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Abstract: This article proposes an efficient and incentive compatible dynamic auction for selling multiple complementary goods to finitely many bidders. The goods are traded in discrete quantities. The seller has a reserve price for every bundle of goods and determines which bundles to sell based on prevailing prices. The auctioneer announces a current price for every bundle of goods and a supply set of goods, every bidder subsequently responds with a set of goods demanded at these prices, and then the auctioneer adjusts prices. We prove that even when bidders can exercise their market power strategically, this dynamic auction always induces them to bid truthfully as price-takers, resulting in an efficient allocation, its supporting Walrasian equilibrium price for every bundle of goods, and a generalized Vickrey-Clarke-Groves payment for every bidder.

Keywords: Dynamic auction, complements, incomplete information, incentive, efficiency, ex post perfect equilibrium, indivisibility.

JEL classification: D44.

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1 Introduction

Auctions are fundamental and common market mechanisms for allocating goods and services. The current article provides an efficient and incentive compatible dynamic auction for allocating complementary goods. The goods are traded in discrete quantities such as houses, cars and machines, or oil measured in barrels. In the last two decades, dynamic auctions for selling multiple items have become phenomenally popular. For example, such auctions have been extensively used by governments to sell spectrum licenses (Klemperer 2004 and Milgrom 2004), to procure goods and services, and to privatize state companies, and by firms to sell virtually all kinds of commodities on the Internet.

Elegant efficient auctions include those proposed by Kelso and Crawford (1982), Demange, Gale and Sotomayor (1986), Gul and Stacchetti (2000), Milgrom (2000), Perry and Reny (2005), and Ausubel (2004, 2006), among others. Auctions developed so far have been especially successful in handling substitute goods. By contrast, how to deal with complementary goods has proved challenging.[3] In a recent survey article, Maskin (2005) presented as his first "open problem" the outstanding issue of how to handle multiple goods with complementarities in a dynamic auction, echoing Milgrom (2000), Jehiel and Moldovanu (2003), Noussair (2003), and Klemperer (2004), who have raised similar points. This article aims to offer a general solution to this problem, a solution which will be shown to have a number of desirable features.

There are several factors which make designing dynamic auctions for complementary items difficult. First, as Milgrom (2000, 2004), Jehiel and Moldovanu (2003), and Klemperer (2004) point out, when there are strong complementarities among items, an exposure problem can occur in a simultaneous ascending auction, resulting in inefficient outcomes. Consider for example the sale of spectrum licenses to several bidding firms. The value to the firms of a license covering one region may be increased significantly if licenses in adjacent regions are also obtained. In this situation, at earlier stages of the auction, because of low prices and synergies among licenses, bidders are likely to bid aggressively, but as prices rise to high levels, some or all licenses can be exposed to a risk that no bidder wishes to demand them anymore, because complementary items have become too expensive. As a result, the auction will get stuck in disequilibrium. Second, because a price

[3]The difficulty of dealing with complementarities has long been known to economists. After Arrow and Hurwicz (1958) prove that the venerable Walrasian tâtonnement process formalized by Samuelson (1941) converges globally to an equilibrium for economies with divisible goods when all goods are gross substitutes, there was optimism that such processes could also apply to any reasonable economies with divisible goods. But Scarf (1960) dashes such hopes by showing that such processes can never tend towards equilibrium in a three complementary goods counter-example. Kelso and Crawford (1982) demonstrate that complementarities in the presence of indivisibilities can even cause problems with existence of Walrasian equilibrium, not just stability.

system is always used in a dynamic auction and the number of bidders is typically not large in practice, it is conceivable that bidders would strategically exercise their market power in this environment. Then a fundamental and intriguing question arises: how to restore such bidders' incentives to bid sincerely as price-takers? Third, a threshold problem also poses a hazard to efficient outcomes in a package auction; see e.g., Noussair (2003), and Porter, Rassenti, Roopnarine and Smith (2003). For illustration, assume that John and Peter want items A and B respectively, but David wants A and B together. In order to win their favorite item, John and Peter must determine what price each should pay in an implicitly coordinated way to make the sum of their bids surpass David's bid on his favorite package. Finally, unlike markets with substitutable indivisible goods where a standard Walrasian equilibrium is well known to exist, it is not totally clear what notion of Walrasian equilibrium applies in the environment with complements.

The objective of this paper is three-fold: (1) to clarify the scope of the problem concerned with (indivisible) complements, (2) to explore an appropriate solution for such an environment, and (3) to devise a dynamic auction that overcomes the potential pitfalls described above, assigns goods efficiently, and at the same time induces bidders to act sincerely as price-takers.

To be precise, we consider an auction market where a seller wishes to sell n complementary indivisible goods to m bidders but has a reserve price for every bundle of the goods. Every bidder has a private valuation of every bundle of the goods and may have an incentive to strategically employ his market power. The goods are complementary in the sense that the value of combining any two disjoint sets of goods equals at least the sum of individual values of the disjoint sets. Such complementarity or more formally, super-additivity, is the most general notion in the literature, subsuming gross complements and supermodularity; see also Samuelson (1974). We explore the notion of nonlinear pricing Walrasian (NPW) equilibrium in this environment. This equilibrium is more general than the standard Walrasian equilibrium and utilizes a nonlinear pricing rule which specifies a price for every bundle of goods. Moreover, the price of every bundle is the same for all bidders, i.e., anonymous pricing. Due to nonlinearity in prices and her own reserve price, the seller has to contemplate how to divide her goods into different groups so as to maximize her revenues. The market is in equilibrium if every bidder receives an optimal bundle and the seller offers an optimal supply, and the market clears. Despite the nonexistence of standard Walrasian equilibrium, we will show that the auction market always has an NPW equilibrium by using nonlinear pricing. We will also analyze the properties of the equilibria and their structure. These display some unusual and interesting features that differ markedly from the familiar lattice structure in the models with substitute goods shown in Shapley and Shubik (1971), Gul and Stacchetti (1999), and Ausubel (2006).

Our focus is on the design of an efficient and incentive compatible (IC) dynamic auction for the environment just described. Our IC dynamic auction is built on a basic ascending auction, which works roughly as follows. Starting with the seller's reserve price, the auctioneer announces the current price for each bundle of items and a supply set of items, then bidders report quantities demanded at these prices, and the auctioneer adjusts prices upwards for the over-demanded bundles and so on. We show that this basic ascending auction always converges to an efficient allocation with a Walrasian equilibrium price for every bundle in finite time, when bidders bid sincerely. Most importantly, it is established that even when bidders are permitted to exercise their market power strategically, our IC dynamic auction always induces bidders to act sincerely as price-takers. And this auction results in an efficient allocation with its supporting Walrasian equilibrium price for every bundle of items and a generalized Vickrey-Clarke-Groves (VCG, hereafter) payment for every bidder. In particular, sincere bidding by every bidder is shown to be an ex post perfect equilibrium in the auction game. This auction can tolerate some dishonest behavior, mistakes, or inaccuracies caused by bidders and gives them chances to adjust and correct.

The current article provides a novel and practical approach for addressing the incentive problem in market models. Traditionally, it has been essential to assume that either agents are price-takers (Samuelson 1941, Arrow and Hurwicz 1958, and Debreu 1959), or there is a countably infinite number of agents (Debreu and Scarf 1962) or a continuum of agents (Aumann 1964) so that each individual agent has only a negligible effect on market prices. Such an assumption does not fit into the current model. Another traditional approach is the VCG mechanism. This is a sealed-bid dominant-strategy direct mechanism. Despite its theoretical appeal, it has rarely been used in practice; see e.g., Rothkopf, Teisberg, and Kahn (1990), Milgrom (2007) and Rothkopf (2007). There are two major criticisms of the VCG mechanism. First, the VCG mechanism lacks a competitive price system.[4] Instead it asks every bidder directly to reveal all his private valuations over every possible bundle of goods in order to determine an allocation and payments. Second, it may yield too low revenues for the seller. In auctions, as in any business, buyers are in general extremely reluctant to expose their true valuations. The current auction utilizes information from every bidder efficiently and judiciously in that it requires him to display demands only on a number of price vectors along the path towards equilibrium prices and nothing more. Thus it keeps the valuations of bidders from being exposed; see also Ausubel (2006). In the language of Hurwicz (1973), it is privacy preserving and informationally efficient. The

[4]A price system is extremely important in any market where there are many rational and strategic agents and each of them possesses private information which plays a role in determining the outcome of the market, as Hayek (1945) argues: "Fundamentally, in a system where the knowledge of the relevant facts is dispersed among many people, prices can act to coordinate the separate actions of different people in the same way as subjective values help the individual to coordinate the parts of his plan."

current auction permits the seller to have a reserve price for each bundle of goods. The seller determines the quantity for sale based upon the bidders' reported demands and will not sell any bundle if the bids are below her reserve price, thus maintaining a lower bound on the seller's revenues; see also Ausubel and Cramton (1999).

Besides, the current dynamic auction also exhibits the following attractive features. First, unlike the auctions proposed in Demange, Gale and Sotomayor (1986), Gul and Stacchetti (2000), Ausubel (2006), and Sun and Yang (2008b, 2009) where every bidder is required to report all his optimal bundles in every round of the auction process, the current auction asks every bidder to declare just one of his optimal bundles. Hence the current auction requests even less information from bidders; the auctions of Kelso and Crawford (1982) and Milgrom (2000) also share this property but can give rise to only an approximate equilibrium in finite time. Second, the current auction can tolerate bidders bidding inaccurately or dishonestly to some extent so long as they do not openly flout the rules or make serious or too many mistakes. In particular, this auction allows bidders to correct their previous mistakes or manipulated bids by withdrawing some of their past bids. Third, as pointed out by Bergemann and Morris (2007), because dynamic auctions create more transparent trading rules and offer opportunities for bidders to learn and adjust, they can reduce payoff uncertainty and strategic uncertainty for bidders.

We also propose a procedure for dividing revenues among several sellers whose goods are all complementary. In this case the sellers prefer to sell their goods jointly rather than separately, but then their revenues are inseparable, because due to synergies among goods, some goods of one seller may be sold together with another seller's as one package. This is in sharp contrast to the case of substitutes in which the revenues of sellers are automatically separable, as substitute goods can be sold at anonymous and linear equilibrium prices.

To close this introductory section, we briefly review several closely related papers. Ausubel and Milgrom (2002) propose a family of ascending package auctions.[5] Their auctions use discriminatory and nonlinear pricing rules, offering a core allocation as the outcome which is weaker than a nonlinear pricing Walrasian equilibrium.[6] However, their auctions were intended to deal with a more complex situation where a bundle of goods can be complements to one bidder but substitutes to another. The current paper finds that an anonymous price dynamic auction is well suited to complementary goods. As argued by Milgrom (2004), discriminatory pricing fails to promote the law of one price and thus may

[5]A sealed-bid package auction is due to Bernheim and Whinston (1986). Levin (1997) proposes an optimal sealed-bid auction for two complementary items, generalizing Myerson (1981). See Day and Milgrom (2008), Erdil and Klemperer (2010) for recent developments on Ausubel and Milgrom (2002). In Wurman and Wellman's (2000) package auction the equilibrium is incompatible with the seller's behavior.

[6]Discriminatory pricing means that the price of the same bundle of goods may differ from one bidder to another.

be psychologically hard for some people to accept.[7] Indeed, in real life auctions, people are more accustomed to anonymous prices. Porter, Rassenti, Roopnarine and Smith (2003) provide laboratory tests on a package auction with anonymous prices in support of the auction over several sets of spectrum licenses which exhibit complementarities. Ausubel (2004) and Perry and Reny (2005) introduce efficient ascending auctions for identical goods in an interdependent value setting. Ausubel (2006) develops an ingenious incentive compatible dynamic auction for selling substitute items.[8] Sun and Yang (2008b, 2009) propose an incentive compatible dynamic auction for a more general environment with two types of goods. Items of the same type are substitutable but are complementary to items of the other type.

This article is organized as follows. Section 2 sets up the model. Section 3 presents the basic ascending auction. Section 4 builds the incentive compatible dynamic mechanism associated with the basic ascending auction. Section 5 deals with several sellers who sell complementary goods. Section 6 discusses applications. Section 7 concludes.

2 The Model

A seller wishes to sell a set of heterogeneous indivisible goods (items) $N = \{1, 2, \cdots, n\}$ through auction to a group of bidders $M = \{1, 2, \cdots, m\}$. Every bidder $i \in M$ attaches a monetary value (units of money) to each bundle of items, namely, each bidder i has a value function $u^i : 2^N \to \mathbb{Z}$ with $u^i(\emptyset) = 0$, where 2^N denotes the family of all bundles of items and $\mathbb{Z}$ ($\mathbb{Z}_+$) is the set of all (nonnegative) integers. Every bidder is endowed with a sufficient amount of money in the sense that he can pay up to his value. The seller (denoted by 0) has a reserve price function $u^0 : 2^N \to \mathbb{Z}$ with $u^0(\emptyset) = 0$. This means that if a bundle were sold to a bidder, the bidder would have to pay at least the reserve price of the bundle. Let $M_0 = M \cup \{0\}$ represent the set of all agents (bidders and seller) in the market. The following assumptions are imposed on the model:

(A1) *Integer Private Values* : Every bidder i knows his own value function $u^i : 2^N \to \mathbb{Z}$ privately and his value function is integer-valued.

[7] However, discriminatory and nonlinear pricing seems natural for situations where a seller is simultaneously and independently negotiating with several buyers for important business, such as Boeing or Airbus selling their major products.

[8] Also for substitutes, the auctions of Kelso and Crawford (1982), Gul and Stacchetti (2000), and Milgorm (2000) ensure convergence to Walrasian equilibria only if bidders are posited to bid truthfully. In a more general setting, Ausubel and Milgrom (2002) require a similar assumption in order to generate a core allocation.

(A2) *Quasilinear Utility* : Every bidder i's net utility or profit is given by $v^i(S, p) = u^i(S) - p(S)$, when he receives the bundle S and pays the price $p(S)$.

(A3) *Superadditivity* : For every agent $i \in M_0$, and for any disjoint sets $S_1, S_2 \in 2^N$, (i.e., $S_1 \cap S_2 = \emptyset$), agent i's utility function u^i satisfies $u^i(S_1 \cup S_2) \geq u^i(S_1) + u^i(S_2)$.

Assumptions (A1) and (A2) are standard and have been extensively used in the literature. Assumption (A3) says that both bidders and the seller view all items as complements in the sense that they get at least as much value from combining two disjoint sets of items as the sum of the values they get from each set individually. This condition describes the most general form of complementarity including both gross complements based on Marshallian demand, and supermodularity as special cases. Recall that a value function $u^i : 2^N \to \mathbb{R}$ is said to be *supermodular* if we have $u^i(S_1 \cup S_2) + u^i(S_1 \cap S_2) \geq u^i(S_1) + u^i(S_2)$ for any $S_1, S_2 \in 2^N$. Superadditivity[9] is more general than supermodularity, because the former applies to only disjoints sets, while the latter applies also to intersecting sets. Supermodularity requires that the value function exhibit increasing marginal returns.

The conventional approach of handling complementary items is to bundle them all together in advance and to sell them as one package. Unfortunately, this approach may yield inefficient outcomes. In contrast, our analysis will show that we do not need to exogenously bundle complementary items in advance, and instead items can be endogenously and efficiently allocated among agents via a dynamic auction. As a result, the market realizes its full potential. The following example illustrates this point and also shows that unlike gross substitutes (GS), superadditivity is insufficient to guarantee the existence of a standard Walrasian equilibrium.

Example 1: Suppose that three firms (bidders 1, 2, 3) compete for a seller's three complementary objects (A, B, C). Bidders' values and seller's reserve prices are given in Table 6. Clearly, all items are viewed by every bidder and the seller as complements. Observe that u^1 is superadditive but not supermodular, because $u^1(AB) + u^1(AC) = 10 > u^1(ABC) + u^1(A) = 9$. Neither is u^2 nor u^3 supermodular. In this economy, there are two efficient allocations: One allocation assigns the bundle AB to bidder 1 and item C to bidder 2; another allocation assigns AB to bidder 1 and item C to bidder 3. We will show that these allocations cannot be supported by a price vector (p_A, p_B, p_C). Consider,

[9] Superadditivity is an analog in this problem to the well-known gross substitutes (GS) condition of Kelso and Crawford (1982). The GS condition requires that every bidder regard all items as substitutes. This condition provides a general sufficient condition for the existence of a standard Walrasian equilibrium and has been widely used in auction design; see e.g., Gul and Stacchetti (1999, 2000), Milgrom (2000) and Ausubel (2006). It is known that the class of utility functions satisfying the GS condition is a proper subset of submodular functions; see Gul and Stacchetti (1999) and Fujishige and Yang (2003). A function u is submodular if $-u$ is supermodular.

for instance, the first efficient allocation. Suppose that the price vector $(p(A), p(B), p(C))$ supports this allocation. Then for the seller, we must have $p(A) \geq 1$, $p(B) \geq 1$ and $p(C) \geq 1$. For bidder 1, we must have $p(A) + p(B) \leq 7$. For bidder 2, we must have $p(C) \leq 2$ and $p(A) \geq 4$. For bidder 3, we must have $p(C) \geq 2$ and $p(B) + p(C) \geq 6$. From $p(C) = 2$, we have $p(B) \geq 4$. Combining the inequalities $p(A) \geq 4$ and $p(B) \geq 4$ yields $p(A) + p(B) \geq 8$, which contradicts $p(A) + p(B) \leq 7$. A similar argument applies to the other efficient allocation. Thus, there exists no standard Walrasian equilibrium for this economy. Nevertheless, there exists a Walrasian equilibrium if the auctioneer adopts the following anonymous and nonlinear pricing rule $p(\emptyset) = 0$, $p(A) = 2$, $p(B) = 2$, $p(C) = 2$, $p(AB) = 6$, $p(AC) = 6$, $p(BC) = 6$, and $p(ABC) = 7$. For this pricing rule, the auctioneer can assign the bundle AB to bidder 1 by asking a price of 6 and the bundle C to bidder 2 by asking a price of 2. As will be shown in Section 3, this equilibrium can be found by a dynamic auction. As a result, the market realizes a total value of 9 and the seller gains a revenue of 8. On the other hand, however, if the seller were to adopt the conventional strategy of selling three complementary objects as a single bundle, she could gain a revenue of only 7 and the market could achieve a total value of only 7.

Table 1: Bidders' and seller's values over items.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	2	2	0	7	3	4	7
Bidder 2	0	2	0	2	3	6	3	6
Bidder 3	0	0	2	2	4	3	6	7
Seller	0	1	1	1	2	2	2	3

This example suggests that a possible way of developing a meaningful theory for coping with complementarity is to modify the standard notion of Walrasian equilibrium by adopting an anonymous and nonlinear pricing rule. A function $p : 2^N \to \mathbb{R}$ is a pricing function if $p(\emptyset) = 0$. A pricing function indicates a price $p(S)$ for each bundle S and will be also viewed as a vector. For instance, in Example 1, a pricing function p means the vector $(p(\emptyset), p(A), p(B), p(C), p(AB), p(AC), p(ABC))$. A pricing function p is *linear* if for any bundle S, we can write $p(S) = \sum_{k \in S} p(\{k\})$. Otherwise, it is *nonlinear*.[10] A pricing function p is *feasible* if $p(S) \geq u^0(S)$ for every $S \in 2^N$. Since we consider only feasible pricing functions, in the following a pricing function always means a feasible one. For a

[10]Nonlinear pricing has a long history in business practice and is widely used in many industries; see Makowski (1979), Wilson (1993), Bikhchandani and Ostroy (2002), and Png (2002).

pricing function $p : 2^N \to \mathbb{R}$, bidder i's demand correspondence $D^i(p)$ and indirect utility function $V^i(p)$ are defined from his net utility function v^i, respectively, by

$$D^i(p) = \arg\max_{S\subseteq N} v^i(S,p) \text{ and } V^i(p) = \max_{S\subseteq N} v^i(S,p).$$

In nonlinear pricing settings, a seller has to consider not only which items to supply, but also how to supply them. For example, when the seller has two items for sale, she needs to decide whether she should supply the two items as two separate bundles or as a single bundle. In general, when facing a feasible pricing function p, the seller will choose those partitions of the items in N that maximize her revenues. We call $\gamma = \{A_1, \cdots, A_k\}$ *a partition* of the items in N for some $k \leq n$ if each bundle $A \in \gamma$ is nonempty, $\cup_{A\in\gamma} A = N$, and $A \cap A' = \emptyset$ for any two different bundles A and A' in γ. Let $\mathcal{B}$ represent the family of all partitions of N.

The seller's supply correspondence $S(p)$ whose each element is called *a supply set*, and her revenue function $Re(p)$, are defined by, respectively,

$$S(p) = \arg\max_{\gamma\in\mathcal{B}}\Big\{\sum_{A\in\gamma} p(A)\Big\} \text{ and } Re(p) = \max_{\gamma\in\mathcal{B}}\Big\{\sum_{A\in\gamma} p(A)\Big\}.$$

An allocation of items in N is a redistribution $\pi = (\pi(i), i \in M_0)$ of items among all agents in M_0 such that $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{i\in M_0}\pi(i) = N$. An allocation π assigns the bundle $\pi(i)$ to agent i. Note that $\pi(i) = \emptyset$ is allowed. If $\pi(0) \neq \emptyset$, then the bundle $\pi(0)$ is not sold and thus stays with the seller. Let $\mathcal{A}$ denote the family of all allocations. We also view every allocation $\pi = (\pi(0), \pi(1), \cdots, \pi(m))$ as a partition γ of N in the sense that $\gamma = \{\pi(i) \mid \pi(i) \neq \emptyset \text{ and } i \in M_0\}$. Note that $\pi(i)$ disappears in γ if $\pi(i)$ is empty. An allocation π is *efficient* if for every allocation ρ we have $\sum_{i\in M_0} u^i(\pi(i)) \geq \sum_{i\in M_0} u^i(\rho(i))$. Given an efficient allocation π^*, let $V(N) = \sum_{i\in M_0} u^i(\pi^*(i))$. We call $V(N)$ *the market value*. Clearly, this value is the same for any efficient allocation.

Definition 2.1 *A nonlinear pricing Walrasian* (NPW) *equilibrium consists of a price function $p^* : 2^N \to \mathbb{R}$ and an allocation π^* such that $\pi^* \in S(p^*)$ for the seller and $\pi^*(i) \in D^i(p^*)$ for every bidder $i \in M$.*

For a nonlinear pricing Walrasian equilibrium (p^*, π^*), we call p^* *an equilibrium pricing function* and π^* *an equilibrium allocation*, and we say π^* is supported by p^*. Clearly, a nonlinear pricing Walrasian equilibrium (p^*, π^*) becomes the standard Walrasian equilibrium if p^* is linear.

It is well known that the standard Walrasian equilibrium leads to the most efficient allocation of scarce resources. The following simple lemma shows that the notion of NPW equilibrium possesses the same property.

Lemma 2.2 *If (p^*, π^*) is a nonlinear pricing Walrasian equilibrium, π^* is efficient; and if ρ is an efficient allocation, (p^*, ρ) is also a nonlinear pricing Walrasian equilibrium.*

Some might worry that the concept of NPW equilibrium could be too general to be interesting. The following example is intended to dispel such concern and shows that Assumptions (A1) and (A2) without (A3) do not guarantee the existence of NPW equilibrium.

Example 2: Assume that there are three bidders (1, 2, 3) and three items (A, B, C) in a market. The seller's reserve prices are all zero. Bidders' values over all bundles are given in Table 2. This example is taken from Bevia, Quinzii and Silva (1999), who show that there exists no (standard) Walrasian equilibrium in this market. We will see that this market does not have any NPW equilibrium, either. Observe that $(\pi(0), \pi(1), \pi(2), \pi(3)) = (\emptyset, B, A, C)$ is the unique efficient allocation at which every bidder gets one item. To have an NPW equilibrium, it follows from Lemma 2.2 that π must be supported by a nonlinear pricing function p. Assume without loss of generality that $p(AB) > p(A) + p(B)$. Then the seller will choose $\{AB, C\}$ instead of π, i.e., $\pi \notin S(p)$. So there is no NPW equilibrium.

Table 2: Bidders' values over items.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	10	8	2	13	11	9	14
Bidder 2	0	8	5	10	13	14	13	15
Bidder 3	0	1	1	8	2	9	9	10

Observe that the value functions in this example are submodular and so do not satisfy Assumption (A3). In short, this example demonstrates a striking fact that even if every bidder's value function is submodular and thus exhibits decreasing marginal returns, an NPW equilibrium may still not exist. In contrast, we will show that superadditivity ensures the existence of NPW equilibrium. However, unlike the auction models with gross substitutes, the set of equilibrium pricing functions in the current model with complements fails to be a lattice.[11] To see this, consider two NPW equilibria (p, π) and (q, π) in Example 1, where $\pi = (\pi(0), \pi(1), \pi(2), \pi(3)) = (\emptyset, AB, C, \emptyset)$, $p = (0, 2.4, 2, 2, 6.5, 6, 6, 7)$,

[11] For any functions p and q, $p \vee q$ means $p \vee q(S) = \max\{p(S), q(S)\}$ for every $S \subseteq N$, and $p \wedge q$ means $p \wedge q(S) = \min\{p(S), q(S)\}$ for every $S \subseteq N$. A set W of pricing functions is a *lower semi-lattice* if $p \wedge q \in W$ for any $p, q \in W$; W is an *upper semi-lattice* if $p \vee q \in W$ for any $p, q \in W$; W is a *lattice* if W is both a lower and an upper semi-lattice. We use the threshold problem to illustrate these concepts. Assume that John's value function is $(u(A), u(B), u(AB)) = (3, 0, 3)$, Peter's is $(0, 3, 3)$, and David's is $(0, 0, 4)$. Then the set of proper equilibrium pricing functions is $\{\, (p(A), p(B), p(AB)) \mid p(AB) = 4, p(A) \leq 3, p(B) \leq 3, \text{ and } p(A) + p(B) \geq 4 \,\} \subseteq \mathcal{R}^3$, which is an upper semi-lattice but is not a lower semi-lattice.

and $q = (0, 2, 2, 2, 6.5, 6, 6.4, 7)$. It is easy to verify that $(p \vee q, \pi)$ is not an NPW equilibrium. The following notion of proper NPW equilibrium, however, can restore the upper semi-lattice structure.

An equilibrium pricing function p is *proper* if there is no other equilibrium pricing function q such that $q \leq p$, $q \neq p$ and $Re(p) = Re(q)$. In other words, a proper equilibrium pricing function is a minimal equilibrium pricing function among all those equilibrium pricing functions that give the seller the same revenue. An NPW equilibrium (p, π) is *proper* if p is a proper equilibrium pricing function. The following theorem reveals the structure of proper NPW equilibria.

Theorem 2.3 *Suppose that the auction model satisfies Assumptions* (A1)–(A3)*. Then the set of proper equilibrium pricing functions is a nonempty and compact upper semi-lattice.*

First, notice that the theorem holds true only for proper equilibrium pricing functions. One can also construct an example so that (p, π) and (q, π) are (proper) NPW equilibria but $(p \wedge q, \pi)$ is not an NPW equilibrium. Second, the upper semi-lattice has a unique largest equilibrium pricing function in the seller's favor. Furthermore, the theorem implies that under complete information it is possible for the seller to extract a maximum revenue from bidders but impossible to charge every bidder at a lowest price, and that the threshold problem (as described in the introduction) could occur due to lack of lower semi-lattice. This shows a fundamental and inherent difference from the case of substitute goods for which the set of equilibrium price vectors is known to be a lattice, meaning that the threshold problem cannot happen.

3 The Basic Ascending Auction

In this section we first consider the basic case where bidders bid sincerely, and then move to the case where bidders bid quasi-sincerely. In the next section we deal with the case where bidders may bid strategically rather than naively as price-takers. In a dynamic auction, we say that bidder i *bids sincerely* with respect to value function u^i if at every time $t \in \mathbb{Z}_+$ and any price function $p(t)$ at time t, he reports a bid $A_i(t) \in D^i(p(t)) = \arg\max_{S \subseteq N}\{u^i(S) - p(t, S)\}$ with $A_i(t) = \emptyset$ when $\emptyset \in D^i(p(t))$. Notice that when $p(t)$ is a pricing function at time $t \in \mathbb{Z}_+$, then $p(t, S)$ denotes the price of bundle $S \in 2^N$.

3.1 The Sincere-Bidding Case

We now describe the main idea of our basic dynamic auction. The design of this auction process is based on the classical tâtonnement process with several modifications and is quite similar to the salary-adjustment process of Kelso and Crawford (1982). The auctioneer announces the current price function p and the current supply set. Every bidder i responds by reporting a demand bundle $A_i \in D^i(p)$. The auctioneer then examines the aggregated reported demands. If a bundle is over-demanded, the auctioneer raises its price but will not change the price of any bundle which is balanced or over-supplied.[12] This process stops at some price vector p^* when no bundle is over-demanded. Then according to bidders' reported demands and some rules to be specified below, the auctioneer assigns a reported bundle B to each bidder who is asked to pay the current corresponding price $p^*(B)$.

Given a pricing function $p : 2^N \to \mathbb{R}$ and a supply set $\gamma \in S(p)$, each bidder $i \in M$ reports a demand bundle $A_i \in D^i(p)$. With respect to the supply set γ and reported demand bundles $A_i (i \in M)$, we say that a bundle $A \in 2^N \setminus \{\emptyset\}$ is *over-demanded* if it is demanded by more than one bidder (i.e., $A_i = A_j = A$ for at least two bidders i, j) or demanded only by one bidder i (i.e., $A_i = A$) but his bundle A_i is not in the seller's supply set γ (i.e., $A_i \notin \gamma$).[13] Keep in mind that an over-demanded bundle must be nonempty and the empty set can be demanded by many bidders without any cost. Having all the preparations, we now give a formal description of the price adjustment process.

The basic auction process

Step 1: The seller reports her reserve price function u^0 and the auctioneer sets the initial pricing function $p(0) : 2^N \to \mathbb{Z}$ with $p(0, S) = u^0(S)$ for every $S \subseteq N$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \cdots$, the auctioneer announces the current pricing function $p(t)$ and a supply set $\gamma(t) \in S(p(t))$. Then every bidder i is asked to report a demand bundle $A_i(t)$ at the prices $p(t)$. Based on the supply set and reported demand bundles, the auctioneer adjusts prices as follows: If no bundle is over-demanded at $p(t)$, go to Step 3. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged. Set $t := t + 1$ and return to Step 2.

Step 3: The auctioneer assigns the bundle $A_i(t)$ to bidder i who is asked to pay the price $p(t, A_i(t))$ in return, and in addition for any nonempty bundle $B \in \gamma(t)$ which is not demanded by any bidder at $p(t)$, the auctioneer assigns the bundle to the seller if $p(t, B) = u^0(B)$, otherwise, the auctioneer assigns the bundle to some bidder[14] who

[12] This is quite different from the classical tâtonnement process which requires the reduction of the price of an over-supplied good.

[13] This definition of over-demanded sets which depends on the seller's supply set is a crucial and novel point for the price adjustment of the current auction.

[14] If there are several such bidders, the auctioneer randomly chooses one.

previously demanded the bundle and was the last to give up. This bidder is asked to pay $p(t, B)$. Then the process stops.

Three remarks on the auction process are in order. First, because the sequence of generated prices for each bundle is never decreasing even if a bundle is over-supplied, the auction can be viewed as an ascending bid procedure, and so every $p(t)$ is a feasible pricing function. Second, in the final round of the process (Step 3) the auctioneer assigns the bundle $A_i(t)$ to bidder i and may also assign an additional bundle B to this same bidder if he once demanded B and was the last to give up, and moreover the current price of B is greater than its reserve price. This bidder will be asked to pay the sum of $p(t, A)$ and $p(t, B)$. We call this operation *the complementary activity rule*, which is a novel feature of the current process. It will be shown that $p(t, A) + p(t, B)$ is actually equal to $p(t, A \cup B)$, a crucial fact for this operation. Third, in the current auction process, every bidder i only needs to report one bundle $A_i(t)$ from his demand correspondence $D^i(p(t))$ in each round t, whereas in those of Demange et al. (1986), Gul and Stacchetti (2000), Ausubel (2006), and Sun and Yang (2008b, 2009), every bidder i is required to report his entire demand correspondence $D^i(p(t))$. Thus the current auction demands less information from bidders.

To have a better understanding of the auction, let us illustrate it through Example 1. Starting with the reserve price vector $p(0) = (1, 1, 1, 2, 2, 2, 3)$ and the supply set $\gamma(0) = \{AB, C\} \in S(p(0))$, bidder 1 bids AB, bidder 2 AC, and bidder 3 BC. At $p(0)$, the bundles AC and BC are over-demanded and so the auctioneer adjusts the price vector to $p(1)$ by increasing the price of AC and BC each by one unit and keeping the prices of all other bundles unchanged. At $p(1)$, the auctioneer supplies $\gamma(1) = \{AC, B\}$, bidder 1 demands AB, bidder 2 AC and bidder 3 ABC. Now both AB and ABC are over-demanded so the auctioneer adjusts the price vector to $p(2)$ by increasing the price of the two bundles. The auction stops with $p(9)$ at which no bundle is over-demanded. By the auction rule, bidder 1 gets AB and pays 6, bidder 3 gets nothing and pays nothing, but bidder 2 is assigned with C and is asked to pay 2 (note that C is one of the optimal bundles of bidder 2 at $p(9)$!), since bidder 2 demanded C at round 5 and was the last to give up C. Observe that the process ends with the efficient allocation $\pi^* = (\pi(0), \pi(1), \pi(2), \pi(3)) = (\emptyset, AB, C, \emptyset)$. Clearly, $(p(9), \pi^*)$ is an NPW equilibrium. The whole process of demands and supplies at the adjusted price vectors is shown in Table 3, where the price vector is understood as $p = (p(A), p(B), p(C), p(AB), p(AC), p(BC), p(ABC))$ with $p(\emptyset) = 0$, supply column indicates the seller's supplies γ, and each bidder's column indicates his demands.

Observe that in each round t of the auction the price of every bundle contained in the seller's supply set $\gamma(t)$ is at least as much as the sum of the prices of its individual items. Take the bundle AB for instance. It holds that $p(t, AB) \geq p(t, A) + p(t, B)$ for $t = 0, 1, \cdots, 9$. This is because AB has synergies, the seller chooses supplies to maximize her revenues, and the auction can correctly extract price information from the market. In each round t, $p(t)$ shows the price of every bundle S at which some agent is willing to pay for the bundle. Therefore it is natural and also possible for the seller to choose an optimal supply set $\gamma(t)$ against $p(t)$. Consequently, every bundle in this supply set can sell for more than the sum of its individual prices. Consider again AB, say, at prices $p(6)$. Note that bidder 1 is happy to buy AB in spite of $p(A,6)+p(B,6) < p(AB,6)$. It is impossible for anyone else to acquire AB more cheaply by buying A and B separately.

It should be noticed, however, that in retail business, we do not see a bundle of goods sells for more than the sum of the prices of its individual goods. This is because the retail price mechanism differs fundamentally from the auction mechanism. The retail prices are offered by the seller and each consumer can choose any bundle of goods and decides to take it or leave it, and can also buy goods separately. Therefore it makes no sense if the seller sets the price of a bundle higher than the sum of the prices of its individual items. Our theory suggests that, to handle complements in retail, if a seller should have accurate and enough information about the market, she could sell her complementary goods as a partition of goods (i.e., bundling goods into several disjoint packages) to improve her revenues.

Table 3: Demands and supplies at the adjusted price vectors for Example 1.

Price Vector	Supply	Bidder 1	Bidder 2	Bidder 3
$p(0) = (1,1,1,2,2,2,3)$	$\{AB, C\}$	AB	AC	BC
$p(1) = (1,1,1,2,3,3,3)$	$\{AC, B\}$	AB	AC	ABC
$p(2) = (1,1,1,3,3,3,4)$	$\{AC, B\}$	AB	AC	ABC
$p(3) = (1,1,1,4,3,3,5)$	$\{AB, C\}$	AB	AC	BC
$p(4) = (1,1,1,4,4,4,5)$	$\{AB, C\}$	AB	AC	BC
$p(5) = (1,1,1,4,5,5,5)$	$\{AC, B\}$	AB	C	ABC
$p(6) = (1,1,2,5,5,5,6)$	$\{AB, C\}$	AB	A	ABC
$p(7) = (2,1,2,5,5,5,7)$	$\{AB, C\}$	AB	AC	BC
$p(8) = (2,1,2,5,6,6,7)$	$\{A, BC\}$	AB	$\emptyset$	B
$p(9) = (2,2,2,6,6,6,7)$	$\{AB, C\}$	AB	$\emptyset$	$\emptyset$

Theorem 3.1 *Suppose that Assumptions* (A1)–(A3) *hold true for the auction model. When every bidder bids sincerely, the basic auction process yields a nonlinear pricing Walrasian equilibrium, in a finite number of rounds.*

Proof: It is easy to see that the auction process stops at some step t^*, because as soon as the price of any (nonempty) bundle becomes higher than any bidder's valuation for it, no bidder will demand it. Note that the price of the empty set is always fixed at zero.

For ease of notation, let $p^* = p(t^*)$ and let $A_i^* = A_i(t^*)$ that is demanded by bidder i, and let $\gamma^* = \gamma(t^*) \in S(p^*)$ that is the supply set of the seller. Recall that by definition γ^* is a seller's partition of all the items N. We will first construct an allocation π^* so that (p^*, π^*) constitutes an NPW equilibrium. Note that at p^*, no (nonempty) bundle is over-demanded. Thus, for any bidder $i \in M$, if his demand bundle A_i^* is not empty, it must be in the supply set γ^*. Moreover, for any two bidders $i, l \in M$, with $A_i^* \neq \emptyset$ and $A_l^* \neq \emptyset$, we must have $A_i^* \cap A_l^* = \emptyset$. If $\cup_{i \in M} A_i^* = N$, let $\pi^*(i) = A_i^*$ for all $i \in M$ and $\pi^*(0) = \emptyset$, then clearly (p^*, π^*) is an NPW equilibrium and we are done.

Suppose otherwise that there is some bundle B in the supply set γ^* which is not demanded by any bidder at the last round. Such a bundle is called a *squeezed out bundle*. We first consider the case in which $p^*(B) = u^0(B)$. Let $\gamma_0^* = \{B \in \gamma^* \mid p^*(B) = u^0(B) \text{ and } B \neq A_i^* \text{ for all } i \in M\}$ be the collection of all such bundles. Let $\pi^*(0) = \cup_{B \in \gamma_0^*} B$. We assign $\pi^*(0)$ to the seller. By superadditivity, we know that $p^*(\pi^*(0)) \geq u^0(\pi^*(0)) \geq \sum_{B \in \gamma_0^*} u^0(B) = \sum_{B \in \gamma_0^*} p^*(B)$. But we also see that $p^*(\pi^*(0)) \leq \sum_{B \in \gamma_0^*} p^*(B)$ because $\gamma^* \in S(p^*)$. Hence, we have

$$p^*(\pi^*(0)) = u^0(\pi^*(0)) = \sum_{B \in \gamma_0^*} p^*(B) = \sum_{B \in \gamma_0^*} u^0(B). \tag{3.1}$$

Next, we consider the case in which $p^*(B) > u^0(B)$. This implies that the bundle B was demanded by some bidder at some round. Let t be the last round at which B is demanded by some bidder l. By the auction rule B can be assigned to bidder l who is asked to pay the current price $p^*(B)$. We will show that bidder l loses nothing in having the bundle B and paying the price. By the auction rule and Assumption (A2), we must have $V^l(p(t)) = v^l(B, p(t)) = u^l(B) - p(t, B) \geq 1$, and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. For the bidder l, it holds that

$$v^l(B, p^*) = u^l(B) - p^*(B) \geq 0. \tag{3.2}$$

We need to consider the following two situations.

Case 1. When $A_l^* = \emptyset$, let $\pi^*(l) = B$. Because $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$, we have $V^l(p^*) = 0 \geq v^l(B, p^*)$. Recall that $v^l(B, p^*) \geq 0$. These inequalities lead to $v^l(B, p^*) = 0$, which implies $\pi^*(l) \in D^l(p^*)$.

Case 2. When $A_l^* \neq \emptyset$, let $\pi^*(l) = A_l^* \cup B$. For the seller, we know that

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)). \tag{3.3}$$

For the bidder l, superadditivity (A3) implies that

$$u^l(\pi^*(l)) \geq u^l(A_l^*) + u^l(B). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} u^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) \\ &\geq \Big(u^l(A_l^*) - p^*(A_l^*)\Big) + \Big(u^l(B) - p^*(B)\Big) \\ &\geq u^l(A_l^*) - p^*(A_l^*) \end{aligned}$$

where the last inequality is derived from (3.2). Because $A_l^* \in D^l(p^*)$, we have $\pi^*(l) \in D^l(p^*)$ (i.e., $\pi^*(l)$ is also an optimal bundle of bidder l at p^*). Consequently, it further implies that $u^l(\pi^*(l)) - p^*(\pi^*(l)) = u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) = u^l(A_l^*) - p^*(A_l^*)$, yielding

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \tag{3.5}$$

So in both cases bidder l loses nothing from having the nonessential bundle B and paying the price $p^*(B)$. As a result, the indirect utility of bidder l remains unchanged.

We can repeat this adjustment until every such squeezed out bundle B (i.e., $p^*(B) > u^0(B)$) in γ^* is assigned to some bidder. For any bidder i who is not assigned with any squeezed out bundle, let $\pi^*(i) = A_i^*$. So in the end each bidder i gets a bundle $\pi^*(i)$ in his demand set. Because γ^* is a seller's partition of N, $(\pi^*(0), \cdots, \pi^*(m))$ must be an allocation of N. Furthermore, it follows from the formulas (7.13) and (3.5) that $\sum_{i \in M_0} p^*(\pi^*(i)) = \sum_{A \in \gamma^*} p^*(A) = Re(p^*)$. That is, the allocation $\pi^* \in S(p^*)$. Consequently, (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done. □

As a consequence of Theorem 3.1, we have the following corollary immediately.

Corollary 3.2 *The auction model under Assumptions* (A1)–(A3) *has a nonlinear pricing Walrasian equilibrium.*

3.2 The Quasi-Sincere Bidding Case

In practice, bidders may not always submit a precise optimal demand bundle in every step of an auction for whatever reason. Here we investigate to what extent the current auction can tolerate inaccurate bidding behavior of bidders. In a dynamic auction, we say that bidder i *bids quasi-sincerely* with respect to utility function u^i if he reports a bid $A_i(t^*) \in D^i(p(t^*)) = \arg\max_{S \subseteq N}\{u^i(S) - p(t^*, S)\}$ (with $A_i(t^*) = \emptyset$ when $\emptyset \in D^i(p(t^*))$) in every possible last round $t = t^*$ of the auction, and always reports an empty bundle or a nonempty bundle $A_i(t)$ which satisfies $u^i(A_i(t)) - p^i(t, A_i(t)) \geq 1$ for all other rounds t before the termination of the auction. In other words, quasi-sincere bidding requires bidder i to be accurate only in every possible last round of the auction and allows him to submit any bid in any other rounds as long as it gives him a positive profit when the bid is for a nonempty bundle.

Because bidders usually do not know whether the current round is a possible last round or not, we need to modify the basic auction process by requiring that when the auctioneer finds there is no over-demanded bundle, she reminds every bidder that "this may be the last round, please report your optimal demand bundle", and so she gives the bidders a chance to revise their bids. Both 'quasi-sincere bidding' and 'the auctioneer reminding the bidders in every possible last round' imitate what happens in real auction houses where bidders do not submit bids in every round of the auction and the auctioneer does remind the bidders in the last round by e.g., hitting a hammer several times.

We can easily modify the basic auction process so that as long as every bidder bids quasi-sincerely, the modified basic auction will find a nonlinear pricing Walrasian equilibrium in finite time. We leave the detail in the Appendix.

4 The Incentive Compatible Dynamic Auction

We have previously shown that when bidders bid (quasi-)sincerely, the basic auction finds an NPW equilibrium in finite time. However, because bidders' valuations are private information, unobservable and unverifiable, it is conceivable that bidders may not act naively as price-takers but would bid whatever they like as long as it serves their interest and they do not openly defy the rules. In this section, based on the basic ascending auction, we shall design an incentive compatible (IC) dynamic auction which restores bidders' incentive to behave truthfully as price-takers and discovers an NPW equilibrium and generalized VCG payments.

4.1 The IC Dynamic Auction Design and Illustration

As we mentioned earlier, the outcome of our incentive compatible auction is composed of two efficient outcomes: the nonlinear pricing Walrasian equilibrium and the VCG outcome to be introduced shortly. Let $\mathcal{M}$ denote the auction market with the set M of bidders and the set N of items, and for every $i \in M$ let $\mathcal{M}_{-i}$ stand for the market $\mathcal{M}$ *without bidder* i. Furthermore, let $M_{-i} = M \setminus \{i\}$ for every $i \in M$ and for convenience also let $\mathcal{M}_{-0} = \mathcal{M}$ and $M_{-0} = M$.

Definition 4.1 *The VCG mechanism is the following sealed-bid procedure. Every bidder $i \in M$ is asked to report his value function u^i to the auctioneer. Based on bidders' reported value functions u^i and the seller's reserve price function u^0, the auctioneer computes an efficient allocation π of the market $\mathcal{M}$ and assigns bundle $\pi(i)$ to bidder i and charges him a payment of $q_i^* = u^i(\pi(i)) - V(N) + V_{-i}(N)$, where $V(N)$ and $V_{-i}(N)$ are the market values of the markets $\mathcal{M}$ and $\mathcal{M}_{-i}$, respectively.*

For the VCG outcome, we call the payment q_i^* *the generalized VCG payment* of bidder i because the payment here also involves the seller's reserve price function u^0 whereas in the standard VCG mechanism the reserve price is set to be zero. Bidder i's VCG payoff equals $V(N) - V_{-i}(N)$, $i \in M$.

Recall that every bidder i's value function u^i is private information, superadditive and integer-valued. It is natural to assume that all values are distributed in a sufficiently large interval $\mathbb{Z} \cap [0, \bar{U}]$, where $\bar{U}$ is a positive integer; see e.g., Myerson (1981) and Krishna (2002) for similar assumptions. Let $\mathcal{U}$ denote the family of all superadditive value functions $u : 2^N \to \mathbb{Z} \cap [0, \bar{U}]$. Suppose that the auctioneer knows some integer value U^* greater than $\bar{U}$. In this environment, nature according to a joint probability distribution function $F(\cdot)$ draws a profile $\{u^i\}_{i \in M}$ of value functions from $\mathcal{U}$ and reveals to every bidder $i \in M$ only the function u^i. The seller's reserve price function u^0 is also drawn from $\mathcal{U}$ and will be revealed to the bidders when the auction starts. Every bidder i views u^i as his private information and may act strategically. We stress that no one except nature knows the distribution function $F(\cdot)$ and our auction design is independent of $F(\cdot)$.

Before describing our IC dynamic auction, it will be helpful to give a road map for it. In contrast to the VCG mechanism which is static and asks directly every bidder to reveal his utility function and then calculates an efficient allocation of every market $\mathcal{M}_{-i}$, $i \in M_0$ ($m + 1$ markets in total) and payments for all bidders based on reported utility functions, the current auction is dynamic and explores a competitive price system which leads to a Walrasian equilibrium for each of $m + 1$ markets simultaneously, and in addition yields the VCG outcome without requesting any bidder's utility function. The competitive price system consists of two time-dependent price functions called the first and second price functions which might be seen as the generalization of the first and second prices in the Vickrey auction for a single item. The two price functions capture essential information of the $m + 1$ markets and will be used to derive a price function for each bidder $i \in M$, and a price function of each market $\mathcal{M}_{-j}$, $j \in M_0$. As a result, even though every bidder participates in m markets, he will bid as if he faced only a single market, because he only needs to bid according to the price function he faces.

Now we discuss the price functions. Let $p^{-0}(t)$ denote *the first price function* in each round (or time) $t \in \mathbb{Z}_+$ of the auction, which is also the price function of the original market $\mathcal{M}_{-0}(= \mathcal{M})$. The 1st-price $p^{-0}(t, S)$ of each bundle S equals the highest price that bidders in the market $\mathcal{M}_{-0}$ are willing to pay at time t. However, to compute the price function $p^{-j}(t)$ of every sub-market $\mathcal{M}_{-j}$, $j \in M$, we also need *the second price function* which is denoted by $p^0(t)$. For each bundle S, the 2nd-price $p^0(t, S)$ equals the highest price that bidders in the market $\mathcal{M}_{-0}$ but one bidder making the highest offer $p^{-0}(t, S)$ are willing to pay at time t. For every sub-market $\mathcal{M}_{-j}$, $j \in M$, where bidder j is excluded, the price

$p^{-j}(t, S)$ of this sub-market for every bundle S at time t equals the 2nd-price $p^0(t, S)$ if bidder j is the only bidder making the highest offer $p^{-0}(t, S)$, and $p^{-j}(t, S)$ equals the 1st-price $p^{-0}(t, S)$ otherwise. Let $p^i(t)$ denote the price function for every bidder $i \in M$. The price $p^i(t, S)$ of each bundle S that bidder i faces at time t equals the 1st-price $p^{-0}(t, S)$ if bidder i makes the highest offer, and $p^i(t, S)$ equals the 2nd-price $p^0(t, S)$ otherwise.[15] At time $t \in \mathbb{Z}_+$, the auctioneer chooses a supply set $\gamma^{-j}(t) \in S(p^{-j}(t))$ for every market $\mathcal{M}_{-j}$, $j \in M_0$. If a bidder i submits a demand bundle $A_i(t)$ against price $p^i(t, A_i(t))$, we use $b_i(t) = (A_i(t), p^i(t, A_i(t)))$ to denote this bid. As it will soon become apparent, when prices are adjusted from the current round to the next, the auctioneer will have to take all $m + 1$ markets into account.

In our auction design, an important consideration is that the auction must not only induce bidders to bid truthfully, should be easy for them to bid, and should use their information efficiently and judiciously, but also can tolerate certain mistakes, or inaccuracies caused by bidders and gives them opportunities to adjust and correct. The first and second price functions intend to serve the first purpose, while for the second purpose bidders will be allowed to withdraw some of their past bids. Let $\Psi_i(t)$ denote the set of bids that are withdrawn by bidder i at round t. $\Psi_i(t)$ can be empty. Let $\Omega_i(t)$ stand for the set of bidder i's bids in force at round t.[16] At round t, bidder i submits his withdrawal $\Psi_i(t)$ according to the following rules: (i) only bids that contain nonempty bundles and are active up to time t can be withdrawn, i.e., $\Psi_i(t) \subseteq \Omega_i(t-1)$ and $(\emptyset, 0) \notin \Psi_i(t)$; (ii)[17] the price of each withdrawn bid must equal its current price of the underlying bundle or the price minus 1, i.e., $b_i = (S, p(S)) \in \Psi_i(t)$ implies $p(S) = p^i(t, S) - 1$, or $= p^i(t, S)$; (iii) if a bundle with a price (i.e., a bid) is withdrawn, the same bundle with a higher price that is contained in $\Omega_i(t-1)$ must be withdrawn together, i.e., $(S, p^i(t, S) - 1) \in \Psi_i(t)$ and $(S, p^i(t, S)) \in \Omega_i(t-1)$ imply $(S, p^i(t, S)) \in \Psi_i(t)$. In Sun and Yang (2008a), we discuss a simpler auction without allowing bidders to withdraw their past bids. In this case, the equilibrium notion for the underlying auction game is weaker.

[15]It is clear that for every nonempty bundle $S \subseteq N$, the set $\{p^i(t, S) \mid i \in M\}$ contains only $p^{-0}(t, S)$ and $p^0(t, S)$. If $p^{-0}(t, S) > p^0(t, S)$, only one $p^i(t, S)$ equals $p^{-0}(t, S)$.

[16]$\Omega_i(t)$ contains both bids that have been submitted and not been withdrawn by bidder i before time $t + 1$, and bids that are possibly beneficial to bidder i, i.e., $(S, p(S)) \in \Omega_i(t)$ implies $(S, p'(S)) \in \Omega_i(t)$ for all $p'(S) \in \{u^0(S), u^0(S) + 1, \cdots, p(S) - 1, p(S)\}$. $\Omega_i(t)$ can be also recursively defined as follows. If bidder i submits a nonempty bundle $A_i(t)$ at round t, let $\Phi_i(t) = \{(S, p(S)) \mid S = A_i(t) \text{ and } p(S) = u^0(A_i(t)), \cdots, p^i(t, A_i(t))\}$ denote the set of all his actual bids $b_i(t)$ and all possibly beneficial bids at round t; otherwise, let $\Phi_i(t) = \emptyset$. Then we have $\Omega_i(0) = \Phi_i(0)$, $\Omega_i(1) = [\Omega_i(0) \setminus \Psi_i(1)] \cup \Phi_i(1), \cdots, \Omega_i(t) = [\Omega_i(t-1) \setminus \Psi_i(t)] \cup \Phi_i(t)$.

[17]This rule makes it possible in the auction to decrease the price of a bundle by one unit each time opposite the case of increasing the price of a bundle by one unit.

We are now ready to describe the basic idea of how our IC dynamic auction operates. The seller initially announces her reserve price function u^0 and the auctioneer sets the 1st-price and 2nd-price functions $p^{-0}(0)$ and $p^0(0)$ equal to u^0. In each round $t \in \mathbb{Z}_+$, based on the price functions $p^{-0}(t)$ and $p^0(t)$, each bidder i reports a demand bundle against prices $p^i(t)$ and the auctioneer announces a supply set of each market $\mathcal{M}_{-j}$ at its prices $p^{-j}(t)$. Then the auctioneer must take all $m+1$ markets into account and examines the aggregated reported demands for every bundle S. For instance, if a bundle S was previously demanded by two or more bidders but is now demanded by only one bidder, say bidder i^*, and moreover a market containing bidder i^* does not provide the bundle S in its supply set, then we call this bundle S 1st-price over-demanded. The current price of this bundle S becomes the 1st-price of bundle S and its previous price is the 2nd-price. We see $p^{-0}(t,S) > p^0(t,S)$. If a bundle is demanded by two or more bidders, this bundle will be called 2nd-price over-demanded. In the opposite case, a bundle may become 1st-price or 2nd-price over-supplied. In general, the auctioneer increases the price of any over-demanded bundle but decreases the price of any over-supplied bundle. This process is repeated until no bundle is over-demanded or over-supplied. Next we will discuss 1st and 2nd-price over-demanded (or supplied) bundles in detail.

At round $t \in \mathbb{Z}_+$, we say a bidder i is an *active demander* of a bundle S at a price $p(S)$ if his bid $(S, p(S))$ is in force, i.e., $(S, p(S)) \in \Omega_i(t)$. A bidder i is a *crucial demander* of a bundle S if he is an active demander of S at some price $p(S)$ but there is no other active demander of S at a higher price $p'(S) > p(S)$. A bundle S is said to be *1st-price-over-demanded* if there are at least two active demanders of S at its 1st-price $p^{-0}(t,S)$, or if there is only one active demander i of S at its 1st-price $p^i(t,S) = p^{-0}(t,S)$ but S is not supplied in some market $\mathcal{M}_{-l}$ containing bidder i, i.e., $l \in M_0 \setminus \{i\}$, $S \notin \gamma^{-l}(t)$. A bundle S is *2nd-price-over-demanded* if there exist at least two different active demanders of S at its 2nd-price $p^0(t,S)$. Conversely, we say that a bundle S is *1st-price-over-supplied* if its 1st-price $p^{-0}(t,S) > u^0(S)$ and there exists no active demander of S at the price $p^{-0}(t,S) - 1$. A bundle S is *2nd-price-over-supplied* if its 2nd-price $p^0(t,S) > u^0(S)$ and there is at most one active demander of S at the price $p^0(t,S) - 1$. In general, we say a bundle S is *over-demanded* if it is 1st- or 2nd-price-over demanded, and a bundle S is *over-supplied* if it is 1st- or 2nd-price-over supplied.

At round $t \in \mathbb{Z}_+$, the auctioneer adjusts the prices for round $t+1$ as follows: For every bidder i and every bundle S, set

$$p^{-0}(t+1,S) = \begin{cases} p^{-0}(t,S)+1, & \text{if } S \text{ is 1st-price-over-demanded at round } t, \\ p^{-0}(t,S)-1, & \text{if } S \text{ is 1st-price-over-supplied at round } t, \\ p^{-0}(t,S), & \text{otherwise;} \end{cases} \tag{4.6}$$

$$p^{0}(t+1,S) = \begin{cases} p^{0}(t,S)+1, & \text{if } S \text{ is 2nd-price-over-demanded at round } t, \\ p^{0}(t,S)-1, & \text{if } S \text{ is 2nd-price-over-supplied at round } t, \\ p^{0}(t,S), & \text{otherwise;} \end{cases} \tag{4.7}$$

$$p^i(t+1,S) = \begin{cases} p^{-0}(t+1,S), & \text{if bidder } i \text{ is a crucial demander at round } t, \\ p^0(t+1,S), & \text{otherwise;} \end{cases} \tag{4.8}$$

$$p^{-j}(t+1) = \bigvee_{i \in M_{-j}} p^i(t+1), \quad \text{for every } j \in M. \tag{4.9}$$

Note that formula (4.8) uses formulas (4.6) and (4.7). The auction works roughly as follows. In each round t, the auctioneer informs every bidder i of his price function $p^i(t)$. Then, every bidder i withdraws some of his past bids (if necessary) and reports a demand bundle $A_i(t) \in 2^N$. The auctioneer adjusts all price functions through the formulas (4.6)–(4.9). If the auction never terminates, we can punish those bidders who have nullified their bids more than (a prior given large number) L^* times. Formally, we have

The incentive-compatible (IC) dynamic auction mechanism

Step 1: The seller reports her reserve price function $u^0(\cdot)$. Then the auctioneer sets the initial pricing functions $p^i(0) = p^{-i}(0) = u^0(\cdot)$ for every $i \in M_0$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \cdots$, the auctioneer informs every bidder $i \in M$ of his price function $p^i(t)$. Then, every bidder i submits his withdrawal $\Psi_i(t)$ (if any) and reports a bundle $A_i(t) \in 2^N$. The auctioneer chooses a supply set $\gamma^{-j}(t) \in S(p^{-j}(t))$ for every $j \in M_0$ and adjusts prices as follows: If there is neither an over-demanded bundle nor an over-supplied bundle, go to Step 3. Otherwise, if there is any over-demanded bundle or over-supplied bundle, the auctioneer obtains the price functions $p^{-0}(t+1), p^0(t+1), p^i(t+1)$ and $p^{-i}(t+1)$ for all $i \in M$ by formulas (4.6), (4.7), (4.8), and (4.9). If $p^i(t+1, S^*) > U^*$ for some bidder i and some bundle S^*, go to Step 4. Otherwise, set $t := t+1$ and return to Step 2.

Step 3: At the last round $t = t^*$, for every $j \in M_0$ the auctioneer chooses an allocation π^{-j} for the market $\mathcal{M}_{-j}$ as in the basic ascending auction. By the allocation π^{-0} of the original market $\mathcal{M}$, the auctioneer assigns the bundle $\pi^{-0}(0)$ to the seller and the bundle $\pi^{-0}(i)$ to bidder $i \in M$ who is asked to pay the price

$$q_i = Re(p^{-i}(t^*)) - \Big(Re(p^{-0}(t^*)) - \sum_{S \in \gamma_i^{-0}} p^{-0}(t^*, S)\Big), \tag{4.10}$$

where $\gamma_i^{-0} = \{S \mid S \in \gamma^{-0}(t^*) \text{ and } S \subseteq \pi^{-0}(i)\}$ stands for the family of bundles that are in the seller's supply set $\gamma^{-0}(t^*)$ of the market $\mathcal{M}$ and are assigned to bidder i at round t^*. Then the auction stops.

Step 4: The auctioneer assigns the bundle S^* to a bidder[18] i with $p^i(t+1, S^*) > U^*$ and asks him to pay the price U^*. And all other bidders get nothing and pay nothing. The auction stops.

Step 5: If the auction never terminates, then any bidder who has withdrawn his bids more than L^* times has to pay a penalty of U^*, and any other bidder gets nothing and pays nothing.

Note 1: If Step 4 or Step 5 happens, then this is the case in which some bidder has openly defied the rules or made serious or too many mistakes, and thus is punished.

Note 2: Observe that in particular $p^0(0) = p^i(0) = p^{-i}(0)$, in general $p^0(t) \leq p^i(t) \leq p^{-j}(t) \leq p^{-0}(t)$ for all $i, j \in M$ $(i \neq j)$, $t > 0$, and $p^i(t) - p^0(t) = p^{-0}(t) - p^{-i}(t)$, $i \in M$. Moreover, if $p^{-0}(t, S) > p^0(t, S)$ for some bundle S, then there exists a unique bidder i^* such that $p^{i^*}(t, S) = p^{-0}(t, S)$ and $p^j(t, S) = p^0(t, S)$ for any bidder $j \neq i^*$, and $p^{-i^*}(t, S) = p^0(t, S)$ and $p^{-j}(t, S) = p^{-0}(t, S)$ for any market $\mathcal{M}_{-j}$ with $j \neq i^*$.

Note 3: It is important to point out that although every bidder $i \in M$ apparently confronts m markets $\mathcal{M}_{-j}$, $(j \in M_0,\ j \neq i)$, he actually only needs to submit one bundle $A_i(t)$ (the same bundle for all m markets he faces) in each round t of the auction. This is very simple, useful and practical from the viewpoint of bidders, and differs crucially from Ausubel's (2006) auction (also Sun and Yang 2008b) where every bidder $i \in M$ faces also m markets $\mathcal{M}_{-j}$, $(j \in M_0,\ j \neq i)$, but needs to submit a bid $A_i^j(t)$ for every market $\mathcal{M}_{-j}$, $j \in M_0$, $j \neq i$, and thus in total has to report m possibly different bids for m markets in the auction. This means that the current auction requires far less communication and information from bidders.

Note 4: Observe that bidder i's payment q_i given by (4.10) involves only the prices and allocations of the last round t^* of the auction. This payment is simple and easy to compute[19] and moreover has an intuitive interpretation: it equals the seller's total revenue $Re(p^{-i}(t^*))$ in the market $\mathcal{M}_{-i}$ minus her total revenue $Re(p^{-0}(t^*))$ in the original market $\mathcal{M}$ plus the prices $\sum_{S \in \gamma_i^{-0}} p^{-0}(t^*, S)$ of all bundles assigned to bidder i in the original market $\mathcal{M}$ at round t^*. It will be shown when all bidders bid truthfully, the pair $(p^{-j}(t^*), \pi^{-j})$ is an NPW equilibrium in every market $\mathcal{M}_{-j}$, and q_i equals the difference between the total equilibrium payments of bidder i's opponents (including the seller) in $\mathcal{M}_{-i}$ and the total equilibrium payments of bidder i's opponents in $\mathcal{M}$.

Before studying the strategic aspect of the auction, we demonstrate how the auction actually works via Example 1. To save space, we assume that the seller's reserve price

[18]If there are several such bidders, the auctioneer randomly chooses one.

[19]This also contrasts with that of Ausubel (2006) which is calculated along the entire path generated by his auction.

function is $u^0(\cdot) = (0, 1, 2, 1, 3, 3, 4, 5)$. The demands, supplies and withdrawals generated in the auction are shown in Table 4, where the price vectors are understood as $p = (p(A), p(B), p(C), p(AB), p(AC), p(BC), p(ABC))$ with $p(\emptyset) = 0$, the bidder's column indicates each bidder's withdrawals and demands, the last column indicates the seller's supplies γ, and the last row gives the final outcome. Note that in the entire process every bidder i reports only $A_i(t)$ against prices $p^i(t)$ (and a withdrawal $\Psi_i(t)$ if any).

The auctioneer starts with the reserve prices $p^0(0) = (1, 2, 1, 3, 3, 4, 5)$. In each round $t(= 0, 1, \cdots, 7)$, every bidder $i(= 1, 2, 3)$ faces his price function $p^i(t)$ and reports a bid $A_i(t)$ (and a withdrawal $\Psi_i(t)$ if any). We see that at every round t every bidder i reports an optimal bundle $A_i(t) \in D^i(p^i(t))$, except that bidder 3 reports the bundle $AB \notin D^3(p^3(0))$ at round 0, reports the bundle $B \notin D^3(p^3(4))$ at round 4, and bidder 2 reports the bundle $AB \notin D^2(p^2(5))$ at round 5. In addition, at round $t = 5$ bidder 3 withdraws his bid $b_3(4)$ which yields a non-positive profit for him but bidder 3 does not withdraw his bid $b_3(0)$ giving him positive profit, and at round $t = 6$ bidder 2 withdraws his bid $b_2(5)$ which yields a non-positive profit for him. However, all bidders bid sincerely from round $t = 6$ on.

Table 4: The illustration of the IC dynamic auction for Example 1.

	$p^0(0) = (1,2,1,3,3,4,5)$		$p^{-0}(0) = (1,2,1,3,3,4,5)$	$\gamma^{-0}(0) = \{ABC\}$
$t=0$	$p^1(0) = (1,2,1,3,3,4,5)$	$A_1(0) = AB$	$p^{-1}(0) = (1,2,1,3,3,4,5)$	$\gamma^{-1}(0) = \{ABC\}$
	$p^2(0) = (1,2,1,3,3,4,5)$	$A_2(0) = AC$	$p^{-2}(0) = (1,2,1,3,3,4,5)$	$\gamma^{-2}(0) = \{ABC\}$
	$p^3(0) = (1,2,1,3,3,4,5)$	$\mathbf{A_3(0) = AB}$	$p^{-3}(0) = (1,2,1,3,3,4,5)$	$\gamma^{-3}(0) = \{ABC\}$
	$p^0(1) = (1,2,1,4,3,4,5)$		$p^{-0}(1) = (1,2,1,4,4,4,5)$	$\gamma^{-0}(1) = \{AC,B\}$
$t=1$	$p^1(1) = (1,2,1,4,3,4,5)$	$A_1(1) = AB$	$p^{-1}(1) = (1,2,1,4,4,4,5)$	$\gamma^{-1}(1) = \{AC,B\}$
	$p^2(1) = (1,2,1,4,4,4,5)$	$A_2(1) = AC$	$p^{-2}(1) = (1,2,1,4,3,4,5)$	$\gamma^{-2}(1) = \{AC,B\}$
	$p^3(1) = (1,2,1,4,3,4,5)$	$A_3(1) = ABC$	$p^{-3}(1) = (1,2,1,4,4,4,5)$	$\gamma^{-3}(1) = \{AC,B\}$
	$p^0(2) = (1,2,1,4,3,4,5)$		$p^{-0}(2) = (1,2,1,5,4,4,6)$	$\gamma^{-0}(2) = \{AB,C\}$
$t=2$	$p^1(2) = (1,2,1,5,3,4,5)$	$A_1(2) = AB$	$p^{-1}(2) = (1,2,1,4,4,4,6)$	$\gamma^{-1}(2) = \{AC,B\}$
	$p^2(2) = (1,2,1,4,4,4,5)$	$A_2(2) = AC$	$p^{-2}(2) = (1,2,1,5,3,4,6)$	$\gamma^{-2}(2) = \{AB,C\}$
	$p^3(2) = (1,2,1,4,3,4,6)$	$A_3(2) = BC$	$p^{-3}(2) = (1,2,1,5,4,4,5)$	$\gamma^{-3}(2) = \{AB,C\}$
	$p^0(3) = (1,2,1,4,3,4,5)$		$p^{-0}(3) = (1,2,1,5,5,5,6)$	$\gamma^{-0}(3) = \{AC,B\}$
$t=3$	$p^1(3) = (1,2,1,5,3,4,5)$	$A_1(3) = AB$	$p^{-1}(3) = (1,2,1,4,5,5,6)$	$\gamma^{-1}(3) = \{AC,B\}$
	$p^2(3) = (1,2,1,4,5,4,5)$	$A_2(3) = C$	$p^{-2}(3) = (1,2,1,5,3,5,6)$	$\gamma^{-2}(3) = \{AB,C\}$
	$p^3(3) = (1,2,1,4,3,5,6)$	$A_3(3) = C$	$p^{-3}(3) = (1,2,1,5,5,4,5)$	$\gamma^{-3}(3) = \{AC,B\}$
	$p^0(4) = (1,2,2,4,3,4,5)$		$p^{-0}(4) = (1,2,2,6,5,5,6)$	$\gamma^{-0}(4) = \{AB,C\}$
$t=4$	$p^1(4) = (1,2,2,6,3,4,5)$	$A_1(4) = ABC$	$p^{-1}(4) = (1,2,2,4,5,5,6)$	$\gamma^{-1}(4) = \{AC,B\}$
	$p^2(4) = (1,2,2,4,5,4,5)$	$A_2(4) = AC$	$p^{-2}(4) = (1,2,2,6,3,5,6)$	$\gamma^{-2}(4) = \{AB,C\}$
	$p^3(4) = (1,2,2,4,3,5,6)$	$\mathbf{A_3(4) = B}$	$p^{-3}(4) = (1,2,2,6,5,4,5)$	$\gamma^{-3}(4) = \{AB,C\}$
	$p^0(5) = (1,2,2,4,3,4,6)$	$A_1(4) = A$	$p^{-0}(5) = (1,3,2,6,6,5,6)$	$\gamma^{-0}(5) = \{AC,B\}$
$t=5$	$p^1(5) = (1,2,2,6,3,4,6)$	$\mathbf{A_2(5) = AB}$	$p^{-1}(5) = (1,3,2,4,6,5,6)$	$\gamma^{-1}(5) = \{AC,B\}$
	$p^2(5) = (1,2,2,4,6,4,6)$	$\mathbf{\Psi_3(5) = \{b_3(4)\}}$	$p^{-2}(5) = (1,3,2,6,3,5,6)$	$\gamma^{-2}(5) = \{AB,C\}$
	$p^3(5) = (1,3,2,4,3,5,6)$	$A_3(5) = ABC$	$p^{-3}(5) = (1,2,2,6,6,4,6)$	$\gamma^{-3}(5) = \{AC,B\}$
	$p^0(6) = (1,2,2,5,3,4,6)$	$A_1(6) = AB$	$p^{-0}(6) = (2,2,2,6,6,5,7)$	$\gamma^{-0}(6) = \{AB,C\}$
$t=6$	$p^1(6) = (2,2,2,6,3,4,6)$	$\mathbf{\Psi_2(6) = \{b_2(5)\}}$	$p^{-1}(6) = (1,2,2,5,6,5,7)$	$\gamma^{-1}(6) = \{AC,B\}$
	$p^2(6) = (1,2,2,5,6,4,6)$	$A_2(6) = A$	$p^{-2}(6) = (2,2,2,6,3,5,7)$	$\gamma^{-2}(6) = \{AB,C\}$
	$p^3(6) = (1,2,2,5,3,5,7)$	$A_3(6) = BC$	$p^{-3}(6) = (2,2,2,6,6,4,6)$	$\gamma^{-3}(6) = \{AB,C\}$
	$p^0(7) = (2,2,2,4,3,4,6)$		$p^{-0}(7) = (2,2,2,6,6,6,7)$	$\gamma^{-0}(7) = \{AB,C\}$
$t=7$	$p^1(7) = (2,2,2,6,3,4,6)$	$A_1(7) = AB$	$p^{-1}(7) = (2,2,2,4,6,6,7)$	$\gamma^{-1}(7) = \{AC,B\}$
	$p^2(7) = (2,2,2,4,6,4,6)$	$A_2(7) = \emptyset$	$p^{-2}(7) = (2,2,2,6,3,6,7)$	$\gamma^{-2}(7) = \{AB,C\}$
	$p^3(7) = (2,2,2,4,3,6,7)$	$A_3(7) = \emptyset$	$p^{-3}(7) = (2,2,2,6,6,4,6)$	$\gamma^{-3}(7) = \{AB,C\}$
		$\pi(0) = \emptyset$	$Re(p^{-0}(7)) = 8$	$\pi^{-0} = (\emptyset, AB, C, \emptyset)$
	$q_1 = 6$	$\pi(1) = AB$	$Re(p^{-1}(7)) = 8$	$\pi^{-1} = (B, AC, \emptyset)$
	$q_2 = 2$	$\pi(2) = C$	$Re(p^{-2}(7)) = 8$	$\pi^{-2} = (\emptyset, AB, C)$
	$q_3 = 0$	$\pi(3) = \emptyset$	$Re(p^{-3}(7)) = 8$	$\pi^{-3} = (\emptyset, AB, C)$

At round 7 there is neither an over-demanded set nor an over-supplied set. The auction terminates with the allocations $\pi^{-0} = (\emptyset, AB, C, \emptyset)$, $\pi^{-1} = (B, AC, \emptyset)$, $\pi^{-2} = (\emptyset, AB, C)$, $\pi^{-3} = (\emptyset, AB, C)$. In the last row of the table, $\pi = \pi^{-0}$. By the auction rule, bidder 1 gets $\pi(1) = AB$ and pays $q_1 = Re(p^{-1}(7)) - (Re(p^{-0}(7)) - p^{-0}(7, AB)) = 8-(8-6) = 6$, bidders 2 gets $\pi(2) = C$ and pays $q_2 = Re(p^{-2}(7)) - (Re(p^{-0}(7)) - p^{-0}(7, C)) = 8-(8-2) = 2$, but bidder 3 gets ($\pi(3) = \emptyset$) nothing and pays nothing. The pairs $(p^{-0}(7), \pi^{-0})$, $(p^{-1}(7), \pi^{-1})$, $(p^{-2}(7), \pi^{-2})$, and $(p^{-3}(7), \pi^{-3})$ are NPW equilibria for the markets $\mathcal{M}$, $\mathcal{M}_{-1}$, $\mathcal{M}_{-2}$ and $\mathcal{M}_{-3}$, respectively. Finally, notice that in equilibrium (i.e., the last round), every bidder i's optimal bundle $\pi(i) \in D^i(p^i(7))$ with respect to his own prices $p^i(7)$ is also optimal $\pi(i) \in D^i(p^{-j}(7))$ to him with respect to the prices $p^{-j}(7)$ of every market $\mathcal{M}_{-j}$ containing him.

4.2 The Auction Game and Its Properties

To investigate the incentive properties of this dynamic auction mechanism, we need to study the extensive-form dynamic game of incomplete information induced by the IC dynamic auction. We simply call this game *the auction game*. In the game, bidders are the players. Prior to the start of the auction, nature according to a joint probability distribution function $F(\cdot)$ draws a profile $\{u^i\}_{i\in M}$ with $u^i \in \mathcal{U}$ for all $i \in M$, and reveals to every player $i \in M$ only his own value function u^i of private information. Let $H_i(t)$ be the part of the information (or history) of play that player i has observed just before he takes action at time $t \in \mathbb{Z}_+$. According to the auction rules, a natural and sensible specification can be that $H_i(t)$ comprises all observable price functions and his own past actions, i.e.,

$$H_i(t) = \{p^i(t), p^i(s), b_i(s), \Psi_i(s) \mid 0 \le s < t\},\ H_i(s) \subseteq H_i(t),\ \text{for } 0 \le s \le t.$$

Following any history $H_i(t)$, $t \in \mathbb{Z}_+$, every player i updates his posterior beliefs $\mu_i(\cdot \mid t, H_i(t), u^i)$ over opponents' value functions. We stress that even after the auction is finished, player i may not know his opponents' value functions precisely.

A (dynamic) *strategy* σ_i of player i is a function $\{(t, H_i(t), u^i) \mid t \in \mathbb{Z}_+, u^i \in \mathcal{U}\} \to 2^{\Omega_i(t-1)} \times 2^N$, which tells him to submit a withdrawal and a bid $\sigma_i(t, H_i(t), u^i) = (\Psi_i(t), A_i(t)) \in 2^{\Omega_i(t-1)} \times 2^N$ at each time $t \in \mathbb{Z}_+$ when he observes $H_i(t)$. Note that when bidder i observes the history $H_i(t)$, he knows the set $\Omega_i(t-1)$. Let Σ_i denote player $i's$ strategy space of all such strategies σ_i. Given time $t \in \mathbb{Z}_+$, we say that bidder i *bids sincerely at time* t relative to utility function u^i if, for any history $H_i(t)$, he withdraws $\Psi_i(t)$ (if any) according to the withdrawing rules those of his past bids that give him non-positive profits, i.e., $u^i(S) - p(S) \le 0$ for all $(S, p(S)) \in \Psi_i(t)$, and reports a *demand bundle* $A_i(t) \in D^i(p^i(t)) = \arg\max_{S\subseteq N}\{u^i(S) - p^i(t, S)\}$ with $A_i(t) = \emptyset$ when $\emptyset \in D^i(p^i(t))$. A strategy σ_i of bidder i is *sincere* if he always bids sincerely. Observe that a sincere bidding strategy may not be unique, and the strategy space Σ_i of player i contains sincere bidding strategies and also various other strategies.

Given the auction rules, the outcome of this auction game depends entirely upon the realization $\{u^i\}_{i\in M}$ of value functions and the strategies $\{\sigma_i\}_{i\in M}$, the bidders take. Let $W_i\Big(\{\sigma_l\}_{l\in M}, \{u^l\}_{l\in M}\Big)$ denote the payoff of player $i \in M$. Then this payoff is determined as follows: (i) If the IC dynamic auction terminates in Step 3, then each bidder i receives bundle $\pi^{-0}(i)$ and pays q_i in formula (4.10) and his payoff is given by

$$W_i\Big(\{\sigma_l\}_{l\in M}, \{u^l\}_{l\in M}\Big) = u^i(\pi^{-0}(i)) - q_i. \tag{4.11}$$

(ii) If the dynamic auction stops in Step 4, the player i^* who receives the bundle S^* is punished and receives a strictly negative payoff, while the payoff of all other players is zero. (iii) If the dynamic auction does not terminate (i.e., in Step 5), then every bidder i

who has withdrawn his bids more than L^* times is punished with a penalty of U^*, but the payoff of any other player is zero.

For auction games of incomplete information, the ex post equilibrium has been used by Crémer and McLean (1985) for a sealed-bid auction (see also Krishna 2002) and the ex post perfect equilibrium by Ausubel (2004, 2006) for dynamic auctions. Stronger than ex post equilibrium, Bayesian equilibrium, and perfect Bayesian equilibrium, the notion of ex post perfect equilibrium has a number of additional desirable properties, i.e., it is not only robust against any regret but also independent of any probability distribution, and furthermore it requires that the equilibrium strategy for every player should remain optimal at every node of the auction game even if the player were to learn his opponents' private values. Following Ausubel (2006), the m-tuple $\{\sigma_i\}_{i\in M}$ is an *ex post perfect equilibrium* if for any time $t \in \mathbb{Z}_+$, following any history profile $\{H_i(t)\}_{i\in M}$, and for any realization $\{u^i\}_{i\in M}$ of profile of value functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i(t), u^i)$ of every player $i \in M$ (i.e., $\sigma_i(s, H_i(s) \mid t, H_i(t), u^i) \subseteq 2^{\Omega_i(s-1)} \times 2^N$ for all $s \geq t$) constitutes his best response against the continuation strategies $\{\sigma_l(\cdot \mid t, H_l(t), u^l)\}_{l\in M_{-i}}$ of player i's opponents of the game even if the realization $\{u^i\}_{i\in M}$ becomes common knowledge. In addition, we say that an auction mechanism is *ex post individually rational*, if, for every bidder, no matter how his opponents bid, as long as he always bids sincerely he will never end up with a negative payoff.

The next result shows a useful and interesting property of the IC dynamic auction that although each bidder i submits the same bid for all markets according to his price function $p^i(t)$ rather than every market price function $p^{-j}(t)$, after any history, as long as all bidders bid sincerely, the auction simultaneously finds an NPW equilibrium in every market.

Lemma 4.2 *Suppose that Assumptions* (A1)–(A3) *hold for the auction model. For any time $t_0 \in \mathbb{Z}_+$ when the IC dynamic auction has not stopped, following any history profile $\{H_i(t_0)\}_{i\in M}$, if every bidder bids sincerely from t_0 on, the IC dynamic auction must stop in Step 3 at some round t^* and for every $j \in M_0$, the pair $(p^{-j}(t^*), \pi^{-j})$ is a nonlinear pricing Walrasian equilibrium for the market $\mathcal{M}_{-j}$.*

The following theorem establishes that sincere bidding is an ex post perfect equilibrium in the auction game.[20] That is, at any time t_0 and following any history up to t_0, bidding truthfully is a best response for every bidder from t_0 on, no matter whether he is now at time t_0 on the equilibrium path or off the equilibrium path, as long as all his opponents bid sincerely from time t_0 on. The auction also guarantees that a bidder will never get a negative payoff as long as he always bids sincerely no matter how others do, and moreover no bundle of goods would be sold below the seller's reserve price.

[20]To our best knowledge, the ex post perfect equilibrium is the most desirable property that a dynamic auction could possibly achieve. See also Ausubel (2004, 2006), and Sun and Yang (2008b).

Theorem 4.3 *Suppose that the market* $\mathcal{M}$ *satisfies Assumptions* (A1)–(A3).
(i) *Sincere bidding by every bidder is an ex post perfect equilibrium in the auction game.*
(ii) *When bidders bid sincerely, the IC dynamic auction yields an efficient allocation* π^{-0} *with its Walrasian equilibrium price for every bundle of goods and a generalized VCG payment* $q_i \geq u^0(\pi^{-0}(i))$ *for every bidder* $i \in M$ *in finite time.*
(iii) *The IC dynamic auction is ex post individually rational.*

5 The Model of Multiple Sellers

In this section we extend our approach to deal with multiple sellers. Suppose that there are K sellers and m buyers, denoted by $\mathcal{S} = \{1, \cdots, K\}$ and $M = \{K+1, \cdots, K+m\}$, respectively. Each seller $i \in \mathcal{S}$ owns a set W_i of indivisible goods. The sellers wish to sell their goods to the buyers through auction. Let $N = \{1, 2, \cdots, n\}$ denote the collection of all goods, i.e., $N = \cup_{i=1}^{K} W_i$, where $W_h \cap W_i = \emptyset$ for $h \neq i$. Every bidder $j \in M$ has a utility function $u^j : 2^N \to \mathbb{Z}$ with $u^j(\emptyset) = 0$, and is endowed with a sufficient amount of money in the sense that he can pay up to his value. Each seller i has a reserve price function $u^i : 2^{W_i} \to \mathbb{Z}$ with $u^i(\emptyset) = 0$ and is interested only in her own goods.

The same Assumptions (A1), (A2), and (A3) in Section 2 are used for this extended model, i.e., all buyers and sellers regard all goods as complements. Two basic questions arise immediately in the study of this model. Should the sellers sell their goods separately or jointly? If they sell their goods jointly, how should they divide their joint revenues? because the goods are complementary, some goods of one seller may be sold with another seller's goods as one package and thus the revenues of the sellers are not separable in general. This is in sharp contrast with models of Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), Milgrom (2000), and Ausubel (2006) in which goods are substitutes and can be sold through competitive anonymous and linear prices. As a result, the revenues of the sellers in these models are automatically separable. We use the following example to illustrate the two issues and indicate how they may be resolved.

Example 3: Suppose that there are two sellers (1 and 2) and three bidders (3, 4, 5). Seller 1 owns item A and seller 2 owns items B and C. Bidders' values and sellers' reserve prices are given in Table 5. Clearly, all items are complementary. We consider and compare two scenarios in which items will be allocated.

First, if the sellers decide to sell their goods separately, item A will go to buyer 5 and seller 1's revenue is between 1 and 2, while items B and C will go to buyer 4 and seller 2's revenue is between 5 and 6. The allocation of the items achieves a value of 8.

Second, if the sellers were to sell their goods jointly, then item C would go to buyer 3, and items A and B to buyer 5 with (superadditive) competitive equilibrium prices, e.g., $p^* = (0, 2, 1, 3, 13, 5, 6, 16)$. Let $\pi^* = (C, \emptyset, AB)$ denote the allocation of items to the buyers. This is the unique efficient allocation and gives the total market value of 22, much higher than 8 in the first case! The total joint revenues of the sellers are 16 and inseparable because items A and B are sold as a bundle at the price of 13. How should the sellers divide their joint revenues?

We may view the problem as a cooperative game for the sellers and adopt the Shapley value (see Shapley 1953 and Myerson 1991) as a fair division rule. To do this, the equilibrium price of A can be seen as the value of seller 1, the equilibrium price of BC as the value of seller 2, and the equilibrium price of A, B and C as the value of the coalition of sellers 1 and 2. Let $v(\{1\}) = 2$, $v(\{2\}) = 6$ and $v(\{1, 2\}) = 16$ be the value of the coalitions $\{1\}$, $\{2\}$, and $\{1, 2\}$, respectively. For the ordering $(1, 2)$, the marginal contributions of sellers 1 and 2 are 2 and 14, respectively, while for the ordering $(2, 1)$, the marginal contributions of sellers 1 and 2 are 10 and 6, respectively. The average of the two possibilities gives the Shapley value $\phi(v) = (\phi_1(v), \phi_2(v)) = (6, 10)$ by which seller 1 receives the revenue of 6 and seller 2 the revenue of 10. This shows that if the sellers sell their goods jointly, the equilibrium allocation π^* achieves the maximal market efficiency, far better than they sell separately, and also the sellers gain higher revenues than if they act independently.

Table 5: Bidders' and seller's values over items.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Seller 1	0	1	0	0	1	1	0	1
Seller 2	0	0	1	1	1	1	3	3
Bidder 3	0	0	1	4	6	5	5	12
Bidder 4	0	1	0	1	12	3	6	14
Bidder 5	0	2	1	1	18	4	3	20

Let 0 represent the auctioneer (agent 0) who acts in the interests of sellers. Then the auctioneer's reserve price function $u^0 : 2^N \to \mathbb{R}$ is given by

$$u^0(L) = \sum_{i \in \mathcal{S}} u^i(L \cap W_i), \quad \text{for any } \; L \subseteq N = \cup_{h \in \mathcal{S}} W_h.$$

It is easy to see that u^0 is superadditive if u^i is superadditive for every $i \in \mathcal{S}$. As in Section 2, let $M_0 = M \cup \{0\}$. We can analogously work on M_0 and define the corresponding concepts such as allocation, efficient allocation, market value, and nonlinear pricing Walrasian equilibrium for the extended model. Recall that allocation $\pi = (\pi(i), i \in M_0)$ assigns the bundle $\pi(i)$ to agent i. If $\pi(0) \neq \emptyset$, the bundle $\pi(0)$ is not sold and thus stays with the auctioneer who actually returns the bundle $\pi(0) \cap W_i \neq \emptyset$ to seller $i \in \mathcal{S}$.

We say that two sellers are *symmetric* if they have the same heterogeneous goods and the same reserve price function. A seller is *dummy* if the value of any bundle of her goods to every bidder is worth no more than her own reservation value of the bundle. More precisely, a seller $i \in \mathcal{S}$ is dummy if one has $u^j(A \cup B) - u^j(A) < u^i(B)$ for every bidder $j \in M$, every nonempty bundle $A \subseteq \cup_{h \in \mathcal{S} \setminus \{i\}} W_h$ and every nonempty bundle $B \subseteq W_i$.

An allocation rule specifies a price function p, an allocation π of goods and a payoff vector $\psi \in \mathbb{R}^K$ at which buyer j receives bundle $\pi(j)$ and pays $p(\pi(j))$, while seller i obtains a total payoff of ψ_i. So at (p, π, ψ), buyer j achieves a profit of $u^j(\pi(j)) - p(\pi(j))$, and seller i gains a total payoff of ψ_i (Note that if seller i retains a bundle $\pi(0) \cap W_i \neq \emptyset$ of unsold goods, she receives a revenue of $\psi_i - p(\pi(0) \cap W_i))$.

An allocation rule is *Pareto efficient* if there does not exist another allocation rule which makes no agent worse off but at least one strictly better off. An allocation rule is *individually rational* if it makes every agent better off than she/he stands alone. An allocation rule is *symmetric* if it gives any two symmetric agents the same payoff. An allocation rule has the *dummy property* if it gives a dummy agent its own value.

The Shapley value has many interesting properties including the above four and is the most widely used single value solution for cooperative games. Unlike the core that can be quite large or empty, the Shapley value always exists and prescribes a unique payoff allocation for every transferable utility (TU) game and is easy to apply as a predictive theory. We will show how this solution can be implemented by using our auction.

By Corollary 3.2, the extended model has a nonlinear pricing Walrasian equilibrium (p^*, π^*). From (p^*, π^*) we can construct a new nonlinear pricing Walrasian equilibrium (q^*, π^*) such that q^* is superadditive. We define

$$q^*(L) = \max_{\gamma \in \mathcal{B}(L)} \Big\{ \sum_{A \in \gamma} p^*(A) \Big\}, \text{for every } L \in 2^N, \tag{5.12}$$

where $\mathcal{B}(L)$ denotes the family of all partitions of the elements in L. Then we have the following useful observation.

Lemma 5.1 *If (p^*, π^*) is a nonlinear pricing Walrasian equilibrium, then (q^*, π^*) is also a nonlinear pricing Walrasian equilibrium.*

From (q^*, π^*), we determine the value of each coalition of sellers. For any $T \subseteq \mathcal{S}$, let $v(T) = q^*(\cup_{i \in T} W_i)$, i.e., the equilibrium price of the bundle $\cup_{i \in T} W_i$ owned by the sellers in T. Clearly, $v(\emptyset) = 0$, $v(\{i\}) \geq u^i(W_i)$ for every $i \in \mathcal{S}$, and v is superadditive. This defines a superadditive TU game $(\mathcal{S}, v)$. Let $\beta = (\beta(1), \cdots, \beta(K))$ be a sequence of the K sellers by which the grand coalition $\mathcal{S}$ could be built up from nothing by adding one member each time, and let Ω be the family of all such sequences β. In each sequence β, the marginal contribution of seller i equals $v(P_\beta(i) \cup \{i\}) - v(P_\beta(i))$ where $P_\beta(i)$ is the set of players that precede i in β. According to Shapley (1953), every agent $i \in \mathcal{S}$ should receive the average of all her marginal contributions

$$\phi_i(v) = \sum_{\beta \in \Omega} \frac{v(P_\beta(i) \cup \{i\}) - v(P_\beta(i))}{K!}$$

We denote this rule together with (q^*, π^*) by (q^*, π^*, ϕ) and call it *the equilibrium Shapley value allocation rule.*

Theorem 5.2 *For the extended model, the equilibrium Shapley value allocation rule satisfies Pareto efficiency, individual rationality, dummy, and symmetry properties.*

6 Applications

Situations involving complementarities abound. Economies of scale and scope generate complementarities. Different segments of a railroad network are typically complements, and so are different parts of a machine, players of a soccer team and books in several volumes. In this section we discuss two major practical applications. For related models, we refer to Rassenti et al. (1982), Brewer and Plott (1996), Sun and Yang (2006), and Ostrovsky (2008).

As the first application, we examine a multi-sided matching equilibrium model. Suppose that a seller wishes to sell three complementary sets A, B and C of indivisible goods to a number of buyers $M = \{1, 2, \cdots, m\}$. (The analysis also applies to four or more complementary sets of goods). For instance, one may think of A as main processors (CPUs), of B as displays, and of C as keyboards. Let $A = \{a_1, a_2, \cdots, a_r\}$, $B = \{b_1, b_2, \cdots, b_s\}$, and $C = \{c_1, c_2, \cdots, c_t\}$. Without loss of generality, we assume that the seller values every bundle of goods at zero. Each buyer has a sufficient amount of money and can buy as many goods as he wishes. The value of buyer $i \in M$ depends on the combination of a CPU $a \in A$, a display $b \in B$ and a keyboard $c \in C$ and is denoted by $v_i(a, b, c)$, which is assumed to be a nonnegative integer value. If buyer $i \in M$ uses a set S of CPUs, displays and keyboards, the value $u^i(S)$ of these CPUs, displays and keyboards to the buyer totally depends on the combination of CPU, display and keyboard that the goods in S can generate, and can be explicitly formulated as

$$u^i(S) = \max\{0, v_i(a_{i_1}, b_{j_1}, c_{k_1}) + v_i(a_{i_2}, b_{j_2}, c_{k_2}) + \cdots + v_i(a_{i_l}, b_{j_l}, c_{k_l})\}$$

with the maximum to be taken over all sets $\{(a_{i_1}, b_{j_1}, c_{k_1}), (a_{i_2}, b_{j_2}, c_{k_2}), \cdots, (a_{i_l}, b_{j_l}, c_{k_l})\}$ of distinct CPU-display-keyboard matchings in S. The valuation formula u^i for every buyer $i \in M$ can be seen as an extension of the worker-machine assignment valuations derived by Shapley (1962). Hatfield and Milgrom (2005) give a different extension of Shapley's assignment model.

So in this model, every buyer faces internally an optimal three-sided (CPU-display-keyboard) assignment problem and the whole economy faces a more complex optimal four-sided (buyer-CPUs-displays-keyboards) assignment problem. The following theorem shows that for this model there is a nonlinear system of competitive prices through which all CPUs, displays and keyboards can be efficiently allocated to the buyers. But the model may not have a standard Walrasian equilibrium.

Theorem 6.1 *The value function u^i of every buyer $i \in M$ satisfies Assumptions* (A1)–(A3) *and thus the model has a nonlinear pricing Walrasian equilibrium. But the model may not have a standard Walrasian equilibrium.*

Our second application concerns several firms competing for various parts of a transportation or telecommunication network. A government owns the network that it wishes to sell or rent out to the firms.[21] The fundamental issue here is which part of the network should be allocated to which firm at what price. We will show that this problem fits well into the general model presented in Section 2 and can be therefore solved by our dynamic auction. To see this, it is sufficient to observe that a general value (or utility) function of every firm or the government over the network is actually superadditive. In the sequel, we explain how such value functions can be constructed.

Let $G = (N, E)$ denote the network or graph, where N is called the vertex set representing the collection of all stations and E is the edge set standing for the collection of roads linking stations. We use $\{i, j\}$ to represent a road that connects two distinct stations i and j directly not via any other station. The edge set is a family of such roads. A *subnetwork* $G' = (N', E')$ of the network $G = (N, E)$ is a network whose vertex set N' is a subset of N, and whose edge set E' is a subset of E restricted to the subset N', i.e., E' is a subset of the set $\{\{i, j\} \in E \mid i, j \in N'\}$. Clearly, when N' equals N with $E' = E$, the subnetwork G' will become the original network G. A sequence of stations $(i_1, \ldots, i_{k'})$ yields a *path* between i_1 and $i_{k'}$ in the network (N, E) if all roads $\{i_k, i_{k+1}\} \in E$, $k = 1, \cdots, k' - 1$, are different, and if one can walk from station i_1 to station $i_{k'}$ following each road $\{i_k, i_{k+1}\}$ and passing every station i_k for $k = 1, \ldots, k' - 1$. A path forms a *cycle or loop* if one starts with a station on the path and follows along the path, one will return to the station.

A subnetwork $G' = (N', E')$ of the network G is *connected* if for any two stations $i, j \in N'$, $i \neq j$, there is a path in the subnetwork G' between i and j; G' is a *tree* if it contains no cycle and is connected; G' is a *spanning tree* of the network G if G' is a tree and its vertex set N' equals the vertex set N of the original network; G' is called a

[21]How to sell public utilities is an important issue in privatization (Maskin 1992 and Janssen 2004).

component of the network G if G' is a maximally connected subnetwork in the network G. Analogously one can define spanning trees or components of any subnetwork of the network. It is well-known that every (sub)network is the union of disjoint components. We refer to Schrijver (2004) on graphs and networks in detail.

When a firm l intends to use the network $G = (N, E)$, it has its own estimated cost $c^l(e) \in \mathbb{Z}_+$ for each road $e = \{i, j\} \in E$. For any subnetwork $G' = (N', E')$ of the network G, firm l's cost of operating it equals the sum of the minimum cost among all spanning trees in each component of G', denoted by $C^l(G')$. Formally, we can write $G' = (N', E')$ as the union of disjoint components $G_i = (N_i, E_i)$, $i = 1, \cdots, k$, in $G' = (N', E')$. Notice that each component G_i is a subnetwork of G', of course, of G as well. Each component G_i, $i = 1, \cdots, k$, has at least one spanning tree because every component is connected. Firm l's cost of operating component G_i is the minimum cost among all spanning trees in G_i and is denoted by $C^l(G_i)$. Let $T_i = (N_i, E_i^m)$ stand for a minimum cost spanning tree of the component $G_i = (N_i, E_i)$. Observe that T_i is a subnetwork of G_i and E_i^m is a subset of E_i. Then firm l's total cost $C^l(G')$ of operating G' is the sum of operating all components of G' separately, i.e., $C^l(G') = \sum_{i=1}^k C^l(G_i)$ where $C^l(G_i) = \sum_{e \in E_i^m} c^l(e)$ for $i = 1, \cdots, k$. Minimum cost spanning trees are frequently used in projects concerning transportation or telecommunication networks, electrical power lines, and pipe lines, etc, and have at least two important properties: their cost is minimum, and they connect all stations in the underlying component so that all stations can be served. The utility function of firm l using the network can be the cost-saving function induced by the firm's cost function and is given by $u^l(G') = \sum_{e \in E'} c^l(e) - C^l(G')$ for every subnetwork $G' = (N', E')$ of the network G. Similarly, the government (i.e., the seller) also has an estimated cost $c^0(e) \in \mathbb{Z}_+$ for each road $e \in E$ and the corresponding cost function C^0 and the corresponding reserve price function u^0 defined by $u^0(G') = \sum_{e \in E'} c^0(e) - C^0(G')$ for every subnetwork $G' = (N', E')$ of the network G.

The following theorem shows that it is possible for the government to sell the network to the firms through a competitive and nonlinear pricing system.

Theorem 6.2 *Both the value function u^l of every firm l and the reserve price function u^0 of the government are superadditive. Thus, the model has a nonlinear pricing Walrasian equilibrium.*

7 Conclusion

We have presented a general auction model with heterogenous items. The items are sold in discrete quantities. Both bidders and the seller view all items as complements. Every bidder has a private valuation over each bundle of goods that the seller does not know. The essential features of this model are asymmetric information, complementarity, indivisibility, heterogeneity and multiplicity. These factors have been identified in the literature as major obstacles to the design of dynamic auctions, because complementarity induces both the exposure problem and the threshold problem, asymmetric information creates incentive issues, and complementarity, indivisibility, heterogeneity and multiplicity aggravate the design issue even further by causing problems with existence of Walrasian equilibrium.

Since we have only alluded to indivisibility, it is appropriate and necessary to elaborate on this here. Indivisibility is an extreme form of non-convexity and is hard to tackle. It is well-known that the assumption of convexity and perfect divisibility has played a central role in most economic analyses. However, this stringent assumption does not always fit well into reality where indivisible commodities are pervasive and significant, such as houses, cars, computers, machines, networks, and airplanes, to name but a few. In practice, virtually all divisible goods are also traded in discrete quantities, such as oil sold in barrels. Obviously, modeling economies with indivisibility is more meaningful and more realistic. The importance and difficulty of studying such economies have long been recognized in the literature (Koopmans and Beckmann 1957, Debreu 1959, Arrow and Hahn 1971, and Kelso and Crawford 1982).

The existing auction design literature has succeeded in dealing with substitute goods but cannot be equally applied to the current environment where goods exhibit complementarities. The current article goes towards filling this outstanding theoretical gap and has proposed a new dynamic auction which successfully resolves all the potential problems described above. Besides its theoretical significance, the current approach may also have far-reaching implications for practical auction design, because firstly it deals with a general and practical market model in which each bidder possesses payoff-relevant critical private information and may have an incentive to economize on such information; secondly it has taken the complex nature of rational and strategic economic agents into account in the design of the auction. For instance, economic agents are reluctant to reveal their private value or cost, prefer simple, detail-free, error-tolerant, robust and transparent trading rules, and are accustomed to a competitive price system rather than a non-price system. The current auction mechanism is capable of handling such agents. Another attractive feature of this mechanism is that it is robust against bidders' regret and also independent of the probability distribution of their values.

The conventional practice for handling complementary items is to bundle all those items together in advance and sell them as a single package, often resulting in inefficient outcomes. The current auction design offers an efficient and incentive compatible alternative approach in tackling this challenging problem. Because the auction rules are quite intuitive, detail-free and transparent, and the sufficient conditions are extremely general and simple, we believe that it will not be difficult to put the auction on test in the laboratory, and ultimately we hope that this auction may someday find its way into practical use.

As in most of the auction literature we have concentrated on a private value model. While the private value assumption may not be satisfied in some auction markets, we believe that the analysis developed here for this basic model provides a useful and necessary basis for the study of efficient auction design in more complex environments. Our model is very general in almost all respects but its private value assumption. How to relax this assumption remains widely open. For instance, a natural and important question is whether and how it can be extended to an interdependent value setting. This is the situation where each bidder has only partial information about the value of every bundle of items for sale and his valuation may be affected by information possessed by other bidders. For readers interested in pursuing this direction, we recommend starting with Milgrom and Weber (1982), Cremér and McLean (1985), and Maskin (1992) on auctions for a single item, Ausubel (2004), and Perry and Reny (2002, 2005) for homogeneous goods.

We hope that the current study will prove to be useful in coping with practical and complex resource allocation problems.

APPENDIX

This appendix contains all omitted proofs and details in the main body of the paper.

Proof of Lemma 2.2 The proof of this lemma is analogous to the proof of a similar but simpler lemma in Bevia, Quinzii and Silva (1999), Gul and Stacchetti (1999). Because (p^*, π^*) is a nonlinear pricing Walrasian equilibrium, for any bidder $i \in M$ and any allocation $\rho \in \mathcal{A}$, it holds

$$u^i(\pi^*(i)) - p^*(\pi^*(i)) \geq u^i(\rho(i)) - p^*(\rho(i)).$$

It follows that

$$\begin{aligned}&\textstyle\sum_{i\in M_0} u^i(\pi^*(i)) - \sum_{i\in M_0} u^i(\rho(i)))\\ &\geq u^0(\pi^*(0)) + \textstyle\sum_{i\in M} p^*(\pi^*(i)) - \Big(u^0(\rho(0)) + \sum_{i\in M} p^*(\rho(i))\Big).\end{aligned}$$

It follows from $\pi^* \in S(p^*)$ that $p^*(\pi^*(0)) = u^0(\pi^*(0))$ when $\pi^*(0) \neq \emptyset$. Thus we have

$$\begin{aligned}&u^0(\pi^*(0)) + \textstyle\sum_{i\in M} p^*(\pi^*(i)) = \sum_{i\in M_0} p^*(\pi^*(i)) = Re(p^*)\\ &\geq \textstyle\sum_{i\in M_0} p^*(\rho(i))) \geq u^0(\rho^*(0)) + \sum_{i\in M_0} p^*(\rho(i))).\end{aligned}$$

Consequently, we have

$$\sum_{i\in M_0} u^i(\pi^*(i)) \geq \sum_{i\in M_0} u^i(\rho(i)),$$

for all $\rho \in \mathcal{A}$. This implies that π^* is efficient.

Suppose that ρ is an efficient allocation. Then we have $V(N) = \sum_{i \in M_0} u^i(\rho(i))$. Because (p^*, π^*) is an NPW equilibrium, π^* is an efficient allocation and thus $V(N) = \sum_{i \in M_0} u^i(\pi^*(i))$. Furthermore it holds

$$\begin{array}{l} V^i(p^*) \geq u^i(\rho(i)) - p^*(\rho(i)), \qquad \text{for all } i \in M \\ u^0(\pi^*(0)) + \sum_{i \in M} p^*(\pi^*(i)) = \sum_{i \in M_0} p^*(\pi^*(i)) = Re(p^*) \\ \geq \sum_{i \in M_0} p^*(\rho(i))) \geq u^0(\rho^*(0)) + \sum_{i \in M_0} p^*(\rho(i))). \end{array}$$

If one of the above inequalities were strict, we would have

$$\begin{array}{rcl} V(N) & = & \sum_{i \in M_0} u^i(\pi^*(i)) \\ & = & u^0(\pi^*(0)) + \sum_{i \in M}\Big(u^i(\pi^*(i)) - p^*(\pi^*(i))\Big) + \sum_{i \in M} p^*(\pi^*(i)) \\ & = & \sum_{i \in M} V^i(p^*) + Re(p^*) \\ & > & \sum_{i \in M}\Big(u^i(\rho(i)) - p^*(\rho(i))\Big) + u^0(\rho(0)) + \sum_{i \in M} p^*(\rho(i)) \\ & = & \sum_{i \in M_0} u^i(\rho(i)) \\ & = & V(N), \end{array}$$

yielding a contradiction. We therefore have

$$\begin{array}{l} V^i(p^*) = u^i(\rho(i)) - p^*(\rho(i)), \qquad \text{for all } \ i \in M, \\ u^0(\rho(0)) + \sum_{i \in M} p^*(\rho(i)) = Re(p^*), \quad \text{i.e.,} \quad \rho \in S(p^*). \end{array}$$

This shows that (p^*, ρ) is also an NPW equilibrium. □

The following lemma establishes the important properties of proper NPW equilibria and their structure. It implies that several NPW equilibria may just correspond to one proper NPW equilibrium. The proper NPW equilibrium captures the essential property of all its corresponding NPW equilibria. As a result the set of proper NPW equilibria reveals the inner structure of all NPW equilibria.

Lemma A (i) An equilibrium pricing function p is proper if and only if, for every nonempty bundle A with $p(A) > u^0(A)$, there is a bidder $i \in M$ such that $A \in D^i(p)$, i.e., $u^i(A) - p(A) = V^i(p)$. (ii) For each equilibrium pricing function p there exists a proper equilibrium pricing function q such that $V^i(p) = V^i(q)$ for all $i \in M$ and $Re(p) = Re(q)$.

Proof: (i) Suppose that p is a proper equilibrium pricing function, but there is a bundle A with $p(A) > u^0(A)$ satisfying $u^i(A) - p(A) < V^i(p)$ for all $i \in M$. Let π be an efficient allocation. Then, we see $A \neq \pi(i)$ for all $i \in M$ because $V^i(p) = u^i(\pi(i)) - p(\pi(i))$ for all $i \in M$. Let $\delta = \min\Big\{ \frac{1}{2}(p(A) - u^0(A)), \ \frac{1}{2}\min\{V^i(p) - u^i(A) + p(A) \mid i \in M\} \Big\}$, and define a different feasible pricing function $q(\leq p)$ by

$$q(S) = \begin{cases} p(S) - \delta & \text{if } S = A \\ p(S) & \text{if } S \neq A. \end{cases}$$

Then, by the definition of δ, for every $i \in M$ we have $u^i(\pi(i)) - q(\pi(i)) \geq u^i(S) - q(S)$ for all $S \in 2^N$. On the other hand, since $A \neq \pi(i)$ for all $i \in M$, we have $Re(q) =$

$u^0(\pi(0)) + \sum_{i\in M} p(\pi(i)) = Re(p)$. This shows that (q, π) is also an NPW equilibrium, contradicting the assumption that p is a proper equilibrium pricing function.

Assume that p is an equilibrium pricing function, and for every bundle A with $p(A) > u^0(A)$ there is a bidder $i \in M$ such that $u^i(A) - p(A) = V^i(p)$, but p is not a proper equilibrium pricing function. Namely, there is a different equilibrium pricing function $q \le p$ satisfying $Re(q) = Re(p)$. Let π be an efficient allocation. Then, we see (p, π) and (q, π) are both NPW equilibria, and so $\sum_{i\in M_0} q(\pi(i)) = Re(q) = Re(p) = \sum_{i\in M_0} p(\pi(i))$. This implies that $q(\pi(i)) = p(\pi(i))$ for all $i \in M$. But, since $p \ge q \ge u^0(\cdot)$ and $p \ne q$, there must be a nonempty bundle $\bar{A} \subseteq N$ such that $p(\bar{A}) > q(\bar{A}) \ge u^0(\bar{A})$. It is clear that $\bar{A} \ne \pi(i)$ for all $i \in M$. However, by hypothesis, for the bundle $\bar{A}$ there is a bidder i such that $u^i(\bar{A}) - p(\bar{A}) = V^i(p) = u^i(\pi(i)) - p(\pi(i))$. It follows that $u^i(A) - q(A) > u^i(A) - p(A) = u^i(\pi(i)) - p(\pi(i)) = u^i(\pi(i)) - q(\pi(i))$, contradicting the fact that (q, π) is an NPW equilibrium.

(ii) Let (p, π) be an NPW equilibrium. We shall construct a corresponding proper equilibrium (p^*, π) as follows: Let $p^*(\pi(i)) = p(\pi(i))$ for every $i \in M$. For any other nonempty bundle A, if there is a bidder i with $A \in D^i(p)$, then let $p^*(A) = p(A)$; otherwise we decrease continuously the price of A until there is a bidder i whose demand set contains A or until it becomes $u^0(A)$, and we set $p^*(A)$ equal to the current price of A. We repeat this process sequentially for every such bundle. Then we obtain a proper equilibrium pricing function p^* from p. □

The following example shows that the set of NPW equilibrium pricing functions is not a lattice.

Example 4: Suppose that there are two bidders (1, 2) and three items (A, B, C) in a market. Bidders' values and seller's reserve prices are given in Table 6. All items are viewed by every bidder and the seller as complements. In the table, both p and q are (proper) pricing equilibrium functions. It is easy to see that $p \wedge q$ is not an equilibrium pricing function!

Proof of Theorem 2.3: It follows immediately from Corollary 3.2 that there is an equilibrium pricing function. Then Lemma A given above implies that there is a proper equilibrium pricing function associated with the equilibrium pricing function. We now prove the second statement. Suppose that π is an efficient allocation and p, q are two proper equilibrium pricing functions. Then by Lemma 2.2, (p, π) and (q, π) are NPW equilibria. We need to prove that (p^*, π) is also an NPW equilibrium, where $p^*(S) = \max\{p(S), q(S)\}$ for every $S \in 2^N$. For every bidder $i \in M$, we have $\pi(i) \in D^i(p) \cap D^i(q)$. Thus,

$$u^i(\pi(i)) - p(\pi(i)) \ge u^i(S) - p(S), \ \forall \ S \in 2^N$$

Table 6: Bidders' and seller's values over items.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	0	2	0	6	3	4	7.5
Bidder 2	0	0	0	2	3	6	3	7.5
Seller	0	0	0	0	0	0	0	0
Prices p	0	0	0	2	6	6	4	7.5
Prices q	0	0	2	0	6	6	4	7.5
$p \wedge q$	0	0	0	0	6	6	4	7.5

$$u^i(\pi(i)) - q(\pi(i)) \geq u^i(S) - q(S), \ \forall\ S \in 2^N$$

For simplicity, we assume that $p^*(\pi(i)) = \max\{p(\pi(i)), q(\pi(i))\} = p(\pi(i))$. Then we have

$$\begin{aligned} u^i(\pi(i)) - p^*(\pi(i)) &= u^i(\pi(i)) - p(\pi(i)) \\ &\geq u^i(S) - p(S) \\ &\geq u^i(S) - \max\{p(S), q(S)\} \\ &= u^i(S) - p^*(S), \ \forall\ S \in 2^N \end{aligned}$$

This proves $\pi(i) \in D^i(p^*)$ for every bidder $i \in M$.

Next we show that $\pi \in S(p^*)$. For p and q, define Δp by $\Delta p(S) = \max\{q(S) - p(S), 0\}$ for all $S \in 2^N$. Then, $p^* = p + \Delta p$. By Lemma A at the proper equilibrium pricing function q for every nonempty bundle $S \in 2^N$ with $q(S) > u^0(S)$, there is a bidder $i \in M$ such that $u^i(\pi(i)) - q(\pi(i)) = u^i(S) - q(S)$. Moreover, $u^i(\pi(i)) - p(\pi(i)) \geq u^i(S) - p(S)$ for the bundle S, because (p, π) is an NPW equilibrium. Therefore for every nonempty $S \in 2^N$ with $\Delta p(S) > 0$ (and so $q(S) > u^0(S)$), there is a bidder $i \in M$ satisfying

$$\Delta p(\pi(i)) = q(\pi(i)) - p(\pi(i)) \geq q(S) - p(S) = \Delta p(S) > 0.$$

Now consider an arbitrary supply set $\gamma \in S(p^*)$ that satisfies $p^*(A \cup B) < p^*(A) + p^*(B)$ for (if any) every pair of different (i.e., disjoint and nonempty) bundles A and B in γ. Then for every pair of different sets A and B in γ with $\Delta p(A) > 0$ and $\Delta p(B) > 0$, we have $p^*(A) = q(A) > p(A)$ and $p^*(B) = q(B) > p(B)$. From the above discussion we know there are bidders k and l such that $u^k(\pi(k)) - q(\pi(k)) = u^k(A) - q(A)$, $p^*(\pi(k)) = q(\pi(k))$, and $u^l(\pi(l)) - q(\pi(l)) = u^l(B) - q(B)$, $p^*(\pi(l)) = q(\pi(l))$. We will show that k and l must be two different bidders. Suppose to the contrary that $k = l$. Then $u^k(\pi(k)) - q(\pi(k)) = u^k(A) - q(A) = u^k(B) - q(B)$. By Assumption (A3) we have

$$\begin{aligned} & u^k(A \cup B) - p^*(A \cup B) \geq u^k(A) + u^k(B) - p^*(A \cup B) \\ > \ & u^k(A) + u^k(B) - p^*(A) - p^*(B) = \Big(u^k(A) - q(A)\Big) + \Big(u^k(B) - q(B)\Big) \\ = \ & 2\Big(u^k(\pi(k)) - q(\pi(k))\Big) \geq u^k(\pi(k)) - p^*(\pi(k)) = V^k(p^*), \end{aligned}$$

yielding a contradiction. Now let $\hat{\gamma} = \{A \in \gamma \mid \Delta p(A) > 0\}$. Then the above discussion shows that for every $A \in \hat{\gamma}$, there is a bidder i_A with $\Delta p(\pi(i_A)) \geq \Delta p(A) > 0$ and that $A \neq B \in \hat{\gamma}$ implies $i_A \neq i_B$. This implies

$$\sum_{i \in M_0} \Delta p(\pi(i)) \geq \sum_{i \in M} \Delta p(\pi(i)) \geq \sum_{A \in \hat{\gamma}} \Delta p(A) = \sum_{A \in \gamma} \Delta p(A).$$

Because (p, π) is an NPW equilibrium, for the seller we have

$$\sum_{i \in M_0} p(\pi(i)) \geq \sum_{A \in \gamma} p(A).$$

Using the two inequalities leads to

$$\begin{array}{rcl} \sum_{i \in M_0} p^*(\pi(i)) & = & \sum_{i \in M_0} \Big(p(\pi(i)) + \Delta p(\pi(i)) \Big) \\ & = & \sum_{i \in M_0} p(\pi(i)) + \sum_{i \in M_0} \Delta p(\pi(i)) \\ & \geq & \sum_{A \in \gamma} p(A) + \sum_{A \in \gamma} \Delta p(A) \\ & = & \sum_{A \in \gamma} \Big(p(A) + \Delta p(A) \Big) \\ & = & \sum_{A \in \gamma} p^*(A) = Re(p^*). \end{array}$$

This shows $\pi \in S(p^*)$. In summary (p^*, π) is an NPW equilibrium.

It remains to show that p^* is proper. Choose any nonempty bundle $S \in 2^N$ with $p^*(S) > u^0(S)$. Without loss of generality, suppose $p^*(S) = q(S) \geq p(S)$. That is, $\Delta p(S) = q(S) - p(S)$. And so $q(S) > u^0(S)$. Then, by the properness of equilibrium pricing function q, there is a bidder $i \in M$ satisfying $u^i(\pi(i)) - q(\pi(i)) = u^i(S) - q(S)$ and $q(\pi(i)) - p(\pi(i)) \geq q(S) - p(S) = \Delta p(S) \geq 0$. This implies that $p^*(\pi(i)) = q(\pi(i))$ and $u^i(\pi(i)) - p^*(\pi(i)) = u^i(S) - p^*(S)$. Then by Lemma A, p^* is a proper equilibrium pricing function. $\square$

We now modify the basic ascending auction in Section 3.1 so that bidders can bid quasi-sincerely as discussed in Section 3.2.

The modified basic auction

Step 1: The seller reports her reserve price function u^0 and the auctioneer sets the initial pricing function $p(0) : 2^N \to \mathbb{Z}$ with $p(0, S) = u^0(S)$ for every bundle $S \subseteq N$. Set $t := 0$ and go to Step 2.

Step 2: At each round $t = 0, 1, 2, \cdots$, the auctioneer announces the current pricing function $p(t)$ and a supply set $\gamma(t) \in S(p(t))$. Then every bidder i reports a bid $A_i(t)$ at the announced prices $p(t)$. Based on the supply set and demands, the auctioneer checks if there is any over-demanded bundle at $p(t)$. If no bundle is over-demanded, go to Step 3. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged, and obtains a new price $p(t + 1)$. Set $t := t + 1$ and return to Step 2.

Step 3: The auctioneer reminds all bidders that "this may be the last round, please report your optimal demand bundle". Then, every bidder i resubmits a new bid $A_i(t) \in D^i(p(t))$ when his former bid $A_i(t)$ is not an optimal bid. If no bidder revises his bid, go to Step 4. Otherwise, if some bidder revises his bid, the auctioneer checks again if there is any over-demanded bundle at $p(t)$. If no bundle is over-demanded, go to Step 4. But if there is an over-demanded bundle, the auctioneer raises the price of each over-demanded bundle by one unit but holds the price of any other bundle unchanged, and obtains a new price vector $p(t+1)$. Set $t := t+1$ and return to Step 2.

Step 4: The auctioneer assigns the bundle $A_i(t)$ to bidder i who is asked to pay the price $p(t, A_i(t))$ in return, and in addition for any nonempty bundle $B \in \gamma(t)$ which is not demanded by any bidder at $p(t)$, the auctioneer assigns the bundle to the seller if $p(t, B) = u^0(B)$, otherwise, the auctioneer assigns the bundle to some bidder[22] who previously demanded the bundle but was the last to give up, and who is asked to pay $p(t, B)$. Then the process stops.

We have the following convergence theorem for the modified basic auction.

Theorem A Suppose that Assumptions (A1)–(A3) hold for the auction model. When every bidder bids quasi-sincerely, the modified basic auction yields a nonlinear pricing Walrasian equilibrium, in a finite number of rounds.

Proof: It is similar to the proof of Theorem 3.1. Suppose the modified auction stops at some step t^*. Let $p^* = p(t^*)$, $A_i^* = A_i(t^*)$, and $\gamma^* = \gamma(t^*) \in S(p^*)$. Then, $A_i^* \in D^i(p^*)$ for every $i \in M$ and $\gamma^* \in S(p^*)$. We will construct an allocation π^* so that (p^*, π^*) constitutes an NPW equilibrium. Note that for every bidder $i \in M$, if his demand bundle A_i^* is not empty, it must be in the supply set γ^*. Moreover, for any two bidders $i, l \in M$, with $A_i^* \neq \emptyset$ and $A_l^* \neq \emptyset$, we must have $A_i^* \cap A_l^* = \emptyset$. If $\cup_{i \in M} A_i^* = N$, let $\pi^*(i) = A_i^*$, then clearly (p^*, π^*) is an NPW equilibrium and we are done.

Suppose otherwise that there is some nonempty bundle B in the supply set γ^* which is not demanded by any bidder in the last step. Such a bundle is called a *squeezed out bundle*. We first consider the case in which $p^*(B) = u^0(B)$. Let $\gamma_0^* = \{B \in \gamma^* \mid p^*(B) = u^0(B) \text{ and } B \neq A_i^* \text{ for all } i \in M\}$ be the collection of all such bundles. Let $\pi^*(0) = \cup_{B \in \gamma_0^*} B$. We assign $\pi^*(0)$ to the seller. By superadditivity, we know that $p^*(\pi^*(0)) \geq u^0(\pi^*(0)) \geq \sum_{B \in \gamma_0^*} u^0(B) = \sum_{B \in \gamma_0^*} p^*(B)$. But we also see that $p^*(\pi^*(0)) \leq \sum_{B \in \gamma_0^*} p^*(B)$ because $\gamma^* \in S(p^*)$. Hence, we have

$$p^*(\pi^*(0)) = u^0(\pi^*(0)) = \sum_{B \in \gamma_0^*} p^*(B) = \sum_{B \in \gamma_0^*} u^0(B). \tag{7.13}$$

[22]If there are several such bidders, the auctioneer randomly chooses one.

Next, we consider the case in which $p^*(B) > u^0(B)$. This implies that the bundle B was demanded by some bidder at some step. Let t be the last step at which B is demanded at price $p(t) = p^*(B)$ or $p^*(B) - 1$ by some bidder l. By the auction rule B can be assigned to bidder l who is asked to pay the current price $p^*(B)$. We will show that bidder l loses nothing in having the bundle B and paying the price. Since bidder i bids quasi-sincerely, we have $v^l(B, p(t)) = u^l(B) - p(t, B) \geq 1$. Notice that $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. Hence, for the bidder l, it holds that

$$v^l(B, p^*) = u^l(B) - p^*(B) \geq 0. \tag{7.14}$$

We need to consider the following two situations.

Case 1. When $A_l^* = \emptyset$, let $\pi^*(l) = B$. Because $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$, we have $V^l(p^*) = 0 \geq v^l(B, p^*)$. Recall that $v^l(B, p^*) \geq 0$. These inequalities lead to $v^l(B, p^*) = 0$, which implies $\pi^*(l) \in D^l(p^*)$.

Case 2. When $A_l^* \neq \emptyset$, let $\pi^*(l) = A_l^* \cup B$. For the seller, we know that

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)). \tag{7.15}$$

For the bidder l, the Superadditivity Assumption (A3) implies that

$$u^l(\pi^*(l)) \geq u^l(A_l^*) + u^l(B). \tag{7.16}$$

It follows from (7.15) and (7.16) that

$$\begin{aligned} u^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) \\ &\geq \Big(u^l(A_l^*) - p^*(A_l^*)\Big) + \Big(u^l(B) - p^*(B)\Big) \\ &\geq u^l(A_l^*) - p^*(A_l^*) \end{aligned}$$

where the last inequality is derived from (7.14). Because $A_l^* \in D^l(p^*)$, we have $\pi^*(l) \in D^l(p^*)$ (i.e., $\pi^*(l)$ is also an optimal bundle of bidder l at p^*). Consequently, it further implies that

$$u^l(\pi^*(l)) - p^*(\pi^*(l)) = u^l(\pi^*(l)) - (p^*(A_l^*) + p^*(B)) = u^l(A_l^*) - p^*(A_l^*)$$

which leads to

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \tag{7.17}$$

So in both cases bidder l loses nothing from having the nonessential bundle B and paying the price $p^*(B)$. As a result, the indirect utility of bidder l remains unchanged.

We can repeat this adjustment until every such squeezed out bundle B (i.e., $p^*(B) > u^0(B)$) in γ^* is assigned to some bidder. For any bidder i who is not assigned with any squeezed out bundle, let $\pi^*(i) = A_i^*$. So in the end each bidder i gets a bundle $\pi^*(i)$ in his demand set. Because γ^* is a seller's partition of N, $(\pi^*(0), \cdots, \pi^*(m))$ must be an allocation of N. Furthermore, it follows from the formulas (7.13) and (7.17) that $\sum_{i\in M_0} p^*(\pi^*(i)) = \sum_{A\in\gamma^*} p^*(A) = Re(p^*)$. That is, the allocation $\pi^* \in S(p^*)$. Consequently, (p^*, π^*) is a nonlinear pricing Walrasian equilibrium and we are done. □

The following result establishes a relation between the number of different active demanders at the current round and the first and the second prices of the next round in the IC dynamic auction described in Section 4. In particular, R3 concerns the existence of a unique crucial demander stated in Note 2 of Section 4.1.

Proposition A: *At every round t of the IC dynamic auction, for each nonempty bundle $S \subseteq N$ we have:*

(R1) there exists no active demander at prices $p(S) > p^{-0}(t+1, S)$;

(R2) there exists at most one active demander at price $p^{-0}(t+1, S)$;

(R3) there exists one and only one active demander at every price $p(S) \in \{p^0(t+1, S), \cdots, p^{-0}(t+1, S) - 1\}$ in the case of $p^{-0}(t+1, S) > p^0(t+1, S)$. In particular, if $p^{-0}(t+1, S) > p^0(t+1, S)$, then at round t there is a unique crucial demander of S at $p^{-0}(t+1, S) - 1$ or $p^{-0}(t+1, S)$.

(R4) there exist at least two active demanders at price $p^0(t+1, S) - 1$ in the case of $p^0(t+1, S) > u^0(S)$.

Proof: At first, by the definition of active demander, we see that if at round t bidder i is an active demander of a bundle S at price $p(S)$, he must also be an active demander of S at any price $q(S)$, $u^0(S) \leq q(S) < p(S)$ at that round, due to possibly beneficial bids in $\Omega_i(t)$. And so, if there are k active demanders of S at price $p(S)$, then there are no less than k active demanders of S at every price $q(S)$, $u^0(S) \leq q(S) < p(S)$. Observe that for every bidder i, his possibly beneficial bids in $\Omega_i(t)$ play no role when the auction is ascending, but can be used when the auction is descending due to bidders' withdrawing their bids. In other words, possibly beneficial bids enable the IC auction to decrease the price of any over-supplied bundle in each round one unit by one unit without big jump.

In the following, we will show this lemma by induction in each round t. If $t = 0$, these results are trivial. So let $t > 0$ and suppose that these results (R1)-(R4) hold up to $t-1$. We will fix a non-empty bundle $S \subseteq N$.

We first show that for S the results (R1) and (R2) are true at round t. From the assumption that (R1) and (R2) are true at $t-1$, we see that at round $t-1$ there is no active demander of S at a higher price $p(S) > p^{-0}(t, S)$, and there is at most one active demander of S at price $p^{-0}(t, S)$. On the other hand, from the bidding rule, we see that at round t

bidders can only bid S at prices $p(S) \le p^{-0}(t,S)$. Thus, if $p^{-0}(t+1,S) = p^{-0}(t,S)+1$, then there exists no active demander at any price $p(S) \ge p^{-0}(t,S)+1 = p^{-0}(t+1,S)$ at t. If $p^{-0}(t+1,S) = p^{-0}(t,S)$, then there exists at most one active demander at $p^{-0}(t,S) = p^{-0}(t+1,S)$, and no active demander at any higher price $p(S) > p^{-0}(t,S) = p^{-0}(t+1,S)$. Otherwise, i.e., in the case of $p^{-0}(t+1,S) = p^{-0}(t,S)-1$, by the definition of 1st-price-over-supplied and Formula (4.6), there exists no active demander at prices $p(S) \ge p^{-0}(t,S)-1 = p^{-0}(t+1,S)$. In summary, the results (R1) and (R2) hold at round t.

We now show that the result (R4) is true at round t. From the assumption that (R4) is true at round $t-1$, we see that at round $t-1$ there are at least two active demanders at price $p^0(t,S)-1$. Next, by the definition of 2nd-price-over-demanded and -supplied, we see that at round t there are at least two active demanders at price $p^0(t,S) = p^0(t+1,S)-1$ in the case of $p^0(t+1,S) = p^0(t,S)+1$, and there are at least two active demanders at $p^0(t,S)-1 = p^0(t+1,S)-1$ in the case of $p^0(t+1,S) = p^0(t,S)$. In addition, by the rule of withdrawal, at round t bidders can only withdraw some bids of S at prices $p(S) \ge p^0(t,S)-1$. Thus, at round t there are at least two active demanders at $p^0(t,S)-2$ (if $\ge u^0(S)$). Therefore, in the case of $p^0(t+1,S) = p^0(t,S)-1 > u^0(S)$ there are also at least two active demanders at price $p^0(t,S)-2 = p^0(t+1,S)-1$. Consequently, the result (R4) is true at round t.

Finally, we show that the result (R3) is true at round t, and so we assume that $p^{-0}(t+1,S) > p^0(t+1,S)$. From the assumption that (R3) is true at round $t-1$, we see that at round $t-1$ there exists one and only one active demander at each price $p(S) \in \{p^0(t,S), \cdots, p^{-0}(t,S)-1)\}$ in the case of $p^{-0}(t,S) > p^0(t,S)$. Thus, by the rules of bidding and withdrawal and the definition of 2nd-price-over-demanded and -supplied, we see that at round t there is at most one active demander at price $p^0(t+1,S)$ (including three cases: (i) $p^0(t+1,S) = p^0(t,S)+1$; (ii) $p^0(t+1,S) = p^0(t,S)$, (iii) $p^0(t+1,S) = p^0(t,S)-1$.). In addition, by the rules of bidding and withdrawal and the definition of 1st-price-over-demanded and -supplied, we see that at round t there is at least one active demander at price $p^{-0}(t+1,S)-1$ (including three cases: (i) $p^{-0}(t+1,S) = p^{-0}(t,S)+1$; (ii) $p^{-0}(t+1,S) = p^{-0}(t,S)$, (iii) $p^{-0}(t+1,S) = p^{-0}(t,S)-1$.). In summary, at round t there exists one and only one active demander of S at each price $p(S) \in \{p^0(t+1,S), \cdots, p^{-0}(t+1,S)-1)\}$. Consequently, we see that there exists a unique crucial demander of S at price $p^{-0}(t+1,S)-1$ or at $p^{-0}(t+1,S)$ in every round t with $p^{-0}(t+1,S) > p^0(t+1,S)$. □

Proof of Lemma 4.2: By definition of sincere bidding, we know that from any given history profile $\{H_i(t_0)\}_{i\in M}$, there are only finitely many bids for bidders to withdraw, i.e., all non-profitable bids will be eliminated in finitely many rounds. Thus, the IC dynamic auction for every market $\mathcal{M}_{-j}$, $j \in M_0$, becomes the basic auction process of Section 3 after some rounds and must therefore terminate in Step 3 at some time t^*.

Consider an arbitrary market $\mathcal{M}_{-j}$, $j \in M_0$. Let $p^{-j}(t^*)$ be the price function of the auctioneer for the market $\mathcal{M}_{-j}$ and let π^{-j} be the allocation chosen by the auctioneer for the market $\mathcal{M}_{-j}$ in the last round t^*. For every agent $i \in M_{-j} \cup \{0\}$, let $\gamma_i^{-j} = \{S \mid S \in \gamma^{-j}(t^*) \,\text{and}\, S \subseteq \pi^{-j}(i)\}$ denote the set of bundles in the supply set $\gamma^{-j}(t^*)$ that are assigned to agent i by the auctioneer at step 3. Then, we have $\pi^{-j}(i) = \cup_{S\in\gamma_i^{-j}} S$. Notice that, by the assignment rule of the basic ascending auction, the auctioneer always assigns each bundle $S \in \gamma^{-j}(t^*)$ with $p^{-j}(t^*, S) > u^0(S)$ to a bidder $i \in M_{-j}$, who faces the price $p^{-j}(t^*, S)$ and retains the bid $(S,\, p^{-j}(t^*, S) - 1) \in \Omega_i(t^* - 1)$. We see that for every $S \in \gamma_i^{-j}$, it holds $p^i(t^*, S) = p^{-j}(t^*, S)$ and $(S,\, p^i(t^*, S) - 1) \in \Omega_i(t^* - 1)$. This also means that according to the withdrawing rule bidder i can withdraw this bid $(S, p^i(t^*, S) - 1)$ at the last round. However, because he bids sincerely after round t_0, and does not withdraw this bid at time t^*, we must have $u^i(S) - (p^i(t^*, S) - 1) \geq 1$. Thus we proved that for every $i \in M_{-j}$, $u^i(S) - p^i(t^*, S) \geq 0$ for all $S \in \gamma_i^{-j}$. In addition, since at time t^* there is no over-demanded bundle in $\mathcal{M}_{-j}$, we must have $A_i(t^*) \in \gamma_i^{-j}$ and $p^i(t^*, A_i(t^*)) = p^{-j}(t^*, A_i(t^*))$ when bidder i reports $A_i(t^*)$ at round t^*. Furthermore, we know that $A_i(t^*) \neq \emptyset$ when $V^i(p^i(t^*)) > 0$, because bidder i bids sincerely at time t^*.

We will show that if a bidder $i \in M_{-j}$ bids sincerely at time t^*, then $p^{-j}(t^*, \pi^{-j}(i)) = p^i(t^*, \pi^{-j}(i)) = \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$, and $\pi^{-j}(i) \in D^i(p^{-j}(t^*))$. First, it is clear that $p^{-j}(t^*, \pi^{-j}(i)) \geq p^i(t^*, \pi^{-j}(i))$. Next, $p^{-j}(t^*, \pi^{-j}(i)) \leq \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$ because $\gamma^{-j}(t^*) \in S(p^{-j}(t^*))$ is a supply set of the market $\mathcal{M}_{-j}$ against the price function $p^{-j}(t^*)$. Finally, to show $p^i(t^*, \pi^{-j}(i)) \geq \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$, assume by way of contradiction that $p^i(t^*, \pi^{-j}(i)) < \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$. Then, it follows from Assumption (A3) that

$$
\begin{array}{rl}
 & u^i(\pi^{-j}(i)) - p^i(t^*, \pi^{-j}(i)) \\
> & u^i(\pi^{-j}(i)) - \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S) \\
\geq & \sum_{S\in\gamma_i^{-j}} u^i(S) - \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S) \\
= & \sum_{S\in\gamma_i^{-j}} \big(u^i(S) - p^i(t^*, S)\big) \\
\geq & \begin{cases} u^i(A_i(t^*)) - p^i(t^*, A_i(t^*)), & \text{if } V^i(p^i(t^*)) > 0 \\ 0, & \text{if } V^i(p^i(t^*)) = 0 \end{cases} \\
= & V^i(p^i(t^*)).
\end{array}
\tag{7.18}
$$

This implies that at time t^*, bidder i prefers $\pi^{-j}(i)$ to $A_i(t^*)$ or the empty bundle. This contradicts the fact that bidder i bids sincerely at t^*. Thus, we obtain $p^{-j}(t^*, \pi^{-j}(i)) = p^i(t^*, \pi^{-j}(i)) = \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$ for all bidder $i \in M_{-j}$. Moreover, using this equality yields

$$
\begin{array}{rl}
 & u^i(\pi^{-j}(i)) - p^{-j}(t^*, \pi^{-j}(i)) = u^i(\pi^{-j}(i)) - p^i(t^*, \pi^{-j}(i)) \\
= & u^i(\pi^{-j}(i)) - \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S) \;\geq\; \sum_{S\in\gamma_i^{-j}} \big(u^i(S) - p^i(t^*, S)\big) \\
\geq & V^i(p^i(t^*)) \geq V^i(p^{-j}(t^*)).
\end{array}
\tag{7.19}
$$

The last inequality is due to $p^i(t^*) \leq p^{-j}(t^*)$. This implies

$$\begin{aligned} u^i(\pi^{-j}(i)) - p^{-j}(t^*, \pi^{-j}(i)) &= u^i(\pi^{-j}(i)) - \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S) \\ &= V^i(p^i(t^*)) = V^i(p^{-j}(t^*)). \end{aligned} \tag{7.20}$$

Consequently, we have $\pi^{-j}(i) \in D^i(p^{-j}(t^*))$ for all $i \in M_{-j}$.

Finally, note that $p^{-j}(t^*, \pi^{-j}(0)) \geq u^0(\pi^{-j}(0)) \geq \sum_{S\in\gamma_0^{-j}} u^0(S) = \sum_{S\in\gamma_0^{-j}} p^{-j}(t^*, S)$ because $p^{-j}(t^*, S) = u^0(S)$ for all $S \in \gamma_0^{-j}$. Using these equalities and inequalities, we obtain that

$$\begin{aligned} \sum_{i\in M_{-j}\cup\{0\}} p^{-j}(t^*, \pi^{-j}(i)) &\geq \sum_{i\in M_{-j}\cup\{0\}} \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S) \\ &= \sum_{S\in\gamma^{-j}} p^{-j}(t^*, S) = Re(p^{-j}(t^*)). \end{aligned}$$

This implies $\sum_{i\in M_{-j}\cup\{0\}} p^{-j}(t^*, \pi^{-j}(i)) = Re(p^{-j}(t^*))$, i.e., $\pi^{-j} \in S(p^{-j}(t^*))$.

So far we have proved $\pi^{-j}(i) \in D^i(p^{-j}(t^*))$ for all $i \in M_{-j}$, and $\pi^{-j} \in S(p^{-j}(t^*))$. In other words, $(p^{-j}(t^*), \pi^{-j})$ is an NPW equilibrium of the market M_{-j}. □

Proof of Theorem 4.3: To save space, we prove (iii), (ii) and (i) in the reverse order. First note that if the auction stops at Step 5, then some bidder must have withdrawn his bids more than L^* times. To see this, because there are only finitely many bidders, if bidders have withdrawn only finitely many times, the auction is ascending and then must have stopped either at Step 3 or at Step 4 which is not the case. Hence some bidder must have withdrawn infinitely many times, certainly larger than L^* times.

To prove (iii), suppose that bidder i bids sincerely, but his opponents use some other strategies $\{\hat{\sigma}_l\}_{l\in M_{-i}}$ which may not be sincere. If the auction stops in Step 4 or in Step 5, it is obvious that bidder i gets nothing and pays nothing, and so $W_i\Big(\sigma_i, \{\hat{\sigma}_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big) = 0$. Now assume that the auction stops in Step 3 at round $\hat{t}$, and finds an allocation $\hat{\pi}^{-j}$ in each market $\mathcal{M}_{-j}$ respectively. Let $\hat{p}^j(\hat{t})$, $\hat{p}^{-j}(\hat{t})$, $\hat{\gamma}^{-j}(\hat{t})$, and $\hat{\pi}^{-j}$ be the price functions, the supply sets, and the allocations generated by the auction. First, note that $\hat{p}^{-0}(\hat{t}) \geq \hat{p}^{-i}(\hat{t})$. Hence, the seller's revenue function satisfies

$$Re(\hat{p}^{-0}(\hat{t})) \geq Re(\hat{p}^{-i}(\hat{t})).$$

Let $\hat{\gamma}_i^{-0} = \{S \mid S \in \hat{\gamma}^{-0}(\hat{t}) \,\text{and}\, S \subseteq \hat{\pi}^{-0}(i)\}$ denote the set of bundles in the supply set $\hat{\gamma}^{-0}(\hat{t})$ that are assigned to agent i in Step 3. Then, exactly as in the proof of Lemma 4.2, we see that $\hat{\pi}^{-0}(i) = \cup_{S\in\hat{\gamma}_i^{-0}} S$, and $\hat{p}^i(\hat{t}, S) = \hat{p}^{-0}(\hat{t}, S)$ for all $S \in \hat{\gamma}_i^{-0}$. Moreover, because bidder i bids sincerely, we also have $u^i(S) - \hat{p}^i(\hat{t}, S) \geq 0$ for all $S \in \hat{\gamma}_i^{-0}$.

Consequently, by Assumption (A3), we see that for every realization $\{u^l\}_{l\in M}$

$$\begin{aligned}
& W_i\Big(\sigma_i, \{\hat{\sigma}_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big) \\
= \ & u^i(\hat{\pi}^{-0}(i)) - Re(\hat{p}^{-i}(\hat{t})) + Re(\hat{p}^{-0}(\hat{t})) - \textstyle\sum_{S\in\hat{\gamma}_i^{-0}} \hat{p}^{-0}(\hat{t}, S) \\
\geq \ & u^i(\hat{\pi}^{-0}(i)) - \textstyle\sum_{S\in\hat{\gamma}_i^{-0}} \hat{p}^{-0}(\hat{t}, S) \\
\geq \ & \textstyle\sum_{S\in\hat{\gamma}_i^{-0}} u^i(S) - \sum_{S\in\hat{\gamma}_i^{-0}} \hat{p}^i(\hat{t}, S) \\
\geq \ & \textstyle\sum_{S\in\hat{\gamma}_i^{-0}} \Big(u^i(S) - \hat{p}^i(\hat{t}, S)\Big) \\
\geq \ & 0.
\end{aligned}$$

Next we prove (ii). From Lemma 4.2 we know that when all bidders bid sincerely from an open round t_0, the auction must terminate at some round $t = t^*$ in Step 3. Let $p^j(t^*), p^{-j}(t^*)$, $\gamma^{-j}(t^*)$, γ_i^{-j}, and π^{-j} denote the generated results by the auction. Now it is sufficient to prove that q_i coincides with the generalized VCG payment $q_i^* = u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N)$. Notice that because all bidders bid sincerely, from the result and the proof of Lemma 4.2, for every $j \in M_0$ and every $i \in M_{-j} \cup \{0\}$, we have: (i) $p^{-j}(t^*, \pi^{-j}(i)) = \sum_{S\in\gamma_i^{-j}} p^{-j}(t^*, S)$; (ii) $p^{-j}(t^*, \pi^{-j}(0)) = u^0(\pi^{-j}(0))$; (iii) $\sum_{i\in M_{-j}\cup\{0\}} p^{-j}(t^*, \pi^{-j}(i)) = Re(p^{-j}(t^*))$; (iv) $u^i(\pi^{-j}(i)) = V^i(p^{-j}(t^*)) + p^{-j}(t^*, \pi^{-j}(i)) = V^i(p^i(t^*)) + p^{-j}(t^*, \pi^{-j}(i))$. Using these results we have:

$$\begin{aligned}
q_i^* &= u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N) \\
&= u^i(\pi^{-0}(i)) - \textstyle\sum_{l\in M_0} u^l(\pi^{-0}(l)) + \sum_{l\in M_{-i}\cup\{0\}} u^l(\pi^{-i}(l)) \\
&= \textstyle\sum_{l\in M_{-i}\cup\{0\}} u^l(\pi^{-i}(l)) - \sum_{l\in M_{-i}\cup\{0\}} u^l(\pi^{-0}(l)) \\
&= \Big(u^0(\pi^{-i}(0)) + \textstyle\sum_{l\in M_{-i}} [V^l(p^l(t^*)) + p^{-i}(t^*, \pi^{-i}(l))]\Big) \\
&\qquad -\Big(u^0(\pi^{-0}(0)) + \textstyle\sum_{l\in M_{-i}} [V^l(p^l(t^*)) + p^{-0}(t^*, \pi^{-0}(l))]\Big) \\
&= \textstyle\sum_{l\in M_{-i}\cup\{0\}} p^{-i}(t^*, \pi^{-i}(l)) - \sum_{l\in M_{-i}\cup\{0\}} p^{-0}(t^*, \pi^{-0}(l)) \\
&= Re(p^{-i}(t^*)) - \Big(Re(p^{-0}(t^*)) - \textstyle\sum_{S\in\gamma_i^{-0}} p^{-0}(t^*, S)\Big) \\
&= q_i.
\end{aligned}$$

Thus, the payoff W_i of every bidder i is equal to his VCG payoff, i.e.,

$$W_i\Big(\sigma_i, \{\sigma_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big) = u^i(\pi^{-0}(i)) - q_i^* = V(N) - V_{-i}(N) \geq 0.$$

Next, note that $V_{-i}(N) \geq u^0(\pi^{-0}(0) \cup \pi^{-0}(i)) + \sum_{l\in M_{-i}} u^l(\pi^{-0}(l))$, since $V_{-i}(N)$ is the market value of the market $\mathcal{M}_{-i}$. Also by Assumption (A3), we have $u^0\Big(\pi^{-0}(0) \cup \pi^{-0}(i)\Big) \geq u^0(\pi^{-0}(0)) + u^0(\pi^{-0}(i))$. Therefore, it follows from $q_i = q_i^* = u^i(\pi^{-0}(i)) - V(N) + V_{-i}(N)$ that

$$\begin{aligned}
q_i &= u^i(\pi^{-0}(i)) - \Big(u^0(\pi^{-0}(0)) + \textstyle\sum_{l\in M} u^l(\pi^{-0}(l))\Big) + V_{-i}(N) \\
&= V_{-i}(N) - \Big(u^0(\pi^{-0}(0)) + \textstyle\sum_{l\in M_{-i}} u^l(\pi^{-0}(l))\Big) \\
&= u^0(\pi^{-0}(i)) + V_{-i}(N) - \Big(u^0(\pi^{-0}(0)) + u^0(\pi^{-0}(i)) + \textstyle\sum_{l\in M_{-i}} u^l(\pi^{-0}(l))\Big) \\
&\geq u^0(\pi^{-0}(i)) + V_{-i}(N) - \Big(u^0\Big(\pi^{-0}(0) \cup \pi^{-0}(i)\Big) + \textstyle\sum_{l\in M_{-i}} u^l(\pi^{-0}(l))\Big) \\
&\geq u^0(\pi^{-0}(i)) + V_{-i}(N) - V_{-i}(N) \\
&= u^0(\pi^{-0}(i)).
\end{aligned}$$

Finally we prove (i). Consider any time $t_0 \in \mathbb{Z}_+$, any history profile $\{H_i(t_0)\}_{i\in M}$ (which may be on or off the equilibrium path), and any realization $\{u^i\}_{i\in M}$ of profile of value functions in $\mathcal{U}$ of private information. In this case, the outcome of the auction game depends on the histories $H_i(t_0)$ and the strategies that all bidders will take in the continuation game starting from time t_0. Bidders cannot change histories but can influence the path of the future from t_0 on. Take any bidder $i \in M$. Suppose that in the continuation game from time t_0 on, bidder i exploits a strategy $\bar{\sigma}_i$ which may deviate from his sincere bidding strategy σ_i, but all his opponents follow their sincere bidding strategies $\{\sigma_l\}_{l\in M_{-i}}$. If this strategic behavior of bidder i makes the auction stop in Step 4 or in Step 5, then it is obvious that

$$W_i\Big(\bar{\sigma}_i, \{\sigma_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big) \le 0 \le W_i\Big(\sigma_i, \{\sigma_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big),$$

where the second inequality follows from the proof of (ii). In this case, bidder i gains nothing from using the strategy $\bar{\sigma}_i$. It remains to consider the major case in which the auction stops in Step 3. Assume that the auction terminates at time $t = \bar{t}$. Let $\bar{p}^j(\bar{t}), \bar{p}^{-j}(\bar{t})$, $\bar{\gamma}^{-j}(\bar{t})$, $\bar{\gamma}_i^{-j}$, and $\bar{\pi}^{-j}$ denote the results yielded by the auction. By Lemma 4.2, we see that when every bidder $l \in M_{-i}$ bids sincerely from round t_0, the pair $(\bar{p}^{-i}(\bar{t}), \bar{\pi}^{-i})$ is an NPW equilibrium for the market $\mathcal{M}_{-i}$. And by Lemma 2.2 the allocation $\bar{\pi}^{-i}$ is efficient for $\mathcal{M}_{-i}$. Thus for every realization $\{u^l\}_{l\in M}$, we have

$$\sum_{l\in M_{-i}\cup\{0\}} u^l(\bar{\pi}^{-i}(l)) = V_{-i}(N) \quad \text{and} \quad \sum_{l\in M_0} u^l(\bar{\pi}^{-0}(l)) \le V(N),$$

because $\bar{\pi}^{-i}$ is efficient for $\mathcal{M}_{-i}$ but $\bar{\pi}^{-0}$ need not be efficient for $\mathcal{M}_{-0}$. Recall that because every opponent of bidder i bids sincerely from t_0 on, for $j = i$, 0 and $l \in M_{-j} \cup \{0\}$ we have : (i) $\bar{p}^{-j}(\bar{t}, \pi^{-j}(l)) = \sum_{S\in\bar{\gamma}_l^{-j}} \bar{p}^{-j}(\bar{t}, S)$; (ii) $\bar{p}^{-j}(\bar{t}, \pi^{-j}(0)) = u^0(\pi^{-j}(0))$; (iii) $\sum_{l\in M_{-i}\cup\{0\}} \bar{p}^{-j}(\bar{t}, \pi^{-j}(l)) = Re(\bar{p}^{-j}(\bar{t}))$; (iv) $u^l(\bar{\pi}^{-j}(l)) = V^l(\bar{p}^l(\bar{t})) + \bar{p}^{-j}(\bar{t}, \pi^{-j}(l))$. It follows that

$$\begin{aligned}
& W_i\Big(\bar{\sigma}_i, \{\sigma_l\}_{l\in M_{-i}}, \{u^l\}_{l\in M}\Big) \\
= \;& u^i(\bar{\pi}^{-0}(i)) - Re(\bar{p}^{-i}(\bar{t})) + \Big(Re(\bar{p}^{-0}(\bar{t})) - \textstyle\sum_{S\in\bar{\gamma}_i^{-0}} \bar{p}^{-0}(\bar{t}, S)\Big) \\
= \;& u^i(\bar{\pi}^{-0}(i)) - \textstyle\sum_{l\in M_{-i}\cup\{0\}} \bar{p}^{-i}(\bar{t}, \bar{\pi}^{-i}(l)) + \sum_{l\in M_{-i}\cup\{0\}} \bar{p}^{-0}(\bar{t}, \bar{\pi}^{-0}(l)) \\
= \;& u^i(\bar{\pi}^{-0}(i)) - \Big(u^0(\bar{\pi}^{-i}(0)) + \textstyle\sum_{l\in M_{-i}}[u^l(\bar{\pi}^{-i}(l)) - V^l(\bar{p}^l(\bar{t}))]\Big) \\
& \quad + \Big(u^0(\bar{\pi}^{-0}(0)) + \textstyle\sum_{l\in M_{-i}}[u^l(\bar{\pi}^{-0}(l)) - V^l(\bar{p}^l(\bar{t}))]\Big) \\
= \;& \textstyle\sum_{l\in M_0} u^l(\bar{\pi}^{-0}(l)) - \sum_{l\in M_{-i}\cup\{0\}} u^l(\bar{\pi}^{-i}(l)) \\
= \;& \textstyle\sum_{l\in M_0} u^l(\bar{\pi}^{-0}(l)) - V_{-i}(N) \\
\le \;& V(N) - V_{-i}(N) \\
= \;& W_i\Big(\sigma_i, \{\sigma_j\}_{j\in M_{-i}}, \{u^l\}_{l\in M}\Big).
\end{aligned}$$

This shows that sincere bidding is indeed an ex post perfect equilibrium in the auction game. □

Proof of Lemma 5.1: Since $\pi \in S(p^*)$, by the definition of supply set, we have

$$p^*(\pi(i)) = \max_{\gamma \in \mathcal{B}(\pi(i))} \Big\{ \sum_{A \in \gamma} p^*(A) \Big\}$$

for every $i \in M_0$, i.e., $q^*(\pi(i)) = p^*(\pi(i))$ for all $i \in M_0$. Similarly, it can be shown that $\pi \in S(q^*)$. Moreover, observe that $q^*(L) \geq p^*(L)$ for any bundle $L \subseteq N$, we see $\pi(i) \in D^i(q^*)$ for every $i \in M$. This shows that (q^*, π^*) is indeed a nonlinear pricing Walrasian equilibrium. $\square$

Proof of Theorem 5.2: Pareto efficiency follows easily from the Shapley value. Superaddivity of the coalition value v leads to individual rationality. Symmetry is readily seen from the fact that the Shapley value is constructed from an equilibrium. We need to show the dummy property. Suppose that $i \in \mathcal{S}$ is a dummy seller. Then for the given equilibrium (q^*, π^*), we must have $q^*(L) \geq u^i(L)$ for any bundle $L \subseteq W_i$ from the viewpoint of seller i. Since agent i is a dummy seller, then we must also have $q^*(L) \leq u^i(L)$ from the viewpoint of bidders. Thus $q^*(L) = u^i(L)$ for every $L \subseteq W_i$. By superadditivity of u^i, the auction rule implies that the whole bundle W_i will not be sold and thus be kept by seller i. This means that adding the dummy seller i to any coalition of sellers does not yield any extra value except seller i's own value. Therefore the coalition value v satisfies the equality: $v(T \cup \{i\}) - v(T) = u^i(W_i)$ for any $T \subseteq \mathcal{S} \setminus \{i\}$. Then the dummy property follows immediately from the definition of the Shapley value. $\square$

Proof of Theorem 6.1: It is obvious that the value function u^i of every buyer $i \in M$ satisfies Assumptions (A1) and (A2). Now let us show that u^i satisfies Assumption (A3). Let $N = A \cup B \cup C$. For any $S, S' \in 2^N$ with $S \cap S' = \emptyset$, by the definition of bidder i's value function, we have distinct CPU-display-keyboard matchings $\{(a_{i_1}, b_{j_1}, c_{k_1}), \cdots, (a_{i_l}, b_{j_l}, c_{k_l})\}$ in S satisfying $u^i(S) = v_i(a_{i_1}, b_{j_1}, c_{k_1}) + \cdots + v_i(a_{i_l}, b_{j_l}, c_{k_l})$, and distinct CPU-display-keyboard matchings $\{(a_{i'_1}, b_{j'_1}, c_{k'_1}), \cdots, (a_{i'_{l'}}, b_{j'_{l'}}, c_{k'_{l'}})\}$ in S' satisfying $u^i(S') = v_i(a_{i'_1}, b_{j'_1}, c_{k'_1}) + \cdots + v_i(a_{i'_{l'}}, b_{j'_{l'}}, c_{k'_{l'}})$. Observe that $\{(a_{i_1}, b_{j_1}, c_{k_1}), \cdots, (a_{i_l}, b_{j_l}, c_{k_l}),\ (a_{i'_1}, b_{j'_1}, c_{k'_1}), \cdots, (a_{i'_{l'}}, b_{j'_{l'}}, c_{k'_{l'}})\}$ are also distinct CPU-display-keyboard matchings in $S \cup S'$. Thus, by definition of u^i, we have

$$\begin{aligned} u^i(S \cup S') \;\; \geq \;\; & v_i(a_{i_1}, b_{j_1}, c_{k_1}) + \cdots + v_i(a_{i_l}, b_{j_l}, c_{k_l}) + \\ & + v_i(a_{i'_1}, b_{j'_1}, c_{k'_1}) + \cdots + v_i(a_{i'_{l'}}, b_{j'_{l'}}, c_{k'_{l'}}) = u^i(S) + u^i(S'). \end{aligned}$$

This shows that u^i satisfies Assumption (A3). Then it follows from Theorem 2.3 that the model has a nonlinear pricing Walrasian equilibrium.

We will show the last statement by an example. Consider an economy in which a seller wants to sell two CPUs $\{a_1, a_2\}$, two displays $\{b_1, b_2\}$ and two keyboards $\{c_1, c_2\}$ to two buyers $\{1, 2\}$. The values of the buyers on CPU-display-keyboard matchings are given by $v_1(a_2, b, c) = v_2(a_1, b, c) = 1$ for any $b \in \{b_1, b_2\}$ and $c \in \{c_1, c_2\}$ and the rest are given in Table 7. In this economy there is only one efficient allocation, namely, buyer 1 is assigned the bundle $\{a_1, b_1, c_1\}$ and buyer 2 the bundle $\{a_2, b_2, c_2\}$. We will prove that this allocation cannot be supported by a price vector $(p_{a_1}, p_{a_2}, p_{b_1}, p_{b_2}, p_{c_1}, p_{c_2})$. Suppose to the contrary that such a price vector does exist. Then, we must have the following system of inequalities:

$$\begin{aligned} 12 - p_{a_1} - p_{b_1} - p_{c_1} &\geq 20 - p_{a_1} - p_{b_2} - p_{c_2} \\ 12 - p_{a_2} - p_{b_2} - p_{c_2} &\geq 11 - p_{a_2} - p_{b_2} - p_{c_1} \\ 12 - p_{a_2} - p_{b_2} - p_{c_2} &\geq 11 - p_{a_2} - p_{b_1} - p_{c_2} \end{aligned}$$

where the 1st inequality holds for buyer 1 and other two hold for buyer 2. It follows that $p_{b_2} - p_{b_1} \geq 8 + p_{c_1} - p_{c_2}$, $p_{c_1} - p_{c_2} \geq -1$, and $1 \geq p_{b_2} - p_{b_1}$. These inequalities imply $1 \geq p_{b_2} - p_{b_1} \geq 8 + p_{c_1} - p_{c_2} \geq 7$, which is impossible. Thus the economy has no standard Walrasian equilibrium. □

Table 7: The values of the buyer on matchings.

$v_1(a_1, b, c)$	c_1	c_2	$v_2(a_2, b, c)$	c_1	c_2
b_1	$\underline{12}$	11	b_1	2	11
b_2	11	20	b_2	11	$\underline{12}$

Proof of Theorem 6.2: It suffices to show that for any disjoint subnetworks $G^1 = (N^1, E^1)$ and $G^2 = (N^2, E^2)$ of the network $G = (N, E)$, i.e., $E^1 \cap E^2 = \emptyset$, $C^l(G^1) + C^l(G^2) \geq C^l(G^1 \cup G^2)$, i.e., C^l is subadditive. Here $G^1 \cup G^2 = (N^1 \cup N^2, E^1 \cup E^2)$. Let $G^1_1, G^1_2, \cdots, G^1_k$ be the disjoint components of G^1, and $G^2_1, G^2_2, \cdots, G^2_l$ the disjoint components of G^2. When G^1 and G^2 merge, we have the following two cases to consider. Case 1: $G^1_1, G^2_2, \cdots, G^1_k$, $G^2_1, G^2_2, \cdots, G^2_l$ continue to be components of $G^1 \cup G^2$. Case 2: Some component G^1_i and some component G^2_j connect and become one component of $G^1 \cup G^2$, and other components of G^1 and of G^2 continue to be components of $G^1 \cup G^2$. Then in Case 1, we have $C^l(G^1 \cup G^2) = C^l(G^1) + C^l(G^2)$. In Case 2, we have $C^l(G^1 \cup G^2) \leq C^l(G^1) + C^l(G^2)$. Notice that in $u^l(G') = \sum_{e \in E'} c^l(e) - C^l(G')$ for any subnetwork $G' = (N', E')$, the first term is linear and thus superadditive, and the second term $-C^l$ is also superadditive. Their sum is clearly superadditive. Then by Theorem 2.3, the model has a nonlinear pricing Walrasian equilibrium. □

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