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**An Efficient Multi-Item Dynamic Auction with Budget Constrained Bidders**

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# An Efficient Multi-Item Dynamic Auction with Budget Constrained Bidders[1]

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**Abstract:** An auctioneer wishes to sell several heterogeneous indivisible items to a group of potential bidders. Each bidder has valuations over the items but might face a budget constraint and may therefore not be able to pay up to his values. In such markets, a competitive equilibrium typically fails to exist. We develop a novel dynamic auction and prove that the auction always finds a core allocation. The core allocation consists of an assignment of the items and its associated supporting price vector and is Pareto efficient.

**Keywords:** Dynamic auction, multi-item auction, budget constraint, core, efficiency.

JEL classification: D44.

## 1 Introduction

Auction theory typically assumes that all bidders can pay up to their values on the goods for sale. However, in reality buyers often face budget or liquidity constraints and may therefore be unable to afford what the goods are worth to them. Financial constraints arise in a variety of situations, such as less developed countries, business downturns, and financial crises; see e.g., Che and Gale (1998), Laffont and Robert (1996), Maskin (2000), Krishna (2002), Klemperer (2004), and Milgrom (2004). Financial constraints may pose a serious obstacle to the efficient allocation of the goods, resulting in loss of market efficiency. It is known that even when a single item is sold, it is generally impossible to have a mechanism for achieving full market efficiency when bidders face budget constraints, because budget constraints can fail the existence of a Walrasian equilibrium. Moreover, auctions that do very well without budget constraints, if implemented under budget constraints, often produce highly inefficient outcomes.

Although budget constraints make full market efficiency unattainable due to nonexistence of a Walrasian equilibrium, it is natural to ask whether there exist mechanisms that

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can achieve the allocation of goods as efficient as possible. This paper aims to solve this problem. To be precise, we examine a general model in which a finite number of (indivisible) items are sold to a finite number of budget constrained bidders. Each bidder wants to consume at most one item. When bidders face no budget constraints, the model reduces to the famous assignment model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), and Demange et al. (1986), among others. We propose a novel dynamic auction and prove that the auction always finds a core allocation of the goods among bidders in finitely many steps and thus achieves Pareto efficiency. The notion of core is more general than that of competitive equilibrium and is a widely used solution concept for general exchange economies and nontransferable utility (NTU) games; see Scarf (1967).

In early literature, to cite but a few, Che and Gale (1998), Laffont and Robert (1996), Maskin (2000), Krishna (2002), and Zheng (2001) have analysed various auctions for selling a single item when bidders are financially constrained. Moreover, Benoît and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009) have studied auctions for selling two items to budget constrained bidders. Recently, van der Laan and Yang (2008) have examined a similar model as the current one and developed an ascending auction that always finds a constrained equilibrium. The constrained equilibrium possesses several interesting properties but does not necessarily yield a core allocation.

Our dynamic auction is most closely related to Ausubel and Milgrom (2002). In their Section 8, Ausubel and Milgrom briefly discuss a dynamic auction with budget constrained bidders after they have extensively examined a family of core-selecting package auctions without budget constrained bidders; see Bernheim and Whinston (1986) for a static setting. Ausubel-Milgrom auction with budget constrained bidders uses the core of an NTU game as solution and is shown to find a core element in their auction model. However, their auction mechanism cannot be applied to our current model, because their auction requires that every bidder should have a strict preference relation over a finite set of choices, whereas in our model every agent has only a weak preference relation over a continuum of choices and is typically indifferent among many choices, and tie-breaking rules cannot be used. We also point out that Day and Milgrom (2008) and Erdil and Klemperer (2010) have refined and improved the core-selecting package auction of Ausubel and Milgrom (2002) for the case without budget constrained bidders.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 describes and analyzes the auction. Section 4 concludes.

# 2 The Model

Consider an auction model consisting of a seller (i.e., auctioneer) and $m$ potential bidders. The seller has $n$ indivisible goods for sale. Let $N = \{1, \ldots, n\}$ denote the set of items and $M = \{1, 2, \cdots, m\}$ the set of bidders. Item 0 denotes a *dummy good* which has no value and costs nothing for any agent. Every real item $j \in N$ is inherently indivisible and thus can be assigned to at most one bidder. The seller has for each real item $j \in N$ a reservation price $c(j) \in \mathbb{Z}_+$ below which the item will not be sold, where $\mathbb{Z}$ and $\mathbb{Z}_+$ are the set of integers and the set of nonnegative integers, respectively. The reservation price of the dummy good is known to be $c(0) = 0$. Every bidder $i \in M$ demands at most one item and has a (possibly negative) monetary valuation to each item in $N \cup \{0\}$ given by the function $V^i : N \cup \{0\} \to \mathbb{Z}$ with $V^i(0) = 0$, and is endowed with a budget $m^i \in \mathbb{Z}_+$ units of money. For any $i \in M$, all values $V^i(j)$, $j \neq 0$, and $m^i$ are private information and thus only bidder $i$ knows these values.

It is assumed that buying an item $j \in N \cup \{0\}$ against price $p(j)$ by bidder $i$ yields him a utility $U^i$ equal to

$$U^i = \begin{cases} V^i(j) + m^i - p(j) & \text{if } p(j) \leq m^i, \\ -\infty & \text{if } p(j) > m^i. \end{cases}$$

That is, bidders are not allowed to have a deficit of money. By this assumption, no bidder is willing to pay a price for any item above his budget $m^i$. We say that bidder $i$ is *financially constrained* if $m^i < \max_{j \in N} V^i(j)$, i.e., the valuation of bidder $i$ for some items exceeds what he can afford, and that bidder $i$ faces *no financial constraint* otherwise. When no bidder is financially constrained, the model is equivalent to the classical assignment market model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), and Demange et al. (1986), among others.

A price vector $p \in \mathbb{R}_+^{n+1}$ gives a price $p(j) \geq 0$ for each item $j \in N \cup \{0\}$. A price vector $p \in \mathbb{R}_+^{n+1}$ is *feasible* if $p(j) \geq c(j)$ for every $j \in N$ and $p(0) = 0$. The price of the dummy good is always assumed to be zero. Let $N_0 = N \cup \{0\}$ denote the set of all items in the market and let $M_0 = M \cup \{0\}$ denote the set of all agents, where agent 0 stands for the seller. An assignment is a vector $\pi = (\pi(1), \ldots, \pi(m))$ of items among all bidders in $M$ such that $\pi(i) = \pi(j)$ for $i \neq j$ implies $\pi(i) = 0$. At an assignment $\pi$, any bidder $i \in M$ gets one item, $\pi(i)$, in $N_0$. The dummy good can be assigned to any number of bidders, while a real item is assigned to at most one bidder. With respect to an assignment $\pi$, the set $N_\pi = \{k \in N \mid k \neq \pi(i), \forall i \in M\}$ denotes the set of unsold real items, which will be kept by the seller. Let $\mathcal{A}$ denote the family of all assignments.

At a feasible price vector $p \in \mathbb{R}_+^{n+1}$, the demand set of bidder $i \in M$ is given by

$$D^i(p) = \{j \in N_0 \mid V^i(j) - p(j) = \max\{V^i(k) - p(k) \mid p(k) \leq m^i, \ k \in N_0\}\}.$$

Observe that for any feasible $p$, the demand set $D^i(p) \neq \emptyset$, because $p(0) = 0 \leq m^i$ and thus the dummy item is always in the budget set. This means that the bidder has always the possibility not to buy any real item.

A pair $(p, \pi)$ of a feasible price vector $p$ and an assignment $\pi \in \mathcal{A}$ is said to be *implementable* if $p(\pi(i)) \leq m^i$ for all $i \in M$ and $p(j) = c(j)$ for all $j \in N_\pi$, i.e., every bidder $i$ can afford to buy the item $\pi(i)$ assigned to him, and the price of every unsold item equals its reservation price. A *Walrasian equilibrium* (WE) is an implementable pair $(p^*, \pi^*)$ such that $\pi^*(i) \in D^i(p^*)$ for all $i \in M$. It is well known from Koopmans and Beckmann (1957), and Shapley and Shubik (1971) that, when there is no financial constraint for any bidder, a Walrasian equilibrium always exists. Moreover, Crawford and Knoer (1981) and Demange et al. (1986) have developed auction mechanisms for finding Walrasian equilibria in such markets.

The following example shows that a Walrasian equilibrium may fail to exist if buyers face financial constraints.

**Example 1.** A seller wants to sell item 1 to buyers 1 and 2. The seller's reservation price is zero, $c(1) = 0$. The two buyers' valuations and budgets are $V^1(1) = 4$, $V^2(1) = 6$, and $m^1 = m^2 = 2$. This market has no equilibrium, although both buyers have valuations and budgets above the seller's zero reservation price. To see this, we have to consider two cases for the price of the item. At $p(1) \leq 2$, the item is over-demanded, whereas at $p(1) > 2$ the item is over-supplied.

Although a Walrasian equilibrium mail fail to exist, it is interesting to observe in Example 1 that implementable pairs $(p, \pi^1) = ((0,2),(1,0))$ and $(p, \pi^2) = ((0,2),(0,1))$ are core allocations and Pareto efficient. To verify this, consider $(p, \pi^1) = ((0,2),(1,0))$. At $(p, \pi^1)$, the seller gets utility of $p(1) = 2$, buyer 1 utility of $V(1) + m^1 - p(1) = 4$, and buyer 2 utility of $m^2 = 2$. This allocation is stable and thus in the core in the sense that the seller and buyer 2 cannot block it because it is impossible for the seller to achieve a utility of more than 2. The same argument applies to $(p, \pi^2)$.

The above example shows that when bidders face financial constraints, the Walrasian equilibrium is not guaranteed to exist but the core may be still nonempty. It is well known from Scarf (1965) that an exchange economy can have a nonempty core under fairly general conditions. In the sequel, we shall prove the existence of a nonempty core for the auction model with budget constrained bidders. We establish this result by proposing a dynamic auction, which actually finds a core allocation in finitely many steps.

To introduce the precise notion of core, we first give several definitions. At an implementable pair $(p, \pi)$, the utilities that the bidders and the seller achieve are given by

$$U^i(p, \pi) = V^i(\pi(i)) + m^i - p(\pi(i)), \ i \in M,$$

and

$$U^0(p, \pi) = \sum_{i \in M} p(\pi(i)) + \sum_{j \in N_\pi} c(j),$$

respectively. An implementable pair $(p, \pi)$ is *Pareto efficient* if there does not exist another implementable pair $(p', \pi')$ such that $U^i(p', \pi') > U^i(p, \pi)$ for all $i \in M_0$. A nonempty subset of $M_0$ is called a coalition. Given a coalition $S \subseteq M_0$, a *feasible assignment* $\rho^S$ is an assignment in $\mathcal{A}$ such that $\rho^S(i) = 0$ for every $i \in M \setminus S$. That is, at $\rho^S$, every bidder in $S$ receives at most one real item and every bidder outside $S$ receives the dummy good. A pair $(q, \rho^S)$ of a feasible price vector $q \in \mathbb{R}_+^{n+1}$ and a feasible assignment $\rho^S$ is *implementable* if $q(\rho^S(i)) \leq m^i$ for all $i \in S$ and if $q(j) = c(j)$ for every unsold item $j \in N_{\rho^S}$. An implementable pair $(p, \pi)$ is a *core allocation* if there does not exist any implementable pair $(q, \rho^S)$ for some coalition $S \subseteq M_0$ such that $U^i(q, \rho^S) > U^i(p, \pi)$ for all $i \in S$. Clearly, a core allocation is Pareto efficient. Also a core allocation $(p, \pi)$ ensures individual rationality, that is, $U^i(p, \pi) \geq m^i$ for every bidder $i \in M$ and $U^0(p, \pi) \geq \sum_{h \in N} c(h)$ for the seller. Observe that when we consider any coalition of two or more agents, we only need to concentrate on coalitions that contain at least the seller. Coalitions consisting of two or more bidders can be excluded.

# 3    The dynamic auction

We now establish the existence of a core allocation for the auction model with budget constrained bidders. This result can be seen as a generalization of the classic existence theorem for the assignment markets without budget constraints to the more general case which permits budget constraints. While in the classical model (see Shapley and Shubik (1971)) the set of Walrasian equilibrium allocations coincides with the (strong) core, in the current model the core will be shown to be non-empty and can be strictly larger than the set of Walrasian equilibrium allocations if the latter set is nonempty.

**Theorem 3.1**    *There exists at least one core allocation in the auction model with budget constrained bidders.*

We shall design a dynamic auction that can actually find in finitely many steps a core allocation, thus yielding a constructive proof for this theorem. Briefly speaking, the auction works as follows. Initially, every bidder submits a bid for every item that he is willing and able to pay. Subsequently, taking all the current bids and her own reservation prices into account the auctioneer chooses a profile of bids that can give her the highest revenue and offers a spot market price for each item. Then each bidder updates his bid for every item according to the spot market prices, his current bids and his budget and his valuations.

Again, the auctioneer updates. Repeat this process until no bidder is willing to offer any new bid.

In each round $t \in \mathbb{Z}_+$ of the auction, every bidder $i \in M$ offers a (feasible) bidding function $p_t^i : N_0 \to \mathbb{Z}_+$ with $p_t^i(0) = 0$ and $p_t^i(k) \leq \min\{m^i, V^i(k)\}$ for every $k \in N$. That is, no bidder is willing to bid above his budget or his value for any item. Let $P_t = (p_t^1, \ldots, p_t^m)$ be the bidding system at round $t$. Since the auctioneer wishes to achieve the highest revenue, her choice set at round $t$, is determined by

$$ S(P_t) = \{\pi \in \mathcal{A} \mid \sum_{k \in N_\pi} c(k) + \sum_{i \in M} p_t^i(\pi(i)) = \max_{\rho \in \mathcal{A}}(\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)))\}. $$

We now give a detailed description of the dynamic auction.

**The dynamic auction**

Step 1: Every bidder $i \in M$ offers a bidding function $p_0^i$. Set $t = 0$ and go to Step 2.

Step 2: Based on the current bidding system $P_t = (p_t^1, \ldots, p_t^m)$, the auctioneer announces an assignment $\pi_t \in S(P_t)$ and a spot market price vector $\bar{p}_t \in \mathbb{R}^{n+1}$ as follows. If $t = 0$ or $\pi_{t-1} \notin S(P_t)$, take $\pi_t$ to be any element of $S(P_t)$, and set $\bar{p}_t(0) = 0$, $\bar{p}_t(\pi_t(i)) = p_t^i(\pi_t(i))$ when $\pi_t(i) \in N$ for some $i \in M$, and $\bar{p}_t(k) = c(k)$ when $k \in N_{\pi_t}$, and go to Step 3. If $t > 0$ and $\pi_{t-1} \in S(P_t)$, then set $\pi_t = \pi_{t-1}$, $\bar{p}_t(0) = 0$, for any $\pi_t(i) \in N$ for some $i \in M$, set $\bar{p}_t(\pi_t(i)) = \bar{p}_{t-1}(\pi_t(i)) + 1$ when, at round $t-1$, some bidder $j$ increased his bid for the item $\pi_t(i)$ to $\bar{p}_{t-1}(\pi_t(i)) + 1$, and set $\bar{p}_t(k) = \bar{p}_{t-1}(k)$ otherwise, and go to Step 3.

Step 3: Each bidder $i \in M$ updates his bids by setting $\tilde{p}_t^i(k) = \min\{p_t^i(k), \bar{p}_t(k)\}$ for all $k \in N$. For any bidder $i \in M$, if there exists some item $k \in N$ such that $V^i(k) - \tilde{p}_t^i(k) > V^i(\pi_t(i)) - \tilde{p}_t^i(\pi_t(i))$ and $\tilde{p}_t^i(k) < m^i$, bidder $i$ updates his bidding function by setting $p_{t+1}^i(k) = \tilde{p}_t^i(k) + 1$ for one such $k$, and setting $p_{t+1}^i(h) = \tilde{p}_t^i(h)$ for any other item $h \in N$. Every other bidder $i$ sets $p_{t+1}^i = \tilde{p}_t^i$. When $p_{t+1}^i \neq \tilde{p}_t^i$ for some $i \in M$, then set $t = t + 1$ and go back to Step 2. Otherwise, the auction stops and the output is $(P_t, \pi_t)$.

This auction bears some similarity with pay-as-you-bid auctions used in internet or more traditionally by governments for selling treasury bills, but it differs from them in three crucial aspects: First, in Step 2, the auctioneer does not merely announce the spot price for each item but more importantly she adjusts the spot price upwards when she observes that some bidder increases his bid. Second, in Step 3, unlike the existing pay-as-you-bid auctions in which bidders are not allowed to decrease their bids, the current auction permits bidders to reduce their bids in order to avoid overbidding. Third, in Step

6

3, the current auction has a flexible rule for the bidders to adjust their bids for those items which they can afford and give them higher profits, whereas the existing auctions typically require the bidders to adjust their bids for those items which give the bidders the highest profits.

As we have pointed out earlier, our auction is most closely related to Ausubel and Milgrom (2002); see also Milgrom (2004, Section 8.3.2). Although they focus their analysis on a family of package auctions without budget constrained bidders, they also briefly discuss a dynamic auction with budget constrained bidders in their Section 8. That auction (with budget constrained bidders) uses the core of an NTU game as the solution and is shown to find a core element in their auction model. In their model, bidders are allowed to demand several items. However, their auction mechanism cannot be applied to our current model, because their auction requires every bidder to have a strict preference relation over a finite set of choices, whereas in our model every agent has only a weak preference relation over a continuum of choices $N_0 \times \mathbb{R}$ and is typically indifferent among many choices. Next is a example to show why tie-breaking rules cannot be used to deal with weak preferences.

**Example 2** Suppose that there are bidders 1, 2, 3, and items 1 and 2. The parameters are given by $V^i(j) = 1$, $V^i(0) = 0$, $m^i = 3$, and $c(j) = 0$ for all $i = 1, 2, 3$ and all $j = 1, 2$. Bidders are indifferent between the two items. We break ties as follows. When a bidder is indifferent among bundles, he prefers the bundle with item 1 to the bundle with item 2 and the latter one to the bundle with the dummy good. It is easy to see that there exists a Walrasian equilibrium $(p, \pi)$ where $p(1) = p(2) = 1$ and $\pi(1) = 1$, $\pi(2) = 2$, and $\pi(3) = 0$. At $p^0 = (c(0), c(1), c(2)) = (0, 0, 0)$ we have $D^i(p^0) = \{1\}$ for all $i \in M$. We adjust prices from $p^0$ to $p^1 = (0, 1, 0)$ because item 1 is over-demanded. At $p^1$ we have $D^i(p^1) = \{2\}$ for all $i \in M$. Prices are adjusted to $p^2 = (0, 1, 1)$, because item 2 is over-demanded. At $p^2$ we have $D^i(p^2) = \{1\}$ for all $i \in M$. Prices are adjusted to $p^3 = (0, 2, 1)$ since item 1 is over-demanded. At $p^3$ we have $D^i(p^3) = \{2\}$ for all $i \in M$. Prices are adjusted to $p^4 = (0, 2, 2)$ since item 2 is over-demanded. The auction now ends up with $D^i(p^4) = \{0\}$ for all $i \in M$. But this is not a Walrasian equilibrium price vector.

Now let us return to our dynamic auction. Observe that when the auction stops with some $(P_t, \pi_t)$, it is possible that some bidder $i \in M$ gets an item $\pi_t(i)$ which is not contained in his demand set $D^i(p_t^i)$. In this case, we must have that $p_t^i(k) = m^i$ for any $k \in N$ satisfying $V^i(k) - p_t^i(k) > V^i(\pi_t(i)) - p_t^i(\pi_t(i))$.

**Proposition 3.2** *In every round $t$ of the auction, it holds that $\pi_t \in S(\tilde{P}_t)$.*

**Proof.** In case of $\pi_{t-1} \notin S(P_t)$, we have

$$\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} \tilde{p}_t^i(\rho(i))$$

7

for all $\rho \in \mathcal{A}$. Moreover, it holds that

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} \tilde{p}_t^i(\pi_t(i)) = \sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} p_t^i(\pi_t(i))$$

and that for all $\rho \in \mathcal{A}$

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} p_t^i(\pi_t(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)).$$

Therefore for all $\rho \in \mathcal{A}$, we get

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} \tilde{p}_t^i(\pi_t(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} \tilde{p}_t^i(\rho(i)).$$

Similarly we can show for the case $\pi_{t-1} \in S(P_t)$ that $\pi_t = \pi_{t-1} \in S(\tilde{P}_t)$. $\qquad \square$

The existing auctions are typically either ascending or decreasing. It is relatively easy to prove their finite convergence because of obvious monotonicity. However, this is not the case for the current auction, because the bidders may also decrease some of their bids. This feature makes it not possible to use familiar argument. Instead, we have to explore a different monotonicity argument which makes use of the bids of all bidders and the revenues of the seller.

**Lemma 3.3** *The auction terminates in finitely many rounds.*

**Proof.** Let $R(\pi, P) = \sum_{k \in N_\pi} c(k) + \sum_{i \in M} p^i(\pi(i))$ denote the revenue of the seller at $(\pi, P)$, and let $R(P) = \max\{R(\pi, P) \mid \pi \in \mathcal{A}\}$ be the highest revenue at $P$. In each round $t$, we have $R(\pi_t, P_t) = R(P_t)$. We need to consider two cases. In case of $\pi_{t-1} \notin S(P_t)$, we have $R(P_t) = R(\pi_t, P_t) > R(\pi_{t-1}, P_t)$, by the auction rule we have $R(\pi_{t-1}, P_t) \geq R(\pi_{t-1}, \tilde{P}_{t-1})$, and from Proposition 3.2 it follows that $R(\pi_{t-1}, \tilde{P}_{t-1}) = R(\pi_{t-1}, P_{t-1}) = R(P_{t-1})$. This proves $R(P_t) > R(P_{t-1})$.

In case of $\pi_{t-1} \in S(P_t)$, we have $R(P_t) = R(P_{t-1})$, because $p_t^i(\pi_t(i)) = p_{t-1}^i(\pi_{t-1}(i))$ for all $i \in M$. In this case, at least one bidder increases his bid but no bidder decreases any of his bids.

The two arguments imply that in each round, either the revenue of the seller is strictly increasing, or the revenue of the seller remains constant and no bidder is bidding less but at least one bidder is bidding more. Therefore, because of finiteness of all values and budgets, the auction must stop in finitely many rounds. $\qquad \square$

Let $\pi^* = \pi_t$ and $p^* = \bar{p}_t$ when the auction stops at the final round $t$. The following theorem shows that the outcome $(p^*, \pi^*)$ is a core allocation.

**Theorem 3.4** *The outcome $(p^*, \pi^*)$ found by the auction is in the core and thus Pareto efficient.*

**Proof.** Suppose to the contrary that $(p^*, \pi^*)$ is not in the core. Clearly, the pair $(p^*, \pi^*)$ is individually rational. Then there exist a coalition $S$ consisting of the seller and at least one bidder and an implementable pair $(q, \rho^S)$ such that $U^i(q, \rho^S) > U^i(p^*, \pi^*)$ for all $i \in S$. For the seller this implies

$$
\begin{aligned}
\sum_{j \in N} q(j) &= \sum_{i \in S} q(\rho^S(i)) + \sum_{j \in N_{\rho^S}} c(j) \\
&= U^0(q, \rho^S) \\
&> U^0(p^*, \pi^*) \\
&= \sum_{i \in M} p^*(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \\
&= \sum_{j \in N} p^*(j).
\end{aligned}
\tag{3.1}
$$

Then there exists $j^* \in N$ such that $q(j^*) > p^*(j^*)$. This means that some bidder $i^* \in S$ is assigned item $j^*$ at $\rho^S$, i.e., $\rho^S(i^*) = j^*$, because $\min\{p^*(j), q(j)\} \geq c(j)$ for all $j \in N$ and $q(j) = c(j)$ for every unassigned item $j \in N_{\rho^S}$. Observe that since $(q, \rho^S)$ is implementable and $U^{i^*}(q, \rho^S) > U^{i^*}(p^*, \pi^*)$, we have $V^{i^*}(j^*) - q(j^*) > V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*))$ and $q(j^*) \leq m^{i^*}$. It follows from $q(j^*) > p^*(j^*)$ that

$$
\begin{aligned}
V^{i^*}(j^*) - p^*(j^*) &> V^{i^*}(j^*) - q(j^*) \\
&> V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*)) \\
&= V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)),
\end{aligned}
$$

where $t$ is the round in which the auction stops. Because $\tilde{p}_t^i(k) \leq \bar{p}_t(k) = p^*(k)$ for all $i \in M$ and $k \in N$, we have

$$
\begin{aligned}
V^{i^*}(j^*) - \tilde{p}_t^{i^*}(j^*) &\geq V^{i^*}(j^*) - p^*(j^*) \\
&> V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)).
\end{aligned}
$$

Moreover, $\tilde{p}_t^{i^*}(j^*) \leq \bar{p}_t(j^*) = p^*(j^*) < q(j^*) \leq m^{i^*}$. But then, at prices $\tilde{p}_t^{i^*}$, bidder $i^*$ should have rejected item $\pi^*(i^*)$ and made a new offer, and therefore the auction could not have stopped at round $t$. $\square$

# 4 Concluding remarks

In this article we have proposed a dynamic auction for finding a core allocation in a setting where bidders are budget constrained and each bidder demands at most one item. It is worth pointing out that the auctions developed by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel and Milgrom (2002), Perry and Reny (2005), Ausubel (2004, 2006), Sun and Yang (2009) allow bidders to demand multiple items, albeit in the absence of budget constraint. This more general but also more difficult case remains to be explored when bidders face budget constraints.

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