langle y, \mathcal{B}\nabla v(y) \rangle_H + Ay^0(\mathcal{C}\nabla v(y)) + \frac{\sigma}{1-\sigma}(\mathcal{C}\nabla v(y))^{\frac{\sigma-1}{\sigma}} = 0. \tag{5.8}$$

Definition 5.5. *Let $\Theta \subset H$ be open. A function $v \in C^1(\Theta; \mathbb{R})$ satisfies the HJB equation (5.5) in Θ if*

$$(y, \nabla v(y)) \in G^+, \qquad \rho v(y) - \mathcal{H}\big(y, \nabla v(y)\big) = 0 \quad \forall y \in \Theta.$$

We introduce now some objects that will be used in the sequel:

$$w \in L^2([-d, 0]; \mathbb{R}^+), \quad w(r) \stackrel{def}{=} e^{\xi r}, \quad r \in [-d, 0],$$

where ξ the real positive solution of the characteristic equation (4.3);

$$\varphi \stackrel{def}{=} (1, w) \in D(\mathcal{B}) \subset H; \tag{5.9}$$

We note that

$$\mathcal{B}\varphi = (0, \xi w), \qquad \mathcal{C}\varphi = \int_{-d}^{0} a(r) e^{\xi r} dr = \xi / A. \tag{5.10}$$

Moreover we define

$$\Theta \stackrel{def}{=} \{ y \in H \ : \ \langle \varphi, y \rangle_H > 0 \} \subset H;$$

$$\nu \stackrel{def}{=} \frac{\rho - \xi(1-\sigma)}{\sigma} \frac{A}{\xi}. \tag{5.11}$$

where $\nu > 0$, since Hypothesis 4.5. It is now possible to present an explicit solution of the HJB equation (5.5) in Θ.

Proposition 5.6. *The function*

$$v : \Theta \longrightarrow \mathbb{R}; \qquad v(y) \stackrel{def}{=} \alpha \langle \varphi, y \rangle_H^{1-\sigma}, \qquad \text{where} \qquad \alpha \stackrel{def}{=} \nu^{-\sigma} \frac{1}{(1-\sigma)} \frac{A}{\xi}, \tag{5.12}$$

belongs to $C^1(\Theta)$ and is a solution to the HJB equation (5.5) in Θ in the sense of Definition 5.5.

In the standard AK optimal growth model without time-to-build, which is indeed the special case $d \to 0$ of the problem studied in this paper, it is trivial to show that the solution to the corresponding HJB equation is actually $(1-\sigma)$-homogeneous, i.e. $v(k_0) = \frac{(r-g)^{-\sigma}}{1-\sigma} k_0^{1-\sigma}$. With a finite strictly positive d, the problem is infinite dimensional and the role of capital k is now played by $\langle \varphi, y \rangle_H$, which can be interpreted as the equivalent concept of capital in the case with time-to-build [21]. A similar equivalence was found and discussed extensively in Fabbri and Gozzi [17] in a vintage capital model with linear technology.

The reason why we expect that the value function (and so the solution of the HJB equation) is of the form of v above comes from the following considerations. Firstly the value function has to be $(1-\sigma)$ homogeneous in the state variable due to the structure of the problem, see Proposition 4.6; secondly $\langle \varphi, y \rangle_H$ must be connected linearly with the amount of capital.

To prove that v is the value function we need to prove that the closed loop strategies are admissible. The next subsection is devoted to find parameter's restrictions under which this is indeed the case. It is worth noting untill now that the conditions we will be able to find, are sufficient but not necessary for the closed loop strategies to be admissible.

21. See also Section 5.4 where $\langle \varphi, y \rangle_H$ is explicitly expressed in terms of the economic variables.

5.3 Verification Theorem and Optimal Feedbacks for the Problem $(\tilde{\mathbf{P}}^H)$

The goal of this subsection is to show that the solution v found in Proposition 5.6 coincides indeed with the value function of the associated auxiliary control problem $(\tilde{\mathbf{P}}^H)$. As usual this is done by a verification theorem passing through the study of the so called closed loop equation. From expression (5.6) we see that the candidate optimal feedback map is the linear map

$$\Phi : \Theta \to \mathbb{R}, \qquad \Phi y = Ay^0 - (\mathcal{C}\nabla v(y))^{-1/\sigma} = Ay^0 - \nu\langle \varphi, y\rangle_H, \tag{5.13}$$

and the associated closed loop equation is

$$\begin{cases} Y'(t) = \mathcal{B}^* Y(t) + \mathcal{C}^* \Phi Y(t), \quad t \geq 0, \\ Y(0) = z_0. \end{cases} \tag{5.14}$$

This equation in the space H admits a unique weak solution (see Li and Yong [32]) in the sense that there exists a unique function $Y^*_{z_0} \in C^0([0, +\infty); H)$ such that

$$\langle Y^*_{z_0}(t), h\rangle_H = \langle h, z_0\rangle_H + \int_0^t \langle Bh, Y^*_{z_0}(r)\rangle_H dr + \int_0^t \langle \mathcal{C}^*\Phi Y^*_{z_0}(r), h\rangle_H dr, \quad \forall t \geq 0, \quad \forall h \in D(\mathcal{B}), \tag{5.15}$$

i.e.

$$\langle Y^*_{z_0}(t), h\rangle_H = \langle h, z_0\rangle_H + \int_0^t \langle Bh, Y^*_{z_0}(r)\rangle_H dr + \mathcal{C}h \int_0^t \Phi Y^*_{z_0}(r), dr, \quad \forall t \geq 0, \quad \forall h \in D(\mathcal{B}). \tag{5.16}$$

Consider now this auxiliary problem that we call $(\tilde{\mathbf{P}}^H)$. For a given $z_0 \in H$, the new set of admissible controls is

$$\tilde{\mathcal{I}}^H_{z_0} \overset{def}{=} \{i \in L^2_{\text{loc}}([0, +\infty); \mathbb{R}) \ : \ Y_{z_0}(t) \in \Theta \ \ \forall t \geq 0, \ \ i(t) \leq AY^0_{z_0,i}(t) \text{ for a.e. } t \geq 0\}.$$

The objective functional over $i \in \tilde{\mathcal{I}}^H_{z_0}$ is as in (5.4) and the value function is

$$\tilde{V}^H : H \longrightarrow \mathbb{R}, \qquad \tilde{V}^H(z_0) \overset{def}{=} \sup_{i \in \tilde{\mathcal{I}}^H_{z_0}} J^H(z_0; i),$$

with the convention $\sup \emptyset = -\infty$.

Proposition 5.7. *Let $z_0 \in \Theta$.*

1. *For every $i \in L^2_{loc}([0, +\infty); \mathbb{R})$ be such that $i(t) \leq AY^0_{z_0,i}(t)$ for a.e. $t \geq 0$, we have*

$$\langle \varphi, Y_{z_0,i}(t)\rangle_H \leq \langle \varphi, z_0\rangle_H \, e^{\xi t}, \qquad \forall t \geq 0.$$

2. *For the weak solution $Y^*_{z_0}$ to the closed loop equation* (5.13) *we have*

$$\langle \varphi, Y^*_{z_0}(t)\rangle_H = \langle \varphi, Y^*_{z_0}(0)\rangle_H \, e^{gt} = \langle \varphi, z_0\rangle_H \, e^{gt}, \qquad \forall t \geq 0,$$

where

$$g := \xi\left(1 - \frac{\nu}{A}\right) = \frac{\xi - \rho}{\sigma}. \tag{5.17}$$

In particular, the solution of (5.14) *remains in* Θ [22].

Theorem 5.8 (**Verification Theorem for** $(\tilde{\mathbf{P}}^H)$)**.** *Let v be the function defined in Proposition 5.6.*

1. *$\tilde{V}^H = v$ on Θ.*
2. *Given $z_0 \in \Theta$, the control $i^*_{z_0}(t) := \Phi Y^*_{z_0}(t)$, $t \geq 0$, is optimal for $(\tilde{\mathbf{P}}^H)$ starting at z_0, i.e. $J^H(z_0; i^*_{z_0}) = \tilde{V}^H(z_0)$.*

22. Note the analogy between the first claim of Proposition 5.7 and the statement of Proposition 4.3: Proposition 5.7(1) estimates the maximal growth rate of $\langle \varphi, Y(t)\rangle_H$ by ξ as well as Proposition 4.3 estimates the maximal growth rate of $k(t)$ by ξ.

5.4 The Problem $(\tilde{\mathbf{P}})$ and its Solution

In the previous subsection we have solved $(\tilde{\mathbf{P}}^H)$ over Θ. However $(\tilde{\mathbf{P}}^H)$ is associated to a problem different from our original problem $(\mathbf{P})$. Indeed, let $(k_0, i_0) \in H$, let $z_0 = \mathcal{Q}(k_0, i_0)$, let $i \in L^2_{loc}([0, +\infty); \mathbb{R})$. Then, by Theorem 5.2, we have the equality

$$\langle \varphi, Y_{z_0,i}(t) \rangle_H = k_{(k_0,i_0),i}(t) + \int_{-d}^{0} dr e^{\xi r} \int_{-d}^{r} a(s) \tilde{\imath}(t+s-r) ds, \quad \forall t \geq 0.$$

Thus, defining the variable **equivalent capital** as

$$k^{eq}_{(k_0,i_0);i}(t) \stackrel{def}{=} k_{(k_0,i_0),i}(t) + \int_{-d}^{0} dr e^{\xi r} \int_{-d}^{r} a(s) \tilde{\imath}(t+s-r) ds, \quad t \geq 0, \tag{5.18}$$

we can rephrase $(\tilde{\mathbf{P}}^H)$ in the DDE setting by modifying the set of admissible strategies as follows:

$$\tilde{\mathcal{I}}_{(k_0,i_0)} \stackrel{def}{=} \{ i \in L^2_{\text{loc}}([0,+\infty); \mathbb{R}) \ : \ k^{eq}_{(k_0,i_0);i}(t) > 0 \ \ \forall t \geq 0, \ \ i(t) \leq A k_{(k_0,i_0),i}(t) \text{ for a.e. } t \geq 0 \}.$$

Indeed, denoting by $(\tilde{\mathbf{P}})$ the control problem

$$V(k_0, i_0) \stackrel{def}{=} \sup_{i \in \tilde{\mathcal{I}}_{(k_0,i_0)}} J((k_0, i_0); i),$$

with the usual convention $\sup \emptyset = -\infty$, we have

$$\text{(i)} \ \tilde{\mathcal{I}}_{(k_0,i_0)} = \tilde{\mathcal{I}}^H_{z_0}; \quad \text{(ii)} \ J^H(z_0; i) = J((k_0, i_0); i), \ \forall i \in \tilde{\mathcal{I}}_{(k_0,i_0)} = \tilde{\mathcal{I}}^H_{z_0}; \quad \text{(iii)} \ \tilde{V}^H(z_0) = \tilde{V}(k_0, i_0). \tag{5.19}$$

Let $(k_0, i_0) \in H$ and assume that $z_0 \stackrel{def}{=} \mathcal{Q}(k_0, i_0) \in \Theta$. Given $i \in \tilde{\mathcal{I}}_{(k_0,i_0)}$ we define the associated consumption process

$$c(t) \stackrel{def}{=} A k_{(k_0,i_0),i}(t) - i(t), \quad t \geq 0. \tag{5.20}$$

We now use Theorem 5.8 to deduce explicitly the closed loop formula and the closed loop equation for problem $(\tilde{\mathbf{P}})$ from the corresponding ones for $(\tilde{\mathbf{P}}^H)$.

Let us consider also the optimal investment strategy $i^*_{z_0} \in \tilde{\mathcal{I}}^H_{z_0} = \tilde{\mathcal{I}}_{(k_0,i_0)}$ for $(\tilde{\mathbf{P}}^H)$ provided by Theorem 5.8. Calling

$$k^*_{(k_0,i_0)}(\cdot) \stackrel{def}{=} k_{(k_0,i_0),i^*_{z_0}}(\cdot), \tag{5.21}$$

by Theorem 5.2, we have $k^*_{(k_0,i_0)}(\cdot) = (Y^*_{z_0})^0(\cdot)$. So, setting

$$\Lambda_{(k_0,i_0)} \stackrel{def}{=} \nu \langle \varphi, z_0 \rangle_H = \nu \left(k_0 + \int_{-d}^{0} dr \, e^{\xi r} \int_{-d}^{r} a(s) i_0(s-r) ds \right), \tag{5.22}$$

we see that $i^*_{z_0}(t)$ can be rewritten as

$$i^*_{z_0}(t) = A k^*_{(k_0,i_0)}(t) - \Lambda_{(k_0,i_0)} e^{gt} \stackrel{def}{=} i^*_{(k_0,i_0)}(t), \quad t \geq 0. \tag{5.23}$$

Taking into account Theorem 5.8, next Proposition can be proved.

Proposition 5.9. *Let $(k_0, i_0) \in H$ and assume that $z_0 \stackrel{def}{=} \mathcal{Q}(k_0, i_0) \in \Theta$. Then:*

- $\tilde{V}(k_0, i_0) = v(z_0)$.
- $i^*_{(k_0,i_0)}$ *defined in* (5.23) *is an optimal investment strategy for $(\tilde{\boldsymbol{P}})$ starting at (k_0, i_0), and the associated optimal capital path is $k^*_{(k_0,i_0)}$ defined in* (5.21).

Moreover the optimal investment strategy $i^*_{(k_0,i_0)}$ and the optimal capital path $k^*_{(k_0,i_0)}$ can be characterized as solutions of suitable DDEs as follows:

1. *The optimal capital* $k^*_{(k_0,i_0)}$ *is the unique continuously differentiable solution to the DDE:*

$$\begin{cases} k'(t) = \int_{-d}^{(-d)\vee(-t)} a(r)i_0(t+r)dr + A\int_{(-d)\vee(-t)}^{0} a(r)\big(k(t+r) - \Lambda_{(k_0,i_0)}e^{g(t+r)}\big)dr\ , \quad t\geq 0, \\ k(0) = k_0, \quad i_0(s), \quad s\in[-d,0). \end{cases}$$

2. *The optimal investment* $i^*_{(k_0,i_0)}$ *is the unique continuously differentiable solution to the the DDE*

$$\begin{cases} i'(t) = A\int_{-d}^{0} a(r)i(t+r)dr - g\Lambda_{(k_0,i_0)}e^{gt}, \quad t\geq 0 \\ i(0) = Ak_0 - \Lambda_{(k_0,i_0)}, \quad i(s) = i_0(s), \quad s\in[-d,0). \end{cases} \tag{5.24}$$

We notice the optimal consumption path $c^*_{(k_0,i_0)} \stackrel{def}{=} Ak^*_{(k_0,i_0)} - i^*_{(k_0,i_0)}$ for $(\tilde{\mathbf{P}})$ is exponential: indeed

$$c^*_{(k_0,i_0)}(t) = Ak^*_{(k_0,i_0)}(t) - i^*_{(k_0,i_0)}(t) = \Lambda_{(k_0,i_0)}e^{gt}, \quad t\geq 0. \tag{5.25}$$

Clearly, positive consumption growth requires that $g\geq 0$. In order to guarantee that, we assume, from now on, the following restriction on parameters.

Hypothesis 5.10 (Strictly Positive Consumption Growth). *The parameters are set such as* $\xi > \rho$.

This restriction on parameters corresponds to the requirement in the AK model without time-to-build that the interest rate has to be higher than the intertemporal discount factor, namely $r>\rho$. In fact, as previously observed, in the case without delay $\xi\to r$. Notice that Hypothesis 5.10 together with the already stated Hypothesis 4.5 means that we are considering now the case $\rho\in(\xi(1-\sigma),\xi)$.

5.5 Solution of (P) in a Suitable Set of Initial Conditions

In the previous subsection we have solved $(\tilde{\mathbf{P}})$ over the subset of initial data $\mathcal{Q}^{-1}(\Theta)$. Now we want to address the question of solving $(\mathbf{P})$, which is the original problem we began with, at least on a suitable set $\mathbf{S}\subset\mathcal{Q}^{-1}(\Theta)$ of initial data (i.e. initial stock of capital and initial history of the investment). To do that, we look for sufficient conditions which guarantee that, starting from $(k_0,i_0)\in\mathcal{Q}^{-1}(\Theta)$, the optimal control $i^*_{(k_0,i_0)}$ of $(\tilde{\mathbf{P}})$ provided by Proposition 5.9 belongs to $\mathcal{I}_{(k_0,i_0)}$ and, then, $V(k_0,i_0)=\tilde{V}(k_0,i_0)$ and $i^*_{(k_0,i_0)}$ is optimal also for the original problem $(\mathbf{P})$.

Theorem 5.11 (Unique Optimal Control). *Let* $(k_0,i_0)\in\mathbf{S}$ *with*

$$\mathbf{S} \stackrel{def}{=} \Big\{(k_0,i_0)\in H^{++}\ :\ i_0\in W^{1,2}([-d,0);\mathbb{R}),\quad i_0'(t) - gi_0(t)\geq 0 \text{ for a.e. } t\in[-d,0), \\ \int_{-d}^{0} a(s)i_0(s)ds - gk_0 \geq 0, \quad Ak_0 - \Lambda_{(k_0,i_0)}\geq 0\Big\}.$$

and consider the investment strategy $i^*_{(k_0,i_0)}$ *defined in Proposition 5.9. Then* $V(k_0,i_0) = v(\mathcal{Q}(k_0,i_0))$ *and* $i^*_{(k_0,i_0)}$ *is the unique optimal investment strategy for* $(\mathbf{P})$.

Some considerations are useful before moving to the next section. First, it is worth noting that $\mathcal{Q}(H^{++})\subset\Theta$, so $\mathbf{S}\subset\mathcal{Q}^{-1}(\Theta)\cap H^{++}$. Of course, the set $\mathbf{S}$ is not the largest set of admissible initial data. For example, it excludes the case of a history of past investments with reversibility which is indeed allowed by our original model setup.[23] Reversibility does not necessarily violate the initial inequality constraints as it will be shown in the section on quantitative analysis. Yet it is more difficult to characterize analytically a set of initial condition which contains this case. Secondly the optimal investment strategy $i^*_{(k_0,i_0)}$ and the optimal capital path $k^*_{(k_0,i_0)}$ are found in the next section as solutions of the DDEs of Proposition 5.9, while the optimal consumption path $c^*_{(k_0,i_0)}$ is given by (5.25). Lastly, the set $\mathbf{S}$ is empty, and therefore no optimal control exists on this set of initial conditions, if Hypothesis 5.10 does not hold and then $g<0$.

23. Clearly, this doe not preclude the possibility of reversible investment for $t>0$.

6 Balanced Growth Paths

We are now ready to define the couple $\mathbf{E}_b$ of initial data which puts the economy on a balanced growth path from $t = 0$ on. Let us consider any $b \in \mathbb{R}$, and define $\mathbf{E}_b$ as follows:

$$\mathbf{E}_b = (k_0, i_0), \quad \text{where:} \quad i_0(s) \stackrel{def}{=} be^{gs}, \ \text{for a.e. } s \in [-d, 0); \quad k_0 \stackrel{def}{=} \frac{b}{g} \int_{-d}^{0} a(s) e^{gs} ds.$$

Of course, the initial choice of the control variable, and therefore of initial consumption, $\Lambda_{(k_0,i_0)}$, takes a very specific form under this specification of the initial data. This is indeed stated in the following Lemma, which simply uses the general expression of $\Lambda_{(k_0,i_0)}$, as stated in (5.22), and rewrites it when the initial conditions are $\mathbf{E}_b$

Lemma 6.1. *Let* $(k_0, i_0) = \mathbf{E}_b$. *Then* $\Lambda_{(k_0,i_0)} = Ak_0 - b$.

In the next proposition, we prove that the initial data $\mathbf{E}_b$ belongs to $\mathbf{S}$. This result holds under the restriction on parameters imposed in Hypothesis 5.10.

Proposition 6.2. *For any* $b > 0$ *we have that* $\mathbf{E}_b \in \mathbf{S}$ [24].

Using the above results we can now characterize the optimal paths k^*, i^*, c^* when the initial conditions are $\mathbf{E}_b$. We already know from expression (5.25) that the optimal consumption path is purely exponential for each $(k_0, i_0) \in \mathbf{S}$ which includes the case $\mathbf{E}_b$ for Proposition 6.2. Therefore, the optimal consumption path is

$$c^*(t) = (Ak_0 - b)e^{gt} \qquad \forall t \geq 0.$$

In the next proposition we prove under which condition the optimal paths $k^*_{(k_0,i_0)}$ and $i^*_{(k_0,i_0)}$ are purely exponential as well.

Proposition 6.3. *The optimal capital and investment paths are purely exponential if and only if* $(k_0, i_0) = \mathbf{E}_b$ *for any* $b > 0$. *Formally we have that*

$$k^*_{(k_0,i_0)}(t) = k_0 e^{gt}, \qquad i^*_{(k_0,i_0)}(t) = be^{gt}, \qquad \forall t \geq 0.$$

If all the aggregate variables grow at the same rate g, their optimal path is purely exponential, and the inequality constraints are respected then we say that the economy is on a **balanced growth path**. It is also worth noting how the closed loop policy function writes when the economy is on a BGP and compare it with the case without time-to-build. It is easy to verify that such function writes

$$c(t) = \left(A - \frac{g}{\int_{-d}^{0} a(s) e^{gs} ds} \right) k(t) \qquad \forall t \geq 0.$$

Interestingly enough, the derivative of the second term in parenthesis with respect to d is always negative. Therefore, a longer projects' length implies a higher initial consumption and therefore a lower initial investment since at $t = 0$ the initial output is Ak_0 with k_0 exogenously given. Consistently with $g'(d) < 0$ lower investment implies slower capital accumulation and therefore lower growth. Observe also that this mechanism does not pass through different interest rates, which is always $r = A$ independently on d and $a(\cdot)$, but it depends on the resource left unproductive by the different time-to-build specifications. Consistently with this observation the case without time to build implies an initial consumption $c_0 = (A - g)k_0$ which is the lowest among the possible specification of d and $a(\cdot)$ and therefore it implies the highest investment, the fastest accumulation of capital and the highest growth rate.

Moreover, Proposition 6.3 tells us that the economy is on a balanced growth path till the very beginning, i.e. from $t \geq 0$, if and only if it was already there in the past: in this extent, history matters. To

24. In the proof of this proposition we also show that, in the case of zero economic growth, $(k_0, i_0) \in \mathbf{S}$ if and only if $i_0 \equiv 0$.

understand better this result, it is worth to read the restriction on the initial conditions in an alternative way.

Consider an exogenously given k_0, then the economy is on a BGP till $t = 0$ if and only if the initial history of investment is $i_0(s) = \frac{k_0 g}{\int_{-d}^{0} a(s)e^{gs}ds} e^{gs}$ with $s \in [-d, 0)$. Observe that this past history of investment becomes $i_0(0) = gk_0$ in the case without time-to-build (Dirac's delta in 0). As well known, this condition implies an initial "jump" of the control variable in order to put the economy on its BGP till the very beginning. With a generic form of time-to-build, the "jump" is not on initial investment but it is rather a very specific choice of the past history of investment among the admissible ones.

On the other hand, an economy with a past history of the investment, $i_0(s)$ with $s \in [-d, 0)$, set as in $\mathbf{E}_b$, but with a capital stock different from $\frac{b}{g}\int_{-d}^{0} a(s)e^{gs}ds$ but still in the feasible set of initial condition S, will not be on its balanced growth path. Under these initial conditions, the optimal path of investment and capital are no more purely exponential, and converge over time to the balanced growth path. The next section is dedicated to find the explicit form of these optimal paths and to prove that an economy which, generically, starts out of the BGP, meaning that the initial conditions are in S but are not E_b, will converge to it over time by damping fluctuations.

7 Transitional Dynamics

Now we look at optimal trajectories other than balanced growth paths. The following assumption will be done from now on.

Hypothesis 7.1. *Concerning all the complex roots of the characteristic equation* (4.3) *we assume:*

(i) the real part is smaller than g;

(ii) they are simple[25].

Remark 7.2. *It is indeed theoretically viable to provide restrictions on parameters and on the distribution $a(\cdot)$ such that (i) and (ii) hold. However we do not address this issue analytically but rather numerically by checking Hypothesis 7.1 - case by case - in Section 8. Also, (ii) occurs generically and, while not essential, it is useful to simplify the analysis of detrended optimal paths in the subsequent Proposition 7.3.*

Let $\{\lambda_j\}$ and $\{\bar{\lambda}_j\}$ as in Proposition 4.2, item 3. Applying Corollary 6.4 in Diekmann et al [15], the solution of (4.5) can be written as

$$\gamma(t) = \alpha_\xi e^{\xi t} + \sum_{j=1}^{\infty} \left(e^{\lambda_j t} p_j(t) + e^{\bar{\lambda}_j t} \bar{p}_j(t) \right) = \alpha_\xi e^{\xi t} + \sum_{j=1}^{\infty} 2Re\left(p_j(t)e^{\lambda_j t}\right),$$

since $\overline{ae^{\lambda}} = \bar{a}e^{\bar{\lambda}}$ and where $p_j(t)$ are complex polynomial of degree $m_{\lambda_j} - 1$, with m_{λ_j} denoting the multiplicity of the root λ_j. Also, the series above converges uniformly on compact subsets of $(0, +\infty)$. Under Hypothesis 7.1(ii), the polynomials $p_j(\cdot)$ are constants and we denote them simply by p_j, i.e.

$$\gamma(t) = \alpha_\xi e^{\xi t} + \sum_{j=1}^{\infty} 2Re(p_j e^{\lambda_j t}). \tag{7.1}$$

Proposition 7.3. *Consider a $(k_0, i_0) \in \mathbf{S}$. Then the optimal paths are:*

$$c^*_{(k_0,i_0)}(t) = \Lambda_{(k_0,i_0)} e^{gt}, \quad t \geq 0, \tag{7.2}$$

$$k^*_{(k_0,i_0)}(t) = \frac{1}{A}(i^*_{(k_0,i_0)}(t) + c^*_{(k_0,i_0)}(t)), \quad t \geq 0,$$

$$i^*_{(k_0,i_0)}(t) = \alpha_\xi \left(-\frac{\Lambda_{(k_0,i_0)} g}{g - \xi} \right) e^{gt} + \sum_{j=1}^{\infty} p_j e^{\lambda_j t} a_j + p_j e^{gt} b_j, \qquad t \geq 0, \tag{7.3}$$

25. It can be proved that this event generically occurs.

where $\alpha_\xi > 0$ *is the real constant in* (4.7)*,* a_j *and* b_j *are the complex numbers*

$$a_j \stackrel{def}{=} A\Gamma_j(k_0, i_0) - \Lambda_{(k_0,i_0)} + \frac{g\Lambda_{(k_0,i_0)}}{g-\lambda_j}, \qquad b_j \stackrel{def}{=} -\frac{g\Lambda_{(k_0,i_0)}}{g-\lambda_j}, \tag{7.4}$$

and

$$\Gamma_j(k_0, i_0) := k_0 + \int_{-d}^{0} e^{\lambda_j r} \int_{-d}^{r} a(s) i_0(s-r) ds\, dr.$$

Moreover, defining the optimal detrended paths as:

$$k^*_{(k_0,i_0),g}(t) \stackrel{def}{=} e^{-gt} k^*_{(k_0,i_0)}(t)\ , \quad i^*_{(k_0,i_0),g}(t) \stackrel{def}{=} e^{-gt} i^*_{(k_0,i_0)}(t)\ , \quad c^*_{(k_0,i_0),g}(t) \stackrel{def}{=} e^{-gt} c^*_{(k_0,i_0)}(t), \quad t \geq 0,$$

we have that the optimal detrended consumption path, see definition (5.20)*,* $c^*_{(k_0,i_0),g}(t) \stackrel{def}{=} Ak^*_{(k_0,i_0),g}(t) - i^*_{(k_0,i_0),g}(t)$ *is constant and equal to* $\Lambda_{(k_0,i_0)}$ *by* (7.2)*, and there exist positive constants* k_l *and* i_l *such that*

$$\lim_{t\to+\infty} k^*_{(k_0,i_0),g}(t) = k_l, \qquad \lim_{t\to+\infty} i^*_{(k_0,i_0),g}(t) = i_l,$$

where k_l *and* i_l *satisfy the algebraic system*

$$Ak_l - i_l = \Lambda_{(k_0,i_0)}, \qquad i_l \left(\frac{A}{\xi} \int_{-d}^{0} a(\eta) e^{g\eta} d\eta \right) = (A-\nu) k_l.$$

Explicitely

$$k_l = \frac{\Lambda_{(k_0,i_0)} \int_{-d}^{0} e^{gr} a(r) dr}{A \int_{-d}^{0} e^{gr} a(r) dr - g}, \qquad i_l = \frac{g\Lambda_{(k_0,i_0)}}{A \int_{-d}^{0} e^{gr} a(r) dr - g}.$$

From these reasonings, it is clear that countries with the same past history of investment and initial stock of capital behave very differently if they differs in the projects' structure. In fact, different investment distributions over the projects and different projects' length across countries determine different asymptotic growth rates, different balance growth paths and different adjustments over time to them.

8 Quantitative Analysis

In this section, we perform two exercises. The first evaluates the growth rate differentials across countries which are on their respective balanced growth path, as defined in Proposition 6.3, and are identical except for the projects' structure; in fact the projects may be different in length and in term of the investment distributions as defined in (the continuous-time counterpart of) Definition 3.1; the relevance of these two features in affecting the maximal growth rate of capital, ξ, and therefore the growth rate, g, was indeed proved analytically in Propositions 4.2 and 5.17 but their quantitative relevance is assessed in this first exercise. In this context we also propose a welfare evaluation.

The second exercise consists in studying, from a quantitative viewpoint, how the damping fluctuations, which we proved to emerge in Proposition 7.3, are affected by different choices of the projects' length and of the investment's distribution over the projects. To do this assessment, we specify the parameters as in the quantitative exercise on the growth rate differentials, but we also add an initial exogenous shock which makes the economy deviate from its BGP by reducing the initial capital stock of ten percentage points. As observed in Proposition 7.3 the economy will converge back to its BGP by damping fluctuations; therefore our objective is to quantitatively evaluate the output volatility, measured looking at three indicators: the average and maximum absolute deviation from the BGP and the speed of convergence to the BGP. This information will turn out useful to measure how well the different projects' features may explain the negative link between mean output growth and output growth volatility, observed in the data (e.g. Ramey and Ramey [41]).

Before moving to the two exercises we need to clarify how the projects' length, d, and the investment's distribution, $a(r)$, have been chosen. We assign to d a benchmark value of three years. This is slightly higher than the value usually suggested in recent empirical works (e.g. Del Boca et al. [8]); in fact, in our model, the capital stock is broadly defined and includes not only physical capital but also human capital, public infrastructure, etc. which encourages a choice of d larger than the usual setting of one year, for equipment, or two years, for structure. For this reason, we consider a range of values for d, between two and five years. This quite large interval of values accounts for the possible heterogeneity in the length of projects across otherwise similar countries.

The empirical findings on the investment's distributions are rich and sometimes controversial as also explained in the Introduction. In our exercise, we do not focus on a specific contribution but rather we try to account for all the specifications found by the literature even if obtained looking at different countries and using different methodologies. In particular, we consider the following investment's distributions over the projects: Dirac's Delta in 0 (no time to build); Dirac's Delta in $-d$ (i.e. pure gestation lag in investment) and uniform (e.g. Kydland and Prescott [31] among others); decreasing exponential (e.g. Peeters [40]); increasing exponential (e.g. Koeva [29]); U-shaped (e.g. Peeters [40] and Zhou [45]); and hump-shaped (e.g. Altug [2], and Palm et al. [37]).

The decreasing exponential distributions have been introduced through the parametric distribution $a_\mu(r) = \left(\frac{\mu}{e^{\mu d}-1}\right) e^{-\mu r}$ with $\mu > 0$, the increasing exponential through the distribution $a_\mu(r) = \left(\frac{\mu}{1-e^{-\mu d}}\right) e^{\mu r}$ with $\mu > 0$, while a combination of these two parametric distributions has been used to get the hump-shape and U-shape distributions. In all these cases we have properly set the parameter μ to reproduce a specific investment distribution over the projects: for example, in the case of a decreasing exponential distribution, μ was set equal to either 0.3466 or 1.197 to have, respectively, a 75% and 95% of the investment concentrated on the projects which need more than two-years to be completed when the full length of a project is three years. Similarly, when the increasing exponential distribution has been chosen, we have set μ equal to either 0.3466 or 1.197 to have, respectively, a 75% and 95% of the investment concentrated on the projects which need less than two-years to be completed when the full length of a project is three years. In the U-shaped distribution, μ was set to have 50% of the investment allocated to the projects which need more than two years to be completed and finally, in the hump-shaped, μ was set to have 70% of the investment concentrated on the projects requiring less than 2 years to be completed.

Moreover, we have adjusted accordingly the distribution of the investment over the projects when the projects' lengths is different from 3 years.

Finally, all the numerical computations have been done using MATLAB; the section on output volatility has been performed using DDE-BIFTOOL, a MATLAB package developed by Engelborghs and Roose [16].

8.1 First Quantitative Exercise (focus on Economic Growth)

The first quantitative exercise focuses on economies on their respective balanced growth path, see Proposition 6.3, and on their growth rates. The parameters to be decided to perform this assessment are A, ρ, σ, d, and the investment's distribution $a_\mu(r)$. All these parameters enter in the characteristic equation (4.3) and then are relevant to determine the growth rate of the economy, g, as proved in Propositions 4.2 and 5.17. We start considering an economy without time to build − $a_\mu(r)$ is a Dirac's Delta in 0 − and we set $\rho = 0.017$ and $\frac{1}{\sigma} = 0.5$ which are quite standard and non-controversial values for the preference discount rate and the instantaneous intertemporal elasticity of substitution. A choice of the interest rate equal to $R = A = 0.077$ implies a growth rate $g = 0.03$. Then we consider how much the growth rate and welfare are affected by different choices of the delay parameter, d, and of the investment's distribution, $a_\mu(r)$ while keeping unchanged all the other parameters. [26] These values of the growth rates and welfare for the different specifications of the projects' features are reported in Table 1 and Table 3, respectively.

26. It is worth remembering that A is always equal to the real interest rate as observed in the Appendix A, independently on the different projects' features.

Table 1: **Growth Rate, g, on the Balanced Growth Path.**

	$d=2$	$d=3$	$d=4$	$d=5$
Investment's Distributions				
Dirac's Delta in 0	0.03	0.03	0.03	0.03
Increasing Exponential ($\mu = 1.197$)	0.0283	0.0280	0.0279	0.0278
Increasing Exponential ($\mu = 0.3466$)	0.0276	0.0268	0.0262	0.0257
U-Shaped	0.0275	0.0265	0.0252	0.0252
Uniform	0.0274	0.0263	0.0252	0.0243
Hump-Shaped	0.0273	0.0261	0.0253	0.0243
Decreasing Exponential ($\mu = 0.3466$)	0.0271	0.0257	0.0244	0.0231
Decreasing Exponential ($\mu = 1.197$)	0.0265	0.0247	0.0230	0.0215
Dirac's Delta in $-d$	0.02515	0.0233	0.0217	0.0204

Projects' Structure and Growth Rate Differentials

The maximum growth rate differentials are observed when we compare a country with pure investment lags in production – $a_\mu(r)$ is a Dirac's Delta in $-d$ – with another country characterized by time-to-plan – $a_\mu(r)$ is increasing exponential distribution with $\mu = 1.197$. [27] According to our computations the growth differential, due to the different resource distributions over the projects, is around 12.5% when the length of the project is two years. Moreover such differential enlarges to 21.45% when we increase the length of the project to three years. This sharp increase in the growth differential can be immediately explained: the growth rate of the country characterized by time-to-plan is not affected significantly (just around -0.03 percentage points) by the increase in the length of the project because the largest amount of the resources are concentrated on the last stages; on the other hand, in the pure investment lag case all the resources are concentrated at the beginning of the project and therefore a larger amount of resources remains "unproductive" for a longer period of time when the length, d, increases, with a larger negative effect on the growth rate of the economy (around -0.2 percentage points). The growth differentials for the case of $d = 4$ and $d = 5$ years are also computed and they are respectively the 28.6% and the 36.3%. Looking at the income gap after 100 years between two countries, which are identical except the investment distributions, we find that the output of the country having an increasing exponential distribution is around 37% larger than in the other country with a pure investment lag in production when the length of the project is 2 years. This output gap increases to 60%, 86%, and 110% when the length of the project in both countries increase to 3, 4, and 5 years respectively (see Table 2). [28]

Comparisons between other different distributions suggest again how the projects' features may explain a quite significant part of the growth rate differentials across, otherwise identical, countries. Interestingly enough, a comparison of the growth rates when the investment's distribution is hump-shaped and when it is uniform, reveals that the first distribution pins down higher growth rates only when the projects' length is lower or equal than 3 years. Keeping aside this case, a ranking of the distributions in term of the

27. Intuitively the increasing exponential distribution is the distribution "closest" to the Dirac's Delta in 0 (i.e. no time-to-build case), and it indeed converges to it as the resources tends to be concentrated in the last stage of the project. This is the reason why the highest growth rate differential is observed when we compare the time-to-plan economy with a pure-investment lag economy.

28. The income gap has been computed by using the formula $\left(-1 + e^{(\max\{g_i, g_j\} - \min\{g_i, g_j\}) \cdot 100}\right) \cdot 100$ where g_i and g_j indicate respectively the growth rates associated with the investment distribution i and j.

growth rates can be done: given A, ρ, σ, and d, the increasing exponential distribution is characterized by the highest growth rates, followed by the U-shaped distribution, the uniform distribution and the hump-shaped distribution to end with the decreasing exponentials and the Dirac's delta in $-d$, the latter characterized by the lowest growth rate. The robustness of this ranking has been checked for different choices of the parameters σ, ρ, and r. The income gap after 100 years between two countries with different investment's distributions has been computed and reported in Table 2. The comparison has been done focusing on those distributions more often used in the literature and it highlights how differences in the investment's distributions may lead to relevant output differential across countries in the long run.

Table 2: **Income Gap (%) between 2 Countries with Different Investment Distributions after 100 years.**

	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Investment's Distributions				
Incr. Exp. ($\mu = 1.197$) vs. Dirac's Delta $-d$	37.02%	60%	85.9%	109.6%
Uniform vs. Dirac's Delta $-d$	25.2%	35%	41.9%	47.7%
Decr. Exp. ($\mu = 0.3466$) vs. Dirac's Delta $-d$	21.5%	27.1%	31%	31%
Incr. Exp. ($\mu = 1.197$) vs. Decr. Exp. ($\mu = 0.3466$)	12.7%	25.9%	41.9%	60%
Incr. Exp. ($\mu = 1.197$) vs. Uniform	9.4%	18.5%	31%	41.9%

Finally all these findings can be used to shed some light on the growth differentials observed in the data when we compare, for example, Italy with other developed countries. In the period 1990-2012, Italy's average GDP growth rate was around 0.87%, which is almost one percentage point below the Euro Area average of 1.67%, with France, Germany, and United Kingdom growing at an average rate of 1.55%, 1.68%, and 2.15% respectively (World Bank - World Development Indicators). A longer delay in the human capital formation, due for example to the Italian university system, where the average student gets a bachelor degree at around 27 years old in 2001 and 26 in 2009 (see Ruegg [42], and also L'Universita' in Cifre 2009/10 [35] - Gli Studenti graph 2.2.6) against an EU average of around 23 years old, as well as longer delays in the realization of public infrastructures, as documented by the list of the incomplete public projects (see "Elenco Anagrafe Opere Incompiute", Ministero per le Infrastrutture e Transporti), suggest the presence of a longer length of the investment projects in Italy than on average in the other EU countries. Using our quantitative findings, specifically Table 1, we observe that this difference may well explain a relevant part of the growth differential between Italy and the other EU countries and that such differences may be significantly affected if the investment distributions between Italy and the other countries are different.

Projects' Structure and Welfare

The welfare of an economy on its BGP can be easily found by solving the integral of the discounted instantaneous utility:

$$W(d) = -\frac{c_0(d)^{1-\sigma}}{(1-\sigma)[(1-\sigma)g(d)-\rho]}$$

where $W(.)$, $c_0(.)$, and $g(.)$ indicate respectively the welfare, the initial value of consumption and the growth rate of the economy as functions of the projects' length d. Observe also that the term, inside the square brackets, has to be always lower than zero to have bounded utility (i.e. the value function is finite, see Hypothesis 4.5). Consistently with the numerical values in Table 3, the welfare is a positive number

Table 3: **Welfare evaluation when $\sigma = 0.9$ and $\sigma = 2$.**

	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Investment's Distributions				
Increasing Exponential ($\mu = 1.197$)	785.3 (-13.18)	782.5(-13.12)	781.4 (-13.10)	780.9 (-13.08)
Increasing Exponential ($\mu = 0.3466$)	779.2 (-13.03)	771.5 (-12.83)	765.7 (-12.68)	761.2 (-12.56)
U-Shaped	778.1 (-13.01)	768.2 (-12.75)	757.6 (-12.45)	756.9 (-12.44)
Uniform	776.6 (-12.97)	766.5 (-12.70)	757.8 (-12.45)	750.3 (-12.22)
Hump-Shaped	776.2 (-12.95)	765.0 (-12.65)	758.0 (-12.44)	750.5 (-12.21)
Decreasing Exponential ($\mu = 0.3466$)	774.1 (-12.90)	761.7 (-12.56)	750.7 (-12.22)	741.1 (-11.88)
Decreasing Exponential ($\mu = 1.197$)	768.8 (-12.75)	753.4 (-12.30)	740.6 (-11.85)	730.3 (-11.42)
Dirac's Delta in $-d$	757.2 (-12.41)	742.9 (-11.92)	731.9 (-11.48)	723.2 (-11.07)

The welfare values outside the parenthesis refer to the case $\sigma = 0.9$, while those inside to the case $\sigma = 2$.

if and only if $\sigma < 1$. If we now differentiate the welfare function with respect to d, we get

$$\frac{W'(d)}{W(d)} = (1-\sigma)\left[\frac{c_0'(d)}{c_0(d)} + \underbrace{\left(\frac{g'(d)}{\rho - (1-\sigma)g(d)}\right)}_{\equiv \bar{\bar{c}}}\right] \equiv g_W(d)$$

Observe that $\bar{\bar{c}}$ is always negative since $g'(d) < 0$ and the term at the denominator inside the parenthesis is always positive to guarantee bounded utility. Consider the case $\sigma > 1$. According to our numerical simulations reported in Table 3, see values in parenthesis (i.e. case $\sigma = 2$), there is an increase in welfare. Looking at equation (8.1), this is possible only if there is a sufficiently high variation in the initial consumption, formally only if $c_0'(d)$ is positive and sufficiently large. Under these circumstances, $g_W(d) < 0$ and therefore the welfare increases since equal to $W(d) = -|W(0)|e^{g_W(d)d}$.

This reaction of the initial consumption to an increase in d depends on the substitution effect which dominates the income effect. In fact, a higher d implies lower output over time and therefore lower resources to be allocated in consumption and investment; as a result, the households have an incentive to reduce both initial investment and initial consumption (income effect). On the other hand, a longer d implies a disincentive for the households to invest because now the investment is spread over a larger number of projects, some of them taking longer than before, to be completed. This means that a share of current output will be postponed to a further date in the future; therefore the discounted marginal utility associated to the increase in future consumption is lower than in the case with a smaller d. For this reason, the households are less willing to postpone current output when d is larger and they may prefer to invest less and consume more at the beginning (substitution effect).

Consider now the alternative case $\sigma = 0.9$ and an increase in the projects' length d. Since $g'(d) < 0$, and $\bar{\bar{c}} < 0$, a decrease in welfare, as observed in our numerical exercise reported in Table 3, is possible only if there is a sufficiently small variation in the initial consumption, formally only if $c_0'(d)$ is sufficiently small. Under this circumstance, $g_W(d) < 0$ and therefore the welfare decreases since equal to $W(d) = |W(0)|e^{g_W(d)d}$.

Again the adjustment in the initial consumption can be explained in term of income and substitution effect. In particular, a small adjustment as observed in our simulations is coherent with an income effect sufficiently strong when compared to the substitution effect.

It is also worth mentioning that the same ranking of the distributions in term of the growth rates holds when we compare the welfare associated to the different distributions and $\sigma = 0.9$. Therefore the increasing exponential is the distribution which leads to the highest welfare while the Dirac's Delta in $-d$ to the lowest welfare. Also decreasing the length of the project is always welfare enhancing, independently on the choice of the investments' distribution. On the other hand, all these results are reversed when we consider the case with $\sigma = 2$.

Table 4: **Average and maximum absolute deviation from the BGP.**

	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Investment's Distributions				
Dirac's Delta in 0	0	0	0	0
Increasing Exponential ($\mu = 1.197$)	0.3% (0.5%)	0.5%(0.6%)	0.55% (0.6%)	0.6% (0.65%)
Increasing Exponential ($\mu = 0.3466$)	0.5% (0.7%)	0.95% (0.95%)	1.4% (1.1%)	1.85% (1.3%)
Hump-Shaped	0.59% (0.8%)	1.22% (1.18%)	1.8% (1.44%)	2.6% (1.74%)
Uniform	0.6% (0.8%)	1.22% (1.12%)	1.97% (1.44%)	3% (1.75%)
U-Shaped	0.65% (0.72%)	1.27% (1.05%)	2.36% (1.05%)	2.6% (1.45%)
Decreasing Exponential ($\mu = 0.3466$)	0.75% (0.86%)	1.51% (1.3%)	2.58% (1.73%)	3.96% (2.16%)
Decreasing Exponential ($\mu = 1.197$)	0.9% (1%)	2.1% (1.6%)	3.7% (2.2%)	5.86% (2.75%)
Dirac's Delta in $-d$	1.6% (1.5%)	3.3% (2%)	5.5% (2.7%)	8% (3.2%)

The values outside (inside) the parenthesis refer to the average (maximum) deviation.

8.2 Second Quantitative Exercise (focus on Endogenous Fluctuations)

The second quantitative exercise focuses on transitional dynamics. As previously mentioned, the parameters are chosen as in the previous quantitative exercise but we also add an initial exogenous shock which makes the economy deviate from its BGP by reducing the initial capital stock of ten percentage points. More precisely, we consider economies which are identical but the projects' characteristics; each of these economies is assumed to be on its respective balanced growth path, meaning that the initial conditions are exactly $\mathbf{E}_b$. Of course $\mathbf{E}_b$ varies across countries since the differences in the project's characteristics. Also, in a first instance, we set $b = 1$ and we will check later on what happens for different values of b. At $t = 0$, each economy faces an exogenous shock which makes it deviate from its balanced growth path by destroying the 10% of the initial capital. Under our parametrization the past history of the investment and the capital stock after the negative shock are still in the set $\mathbf{S}$ and therefore we know from Proposition 7.3 that each economy will converge by damping fluctuations to its balanced growth path. The output volatility is measured looking at the maximum and average absolute deviation from the BGP and at the speed of convergence. These three indicators have been computed by looking respectively at the following quantities

$$\sup_{t\in(0,T)} \frac{|k^*_{(k_0,i_0),g}(t) - k_l e^{gt}|}{k_l e^{gt}}, \qquad \int_0^T \frac{|k^*_{(k_0,i_0),g}(t) - k_l e^{gt}|}{k_l e^{gt}} dt \qquad and \quad |Re(\lambda_M) - g|$$

These three indicators have been computed for different investment's distributions, $a_\mu(r)$, and projects' length, d, and reported in Tables 4 and 5. Output is said to be more volatile if characterized by a higher maximum and absolute deviation and a lower speed of convergence. Keeping aside the U-shaped and the hump-shaped distributions, we observe that the economy with the projects' investment distribution leading to higher growth rates are also those with higher output volatility. In particular, the same ranking on the investment distributions proposed for the growth rates, holds when we rank the economies from those with lowest to those with highest output volatility.

Therefore these findings show that different investment projects' distribution may explain the negative relation between mean output growth (i.e. the g computed in Table 1) and output volatility documented by several author in the empirical literature (e.g. Ramey and Ramey [41]). The same conclusion can be derived if we look at any specific investment distribution and we increase the project length. In fact, also in this case, the growth rate decreases while the volatility increases.

Interestingly enough our model is also consistent with the Ramey and Ramey finding that the average investment share does not play any role in explaining the negative correlation between mean output growth and output growth volatility. In fact this negative correlation is not affected if we change the average investment share on the BGP, $\frac{b}{A}$. In other words, countries with the same level of technology, A, but different project's features have different negative correlation between mean output growth and output growth volatility, but this correlation does not change if we consider different values of b and therefore different investment shares.

Table 5: **Speed of convergence to the BGP.**

	$d=2$	$d=3$	$d=4$	$d=5$
Investment's Distributions				
Dirac's Delta in 0	∞	∞	∞	∞
Increasing Exponential ($\mu = 1.197$)	3.69	2.61	2.13	1.87
Increasing Exponential ($\mu = 0.3466$)	3.14	2.01	1.48	1.18
Hump-Shaped	3.14	1.91	1.43	1.11
Uniform	2.96	1.81	1.28	0.97
Decreasing Exponential ($\mu = 0.3466$)	2.79	1.66	1.13	0.83
U-Shaped	2.63	1.65	1.08	0.93
Decreasing Exponential ($\mu = 1.197$)	2.49	1.41	0.92	0.66
Dirac's Delta in $-d$	1.79	1.05	0.71	0.53

9 Conclusion

In this paper we have assessed in an endogenous growth model how the investment project's features may affect the growth rate and the transitional dynamics of an economy. The analytical results are tailored to quantify the changes in output growth and output volatility due to different choices of the project's length and of the investment distributions over the projects. Relatively small differences in these features may induce significantly output growth differentials across otherwise identical countries; this finding has implications for policy and empirical work. The policy implication is that countries characterized by long project's length and pure investment lag would increase significantly their output growth by implementing policies acting to reduce the first and abandoning the latter for a different investment distribution. If implemented such policies would also decrease the output volatility. For the purposes of empirical analysis, our quantitative findings encourage empirical studies to collect more information on the projects' features across countries.

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Appendix

A Decentralized Economy

Assume an economy populated by infinitely many atomistic households and firms. The pattern of the ownership right is the following: households own directly the capital stock and the firms while the firms own nothing.

Observe that capital is productive immediately, meaning that $y_t = Ak_t$. Therefore the firms' maximization problem is the usual static one where firms rent the capital stock from the household at each date. The first order condition of this problem pins down the following real interest rate:

$$R_t = A$$

On the other hand the households problem is

$$\max \int_0^\infty e^{-\rho t} \log c_t dt$$

$$s.t. \quad R_t k_t = c_t + i_t \quad and \quad \int_{-d}^0 a_j i_{t-j} dj = \dot{k}_t$$

with k_0 and $i_0(t)$ with $t \in [-d, 0)$ exogenously given.

B Notation

The following notations for functional spaces is based on Brezis [13].

- $L^2([-d,0];\mathbb{R})$ denotes to the space of all functions from $[-d,0]$ to $\mathbb{R}$ that are Lebesgue measurable and square integrable. These functions are identified if they coincide almost everywhere (from now on a.e.) with respect to the Lebesgue measure.
- $L^2_{\text{loc}}([0,+\infty);\mathbb{R})$ denotes the space of all functions from $[0,+\infty)$ to $\mathbb{R}$ that are Lebesgue measurable and square integrable on all bounded intervals. These functions are identified if they coincide a.e. with respect to the Lebesgue measure.
- $W^{1,2}([-d,0];\mathbb{R})$ denotes the space of the functions in $L^2([-d,0];\mathbb{R})$ whose weak first derivative exists and belongs to $L^2([-d,0];\mathbb{R})$ too. These functions admit a (unique) continuous representative.
- $W^{1,2}_{loc}([0,+\infty);\mathbb{R})$ denotes the space of the functions in $L^2_{\text{loc}}([0,+\infty);\mathbb{R})$ whose weak first derivative exists and belongs to $L^2_{\text{loc}}([0,+\infty);\mathbb{R})$ too. These functions admit a (unique) continuous representative
- $C^0([0,+\infty);\mathbb{R})$ and $C^1([0,+\infty);\mathbb{R})$ denote, respectively, the space of continuous and of continuously differentiable functions from $[0,+\infty)$ to $\mathbb{R}$.

Similar definitions are given when $\mathbb{R}$ is replaced by $\mathbb{R}^+ \stackrel{def}{=} [0,+\infty)$: simply, in this case, the functions take values in $\mathbb{R}^+$. Also we define a Hilbert space as

$$H \stackrel{def}{=} \mathbb{R} \times L^2([-d,0];\mathbb{R}),$$

and the subsets of H

$$H^+ \stackrel{def}{=} (0,+\infty) \times L^2([-d,0];\mathbb{R}), \qquad H^{++} \stackrel{def}{=} (0,+\infty) \times L^2([-d,0];\mathbb{R}^+).$$

where given an element $x = (x^0, x^1) \in H$ the scalar product in H is defined as $\langle (x^0,x^1),(y^0,y^1)\rangle_H \stackrel{def}{=} x^0 y^0 + \langle x^1, y^1 \rangle_{L^2([-d,0];\mathbb{R})}$ for all (x^0,x^1), (y^0,y^1).

C Proofs and Technical Results

Proof of Proposition 4.1.

1. Within the setting of [15], the DDE (4.1) is of type $k'(t) = Lk_t + b(t)$ with L a linear operator and $b(t)$ continuous. Hence, the existence and uniqueness of solutions to such DDE follows from Theorem 2.12 in [15]. The continuous differentiability is consequence of the continuity of $t \mapsto b(t)$.

2. By the admissibility constraint (c2), we have for $t \in [0, d)$,

$$
\begin{aligned}
k_{(k_0,i_0),i}(t) &= k_0 + \int_0^t \left(\int_{-d}^{-s} a(r) i_0(s+r) dr + \int_{-s}^{0} a(r) i(s+r) dr \right) ds \\
&\le k_0 + \int_0^t \left(\int_{-d}^{-s} a(r) i_0(s+r) dr + A \int_{-s}^{0} a(r) k_{(k_0,i_0),i}(s+r) dr \right) ds
\end{aligned}
$$

while the function $k^M_{(k_0,i_0)}(t)$ satisfies, for $t \in [0, d)$

$$
k^M_{(k_0,i_0)}(t) = k_0 + \int_0^t \left(\int_{-d}^{-s} a(r) i_0(s+r) dr + A \int_{-s}^{0} a(r) k^M_{(k_0,i_0)}(s+r) dr \right) ds.
$$

Then, by standard comparison results on DDEs (see, e.g. [19] , we get the claim in $[0, d)$. Iterating the argument one gets the claim.

3. Setting $i^M(\cdot) \stackrel{def}{=} k^M_{(k_0,i_0)}(\cdot)$, we have $i^M \in \mathcal{I}_{(k_0,i_0)}$, so the claim follows. □

Proof of Proposition 4.2.

1. Consider the function

$$
h \colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad h(x) \stackrel{def}{=} x - A \int_{-d}^{0} a(r) e^{rx} dr.
$$

It is clear that all real solutions of the characteristic equation (4.3) are zeros of h and viceversa. We observe that

$$
h(0) = -A < 0, \qquad \lim_{x \to +\infty} h(x) = +\infty, \qquad \lim_{x \to -\infty} h(x) = -\infty.
$$

Moreover for all $x \in \mathbb{R}$,

$$
h'(x) = 1 - A \int_{-d}^{0} a(r) r e^{rx} dr > 1, \qquad h''(x) = -A \int_{-d}^{0} a(r) r^2 e^{rx} dr < 0,
$$

so h is strictly increasing and strictly concave. This implies that g admits only one real root $\xi > 0$ which is the only real solution of (4.3). Such solution has multiplicity 1 since $h'(z)$ is never 0.

2. Let $\lambda = \mu + i\nu$ be a solution of (4.3). It is easy to check by direct substitution that, if $\lambda = \mu + i\nu$ solves (4.3), then also $\bar{\lambda} = \mu - i\nu$ solves it. Take the one with $\nu > 0$. Then

$$
\mu + i\nu = A \int_{-d}^{0} a(r) e^{r(\mu + i\nu)} dr = A \left(\int_{-d}^{0} a(r) e^{\mu r} \cos(\nu r) dr + i \int_{-d}^{0} a(r) e^{\mu r} \sin(\nu r) dr \right).
$$

This gives the following two equations:

$$
\mu = A \int_{-d}^{0} a(r) e^{\mu r} \cos(\nu r) dr, \qquad \nu = A \int_{-d}^{0} a(r) e^{\mu r} \sin(\nu r) dr.
$$

Then concerning the real part we clearly get

$$
-A \int_{-d}^{0} a(r) e^{\mu r} dr < \mu < A \int_{-d}^{0} a(r) e^{\mu r} dr = \mu - g(\mu).
$$

So, from the second inequality we get $g(\mu) < 0 = g(\xi)$ which implies, by the fact that g is strictly increasing, that $\mu < \xi$. On the other hand when $\mu < 0$ we get, from the first inequality $Ae^{-\mu d} < \mu$, which give the first of (4.4). Similarly, since $\nu > 0$ we have

$$
\nu < A \int_{-d}^{0} a(r) e^{\mu r} dr < A \left(1 \vee e^{-\mu d} \right).
$$

On the other hand, since $\nu r < 0$ we have that $\sin(\nu r) < 0$ for $\nu r \in (-\xi, 0)$. So, to have $\nu > 0$ in the equation for ν we need to assume $\nu > \xi / d$.

3. First we recall that, by [15, Th. 4.4, Ch. I], all the solutions of (4.3) form a (countable) sequence. So complex roots are at most countable and have the form $\lambda_k = \mu_k \pm i\nu_k$ for two sequences of real numbers $\{\mu_k\}$ and $\{\nu_k\}$.

4. It is enough to prove that

$$\int_{-d}^{0} a_1(r)e^{xr}dr \geq \int_{-d}^{0} a_2(r)e^{xr}dr, \qquad \forall x > 0, \tag{C.1}$$

and then, calling h_1, h_2 the functions defined as h in the first item and associated respectively to a_1, a_2, we get $h_1 \geq h_2$ on $\mathbb{R}^+$ and the claim follows. But (C.1) is a consequence of a simple integration by parts:

$$\int_{-d}^{0} a_1(r)e^{xr}dr = 1 - x\int_{-d}^{0}\left(\int_{-d}^{r} a_1(s)ds\right)e^{xr}dr \geq 1 - x\int_{-d}^{0}\left(\int_{-d}^{r} a_2(s)ds\right)e^{xr}dr = \int_{-d}^{0} a_2(r)e^{xr}dr. \qquad \square$$

Proof of Proposition 4.3. The claim follows from [15, Th. 5.4, p. 34] and using the fact that ξ is the solution to (4.2) with the highest real part (as proved Proposition 4.2).

Proof of Proposition 4.4. Let us show uniqueness. Let $i_1, i_2 \in \mathcal{I}_{(k_0,i_0)}$ and set $i_\lambda = \lambda i_1 + (1-\lambda)i_2$. Then, by linearity of the state equation, one has

$$k_{(k_0,i_0),i_\lambda}(\cdot) = \lambda k_{(k_0,i_0),i_1}(\cdot) + (1-\lambda)k_{(k_0,i_0),i_2}(\cdot). \tag{C.2}$$

First, this implies that the set $\mathcal{I}_{(k_0,i_0)}$ is convex. Moreover, using (C.2) and the strict concavity of the real function $r \mapsto \frac{r^{1-\sigma}}{1-\sigma}$, it is straightforward to show that the functional $J((k_0,i_0);\cdot)$ is strictly concave on its domain. So the claim follows.

Proof of Proposition 4.6

1.a) We could prove this result directly (see e.g. [24]), but for sake of brevity we omit the proof here. The finiteness will be proved a posteriori on a suitable subset of H^{++}.

1.b) Clearly we have $V \leq 0$. To prove that $V(k_0,i_0) > -\infty$ it is enough to exhibit an admissible control $\bar{\imath}$ such that $J((k_0,i_0);\bar{\imath}) > -\infty$. Since $(k_0,i_0) \in H^{++}$, taking $\bar{\imath} \equiv 0$ we have that $k_{(k_0,i_0),\bar{\imath}}(\cdot)$ is nondecreasing. So, we obtain

$$V(k_0,i_0) \geq J((k_0,i_0);\bar{\imath}) = \frac{A^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-\rho t}\left(k_{(k_0,i_0),\bar{\imath}}(t)\right)^{1-\sigma}dt \geq \frac{(Ak_0)^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-\rho t}dt,$$

and the latter integral is finite by Hypothesis 4.5, which in this case reads as $\rho > 0$.

2. Let (k_0,i_0) be such that $V(k_0,i_0)$ is finite. In particular $i \in \mathcal{I}_{(k_0,i_0)} \neq \emptyset$. The linearity of the state equation yields for every $\alpha > 0$

$$i \in \mathcal{I}_{(k_0,i_0)} \iff \alpha i \in \mathcal{I}_{(k_0,i_0)},$$

and $k_{\alpha(k_0,i_0),\alpha i} = \alpha k_{(k_0,i_0),i}$. Then the claim is a straightforward consequence of the homogeneous structure of the functional.

Proof of Lemma 4.7. The fact that if the limit exists it is nonnegative is due to (4.6) and (4.8). The fact that if the limit exists it is smaller than α_0 is due to Proposition 4.3, to (4.10) and to (4.6) and (4.8).

Let us show now the existence of the limit. By (4.7) there exists $T \geq d$ such that

$$\gamma(t) \geq \frac{\alpha_\xi}{2}e^{\xi t}, \qquad \forall t \geq T. \tag{C.3}$$

Setting

$$f(s) := \int_{-d}^{0} a(r)c(s+r)dr \geq 0, \quad s \geq d,$$

and using (4.10) and the admissibility of i, we can write

$$e^{-\xi t}\int_d^t \gamma(t-s)f(s)ds \leq e^{-\xi t}k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d), \qquad t \geq d. \tag{C.4}$$

Now (C.3) and (C.4) yield

$$e^{-\xi t}\int_d^{t-T}\frac{\alpha_\xi}{2}e^{\xi(t-s)}f(s)ds + e^{-\xi t}\int_{t-T}^{t}\gamma(t-s)f(s)ds \leq e^{-\xi t}k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d), \qquad t \geq T.$$

Taking into account (4.6) we get

$$\int_d^{t-T} \frac{\alpha_\xi}{2} e^{-\xi s} f(s) ds \le e^{-\xi t} k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d), \qquad \forall t \ge T.$$

Taking $t \to +\infty$ in the inequality above and considering the nonnegativity of f and Proposition 4.3, we see that the function $s \mapsto e^{\xi s} f(s)$ belongs to $L^1([d,+\infty);\mathbb{R})$ and

$$\lim_{t\to+\infty} \int_d^t e^{-\xi s} f(s) ds = L := \int_d^{+\infty} e^{-\xi s} f(s) ds < +\infty. \tag{C.5}$$

Now, using (4.7) we write for some $C_1, C_2 > 0$

$$\begin{aligned}\left| e^{-\xi t} \int_d^t \gamma(t-s) f(s) ds - \int_d^t \alpha_\xi e^{-\xi s} f(s) ds \right| &\le e^{-\xi t} \int_d^t (C_1 e^{(\xi-\varepsilon)(t-s)} + C_2) f(s) ds \\ &= \int_d^{+\infty} g_t(s) e^{-\xi s} f(s) ds,\end{aligned}$$

where

$$g_t(s) := C_1 e^{-\varepsilon(t-s)} + C_2 e^{-\xi(t-s)}, \quad s,t \ge d.$$

Now notice that $|g_t| \le C_1 + C_2$ and $g_t(s) \to 0$ as $t \to +\infty$ for every $s \ge d$. Hence, taking into account (C.5), passing to the limit for $t \to +\infty$ in the inequality above, we get by dominated convergence in the right hand side

$$\lim_{t\to+\infty} \left| e^{-\xi t} \int_d^t \gamma(t-s) f(s) ds - \int_d^t \alpha_\xi e^{-\xi s} f(s) ds \right| = 0.$$

Taking again into account (C.5), we conclude. □

Proof of Proposition 4.8.

1. This claim follows from Proposition 4.3, Lemma 4.7 and (4.10).
2. Let $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i}(t) = \alpha \ge \beta > 0$. Then, given $\zeta \in (0,\alpha)$ there exists $T \ge d$ such that

$$e^{-\xi t} k_{(k_0,i_0),i}(t) \ge \alpha - \zeta > 0, \qquad \forall t \ge T. \tag{C.7}$$

Let c be defined as in (4.8) and consider the consumption strategy

$$c_{\eta,\delta}(s) := c(s) + \eta e^{\delta s} \mathbf{1}_{[T,+\infty)}(s), \qquad s \ge 0, \qquad \eta > 0, \ \delta \in (0,\xi).$$

Let $k_{\eta,\delta}$ be the solution to

$$\begin{cases} k'(t) = \displaystyle\int_{-d}^{(-d)\vee(-t)} a(r) i_0(t+r) dr + \int_{(-d)\vee(-t)}^0 (Aa(r)k(t+r) - c_{\eta,\delta}(t+r)) dr, \quad t \ge 0, \\ k(0) = k_0, \quad i_0(s), \quad s \in [-d,0). \end{cases}$$

Then, setting $i_{\eta,\delta}(\cdot) := Ak_{\eta,\delta}(\cdot) - c_{\eta,\delta}(\cdot)$, we clearly have $k_{(k_0,i_0),i_{\eta,\delta}}(\cdot) \equiv k_{\eta,\delta}(\cdot)$. Clearly, since $\eta > 0$, we have $k_{(k_0,i_0),i_{\eta,\delta}}(\cdot) \le k_{(k_0,i_0),i}(\cdot)$ and $J((k_0,i_0),i_{\eta,\delta}) > J((k_0,i_0);i)$. So, we need to show that for suitable η, δ one has $i_{\eta,\delta} \in \mathcal{I}_{(k_0,i_0)}$ and $\lim_{t\to+\infty} e^{-\xi t} k_{(k_0,i_0),i_{\eta,\delta}}(t) \le \beta$.

Set $k(\cdot) := k_{(k_0,i_0);i}(\cdot)$ and $k_{\eta,\delta}(\cdot) := k_{(k_0,i_0);i_{\eta,\delta}}(\cdot)$. Since $i \equiv i_{\eta,\delta}$ in $[0,T]$, from the admissibility of i we get

$$k_{\eta,\delta}(t) = k(t) > 0, \quad \forall t \in [0,T]. \tag{C.8}$$

On the other hand, from (4.10) we have

$$e^{-\xi t} k(t) = e^{-\xi t} k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d) - e^{-\xi t} \int_d^t \gamma(t-s) ds \int_{-d}^0 a(r) c(s+r) dr, \quad t \ge d$$

and

$$e^{-\xi t} k_{\eta,\delta}(t) = e^{-\xi t} k^M_{(\bar{k}_0,\bar{\imath}_0)}(t-d) - e^{-\xi t} \int_d^t \gamma(t-s) ds \int_{-d}^0 a(r) c_{\eta,\delta}(s+r) dr, \quad t \ge d.$$

Combining the two equalities above and using (C.7) we get for $t \ge T$

$$e^{-\xi t} k_{\eta,\delta}(t) = e^{-\xi t} k(t) - \eta e^{-\xi t} \int_d^t \gamma(t-s) e^{\delta s} ds \int_{-d}^0 a(r) e^{\delta r} \mathbf{1}_{[T,+\infty)}(s+r) dr.$$

Hence, setting $C_\delta := \int_{-d}^0 a(r)e^{\delta r}dr > 0$, we have

$$e^{-\xi t}k_{\eta,\delta}(t) \;\geq\; e^{-\xi t}k(t) - \eta C_\delta e^{-\xi t}\int_T^t \gamma(t-s)e^{\delta s}ds, \qquad t \geq T, \tag{C.9}$$

$$e^{-\xi t}k_{\eta,\delta}(t) \;\leq\; e^{-\xi t}k(t) - \eta C_\delta e^{-\xi t}\int_{T+d}^t \gamma(t-s)e^{\delta s}ds, \quad\; t \geq T+d. \tag{C.10}$$

Take $\chi \in (0,1)$. Using (4.7) we can assume, without loss of generality, that for the same T fixed in the current proof it holds true

$$(1-\chi)\alpha_\xi e^{\xi t} \;\leq\; \gamma(t) \;\leq\; (1+\chi)\alpha_\xi e^{\xi t}, \qquad \forall t \geq T. \tag{C.11}$$

On the other hand there exists $K_T > 0$ such that

$$\gamma(t) \leq K_T, \quad \forall t \in [0,T]. \tag{C.12}$$

Then (C.9) combined to (C.11), (C.7) and (C.12) yields

$$\begin{aligned}
e^{-\xi t}k_{\eta,\delta}(t) &\geq e^{-\xi t}k(t) - (1+\chi)\alpha_\xi \eta C_\delta \int_T^{t-T} e^{-(\xi-\delta)s}ds - \eta C_\delta K_T e^{-\xi t}\int_{t-T}^t e^{\delta s}ds \\
&\geq \alpha - \zeta - \frac{(1+\chi)\alpha_\xi \eta C_\delta}{\xi-\delta}\left(e^{-(\xi-\delta)T} - e^{-(\xi-\delta)(t-T)}\right) - \eta C_\delta T K_T e^{-(\xi-\delta)t}\frac{1-e^{-\delta T}}{\delta T} \\
&\geq \alpha - \zeta - \frac{(1+\chi)\alpha_\xi \eta C_\delta}{\xi-\delta}e^{-(\xi-\delta)T} - \eta C_\delta T K_T e^{-(\xi-\delta)T} \\
&= \alpha - \zeta - e^{-(\xi-\delta)T}C_\delta \eta\left(\frac{(1+\chi)\alpha_\xi}{\xi-\delta} + TK_T\right), \qquad t \geq T.
\end{aligned}$$

Hence, if η, δ are such that

$$e^{-(\xi-\delta)T}C_\delta \eta\left(\frac{(1+\chi)\alpha_\xi}{\xi-\delta} + TK_T\right) < \alpha - \zeta, \tag{C.13}$$

considering also (C.8), we see that $i_{\eta,\delta} \in \mathcal{I}_{(k_0,i_0)}$. Now we notice that, for any given $\delta \in (0,\xi)$, the inequality (C.13) is fulfilled by choosing

$$\eta = \eta(\delta) \;:=\; (\alpha-\zeta)e^{(\xi-\delta)T}C_\delta^{-1}\left(\frac{(1+\chi)\alpha_\xi}{\xi-\delta} + TK_T\right)^{-1}. \tag{C.14}$$

On the other hand, using (C.10) and (C.11) , we can write

$$e^{-\xi t}k_{\eta,\delta}(t) \;\leq\; e^{-\xi t}k(t) - \eta C_\delta (1-\chi)\alpha_\xi \int_{T+d}^{t-T} e^{-(\xi-\delta)s}ds, \qquad t \geq T+d. \tag{C.15}$$

Taking $\eta = \eta(\delta)$ as in (C.14), we can use part 1 of the present proposition and pass to the limit in (C.15) getting

$$\begin{aligned}
\lim_{t\to+\infty} e^{-\xi t}k_{\eta(\delta),\delta}(t) &\leq \alpha - \eta(\delta)C_\delta(1-\chi)\alpha_\xi \frac{e^{-(\xi-\delta)(T+d)}}{\xi-\delta} \\
&= \alpha - (\alpha-\zeta)(1-\chi)\alpha_\xi e^{-(\xi-\delta)d}\left((1+\chi)\alpha_\xi + TK_T(\xi-\delta)\right)^{-1}.
\end{aligned}$$

Then, setting

$$f(\delta,\chi,\zeta) \;:=\; \alpha - (\alpha-\zeta)(1-\chi)\alpha_\xi e^{-(\xi-\delta)d}\left((1+\chi)\alpha_\xi + TK_T(\xi-\delta)\right)^{-1},$$

since we have freedom in the choice of the parameters $\chi, \zeta > 0$ (they just affect the choice of T), to complete the proof we need to show that

$$\exists\, \chi, \zeta > 0 \text{ such that for suitable } \delta \in (0,\xi) \text{ it is } f(\delta,\chi,\zeta) \leq \beta. \tag{C.16}$$

Now notice that

$$\lim_{\delta\uparrow\xi} f(\delta,\chi,\zeta) = \alpha - (\alpha-\zeta)\frac{1-\chi}{1+\chi}.$$

So, if χ, ζ are sufficiently close to 0, (C.16) can be obtained, and the proof is complete. □

Proof of Theorem 5.2 See Vinter and Kwong [44, Sec. 5].

Proof of Proposition 5.6
Clearly $v \in C^1(\Theta)$ and
$$\nabla v(y) = \left(\alpha(1-\sigma)\langle \varphi, y\rangle_H^{-\sigma}, \alpha(1-\sigma)\langle \varphi, y\rangle_H^{-\sigma} w\right) \ .$$

So $\nabla v(y) \in D(\mathcal{B})$ for every $y \in \Theta$. Moreover $w' = \xi w$ and $\xi = A\int_{-d}^0 a(r)e^{\xi r}dr = A\langle a, w\rangle_{L^2([-d,0];\mathbb{R})}$ since ξ is by definition the solution of the characteristic equation (4.3), and

$$B\,\nabla v(y) = \left(0, \xi\alpha(1-\sigma)w\langle \varphi, y\rangle_H^{-\sigma}\right) \ ,$$
$$C\,\nabla v(y) = \langle a, \alpha(1-\sigma)w\langle \varphi, y\rangle_H^{-\sigma}\rangle_{L^2([-d,0];\mathbb{R})} = \alpha(1-\sigma)\langle \varphi, y\rangle_H^{-\sigma}\langle a, w\rangle_{L^2([-d,0];\mathbb{R})} = \alpha(1-\sigma)\frac{\xi}{A}\langle \varphi, y\rangle_H^{-\sigma}.$$

In particular $(C\,\nabla v(y))^{-1/\sigma} = \nu\langle \varphi, y\rangle_H > 0$. So, plugging these expressions into (5.5) and using (5.7) and (5.8), we obtain the claim.

Proof of Proposition 5.7.
1. We have, taking $h = \varphi$ in (5.3), where φ is defined in (5.9),

$$\frac{d}{dt}\langle \varphi, Y_{z_0,i}(t)\rangle_H = \langle \mathcal{B}\varphi, Y_{z_0,i}(t)\rangle_H + (\mathcal{C}\varphi)i(t), \quad \text{for a.e. } t \geq 0, \quad \forall i \in L^2_{loc}([0,+\infty);\mathbb{R}). \tag{C.17}$$

The right hand-side of (C.17) is

$$\langle \mathcal{B}\varphi, Y_{z_0,i}(t)\rangle_H + (\mathcal{C}\varphi)i(t) = \xi\langle w, (Y^*_{z_0})^1(t)\rangle_{L^2([-d,0];\mathbb{R})} + \frac{\xi}{A}i(t) = \xi\langle \varphi, Y_{z_0,i}(t)\rangle_H - \frac{\xi}{A}(AY^0_{z_0}(t) - i(t)). \tag{C.18}$$

So, if $i \in L^2_{loc}([0,+\infty);\mathbb{R})$ is such that $i(t) \leq AY^0_{z_0,i}(t)$ for a.e. $t \geq 0$, we get from (C.17) and (C.18)

$$\frac{d}{dt}\langle \varphi, Y_{z_0,i}(t)\rangle_H \leq \xi\langle \varphi, Y_{z_0,i}(t)\rangle_H, \qquad \text{for a.e. } t \geq 0.$$

The claim follows by standard comparison results for ODEs.
2. We have, taking $h = \varphi$ in (5.16) where φ is defined in (5.9),

$$\frac{d}{dt}\langle \varphi, Y^*_{z_0}(t)\rangle_H = \frac{d}{dt}\langle \varphi, Y^*_{z_0}(t)\rangle_H = \langle \mathcal{B}\varphi, Y^*_{z_0}(t)\rangle + (\mathcal{C}\varphi)\Phi Y^*_{z_0}(t) \tag{C.19}$$

So, using (5.10), the right-hand side of (C.19) equals

$$\xi\langle w, (Y^*_{z_0})^1(t)\rangle_{L^2([-d,0];\mathbb{R})} + \left(A(Y^*_{z_0})^0(t) - \nu\langle \varphi, Y^*_{z_0}(t)\rangle_H\right)\frac{\xi}{A} = \xi\left(1 - \frac{\nu}{A}\right)\langle \varphi, Y^*_{z_0}(t)\rangle_H,$$

and the claim follows. □

Proof of Theorem 5.8. First of all we notice that, by definition of $i^*_{z_0}$, we have that $Y_{z_0,i^*_{z_0}}$ is a weak solution of (5.15). By uniqueness of weak solution of (5.15), this implies $Y_{z_0,i^*_{z_0}} = Y^*_{z_0}$. So, using Proposition 5.7, we see that $i^*_{z_0} \in \tilde{\mathcal{I}}^H_{z_0}$.
Defining, for $i \in \mathbb{R}$, the differential operator $\mathcal{L}^i$ acting on v as

$$[\mathcal{L}^i v](y) := \langle y, \mathcal{B}\nabla v(y)\rangle_H + (\mathcal{C}\varphi)i.$$

we have (see, e.g., [32, Ch. 2, Prop. 5.5]) the following chain's rule:

$$e^{-\rho t}v(Y_{z_0,i}(t)) - v(z_0) = \int_0^t e^{-\rho s}(-\rho v(Y_{z_0,i}(s)) + [\mathcal{L}^{i(s)}v](Y_{z_0,i}(s)))ds, \quad \forall\, t \geq 0, \quad \forall i \in \tilde{\mathcal{I}}^H_{z_0}.$$

Using the fact that v solves he HJB equation (5.5) in Θ, we can rewrite the equality above as

$$e^{-\rho t}v(Y_{z_0,i}(t)) - v(z_0) = -J^H(z_0; i) + \int_0^t e^{-\rho s}[\mathcal{H}_{CV}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s); i(s)) - \mathcal{H}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s))]ds,$$

i.e.

$$v(z_0) = J^H(z_0; i) + \int_0^t e^{-\rho s}[\mathcal{H}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s)) - \mathcal{H}_{CV}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s); i(s))]ds + e^{-\rho t}v(Y_{z_0,i}(t)), \tag{C.20}$$

for every $t \geq 0$ and every $i \in \tilde{\mathcal{I}}^H_{z_0}$. Taking $t \to +\infty$ in (C.20):

1. Using Hypothesis 4.5, (5.12), and Proposition 5.7(1), we can get rid of the term $e^{-\rho t}v(Y_{z_0,i}(t))$ which converges to 0.
2. Since $\mathcal{H} \geq \mathcal{H}_{CV}$, we can use monotone convergence in the integral term.

So, we can write the equality

$$v(z_0) = J^H(z_0; i) + \int_0^\infty e^{-\rho s}[\mathcal{H}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s)) - \mathcal{H}_{CV}(Y_{z_0,i}(s), \nabla v(Y_{z_0,i}(s); i(s))]ds, \quad \forall t \geq 0, \quad \forall i \in \tilde{\mathcal{I}}^H_{z_0}.$$

Now noticing that $\mathcal{H} \geq \mathcal{H}_{CV}$, we get the inequality $v(z_0) \geq \tilde{V}^H(z_0)$. On the other hand by definition of $i^*_{z_0}$ we also have

$$\mathcal{H}(Y_{z_0,i^*_{z_0}}(s), \nabla v(Y_{z_0,i^*_{z_0}}(s)) - \mathcal{H}_{CV}(Y_{z_0,i^*_{z_0}}(s), \nabla v(Y_{z_0,i^*_{z_0}}(s); i^*_{z_0}(s)) = 0, \quad \forall s \geq 0.$$

Hence $\tilde{V}^H(z_0) \leq v(z_0) = J^H(z_0; i^*_{z_0}) \leq \tilde{V}^H(z_0)$, getting both the claims. □

Proof of Proposition 5.9 The fact that $\tilde{V}(k_0, i_0) = v(z_0)$ the couple $(i^*_{(k_0,i_0)}, k^*_{(k_0,i_0)})$ is an optimal investment/capital couple starting from (k_0, i_0) for $(\tilde{\mathbf{P}})$ is consequence of Theorem 5.8 and of (5.19). Let us show the other claims.

1. This claim follows from (5.23).

2. Differentiating (5.23) and using 1. of the present proposition the differential part of claim follows. The structure of the initial datum $i(0)$ comes from (5.23) as well.

Proof of Theorem 5.11

Let $(k_0, i_0) \in \mathbf{S}$. By Propositions 5.9 and C.1, we have

$$V(k_0, i_0) \geq J((k_0, i_0); i^*_{(k_0,i_0)}) = \tilde{V}(k_0, i_0) = v(\mathcal{Q}(k_0, i_0)).$$

On the other hand, since we have the inequality $V \leq \tilde{V}$ (Proposition C.2), we deduce the optimality of $i^*_{(k_0,i_0)}$ for $(\mathbf{P})$ starting at (k_0, i_0).

Then, uniqueness is stated by Proposition 4.4, and the last claim follows from Proposition 5.9.

Proof of Lemma 6.1. We have

$$\begin{aligned}
\Lambda_{(k_0,i_0)} &= \nu k_0 + \nu b \int_{-d}^0 e^{\xi r} dr \int_{-d}^r a(s) e^{g(s-r)} ds \\
&= \nu k_0 + \nu b \int_{-d}^0 a(s) e^{gs} ds \int_s^0 e^{(\xi - g)r} dr \\
&= \nu k_0 + \nu b \int_{-d}^0 a(s) e^{gs} \frac{1 - e^{(\xi-g)s}}{\xi - g} ds \\
&= \nu k_0 + b \frac{\nu}{\xi - g} \int_{-d}^0 a(s)(e^{gs} - e^{\xi s}) ds \\
&= \nu k_0 + b \frac{\nu}{\xi - g} \left(\frac{g k_0}{b} - \frac{\xi}{A} \right) \\
&= \nu k_0 \frac{\xi}{\xi - g} - b \frac{\nu \xi}{A(\xi - g)}.
\end{aligned}$$

Since $\frac{\nu\xi}{A(\xi-g)} = 1$, see (5.17) and (5.11), the proof is complete.

Proof of Proposition 6.2.

1. If $b > 0$, $g > 0$, then $\mathbf{E}_b \in H^{++}$. Moreover, by Lemma 6.1, we have $Ak_0 - \Lambda_{(k_0,i_0)} = b > 0$. So, all the properties defining the set $\mathbf{S}$ are fulfilled by (k_0, i_0), concluding the proof.

2. Observe that $g = 0$ is equivalent to $A - \nu = 0$. Hence, if $k_0 > 0$ and $i_0 \equiv 0$, then clearly $(k_0, i_0) \in \mathbf{S}$. On the other hand, if $(k_0, i_0) \in \mathbf{S}$, the conditions $(k_0, i_0) \in H^{++}$ and $Ak_0 - \Lambda_{(k_0,i_0)} \geq 0$ imply $k_0 > 0$ and $i_0 \equiv 0$.

3.Observe that $g < 0$ is equivalent to $A - \nu < 0$. Hence, the conditions $(k_0, i_0) \in H^{++}$ and $Ak_0 - \Lambda_{(k_0,i_0)} \geq 0$ are not compatible in this case, so $\mathbf{S}$ is empty.

Proof of Proposition 6.3 If $(k_0, i_0) = \mathbf{E}_b$ for some $b > 0$, using Theorem 5.11 by straightforward computations we get the claim.

Conversely, let us assume that the optimal paths $k^*_{(k_0,i_0)}, i^*_{(k_0,i_0)}$ are exponential. Then, since $Ak^*_{(k_0,i_0)}(t) - \Lambda_{(k_0,i_0)}e^{gt} = i^*_{(k_0,i_0)}(t)$, we see that the common growth rate of $k^*_{(k_0,i_0)}, i^*_{(k_0,i_0)}$ is g. Hence $i^*(t) = be^{gt}$, for some $b \in \mathbb{R}$ and $k^*_{(k_0,i_0)} = k_0 e^{gt}$, with $k_0 > 0$. Defining the function λ as in the proof of Proposition C.1, we see then that $\lambda \equiv 0$ over $\mathbb{R}^+$. Since λ solves (C.23)-(C.24), we see that $(k_0, i_0) = \mathbf{E}_b$, and finally that $b > 0$ since $k_0 > 0$.

Proposition C.1. *Let $(k_0, i_0) \in \mathbf{S}$ and consider the investment strategy $i^*_{(k_0,i_0)}$ defined in Proposition 5.9. Then $i^*_{(k_0,i_0)} \in \mathcal{I}_{(k_0,i_0)}$.*

Proof. Let $(k_0, i_0) \in \mathbf{S}$ and $i^*_{(k_0,i_0)}$ defined as in Proposition 5.9. Consider the function

$$\tilde{\imath}^*(s) = \begin{cases} i_0(s), & s \in [-d, 0), \\ i^*_{(k_0,i_0)}(s), & s \in [0, +\infty). \end{cases}$$

Define the function

$$\lambda(t) \overset{def}{=} (\tilde{\imath}^*)'(t) - g\tilde{\imath}^*(t), \quad t \in [-d, +\infty).$$

Differentiating in (5.23) yields for every $t \geq 0$

$$\begin{aligned} (i^*_{(k_0,i_0)})'(t) - gi^*_{(k_0,i_0)}(t) &= A(k^*_{(k_0,i_0)})'(t) - \nu\Lambda_{(k_0,i_0)}ge^{gt} - gi^*_{(k_0,i_0)}(t) \\ &= A\left((k^*_{(k_0,i_0)})'(t) - gk^*_{(k_0,i_0)}(t)\right) + g\left(Ak^*_{(k_0,i_0)}(t) - \nu\Lambda_{(k_0,i_0)}e^{gt}\right) - gi^*_{(k_0,i_0)}(t) \\ &= A\left((k^*_{(k_0,i_0)})'(t) - gk^*_{(k_0,i_0)}(t)\right). \end{aligned} \tag{C.21}$$

On the other hand we have

$$(k^*_{(k_0,i_0)})'(t) = \int_{-d}^{0} a(r)\tilde{\imath}^*(t+r)dr, \quad \forall t \geq 0.$$

We then see, by definition of $\mathbf{S}$, that $(k^*_{(k_0,i_0)})' \in W^{1,2}_{loc}([0, +\infty); \mathbb{R})$, and we can differentiate the equality above getting

$$(k^*_{(k_0,i_0)})''(t) - g(k^*_{(k_0,i_0)})'(t) = \int_{-d}^{0} a(r)\big((\tilde{\imath}^*)'(t+r) - g\tilde{\imath}^*(t+r)\big)dr, \quad \text{for a.e. } t \geq 0. \tag{C.22}$$

Hence, using (C.21) and (C.22) we see that $\lambda \in W^{1,2}_{loc}([0, +\infty); \mathbb{R})$ and solves the DDE

$$\lambda'(t) = A\int_{-d}^{0} a(r)\lambda(t+r)dr, \quad \text{for a.e. } t \geq 0. \tag{C.23}$$

The initial data for such DDE are

$$\begin{cases} \lambda(s) = i_0'(s) - gi_0(s), \quad \text{for a.e. } s \in [-d, 0), \\ \\ \lambda(0) = (i^*_{(k_0,i_0)})'(0) - gi^*_{(k_0,i_0)}(0) \\ \quad = A\int_{-d}^{0} a(r)i_0(r)dr - g\Lambda_{(k_0,i_0)} - g(Ak_0 - \Lambda_{(k_0,i_0)}) = A\left(\int_{-d}^{0} a(r)i_0(r)dr - gk_0\right). \end{cases} \tag{C.24}$$

By the assumption $(k_0, i_0) \in \mathbf{S}$, the initial data in the DDE above are nonnegative. Since also the constant A and the function $a(\cdot)$ are nonnegative, we get $\lambda(t) \geq 0$ for a.e. $t \geq 0$, i.e. $(i^*_{(k_0,i_0)})'(t) \geq gi^*_{(k_0,i_0)}(t)$ for a.e. $t \geq 0$. Using again the assumption $(k_0, i_0) \in \mathbf{S}$, we also see that $i^*_{(k_0,i_0)}(0) = Ak_0 - \Lambda_{(k_0,i_0)} \geq 0$, so we conclude that $i^*_{(k_0,i_0)}(t) \geq 0$ for every $t \geq 0$ (the passage from a.e. $t \geq 0$ to every $t \geq 0$ is due to continuity of $i^*_{(k_0,i_0)}$).

Now notice that, by (3.4),

$$i \in \tilde{\mathcal{I}}_{(k_0,i_0)}, \ i(\cdot) \geq 0, \ (k_0, i_0) \in H^{++} \implies i \in \mathcal{I}_{(k_0,i_0)},$$

so the proof is complete. ∎

The following proposition, which follows from the work done in Subsection **??**, is crucial to prove our main result Theorem 5.11.

Proposition C.2. *Let $(k_0, i_0) \in H$.*

1. *Let*

$$\tilde{\mathcal{I}}^L_{(k_0,i_0)} \stackrel{def}{=} \{i \in L^2_{\rm loc}([0,+\infty);\mathbb{R}) \ : \ k^{eq}_{(k_0,i_0);i}(t) \geq 0 \ \ \forall t \geq 0, \ \ i(t) \leq Ak_{(k_0,i_0),i}(t) \ \textit{for a.e.} \ t \geq 0\},$$

Then $\mathcal{I}_{(k_0,i_0)} \subset \tilde{\mathcal{I}}^L_{(k_0,i_0)}$.

2. $V(k_0,i_0) \leq \tilde{V}(k_0,i_0)$.

Proof. 1. Let $i \in \mathcal{I}_{(k_0,i_0)}$, assume, by contradiction, that $i \notin \tilde{\mathcal{I}}^L_{(k_0,i_0)}$, and set $k_{eq}(\cdot) := k^{eq}_{(k_0,i_0);i}(\cdot)$ and $k(\cdot) := k_{(k_0,i_0),i}(\cdot)$. Since $i \in \mathcal{I}_{(k_0,i_0)}$, it is $k(\cdot) > 0$ everywhere, whereas, since $i \notin \tilde{\mathcal{I}}_{(k_0,i_0)}$, there exists t_0 such that $k_{eq}(t_0) < 0$. Then, by (**??**), we can deduce that

$$\limsup_{t\to+\infty} e^{-\xi t} k_{eq}(t) = e^{-\xi t_0} k_{eq}(t_0) =: -c_0 < 0. \tag{C.25}$$

Now, we distinguish two cases: I) $\lim_{t\to+\infty} e^{-\xi t}k(t) = 0$; II) $\lim_{t\to+\infty} e^{-\xi t}k(t) = \alpha > 0$.

Case I. From (5.18) and taking into account that $i^+(\cdot) \leq Ak(\cdot)$, we get

$$\begin{aligned}
e^{-\xi t}k_{eq}(t) &= e^{-\xi t}k(t) + \int_t^{t+d} e^{-\xi u} du \int_{-d}^{t-u} a(s) i(u+s) ds \\
&= e^{-\xi t}k(t) + \int_t^{t+d} e^{-\xi u} du \int_{-d}^{t-u} a(s)[i^+(u+s) - i^-(u+s)] ds \\
&\geq e^{-\xi t}k(t) + \int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} a(s)[i^+(u+s) - i^-(u+s)] ds - \int_t^{t+d} e^{-\xi u} du \int_{t-u}^{0} a(s) i^+(u+s) ds \\
&\geq e^{-\xi t}k(t) + \int_t^{t+d} e^{-\xi u} k'(u) du - \int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} a(s) i^+(u+s) ds \\
&\geq e^{-\xi t}k(t) + \int_t^{t+d} e^{-\xi u} k'(u) du - \int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} Aa(s) k(u+s) ds \\
&= e^{-\xi t}k(t) + \Big[e^{-\xi u}k(u)\Big]_t^{t+d} + \xi \int_t^{t+d} e^{-\xi u} k(u) du - \int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} Aa(s) k(u+s) ds \\
&= e^{-\xi(t+d)}k(t+d) + \xi \int_t^{t+d} e^{-\xi u} k(u) du - \int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} Aa(s) k(u+s) ds \\
&\geq -\int_t^{t+d} e^{-\xi u} du \int_{-d}^{0} Aa(s) k(u+s) ds.
\end{aligned}$$

Now consider the sequence $(t_n)_{n\geq 1}$, where $t_n := dn$, and let

$$\bar{t}_n \ \in \ \text{argmax}_{u\in[t_{n-1},t_{n+1}]} k(u), \quad n \geq 2.$$

Then, from the inequality above we get

$$e^{-\xi \bar{t}_n} k_{eq}(\bar{t}_n) \ \geq \ -Ade^{-\xi \bar{t}_n} k(\bar{t}_n). \tag{C.26}$$

Letting $n \to \infty$, since $\bar{t}_n \to +\infty$, from (C.26) and (C.25) we get $-c_0 \geq 0$, absurd.

Case II. Let $(\beta_j)_{j\in\mathbb{N}}$ be a sequence in $(0,\alpha)$ such that $\beta_j \to 0$ when $j \to \infty$. For each $j \in \mathbb{N}$ consider an associated strategy $i_j \in \mathcal{I}_{(k_0,i_0)}$ satisfying the claim of Proposition 4.8(2) with $\beta = \beta_j$, and call $k^j(\cdot) := k_{(k_0,i_0),i_j}(\cdot)$, $k^j_{eq}(\cdot) := k^{eq}_{(k_0,i_0),i_j}(\cdot)$. By construction of i_j [29] and considering (**??**), we see that $k^j_{eq}(\cdot) \leq k_{eq}(\cdot)$, hence, from (C.25), we have

$$\limsup_{t\to+\infty} e^{-\xi t} k^j_{eq}(t) \leq e^{-\xi t_0} k_{eq}(t_0) =: -c_0 < 0, \qquad \forall j \in \mathbb{N}. \tag{C.27}$$

Now, arguing as above in case I, we end up with

$$e^{-\xi \bar{t}^j_n} k_{eq}(\bar{t}^j_n) \ \geq \ -Ade^{-\xi \bar{t}^j_n} k(\bar{t}^j_n), \quad \ \forall j \in \mathbb{N}. \tag{C.28}$$

29. Better is to look at the associated consumption strategy c_j.

Letting $n \to \infty$, from (C.27) and (C.28) we get

$$-c_0 \geq \; -Ad\beta_j, \qquad \forall j \in \mathbb{N}. \tag{C.29}$$

Letting $j \to \infty$ in (C.29), we finally get $-c_0 \geq 0$, absurde.

2. From part 1 of the present proposition we get

$$V(k_0, i_0) \;=\; \sup_{i \in \mathcal{I}_{(k_0,i_0)}} J((k_0, i_0); i) \;\leq\; \sup_{i \in \tilde{\mathcal{I}}^L_{(k_0,i_0)}} J((k_0, i_0); i). \tag{C.30}$$

On the other hand, Remark **??** shows that $(\tilde{\mathbf{P}})$ is a very standard one, for which we know that the solution with the strict and the large state constraint ($k^{eq}(\cdot) > 0$ and $k^{eq}(\cdot) \geq 0$, respectively) has the same solution when starting from $k^{eq}(0) = k_0^{eq} > 0$, hence

$$\sup_{i \in \tilde{\mathcal{I}}^L_{(k_0,i_0)}} J((k_0, i_0); i) \;=\; \sup_{i \in \tilde{\mathcal{I}}_{(k_0,i_0)}} J((k_0, i_0); i) \;=\; \tilde{V}(k_0, i_0).$$

Combining with (C.30), we get the claim. □

Proof of Proposition 7.3. For simplicity of notation we set $i(\cdot) = i^*_{(k_0,i_0)}(\cdot)$, $k(\cdot) = k^*_{(k_0,i_0)}(\cdot)$ and $\Lambda = \Lambda_{(k_0,i_0)}$.

The explicit expression of c is already provided by Theorem 5.11. The expression of k in terms of c, i comes from definition of c. Let us prove (7.3). From Theorem 5.11 we know that i solves (5.24). The solution to this DDE is the sum of the solution of the associated linear homogeneous DDE, i.e. without the forcing term, plus a convolution term (see [25], Chapter 6, pag 170). In our case it means that the solution of (5.24) can be rewritten as (see Example 1.5, pag. 168, and formula (1.18), pag. 172, in [25]),

$$i(t) = \gamma(t)i(0) + \int_{-d}^{0}\int_{0}^{d+r} \gamma(t-s)Aa(r-s)ds\, i_0(r)\, dr - \int_0^t \gamma(t-s)\Lambda g e^{gs} ds \tag{C.31}$$

where γ is defined in series form in (7.1). By the change of variables $s = -w,\ r = z - w$ in the second term of (C.31), i can be rewritten as

$$i(t) = \gamma(t)i(0) + \int_{-d}^{0}\int_{-d}^{w} \gamma(t+w)Aa(z)i_0(z-w)dz\, dw - \int_0^t \gamma(t-s)\Lambda g e^{gs} ds\ . \tag{C.32}$$

We observe that (4.5) is a special case (with special initial data) of (4.2). Plugging (7.1) into (C.32), in view of the linearity of (C.32) with respect to γ, we can analyze the contribution of the real and the complex roots. We start with $\alpha_\xi e^{\xi t}$: its contribution to $i(t)$ is

$$\begin{aligned}
&\alpha_\xi e^{\xi t}\left(Ak_0 - \Lambda\right) + \int_{-d}^{0} \alpha_\xi e^{\xi(t+w)} \int_{-d}^{w} Aa(z)i_0(z-w)dz\, dw - \int_0^t \alpha_\xi e^{\xi(t-s)}\Lambda g e^{gs} ds \\
&= \alpha_\xi e^{\xi t}\left(A\Gamma(x_0) - \Lambda + \frac{\Lambda g}{g-\xi}\right) + \alpha_\xi e^{gt}\left(-\frac{\Lambda g}{g-\xi}\right) \\
&= \alpha_\xi e^{\xi t}\left(\Lambda A \frac{\sigma}{\rho - \xi(1-\sigma)}\frac{\xi}{A} - \Lambda + \frac{\Lambda\frac{\xi-\rho}{\sigma}}{\frac{\xi-\rho}{\sigma} - \xi}\right) + \alpha_\xi e^{gt}\left(-\frac{\Lambda g}{g-\xi}\right) \\
&= \alpha_\xi e^{gt}\left(-\frac{\Lambda g}{g-\xi}\right)
\end{aligned}$$

where the second equality is obtained using (5.22) and (5.17).

Now, to analyze the contribution of the series, we can use the dominated convergence theorem to exchange the series and the integral in (C.32). Then, for each term $p_j e^{\lambda_j t}$, we can develop the integrals as above, obtaining as contribution the sum of two terms:

$$p_j e^{\lambda_j t}\left(A\Gamma_j(k_0, i_0) - \Lambda + \frac{\Lambda g}{g-\lambda_j}\right) + p_j e^{gt}\left(-\frac{\Lambda g}{g-\lambda_j}\right).$$

So by definition of a_j, b_j (7.4), we get (7.3).

Now let us show the second part of the claim, i.e. the existence of the limits for the detrended paths. Let us set, for simplicity of notation,

$$k_g(t) \overset{def}{=} k^*_{(k_0,i_0),g}(t), \quad i_g(t) \overset{def}{=} i^*_{(k_0,i_0),g}(t), \qquad t \geq 0,$$

Being $i(\cdot)$ real (7.3) can be rewritten as

$$i(t) = \alpha_\xi e^{gt}\left(-\frac{\Lambda g}{g-\xi}\right) + \sum_{j=1}^{\infty} Re\left[p_j e^{\lambda_j t} a_j + p_j e^{gt} b_j\right]$$
$$= \alpha_\xi e^{gt}\left(-\frac{\Lambda g}{g-\xi}\right) + \sum_{j=1}^{\infty} e^{\mu_j t} Re\left[p_j e^{i\nu_j t} a_j\right] + e^{gt} Re\left[p_j b_j\right] \ .$$

By Hypothesis 7.1(i), we then have

$$i(t) = C_0 e^{gt} + o(e^{gt}), \qquad \text{where} \qquad C_0 \overset{def}{=} -\Lambda g\left[\frac{\alpha_\xi}{g-\xi} + \sum_{j=1}^{\infty} Re\left(\frac{p_j}{g-\lambda_j}\right)\right],$$

where it can be proved that the last series converges.
This proves that there exists a constant i_l such that $\lim_{t\to+\infty} i_g(t) = i_l$. Of course by relation $Ak_g(\cdot) - i_g(\cdot) \equiv \Lambda$ this implies also that there exists a constant k_l such that $\lim_{t\to+\infty} k_g(t) = k_l$. We now calculate explicitly such i_l and k_l using the explicit form of the optimal feedback provided by (5.22)-(5.23). We have

$$i_g(t) = (A-\nu)k_g(t) - \nu \int_{-d}^{0} e^{(\xi-g)r} dr \int_{-d}^{r} a(s) i_g(t+s-r) e^{gs} ds,$$

and taking the limit for $t \to +\infty$ we obtain

$$i_l = (A-\nu)k_l - \nu i_l \int_{-d}^{0} e^{(\xi-g)r} dr \int_{-d}^{r} a(s) e^{gs} ds,$$

i.e.

$$i_l\left(1 + \nu \int_{-d}^{0} e^{(\xi-g)r} dr \int_{-d}^{r} a(s) e^{gs} ds\right) = (A-\nu)k_l \ .$$

Exchanging tjhe order of integration and using the definitions of ν and ξ, we get

$$i_l\left(\frac{A}{\xi}\int_{-d}^{0} a(\eta) e^{g\eta} d\eta\right) = (A-\nu)k_l. \tag{C.33}$$

Moreover from the relation $Ak_g(t) - i_g(t) = \Lambda$ we have

$$Ak_l - i_l = \Lambda. \tag{C.34}$$

Using (C.33) and (C.34) we find the values i_l and k_l and so the claim. □

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