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Abstract

We employ weak dominance to analyze both first-price and second-price auctions under the discrete private-value setting. We provide a condition under which the expected revenue from second-price auction is higher than that of first-price auction. We also provide implications for large auctions, including the "virtual" revenue equivalence.

Keywords: Discrete Private-Value Auctions, Revenue Comparison, Weak Dominance.

JEL Classification: C72, D44.

1 Introduction

The comparison of the expected revenues from private-value first-price and second-price auctions (FPA and SPA henceforth) has been extensively analyzed, including the revenue equivalence result by Riley and Samuelson (1981) and Myerson (1981). It also has been shown that once some underlying assumptions are relaxed, not only the revenue equivalence result does not necessarily hold, but the comparison results

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become ambiguous.[1] In addition, the analyses often have been limited to the two-player case, implying the lack of implications for large auctions.

In this paper, we revisit the revenue comparison of FPA and SPA. There are two departures from the literature. One is the use of the maximal elimination of weakly dominated bids – *all* weakly dominated bids are eliminated – for *both* FPA and SPA.[2] It has been typically the case that while SPA is analyzed by the maximal elimination of weakly dominated bids, FPA is analyzed by Bayesian Nash equilibrium. It would be ideal to use the same solution concept to assess the differences purely stemming from the comparison of two distinct institutions. Another departure is that we follow a seminal work by Dekel and Wolinsky (2003), which analyze FPA via rationalizability, and adopt the *discrete* sets of bids and values.[3] One advantage of the adoption of weak dominance and discrete setting is that we require minimal assumptions. In particular, our analysis allows asymmetry and an arbitrary number of players.

Our main result provides a condition under which *SPA generates a higher expected revenue compared to FPA*.[4] The key is the comparison of the winning bids in FPA and SPA. The result on the winning bids in FPA is due to Battigalli and Siniscalchi (2003) and Dekel and Wolinsky (2003) who provide *upper bounds* for bids via weak dominance. The maximal elimination of weakly dominated bids implies that the winning bids in SPA is higher than that of FPA. This leads to the comparison of the highest bid in FPA and the second highest bid in SPA. Our condition concerns the case where the highest and the second highest bids are the same in SPA, in which case the price the winner pays in SPA is higher than that of FPA.

We also show that this result is asymptotically robust. Under the assumptions of (i) independently distributed values and (ii) the same highest value (whose probability is bounded below by an arbitrary small number) for every player's support, the expected revenue from SPA is higher than that of FPA *in large auctions*.[5] In addition, if the *iterative* maximal elimination of weakly dominates bids is used with

[1]For example, Maskin and Riley (2000) and Kirkegaard (2012) analyzed the case of asymmetry. See Maskin and Riley (2000), Krishna (2010) and Milgrom (2004) for the overview of related studies.

[2]We use *interim* weak dominance. That is, we apply weak dominance for the realization of each value. We hence use "bids" instead of "strategies." Note also that this does not imply an iterative procedure. We use the iterative maximal elimination of weakly dominated bids later.

[3]Several other studies also analyze FPA via rationalizability, including Battigalli and Siniscalchi (2003), Cho (2005), and Robles and Shimoji (2012).

[4]See also Kim (2013) who compares the revenues under the binary setting.

[5]See also Yu (1999) for the equilibrium analysis of the symmetric case.

the additional assumption of (iii) players' risk-aversion, the difference in the expected revenues converges to the smallest monetary unit, which we denote d, as the number of players increases.[6] This implies the *virtual revenue equivalence* in large auctions for small d.

For asymmetric auctions, Kirkegaard (2012) identified sufficient conditions under which FPA generates a higher expected revenue compared to SPA. Note that our result has a different implication. Both the use of weak dominance for FPA and the discrete setting lead to this difference. We use an example to demonstrate that the discretized version of the condition in Kirkegaard (2012) and ours are not mutually exclusive.

2 Preliminaries

In this paper, we analyze first-price auction (FPA) and second-price auction (SPA). We utilize the private-value setting. The set of players is $N = \{1, \ldots, n\}$ with $n \geq 2$. Player i's utility function is $u_i : \mathbb{R} \to \mathbb{R}$ which is assumed to be strictly increasing. Before the auction starts, each player $i \in N$ observes her value, $v_i \in V_i = \{\underline{v}_i, \underline{v}_i + d, \ldots, \bar{v}_i - d, \bar{v}_i\}$ where $0 \leq \underline{v}_i \leq \bar{v}_i$ and d is an increment. Let $\bar{v} = \max_{j \in N}\{\bar{v}_j\}$ and $\underline{v} = \max_{j \in N}\{\underline{v}_j\}$. Let v be a typical element of $V = \prod_{j \in N} V_j$. We use the subscript "$-i$" to represent player i's opponents. We assume that each player $i \in N$ with any $v_i \in V_i$ assigns a strictly positive probability to every $v_{-i} \in V_{-i}$ and that the auctioneer assigns a strictly positive probability to every $v \in V$.

Each player i chooses her bid $b_i \in B_i = \{0, d, \ldots, \bar{b}_i - d, \bar{b}_i\}$ where $\bar{v} \leq \bar{b}_i$ for each $i \in N$ and hence $V_i \subseteq B_i$. A player wins only if her bid is the highest. If there are multiple players who chose the highest bid, each one of them has an equal chance of winning. If player $i \in N$ is the winner, the price she pays, s, is such that $s = b_i$ for FPA and $s = \max_{j \neq i}\{b_j\}$ for SPA. Player i's utility is $u_i(v_i - s)$ if she wins the object while it is $u_i(0)$ otherwise.

[6] This depends on the uniqueness result for FPA which is a variant of the uniqueness results for FPA via rationalizability in Dekel and Wolinsky (2003) and Robles and Shimoji (2012).

3 Maximal Elimination of Weakly Dominated Bids

In this section, we solve both FPA and SPA via the maximal elimination of weakly dominated bids.

3.1 Winning Bids in FPA and SPA

In SPA, for each $i \in N$ and $v_i \in V_i$, $b_i = v_i$ is the only bid surviving the maximal elimination of weakly dominated bids (i.e., weakly dominant bid). Given $v \in V$, the winning bid in SPA is hence $\max_{j \in N}\{v_j\}$.

For FPA, Battigalli and Siniscalchi (2003, p.41) and Dekel and Wolinsky (2003, Subsection 4.3) show that for each $i \in N$ and $v_i \in V_i$, the highest bid which survives the maximal elimination of weakly dominated bids is strictly lower than v_i.[7]

Lemma 1 (Battigalli and Siniscalchi (2003) and Dekel and Wolinsky (2003)) *For each $i \in N$ and $v_i \in V_i$, the highest bid which survives the maximal elimination of weakly dominated bids in FPA is* $\max\{v_i - d, 0\}$.

We then have the following result.

Corollary 1 *Given $v \in V$, the highest possible winning bid in FPA is* $\max\{\max_{j \in N}\{v_j - d\}, 0\}$.

Note that if $\underline{v} > 0$, the expression is simply $\max_{j \in N}\{v_j - d\}$. Corollary 1 leads to the following result.

Lemma 2 *Given $v \in V$, the winning bid in SPA is weakly higher than the winning bid in FPA. If* $\max_{j \in N}\{v_j\} > 0$*, the winning bid in SPA is strictly higher than the winning bid in FPA.*

If $\underline{v} > 0$, the latter is indeed the case.

[7] Battigalli and Siniscalchi (2003) and Dekel and Wolinsky (2003) assume $V_i = V_j$ for every $i, j \in N$ (i.e., identical support) with $\underline{v} = 0$ and the former uses the continuous bid and value spaces. Their insight remains valid even with the heterogeneous supports.

3.2 Revenue Comparison

Lemma 2 implies that if two highest values are the same and strictly higher than 0, the revenue of SPA is strictly higher than that of FPA. Let P represent the auctioneer's belief over players' values. The following result shows a condition under which this possibility of "ties" outweighs other possibilities, leading to our main result.

Proposition 1 *The expected revenue from SPA is strictly higher than that of FPA if*

$$d \sum_{v' \in \{\max\{\underline{v},d\},\ldots,\bar{v}\}} P\left(\begin{array}{c} \exists i,j \in N \text{ such that } v_i = v_j = v' \\ v' \geq v_k \ \forall k \in N \in \{i,j\} \end{array}\right)$$
$$> \sum_{v' \in \{\max\{\underline{v},2d\},\ldots,\bar{v}\}} \sum_{v'' \in \{0,\ldots,v'-2d\}} [(v'-d)-v'']P\left(\begin{array}{c} \exists i,j \in N \text{ such that } v_i = v' \\ v'-2d \geq v_j = v'' \geq v_k \ \forall k \in N \backslash \{i,j\} \end{array}\right).$$

The expression on the left-hand side concerns the cases in which the first and second highest values are equal, implying that the revenue from SPA is higher than that of FPA. The expression on the right-hand side concerns the cases where the difference between the first and second highest values are at least $2d$, implying that the revenue from FPA *can* be higher than that of SPA.[8] Given that values are linear in d, note also that the expressions on both sides are linear in d, implying that the size of d does not matter for the result.

To visualize the implication of Proposition 1, Figure 1 plots the combination of two order statistics, the highest and the second highest values, v' and v'' respectively.

$A:$ $v' = v''$ corresponding to the left-hand side expression of Proposition 1. In this case, (i) SPA leads to a higher revenue than FPA, and (ii) the difference in the revenues is d.

$B:$ $v' = v'' + d$ which does not appear in the expression. In this case, FPA and SPA generates the same revenue.

$C:$ $v' - v'' \geq 2d$ corresponding to the expression on the right-hand side. They are the cases where (i) FPA generates a higher revenue than SPA and (ii) the difference of revenues is $(v' - d) - v'' \geq d$.

[8] Since we consider the best possible scenario for FPA, even if the condition in Proposition 1 does not hold, this does not necessarily mean that FPA generates a higher expected return than SPA does.

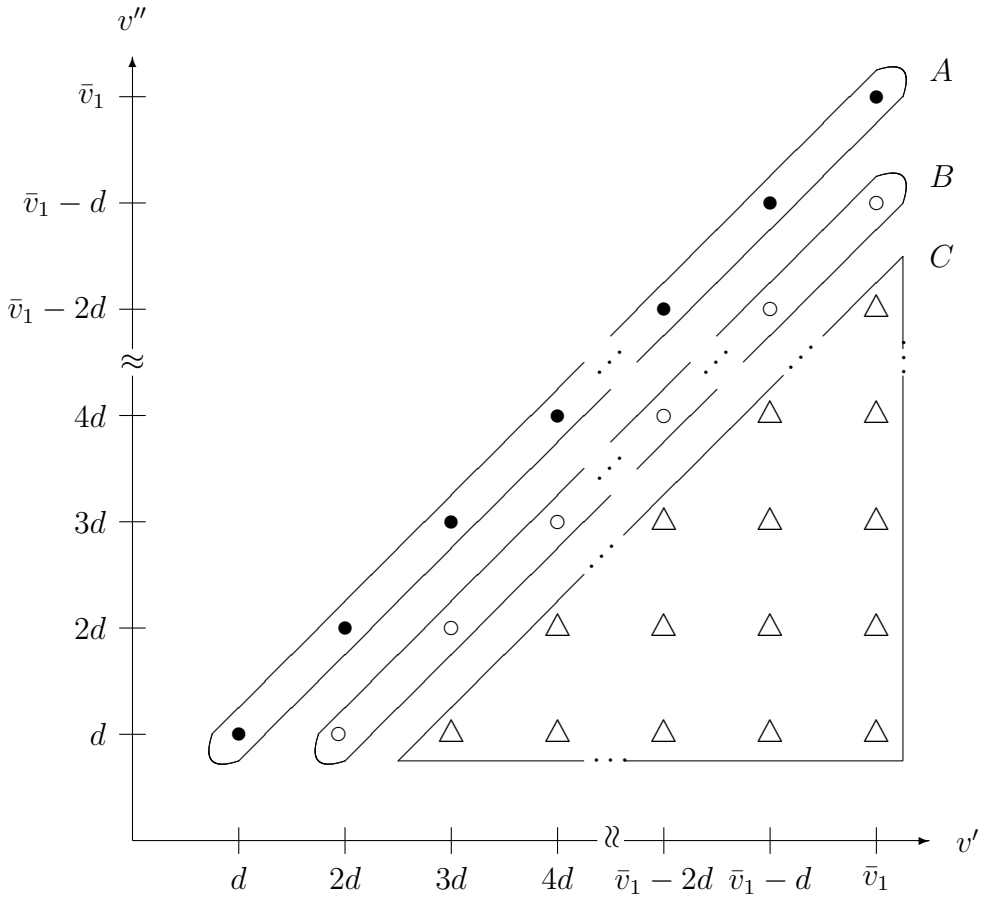


Figure 1: Visualization of Proposition 1

The expression in Proposition 1 says that if the realizations in A are likely, SPA generates a higher expected revenue than FPA.

3.3 Large Auctions

We now provide a condition under which Proposition 1 holds for sufficiently large n. Given n, let

- $\bar{N} = \{i \in N \mid \bar{v} \in V_i\}$ and $\bar{n} = |\bar{N}|$,
- $q_n(\tilde{n})$ be the probability that $\bar{n} = \tilde{n}$ where $\tilde{n} \in \{0, \ldots, \bar{n}\}$, and
- $\rho_i(v_i)$ be the probability that player i's value is $v_i \in V_i$.

We need the following assumptions:

Assumption 1 *Player's values are independently distributed.*

Assumption 2 *There exists* $\underline{\rho} \in (0,1)$ *such that* $\rho_i(\bar{v}) \geq \underline{\rho}$ *for each* $i \in \bar{N}$.

The expression on the left-hand side in Proposition 1 contains the probabilities that the highest and second highest values are the same. The following result identifies a condition under which the expression in Proposition 1 holds as $n \to \infty$.[9]

Proposition 2 *Given Assumptions 1 and 2, if*

$$\lim_{n\to\infty} \left\{ \sum_{\bar{n}=2}^{n} q_n(\bar{n}) \left[1 - \bar{n}\underline{\rho}(1-\underline{\rho})^{\bar{n}-1} - (1-\underline{\rho})^{\bar{n}} \right] \right\} = 1,$$

Proposition 1 holds for sufficiently large n.

The condition implies that as n becomes large, there are a sufficient number of players who has $\bar{v}$ in the support and the chance that there is only one player whose value is $\bar{v}$ diminishes. Note that if $\bar{v}_i = \bar{v}$ for each $i \in N$, $\bar{n} = n$ and hence $q_n(n) = 1$.

Corollary 2 *Given Assumptions 1 and 2, if* $\bar{v}_i = \bar{v}$ *for each* $i \in N$, *Proposition 1 holds for sufficiently large* n.

As an example, consider the case of $V_i = \{0, \ldots, \bar{v}\}$ for each $i \in N$ and each player's value is independently and uniformly distributed. Let $|V| = m+1$ (i.e., $\bar{v} = md$) and hence the probability attached to each value is $\frac{1}{m+1}$. The expression in Proposition 1 becomes

$$\begin{aligned} & d \sum_{i=2}^{m+1} \binom{n}{2} \left(\frac{1}{m+1}\right)^2 \left(\frac{i}{m+1}\right)^{n-2} \\ > & \sum_{i=3}^{m+1} \sum_{j=1}^{i-2} [(i-1)-j] d \binom{n}{1} \left(\frac{1}{m+1}\right) \binom{n-1}{1} \left(\frac{1}{m+1}\right) \left(\frac{j}{m+1}\right)^{n-2} \end{aligned}$$

[9] The proof in Yu (1999, Proposition 13) for the symmetric case carries the same observation; i.e., the probability that the first and second highest values coincide converges to one as $n \to \infty$.

which can be simplified as:

$$\sum_{i=2}^{m+1} i^{n-2} - 2\sum_{i=3}^{m+1}\sum_{j=1}^{i-2}(i-j-1)j^{n-2} > 0. \tag{1}$$

For large m, it is sufficient for the number of players, n, to be approximately 88.2% of m to maintain (1).

4 Iterative Maximal Elimination of Weakly Dominated Bids

The result for FPA is not as sharp as that of SPA. This is because we have focused on (one round of) the maximal elimination of weakly dominated bids and strictly increasing utility functions. If we use the *iterative* maximal elimination of weakly dominated bids and weakly concave (still strictly increasing) utility functions, we obtain a condition under which the uniqueness result is achieved for FPA. This is a variant of the results from Dekel and Wolinsky (2003) and Robles and Shimoji (2012) which use rationalizability. This result leads to the virtual revenue equivalence.

4.1 Uniqueness in FPA

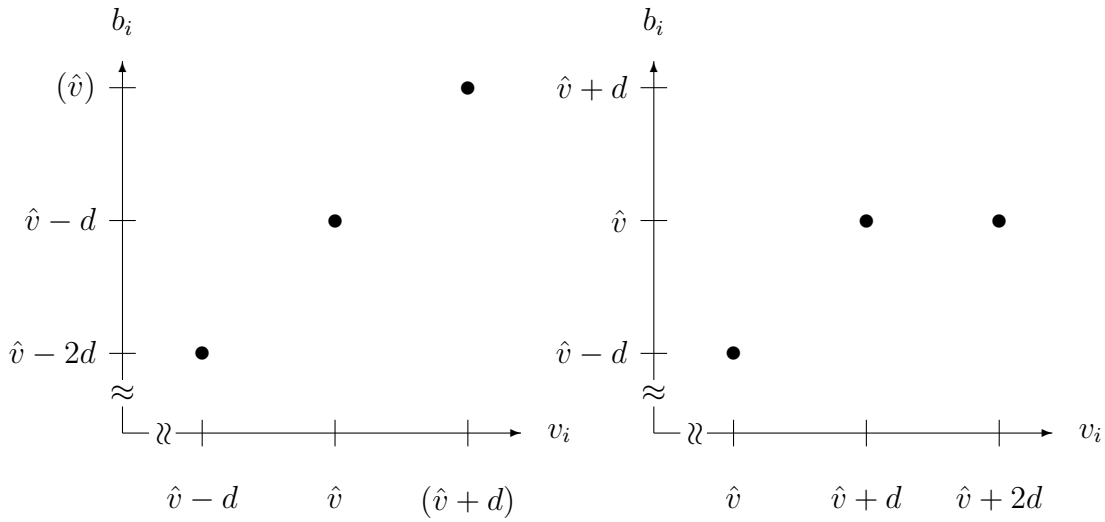
In this subsection, we show a condition under which each player $i \in N$ with $v_i \in V_i$ has a unique bid surviving the iterative maximal elimination of weakly dominated bids. Let

- $p_{v_i}(v_{-i})$ corresponds to player i's belief over the opponents' values when her value is $v_i \in V_i$, and
- player $i' \in N$ be such that $\bar{v}_{i'} = \bar{v}$ and $\hat{v} = \max_{j \neq i'}\{\bar{v}_j\}$ (i.e., the second highest upper bound).

We need the following assumption:

Assumption 3 *For each $i \in N$, u_i is weakly concave.*

We then have the following result:



(i) $\bar{v}-\hat{v} \leq d$ (i) $\bar{v}-\hat{v} \geq 2d$

Figure 2: Uniqueness in FPA

Proposition 3 *Suppose Assumption 3 holds. For each* $\alpha \in \{2, \ldots, \frac{\min\{\bar{v}-d, \hat{v}\}}{d}\}$*, suppose that the following expression holds for each* $i \in N$ *with* $v_i \geq (\alpha+1)d$*:*

$$\sum_{k=0}^{n-1} p_{v_i}\left(|\#\ \textit{of}\ j\textit{'s such that}\ v_j \geq \alpha d| = k\right) \frac{1}{k+1} \leq \frac{v_i - \alpha d}{v_i - (\alpha-1)d}.$$

Then,

- *if* $\bar{v}-\hat{v} \leq d$*, the only bid which survives iterative weak dominance is* $b_i = \max\{v_i - d, 0\}$ *for each* $i \in N$ *and* $v_i \in V_i$*, and*
- *if* $\bar{v}-\hat{v} \geq 2d$*, the only bid which survives iterative weak dominance is*
  1. $b_i = \max\{v_i - d, 0\}$ *for each* i *with* $v_i \leq \hat{v}+d$ *and*
  2. $b_{i'} = \hat{v}$ *for player* i' *with* $v_{i'} \geq \hat{v}+2d$*.*

Note that the right-hand side expression in the condition is at least $\frac{1}{2}$. The result is visualized in Figure 2.

We already know that for $v_i \in \{0, d\}$, $b_i = 0$ is the unique weakly dominant bid. With the assumption that u_i is weakly concave, we can also show that for any $v_i \geq 2d$, $b_i = 0$ is eliminated. Consider $v_i \geq 3d$ and compare $b_i = d$ and $b_i = 2d$:

1. $b_i = 2d$ may win even if $b_i = d$ does not win (but not vice versa). In particular, if the opponents' highest bid is $2d$, the expected utility from $b_i = 2d$ is strictly positive while it is zero for $b_i = d$.

2. If $b_i = d$ wins, the opponents' highest bid is either d or 0. This is the only case where the expected utility from $b_i = 2d$ may be lower than that of $b_i = d$. In this particular scenario, given the argument above,

   - $b_j = d$ for every $j \in N$ with $v_j \geq 2d$, and
   - $b_j = 0$ (only surviving bid) for every $j \in N$ with $v_j \leq d$.

   The corresponding utility is $u_i(v_i - d)$. The left-hand side expression in the condition corresponds to the probability that $b_i = d$ wins. Note that $b_i = 2d$ wins in this case and the corresponding utility is $u_i(v_i - 2d)$.

The condition in Proposition 3 rather states that the expected return from $b_i = 2d$ is higher than that of $b_i = d$ if the opponents' highest bid is either d or 0. This condition is sufficient as long as u_i is weakly concave (Jensen's inequality). The same argument is applied repeatedly to obtain the result.

4.2 Large Auctions

Consider again the case of large auctions with independent distributions from the previous section. Remember the definition of $\hat{v}$ in the precious subsection: the second highest upper bound. Given n, let

- $\hat{N} = \{i \in N \mid \hat{v} \in V_i\}$ and $\hat{n} = |\hat{N}|$,
- $r_{v_i,n}(\tilde{n})$ be the probability from the view point of player i with $v_i \in V_i$ that $\hat{n} = \tilde{n}$ where $\tilde{n} \in \{0, \ldots, \hat{n} - 1\}$ if $i \in \hat{N}$ or $\tilde{n} \in \{0, \ldots, \hat{n}\}$ if $i \notin \hat{N}$, and
- $\tau_i(v_i \geq \tilde{v})$ be the probability that player i's value $v_i \in V_i$ is weakly higher than $\tilde{v}$.

Instead of Assumption 2, we require the following:

Assumption 4 *There exists $\underline{\tau} \in (0, 1)$ such that $\tau_i(v_i \geq \hat{v}) \geq \underline{\tau}$ for each $i \in \hat{N}$.*

We then have the following result.

Proposition 4 *Given Assumptions 1, 3 and 4, if*

$$\lim_{n\to\infty}\left\{\sum_{\hat{n}=1}^{n-1} r_{v_i,n}(\hat{n})\frac{1-(1-\underline{\tau})^{\hat{n}}}{\hat{n}\underline{\tau}}\right\} \leq \frac{1}{2},$$

for each $i \in N$ with $v_i \geq 3d$, the result of Proposition 3 holds for sufficiently large n.

For example, the condition holds if $r_{v_i,n}(\hat{n})$ is lower for smaller $\hat{n}$'s. Again, if $\bar{v}_i = \bar{v}$ for each $i \in N$ (i.e., $r_{v_i,n}(n-1) = 1$), the result immediately holds.

Corollary 3 *Given Assumptions 1, 3 and 4, if $\bar{v}_i = \bar{v}$ for each $i \in N$, Proposition 3 holds for sufficiently large n.*

As an example, consider again the case where $V_i = \{0, \ldots, md\}$ for each $i \in N$ and the values are independently and uniformly distributed. Given $\alpha \in \{2, \ldots, m-1\}$, the left-hand side expression in Proposition 3 becomes

$$\begin{aligned} & \sum_{k=0}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) \left(\frac{m+1-\alpha}{m+1}\right)^k \left(\frac{\alpha}{m+1}\right)^{(n-1)-k} \left(\frac{1}{k+1}\right) \\ = & \frac{m+1}{(m+1-\alpha)n}\left[1-\left(\frac{\alpha}{m+1}\right)^n\right]. \end{aligned}$$

Remember that the right-hand side expression in Proposition 3 is at least $\frac{1}{2}$. Thus, for any $\alpha \in \{2, \ldots, m-1\}$, $n \geq m+1$ suffices.

Given Corollaries 2 and 3 above, we have the following result.[10]

Proposition 5 *Given Assumptions 1, 2, 3 and 4, if $\bar{v}_i = \bar{v}$ for each $i \in N$, the difference of the expected revenues from FPA and SPA via iterative weak dominance converges to d as $n \to \infty$.*

This can be seen as the virtual revenue equivalence result for small d in large auctions.

5 Discussion

In this section, we discuss (i) the order dependence of weak dominance, (ii) the upper bound identified in Lemma 1 and (iii) the comparison of our result to Kirkegaard (2012).

[10]Proposition 4 itself does not state that the expected revenue from FPA converges to $\bar{v} - d$. In addition, since $\bar{v}_i = \bar{v}$ for each $i \in N$, Assumptions 2 and 4 essentially have the same implication.

5.1 Order Dependence

The emphasis on "maximal" elimination is due to the possibility of the order dependence of weak dominance.[11] That is, different orders of elimination could lead to different predictions. As an example, consider the two-player SPA where the supports of their values, $V_i = \{\underline{v}_i, \ldots, \bar{v}_i\}$ for each $i \in \{1, 2\}$, is such that $\bar{v}_2 < \underline{v}_1$. First, eliminate all bids except $b_1 = v_1$ for player 1 with $v_1 \in V_1$ at the first step. Then, for player 2, eliminate every $b_2 \geq \underline{v}_1$ at the second step. No further elimination occurs. In this case, the second highest bid can be higher or lower than v_2, and the price the winner (player 1) pays cannot be uniquely identified for any $v_2 \in V_2$. Note that this applies not only to our result, but also to previous studies which use weak dominance for SPA.

5.2 Upper Bound in FPA

Lemma 1 identifies an upper bound of bid for each value in FPA. This becomes tight once we use the iterative maximal elimination of weakly dominated bids under the condition in Proposition 3. In this subsection, we show a simple example where the bound can even be lower in FPA. This observation is due to Battigalli and Siniscalchi (2003) which use rationalizability and the continuous bid and value spaces.

Consider two-player independent private-value FPA with $d = 1$. The set of values for player $i = 1, 2$ is $V_i = \{0, 1, 2, 3\}$ and let $B_i = V_i$. The chance of each v_i is $\rho(0) = \alpha$, $\rho(1) = \beta$, $\rho(2) = \gamma$, and $\rho(3) = 1 - \alpha - \beta - \gamma$ where for each $v_i \in V_i$, $\rho(v_i) \in (0, 1)$. Lemma 1 says that the set of weakly undominated bids for each $v_i \in V_i$ at the first step is (i) $\{0\}$ for $v_i \in \{0, 1\}$, (ii) $\{0, 1\}$ for $v_i = 2$ and (iii) $\{0, 1, 2\}$ for $v_i = 3$. Furthermore, assuming that u_i is weakly concave for each $i \in N$, $b_i = 0$ is also weakly dominated for each $i \in \{1, 2\}$ and $v_i \in \{2, 3\}$ at the first step: For $v_i \in \{2, 3\}$, the expected utility from $b_i = 0$ is either (i) $\frac{1}{2}(u_i(v_i) + u_i(0))$ if $b_j = 0$ or (ii) $u_i(0)$ otherwise. The expected utility from $b_i = 1$ is (i) $u_i(v_i - 1)$ if $b_j = 0$, (ii) $\frac{1}{2}(u_i(v_i - 1) + u_i(0))$ if $b_j = 1$ or (iii) $u_i(0)$ otherwise. Since

$$u_i(v_i - 1) \geq u_i(\tfrac{v_i}{2}) \geq \tfrac{1}{2}(u_i(v_i) + u_i(0))$$

where the second inequality comes from Jensen's inequality, $b_i = 1$ weakly dominates

[11] See for example Marx and Swinkels (1997).

$b_i = 0$ for each $i \in \{1,2\}$ and $v_i \in \{2,3\}$. For each $i \in \{1,2\}$, at the first step of the iterative maximal elimination of weakly dominated bids, we have (i) $\{0\}$ for $v_i \in \{0,1\}$, (ii) $\{1\}$ for $v_i = 2$, and (iii) $\{1,2\}$ for $v_i = 3$.

Given the result from the first step, we consider $v_i = 3$ at the second step. The expected utility from $b_i = 1$ is

$$\begin{aligned}&(\alpha+\beta)u_i(2) + \gamma\left[\frac{1}{2}(u_i(2)+u_i(0))\right] \\ &+(1-\alpha-\beta-\gamma)\left[\left\{\begin{array}{c}\frac{1}{2}(u_i(2)+u_i(0)) \\ u_i(0)\end{array}\right\} \text{ if } b_j(3) = \left\{\begin{array}{c}1\\2\end{array}\right\}\right] \end{aligned} \tag{2}$$

while the expected utility from $b_i = 2$ is

$$(\alpha+\beta+\gamma)u_i(1) + (1-\alpha-\beta-\gamma)\left[\left\{\begin{array}{c}u_i(1) \\ \frac{1}{2}(u_i(1)+u_i(0))\end{array}\right\} \text{ if } b_j(3) = \left\{\begin{array}{c}1\\2\end{array}\right\}\right]. \tag{3}$$

Then,

$$\begin{aligned} & (2)-(3) \\ = \; & (\alpha+\beta)(u_i(2)-u_i(1)) + \gamma\left\{\frac{1}{2}(u_i(2)+u_i(0)) - u_i(1)\right\} \\ & (1-\alpha-\beta-\gamma)\left[\left\{\begin{array}{c}\frac{1}{2}(u_i(2)+u_i(0)) - u_i(1) \\ u_i(0) - \frac{1}{2}(u_i(1)+u_i(0))\end{array}\right\} \text{ if } b_j(3) = \left\{\begin{array}{c}1\\2\end{array}\right\}\right]. \end{aligned}$$

Note that if $\alpha+\beta$ is close to one, the expression above is strictly positive independent of $b_j(3)$. In this case, $b_i = 1$ strictly dominates $b_i = 2$ for $v_i = 3$.

5.3 On Kirkegaard (2012)

Kirkegaard (2012) identifies two conditions under which FPA leads to a higher expected revenue than SPA. There are several reasons why our result is different from that of Kirkegaard (2012). One reason is that values and bids are discrete in our setting while they are continuous in Kirkegaard (2012) – a tie is not possible in Kirkegaard (2012). Another reason is that while our focus is on weak dominance, Kirkegaard (2012) uses Bayesian Nash Equilibrium for FPA. We now demonstrate that even if (a discretized version of) a condition in Kirkegaard (2012) is satisfied in

our discrete setting, it is possible that our condition still hold.[12]

Kirkegaard (2012) considers the case of (i) two players and (ii) the bid and value spaces are continuous. Take V_i' as a closed interval for $i \in \{1,2\}$, i.e,. $V_i' = [\underline{v}_i, \bar{v}_i]$. Kirkegaard (2012) assumes that the supports for values are such that $0 \leq \underline{v}_2 \leq \underline{v}_1$ and $\bar{v}_2 < \bar{v}_1$ (player 1 is strong and player 2 is weak). Let $F_i(v)$ be the pdf for player i's value. Kirkegaard (2012) assumes

$$\frac{f_1(v)}{F_1(v)} \geq \frac{f_2(v)}{F_2(v)} \text{ and } \frac{f_1(v)}{1-F_1(v)} \leq \frac{f_2(v)}{1-F_2(v)} \text{ for any } v \in V_1' \cap V_2'. \tag{4}$$

That is, F_1 dominates F_2 in terms of not only the reverse hazard rate but also the hazard rate. Given v_2, let $r(v_2) = F_1^{-1}(F_2(v_2))$ (or $F_1(r(v_2)) = F_2(v_2)$). The sufficient condition in Kirkegaard (2012, Expression (9)) is

$$f_2(v_2) \geq f_1(v_1) \text{ for all } v_2 \in V_2' \text{ and } v_1 \in [v_2, r(v_2)]. \tag{5}$$

Note that in the discrete setting, there does not exist the corresponding $r(v_2)$ for each $v_2 \in V_2$ generically. Thus, we instead require (5) to hold for every $v_1 \in \{v_2, \ldots, \bar{v}_1\}$. We also look at the discretized version of (4).

We now turn to an example in the discrete setting. Consider the following example:

$$\begin{aligned} V_1 &= \{\underline{v}_2, \underbrace{\underline{v}_2 + d, \ldots, \bar{v}_2}_{\kappa}, \bar{v}_2 + d\} \text{ and} \\ V_2 &= \{\underline{v}_2, \underbrace{\underline{v}_2 + d, \ldots, \bar{v}_2}_{\kappa}\} \end{aligned}$$

where (i) $\underline{v}_2 > 0$ and (ii) κ is a positive integer. Note that $\bar{v}_1 = \bar{v}_2 + d$ and hence $V_2 \subset V_1$. The players' values are independently distributed. Let

- $\rho_1(v_1) = \varepsilon$ for each $v_1 \in V_1 \backslash \{\underline{v}_2\}$ and $\rho_2(v_2) = \varepsilon$ for each $v \in V_2 \backslash \{\underline{v}_2\}$, and
- $\rho_1(\underline{v}_2) = 1 - (\kappa+1)\varepsilon$ and $\rho_2(\underline{v}_2) = 1 - \kappa\varepsilon$.

We also assume that ε is sufficiently small.

The first inequality in (4) holds with the equality for $\underline{v}_2$ and with the strict inequality for every $v \in V_2 \backslash \{\underline{v}_2\}$. The second inequality in (4) holds with strict inequality for every $v \in V_2$. The inequality in (5) holds with the strict inequality for $\underline{v}_2$ and

[12] Kirkegaard (2012) has two sufficient conditions. We only focus on one of them.

with the equality for every $v_1 \in V_1 \backslash \{\underline{v}_2\}$ and $v_2 \in V_2 \backslash \{\underline{v}_2\}$. Consider the expression in Proposition 1. For sufficiently small ε, the sum of probabilities on the left-hand side is close to one, implying that the inequality is satisfied.[13]

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A Proof of Proposition 1

The highest possible revenue in FPA is $\max\{\max_{i\in N}\{v_i - d\}, 0\}$. The revenue in SPA is the second highest value. Note that if there are multiple highest values which are strictly higher than 0, the revenue from SPA is strictly higher than that of FPA.

Let v' be the highest value and v'' be the second highest value. The expected revenue from SPA (left-hand side) is strictly higher than that of FPA (right-hand side) if

$$
\begin{aligned}
& \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} \sum_{v''\in\{0,\ldots,v'\}} v'' P\left(\begin{array}{c}\exists i,j\in N \text{ such that}\\ v'=v_i\geq v''=v_j\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
> & \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} (v'-d) P\left(\begin{array}{c}\exists i\in N \text{ such that}\\ v_i=v'\geq v_j\ \forall j\in N\backslash\{i\}\end{array}\right) \\
\Leftrightarrow & \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} v' P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v_j=v'\\ v'\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
& + \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} (v'-d) P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v'\\ v_j=v'-d\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
& + \sum_{v'\in\{\max\{\underline{v},2d\},\ldots,\bar{v}\}} \sum_{v''\in\{0,\ldots,v'-2d\}} v'' P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v'\\ v'-2d\geq v_j=v''\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
> & \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} (v'-d) P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v_j=v'\\ v'\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
& + \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} (v'-d) P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v'\\ v_j=v'-d\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
& + \sum_{v'\in\{\max\{\underline{v},2d\},\ldots,\bar{v}\}} (v'-d) \sum_{v''\in\{0,\ldots,v'-2d\}} P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v'\\ v'-2d\geq v_j=v''\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
\Leftrightarrow & d \sum_{v'\in\{\max\{\underline{v},d\},\ldots,\bar{v}\}} P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v_j=v'\\ v'\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right) \\
> & \sum_{v'\in\{\max\{\underline{v},2d\},\ldots,\bar{v}\}} \sum_{v''\in\{0,\ldots,v'-2d\}} [(v'-d)-v''] P\left(\begin{array}{c}\exists i,j\in N \text{ such that } v_i=v'\\ v'-2d\geq v_j=v''\geq v_k\ \forall k\in N\backslash\{i,j\}\end{array}\right).
\end{aligned}
$$

B Proof for Proposition 2

Given n and $\bar{n}$ where $2 \leq \bar{n} \leq n$, take the probability that the second highest value is $\bar{v}$:

$$\sum_{\eta=2}^{\bar{n}} \left[\sum_{i_1 \in \bar{N}} \sum_{i_2 > i_1} \cdots \sum_{i_\eta > i_{\eta-1}} \rho_{i_1}(\bar{v}) \rho_{i_2}(\bar{v}) \cdots \rho_{i_\eta}(\bar{v}) \prod_{k \neq i_1, \ldots, i_\eta} (1 - \rho_k(\bar{v})) \right] \tag{6}$$

where $\{i_1, \ldots, i_\eta\} \subseteq \bar{N}$. The lower bound of the expression above is

$$\begin{aligned} & \sum_{\eta=2}^{\bar{n}} \left(\begin{array}{c} \bar{n} \\ \eta \end{array} \right) \underline{\rho}^\eta (1 - \underline{\rho})^{\bar{n}-\eta} \\ = & \sum_{\eta=0}^{\bar{n}} \left(\begin{array}{c} \bar{n} \\ \eta \end{array} \right) \underline{\rho}^\eta (1 - \underline{\rho})^{\bar{n}-\eta} - \bar{n}\underline{\rho}(1 - \underline{\rho})^{\bar{n}-1} - (1 - \underline{\rho})^{\bar{n}} \\ = & 1 - \bar{n}\underline{\rho}(1 - \underline{\rho})^{\bar{n}-1} - (1 - \underline{\rho})^{\bar{n}}. \end{aligned}$$

Note that the last two expression converge to zero as $\bar{n} \to \infty$ (Corollary 2).

The lower bound for the left-hand expression in Proposition 1 (without d) is hence

$$\sum_{\bar{n}=2}^{n} q_n(\bar{n}) \left[1 - \bar{n}\underline{\rho}(1 - \underline{\rho})^{\bar{n}-1} - (1 - \underline{\rho})^{\bar{n}} \right].$$

Hence, if

$$\lim_{n \to \infty} \left\{ \sum_{\bar{n}=2}^{n} q_n(\bar{n}) \left[1 - \bar{n}\underline{\rho}(1 - \underline{\rho})^{\bar{n}-1} - (1 - \underline{\rho})^{\bar{n}} \right] \right\} = 1,$$

the expression on the left-hand side of Proposition 1 converges to d while the one on the right-hand side converges to zero.

C Revenue Comparison: Uniform Example

Consider again (1):

$$\sum_{i=2}^{m+1} i^{n-2} - 2 \sum_{i=3}^{m+1} \sum_{j=1}^{i-2} (i - j - 1) j^{n-2} > 0,$$

or

$$2^{n-2} + \sum_{i=3}^{m+1} \left[i^{n-2} - 2 \sum_{j=1}^{i-2} (i-j-1) j^{n-2} \right] > 0.$$

We compare the terms in the parentheses for each $i \in \{3, \ldots, m+1\}$.

Let

$$\begin{aligned}
A(i,n) &= i^{n-2} - 2 \sum_{j=1}^{i-2} (i-j-1) j^{n-2} \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - 2 \sum_{j=1}^{i-2} ((i-1)-j-1) j^{n-2} \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - 2 \sum_{j=1}^{i-3} ((i-1)-j-1) j^{n-2} \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - (i-1)^{n-2} + \left[(i-1)^{n-2} - 2 \sum_{j=1}^{i-3} ((i-1)-j-1) j^{n-2} \right] \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - (i-1)^{n-2} + A(i-1,n), \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
D(i,n) &= A(i,n) - A(i-1,n) \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - (i-1)^{n-2} \\
&= i^{n-2} - 2 \sum_{j=1}^{i-2} j^{n-2} - (i-1)^{n-2} - \left[(i-1)^{n-2} - 2 \sum_{j=1}^{i-3} j^{n-2} - (i-2)^{n-2} \right] \\
&\quad + \left[(i-1)^{n-2} - 2 \sum_{j=1}^{i-3} j^{n-2} - (i-2)^{n-2} \right] \\
&= i^{n-2} - (i-2)^{n-2} - 2(i-1)^{n-2} + D(i-1,n). \qquad (8)
\end{aligned}$$

For every $i \geq 3$, (i) $A(i,n)$ is strictly increasing in n if $A(i,n) \geq 0$ and (ii) $D(i,n)$

is strictly increasing in n if $D(i,n) \geq 0$:

$$\begin{aligned}
\frac{\partial A(i,n)}{\partial n} &= i^{n-2}\ln(i) - 2\sum_{j=1}^{i-2}(i-j-1)j^{n-2}\ln(j) > \ln(i)A(i,n) \\
\frac{\partial D(i,n)}{\partial n} &= i^{n-2}\ln(i) - 2\sum_{j=1}^{i-2}j^{n-2}\ln(j) - (i-1)^{n-2}\ln(i-1) > \ln(i)D(i,n).
\end{aligned}$$

Hence, for every $i \geq 3$, if $A(i,n') \geq 0$ for some n', $A(i,n) > 0$ for any $n > n'$. Likewise, if $D(i,n'') \geq 0$ for some n'', $D(i,n) > 0$ for any $n > n''$.

We use induction to show that our claim holds. For the initial step, note that

$$\begin{aligned}
A(3,n) &= 3^{n-2} - (2 \times 1^{n-2}) > 0 \\
D(3,n) &= 3^{n-2} - (2 \times 1^{n-2}) - 2^{n-2} > 0
\end{aligned}$$

for any $n \geq 4$.

Suppose now that there exist $\tilde{i} \geq 3$ and $\tilde{n} \geq 4$ such that for each $i \in \{3,\ldots,\tilde{i}\}$, $A(i,\tilde{n}) > 0$ and $D(i,\tilde{n}) > 0$ (this is the case for $\tilde{i} = 3$ and $\tilde{n} = 4$). If $A(\tilde{i}+1,\tilde{n}) > 0$, choosing $n = \tilde{n}$ suffices. Suppose instead $A(\tilde{i}+1,\tilde{n}) \leq 0$. Given that $A(\tilde{i},\tilde{n}) > 0$ and $D(\tilde{i},\tilde{n}) > 0$, it suffices to find $n^* > \tilde{n}$ such that,

$$(\tilde{i}+1)^{n^*-2} - (\tilde{i}-1)^{n^*-2} - 2\tilde{i}^{n^*-2} \geq 0.$$

This is because $A(\tilde{i},n^*) > 0$ and $D(\tilde{i},n^*) > 0$. The inequality above implies $D(\tilde{i}+1,n^*) > 0$ and hence $A(\tilde{i}+1.n^*) > 0$.

Note that

$$\begin{aligned}
& i^{n-2} - (i-2)^{n-2} - 2(i-1)^{n-2} \\
= \; & (i-1)^{n-2}\left\{\left(\frac{i}{i-1}\right)^{n-2} - \left(\frac{i-2}{i-1}\right)^{n-2} - 2\right\} \\
= \; & (i-1)^{n-2}\left\{\left[\left(1+\frac{1}{i-1}\right)^{i-1}\right]^{\frac{n-2}{i-1}} - \left[\left(1-\frac{1}{i-1}\right)^{i-1}\right]^{\frac{n-2}{i-1}} - 2\right\}
\end{aligned}$$

where (i) the expression in the parentheses is strictly increasing with respect to the power (i.e., $\frac{n-2}{i-1}$) and (ii) we have $\lim_{i\to\infty}(1+\frac{1}{i-1})^{i-1} = e$ and $\lim_{i\to\infty}(1-\frac{1}{i-1})^{i-1} = e^{-1}$.

Let $\ln(y) = \frac{n-2}{i-1}$ and we have

$$e^{ln(y)} - \frac{1}{e^{ln(y)}} \geq 2 \quad \Leftrightarrow \quad y^2 - 2y - 1 \geq 0$$

where (i) $y = 1 + \sqrt{2}$ if the expression is zero and (ii) the expression is strictly increasing if $y > 1$. Note that $\ln(1 + \sqrt{2}) \approx 0.8813$.[14]

D Proof for Proposition 3

First Step:

- For each $i \in N$ and $v_i \in V_i \backslash \{0\}$, every $b_i \geq v_i$ is weakly dominated. None of them leads to a utility strictly higher than $u_i(0)$ while $b_i < v_i$ secures $u_i(0)$ or higher (e.g., the opponents' highest bid is equal to b_i).

- For each $i \in N$ with $v_i = 0$, every $b_i \geq d$ is weakly dominated. Every $b_i \geq d$ leads to a utility of $u_i(0)$ or less (e.g., every opponent bids zero) and $b_i = 0$ guarantees $u_i(0)$.

- For each $i \in N$ with $v_i \geq 2d$, $b_i = d$ weakly dominates $b_i = 0$. The only way $b_i = 0$ wins is that every opponent bids zero as well. In this case, the expected utility with $b_i = 0$ is $\frac{1}{n}u_i(v_i) + \frac{n-1}{n}u_i(0)$ while it is $u_i(v_i - d)$ with $b_i = d$. Since $n \geq 2$, we have

  $$u_i(v_i - d) = u_i\left(\tfrac{v_i - d}{v_i} v_i\right) \geq \left(\tfrac{v_i - d}{v_i}\right) u_i(v_i) + \left(\tfrac{d}{v_i}\right) u_i(0) \geq \left(\tfrac{1}{n}\right) u_i(v_i) + \left(\tfrac{n-1}{n}\right) u_i(0)$$

  for any weakly concave u_i.[15] For the other possibilities, $b_i = d$ leads to a utility weakly higher than that of $b_i = 0$ (i.e., $u_i(0)$). In particular, if the opponents' highest bid is d, the inequality is strict.

The sets of bids surviving the maximal elimination of weakly dominated bids are (i) $\{\max\{v_i - d, 0\}\}$ for $v_i \in \{0, d, 2d\}$ – note that they are unique – and (ii) $\{d, \ldots, v_i - d\}$

[14] Remember that our result ignores the term 2^{n-2} in the first expression. The actual ratio hence can be lower.

[15] Note that this does not necessarily hold without the concavity of u_i. The first inequality uses Jensen's inequality. The second inequality holds since $\frac{v_i - d}{v_i} \geq \frac{1}{2} \geq \frac{1}{n}$.

for $v_i > 2d$. We need to check that for every $v_i > 2d$, no other bid is eliminated. For each $v_i > 2d$, take any $b_i, \check{b}_i \in \{d, \ldots, v_i - d\}$ where $\check{b}_i < b_i$.

1. $\check{b}_i$ does not weakly dominate b_i; if the opponents' highest bid is b_i, the expected utility from b_i is strictly higher than $u_i(0)$ while $\check{b}_i$ only leads to $u_i(0)$.

2. b_i does not weakly dominate $\check{b}_i$; if the opponents bid zero, the utility from $\check{b}_i$ is strictly higher than that of b_i.

This completes the first step.

As in the main text, let $i' \in N$ be such that $\bar{v}_{i'} = \bar{v}$ and $\hat{v} = \max_{j \neq i'}\{\bar{v}_j\}$. The next part is included if $\bar{v} - \hat{v} \geq 2d$.

Second Step: For player $i' \in N$ with $v_{i'} \geq \hat{v} + 2d$, every $b_{i'} \geq \hat{v} + d$ is weakly dominated since the highest possible bid of the opponents is $\hat{v} - d$.

In effect, we treat player i' with $v_{i'} \geq \hat{v} + 2d$ as $\hat{v} + d$ from the third step. The repetition of the next step leads to the result.

α-th Step ($\alpha \geq 2$): Suppose that there exists $\alpha \geq 2$ such that for each $i \in N$, the set of remaining bids are

- $\{\max\{v_i - d, 0\}\}$ for $v_i \leq \alpha d$, and
- $\{(\alpha - 1)d, \ldots, v_i - d\}$ for $v_i \geq (\alpha + 1)d$.[16]

Take player $i \in N$ with $v_i \geq (\alpha + 1)d$. We compare $b_i = (\alpha - 1)d$ and $b_i = \alpha d$. Note that $b_i = (\alpha - 1)d$ wins only if $b_j \leq (\alpha - 1)d$ for each $j \in N$. In this scenario, the expected utility from $b_i = (\alpha - 1)d$ is

$$\sum_{k=0}^{n-1} p_{v_i}(|\#\text{ of } j\text{'s such that } v_j \geq \alpha d| = k) \left[\left(\frac{1}{k+1} \right) u_i(v_i - (\alpha - 1)d) + \left(\frac{k}{k+1} \right) u_i(0) \right] \quad (9)$$

[16]Note that the statement holds for $\alpha = 2$. In addition, note that the latter becomes $\{(\alpha - 1)d, \ldots, \min\{v_i - d, \hat{v}\}\}$ from the third step.

while the utility from $b_i = \alpha d$ is $u_i(v_i - \alpha d)$.[17] Jensen's inequality implies

$$\begin{aligned} & (9) \\ \leq & \sum_{k=0}^{n-1} p_{v_i}(|\#\text{ of } j\text{'s such that } v_j \geq \alpha d| = k) u_i\left(\frac{v_i - (\alpha-1)d}{k+1}\right) \\ \leq & u_i\left(\sum_{k=0}^{n-1} p_{v_i}(|\#\text{ of } j\text{'s such that } v_j \geq \alpha d| = k)\frac{v_i - (\alpha-1)d}{k+1}\right). \end{aligned}$$

Since u_i is strictly increasing, the inequality below is equivalent to $u_i(v_i - \alpha d) \geq (9)$:

$$v_i - \alpha d \geq \sum_{k=0}^{n-1} p_{v_i}(|\#\text{ of } j\text{'s such that } v_j \geq \alpha d| = k)\frac{v_i - (\alpha-1)d}{k+1}$$

or

$$\frac{v_i - \alpha d}{v_i - (\alpha-1)d} \geq \sum_{k=0}^{n-1} p_{v_i}(|\#\text{ of } j\text{'s such that } v_j \geq \alpha d| = k)\frac{1}{k+1}.$$

For the other profiles of the opponents' bids, $b_i = \alpha d$ leads to a utility weakly higher than that of $b_i = (\alpha-1)d$. In particular, if the opponents' highest bid is αd, the inequality is strict. Hence, under the condition above, $b_i = \alpha d$ weakly dominates $b_i = (\alpha-1)d$ for each $i \in N$ with $v_i \geq (\alpha+1)d$.

Then, the remanning bids are

- $\{\max\{v_i - d, 0\}\}$ for $v_i \leq (\alpha+1)d$, and
- $\{\alpha d, \ldots, \min\{v_i - d, \hat{v}\}\}$ for $v_i \geq (\alpha+2)d$.

We need to check that for each $i \in N$ with $v_i \geq (\alpha+2)d$, no other bid is eliminated. For each $v_i \geq (\alpha+2)d$, take any $b_i, \check{b}_i \in \{\alpha d, \ldots, \min\{v_i - d, \hat{v}\}\}$ where $\check{b}_i < b_i$.

1. $\check{b}_i$ does not weakly dominate b_i; if the opponents' highest bid is b_i, the expected utility from b_i is strictly higher than $u_i(0)$ while $\check{b}_i$ only leads to $u_i(0)$.
2. b_i does not weakly dominate $\check{b}_i$; if the opponent's highest bid is $(\alpha-1)d$ or less, the utility from $\check{b}_i$ is strictly higher than that of b_i.

This completes the α-th step. The repetition of the same argument leads to our claim.

[17] Note that any player $j \in N\backslash\{i\}$ with $v_j \leq (\alpha-1)d$ never wins in this scenario.

E Proof of Proposition 4

Given $\hat{N}$, when players' values are independently distributed, the upper bound for the left-hand side of the condition in Proposition 3 (i.e., the chance that $b_i = (\alpha - 1)d$ wins) is

$$\begin{aligned}
& \sum_{k=0}^{\hat{n}-1} \left(\begin{array}{c} \hat{n}-1 \\ k \end{array} \right) \underline{\tau}^k (1-\underline{\tau})^{(\hat{n}-1)-k} \left(\frac{1}{k+1} \right) \\
= & \sum_{k=0}^{\hat{n}-1} \frac{(\hat{n}-1)!}{(k+1)!((\hat{n}-1)-k)!} \underline{\tau}^k (1-\underline{\tau})^{(\hat{n}-1)-k} \\
= & \frac{1}{\hat{n}\underline{\tau}} \sum_{k=0}^{\hat{n}-1} \frac{\hat{n}!}{(k+1)!(\hat{n}-(k+1))!} \underline{\tau}^{k+1} (1-\underline{\tau})^{\hat{n}-(k+1)} \\
= & \frac{1}{\hat{n}\underline{\tau}} \sum_{l=1}^{\hat{n}} \frac{\hat{n}!}{l!(\hat{n}-l)!} \underline{\tau}^{l} (1-\underline{\tau})^{\hat{n}-l} \\
= & \frac{1-(1-\underline{\tau})^{\hat{n}}}{\hat{n}\underline{\tau}}.
\end{aligned}$$

Note that there are at least two players who can have the values weakly higher than $\hat{v}$. Hence, given the definition of $\hat{v}$, $\hat{n} \geq 1$.[18] Therefore, if

$$\lim_{n\to\infty} \left\{ \sum_{\hat{n}=1}^{n-1} r_{v_i,n}(\hat{n}) \frac{1-(1-\underline{\tau})^{\hat{n}}}{\hat{n}\underline{\tau}} \right\} \leq \frac{1}{2}$$

for each $i \in N$ with $v_i \geq 3d$, the condition in Proposition 3 holds. Remember that the expression on the right-hand expression in the condition of Proposition 3 is weakly higher than $\frac{1}{2}$.

[18] This takes into account the possibility that player i can be one of them.

F Virtual Revenue Equivalence: Uniform Example

Given α, we have

$$
\begin{aligned}
& \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{m+1-\alpha}{m+1}\right)^{k} \left(\frac{\alpha}{m+1}\right)^{(n-1)-k} \left(\frac{1}{k+1}\right) \\
= & \left(\frac{1}{m+1}\right)^{n-1} \sum_{k=0}^{n-1} \frac{(n-1)!}{k![(n-1)-k]!} (m+1-\alpha)^{k} \alpha^{(n-1)-k} \left(\frac{1}{k+1}\right) \\
= & \left(\frac{1}{m+1}\right)^{n-1} \sum_{k=0}^{n-1} \frac{(n-1)!}{(k+1)!(n-(k+1))!} (m+1-\alpha)^{k} \alpha^{(n-1)-k} \\
= & \frac{1}{(m+1-\alpha)n} \left(\frac{1}{m+1}\right)^{n-1} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!(n-(k+1))!} (m+1-\alpha)^{k+1} \alpha^{(n-1)-k} \\
= & \frac{1}{(m+1-\alpha)n} \left(\frac{1}{m+1}\right)^{n-1} \sum_{k=0}^{n-1} \binom{n}{k+1} (m+1-\alpha)^{k+1} \alpha^{(n-1)-k} \\
= & \frac{1}{(m+1-\alpha)n} \left(\frac{1}{m+1}\right)^{n-1} \sum_{l=1}^{n} \binom{n}{l} (m+1-\alpha)^{l} \alpha^{n-l} \\
= & \frac{1}{(m+1-\alpha)n} \left(\frac{1}{m+1}\right)^{n-1} \left[\sum_{l=0}^{n} \binom{n}{l} (m+1-\alpha)^{l} \alpha^{n-l} - \alpha^{n}\right] \\
= & \frac{1}{(m+1-\alpha)n} \left(\frac{1}{m+1}\right)^{n-1} \left[(m+1)^{n} - \alpha^{n}\right] \\
= & \frac{m+1}{(m+1-\alpha)n} \left[1 - \left(\frac{\alpha}{m+1}\right)^{n}\right].
\end{aligned}
$$