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Abstract

This paper studies a class of optimal stopping problems that has become popular in the area of investment under uncertainty ("real options"). Necessary conditions for solutions to these problems are that the solution dominates the payoff function and is superharmonic. Neither property is typically verified in the literature. Here, easy-to-check conditions that establish solutions to many optimal stopping problems are provided. Attention is focussed on problems with payoff functions that are monotonic in the state variable (either increasing or decreasing) or payoff functions that are decreasing, then increasing. The state variable can be driven by any one-dimensional time-homogenous diffusion. An application to Bayesian sequential hypothesis testing illustrates the applicability of the approach.

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1 Introduction

In recent decades, applications of optimal stopping theory have been used with great success in the areas of economics and finance. In particular in the theory of investment under uncertainty ("real options") great progress has been made in our understanding of timing decisions under conditions of uncertainty.[1] In most of the literature the resulting optimal stopping problems are solved heuristically and based on economic arguments. In particular, a solution to optimal stopping problems is typically obtained by solving the Bellman equation and the so-called value-matching and smooth-pasting conditions.

This approach implicitly assumes that the solution takes the form of a *trigger policy*: stop as soon as the underlying process reaches a certain, endogenously determined, threshold. This is, however, hardly ever established explicitly. Similarly, following this approach does not establish two necessary conditions for optimal stopping (Peskir and Shiryaev, 2006): (1) the value function must dominate the payoff function, and (2) the value function must be superharmonic.

In this paper I consider optimal stopping problems of the form

$$F^*(y) = \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right].$$

Here $(Y_t)_{t \geq 0}$ follows a time-homogeneous diffusion taking values on some open set (a, b):

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = y,$$

$(B_t)_{t \geq 0}$ is a Wiener process, $\mathcal{M}$ is the set of stopping times, and F is the *payoff function*. The solution F^* is called the *value function*. F^* dominates F if $F^* \geq F$ and F^* is superharmonic if

$$F^*(y) \geq \mathsf{E}_y \left[e^{-r\tau} F^*(Y_\tau) \right],$$

for all $\tau \in \mathcal{M}$.

These problems are solved using the *generator*, $\mathscr{L}_Y$, which on C^2 coincides with the differential operator

$$\mathscr{L}_Y g(y) = \frac{1}{2}\sigma^2(y) g''(y) + \mu(y) g'(y).$$

The kind of payoff functions $F \in C^2$ that I admit are: (1) monotonically non-decreasing, (2) monotonically non-increasing, or (3) monotonically non-increasing on $(a, \bar{y}]$ and monotonically non-decreasing on $[\bar{y}, b)$, for some $\bar{y}$. I call the first two cases *one-sided* problems, because they involve stopping either if Y gets large (case 1) or if Y gets small (case 2). The third case is referred to as a *two-sided* problem, because a decision is taken when Y gets large or small, whichever occurs first.

[1] See, for example, Dixit and Pindyck (1994) for an overview of the early literature.

To describe the contribution of the paper, consider the first case, i.e. $F' \geq 0$. Let $\varphi \in C^2$ be an increasing convex function with $\varphi(a) = 0$.[2] If a solution exists I show that (i) the optimal stopping rule is a trigger policy and (ii) that the trigger is given by the solution y^* to the equation

$$\varphi(y^*)F'(y^*) = \varphi'(y^*)F(y^*). \tag{1}$$

Provided that

$$F''/\varphi'' < F'/\varphi', \tag{2}$$

such a solution is unique and, in fact, maximizes the function F/φ. The solution to the optimal stopping problem is then given by

$$F^*(y) = \begin{cases} \frac{\varphi(y)}{\varphi(y^*)}F(y^*) & \text{if } y < y^* \\ F(y) & \text{if } y \geq y^*, \end{cases}$$

and the optimal stopping time is the first hitting time from below of y^*.

Condition (1) encompasses the traditional "value-matching" and "smooth-pasting" conditions (see Dixit and Pindyck, 1994), whereas (2) ensures that F^* dominates F and is superharmonic. Intuitively, condition (2) ensures that φ is "more convex" than the payoff function. In fact, I show that (1) and (2) are the first and second order conditions, respectively, of an appropriately chosen optimization problem. So, at some level one can say that the traditional real options literature solves the first order condition, but neglects to check the second order condition. An example is provided to show that this can lead to erroneous conclusions. Note, in addition, that rather than solving two equations (value-matching and smooth-pasting) the approach advocated here only requires solving one equation. Similarly, in two-sided problems, the burden is reduced from solving four equations (two value-matching and two smooth-pasting conditions) to two equations.

The paper, therefore, provides easy-to-check conditions that establish a solution to a large class of optimal stopping problems under a wide variety of diffusions. The work is related to recent work by Boyarshenko and Levendorskiĭ (2007). Their approach, however, uses the Wiener-Hopf decomposition of Lévy processes, whereas the approach used here mainly uses Dynkin's formula. In addition, this paper does not aim for full generality. For example, while Boyarshenko and Levendorskiĭ (2011) focus on optimal stopping problems with non-monotonic and discontinuous payoff functions, the approach taken here emphasizes the link between optimal stopping problems and "standard" maximization problems. This implies that we make smoothness assumptions throughout. The reward reaped from paying the price of these stronger assump-

[2] An increasing solution exists for all diffusions, it is convex under certain conditions.

tions is that the resulting sufficient conditions for solutions to optimal stopping problems are easily checked and applicable.

The paper is organized as follows. Section 2 provides some motivating examples to illustrate the kind of problems the results can be applied to. Section 3.1 solves one-sided problems, whereas Section 3.2 solves two-sided problems. In Section 4 the method developed in Section 3.2 is applied to a Bayesian sequential testing problem to illustrate the wide applicability of the results.

2 Motivating Examples

The kind of problems that are considered in this paper often arise in the literature on investment under uncertainty (or "real options").

Example 1 (Optimal investment decision)**.** Consider a firm that can decide to invest in a project that leads to a payoff stream $(Y_t)_{t\geq 0}$ and a constant cost stream $c > 0$, by paying a sunk cost $I > 0$, and suppose that $(Y_t)_{t\geq 0}$ follows some diffusion

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t,$$

where $(B_t)_{t\geq 0}$ is a Wiener process. Suppose that the firm discounts payoffs at the rate $r > 0$.

If the current state of the process $(Y_t)_{t\geq 0}$ is y and the firm decides to invest at that stage the net present value (NPV) of the project is

$$\begin{aligned} F(y) &= \mathsf{E}_y\left[\int_0^\infty e^{-rt}(Y_t - c)dt\right] - I \\ &= \mathsf{E}_y\left[\int_0^\infty e^{-rt}Y_t dt\right] - \left(I + \frac{c}{r}\right), \end{aligned}$$

provided that the integral and the expectation exist.

The firm's problem then is to find a function F^* and a stopping time τ^* that solve the optimal stopping problem

$$F^*(y) := \mathsf{E}_y\left[e^{-r\tau^*}F(Y_{\tau^*})\right] = \sup_{\tau\in\mathcal{M}} \mathsf{E}_y\left[e^{-r\tau}F(Y_\tau)\right],$$

where $\mathcal{M}$ is the set of stopping times.

Intuitively, there will be a trigger y^*, such that investment is optimal as soon as y^* is hit from below. $\triangleleft$

Sufficient conditions for the existence of a unique trigger y^* and a straightforward way of computing it are given in Proposition 3 in Section 3.1.

Example 2 (Optimal liquidation decision)**.** Consider a firm that is currently generating a revenue stream $(Y_t)_{t\geq 0}$ against a constant cost stream $c>0$ and that can decide to liquidate by paying a sunk cost $0<I<c/r$. Suppose that $(Y_t)_{t\geq 0}$ follows some diffusion

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t,$$

where $(B_t)_{t\geq 0}$ is a Wiener process, and suppose that the firm discounts payoffs at the rate $r>0$.

If the current state of the process $(Y_t)_{t\geq 0}$ is y and the firm decides to liquidate at the stopping time τ, then, using the strong Markov property of diffusions, the value of this liquidation policy can be written as

$$\begin{aligned}
G(y) =& \mathsf{E}_y\left[\int_0^\tau e^{-rt}(Y_t-c)dt - e^{-r\tau}I\right]\\
=& \mathsf{E}_y\left[\int_0^\infty e^{-rt}(Y_t-c)dt - e^{-r\tau}\mathsf{E}_{Y_\tau}\left(\int_0^\infty e^{-rt}(c-Y_t)dt - I\right)\right]\\
=& \mathsf{E}_y\left[\int_0^\infty e^{-rt}Y_t dt\right] - \frac{c}{r} + \mathsf{E}_y\left[e^{-r\tau}\left(-\mathsf{E}_{Y_\tau}\left[\int_0^\infty Y_t dt\right] - \left(I-\frac{c}{r}\right)\right)\right]\\
\equiv& \mathsf{E}_y\left[\int_0^\infty e^{-rt}Y_t dt\right] - \frac{c}{r} + \mathsf{E}_y\left[e^{-r\tau}F(Y_\tau)\right],
\end{aligned}$$

assuming that all integrals and expectations exist, and where

$$F(y) = -\mathsf{E}_{Y_\tau}\left[\int_0^\infty Y_t dt\right] - \left(I-\frac{c}{r}\right).$$

The firm's problem then is to find a function F^* and a stopping time τ^* that solve the optimal stopping problem

$$F^*(y) := \mathsf{E}_y\left[e^{-r\tau^*}F(Y_{\tau^*})\right] = \sup_{\tau\in\mathcal{M}} \mathsf{E}_y\left[e^{-r\tau}F(Y_\tau)\right],$$

where $\mathcal{M}$ is the set of stopping times.

Intuitively, there will be a trigger y^*, such that liquidation is optimal as soon as y^* is hit from above. $\lhd$

Sufficient conditions for the existence of a unique trigger y^* and a straightforward way of computing it are given in Proposition 5 in Section 3.1.

Both the investment and liquidation problems are what I call *one-sided* optimal stopping problems, because only one decision needs to be taken: once the optimal stopping time is chosen it is obvious what action is taken at that time. In many realistic problems a firm has to decide not only when to stop, but also what to do at that time. In the simplest case, the firm has to choose between two different actions. I call such problems *two-sided* optimal stopping problems.

Example 3 (Adoption or abandonment of an investment opportunity)**.** Suppose that a firm has an option to invest in a particular project, the profits of which follow a diffusion

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t,$$

where $(B_t)_{t\geq 0}$ is a Wiener process. The sunk costs of investment are $I > 0$. If the project is infinitely-lived, then the expected net present value (NPV) of investing at a time when the process $(Y_t)_{t\geq 0}$ takes the value y is

$$F(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} Y_t dt \right] - I,$$

again assuming that the integral and the expectation exist.

In addition, suppose that there is a constant cost stream $c > 0$ for keeping the investment opportunity alive (for example, in the case of real estate investment the lease of a plot of land on which to build a development). The firm then has to decide (i) when to make a decision (and stop paying the cost stream c) and (ii) at the time of decision whether to adopt the project (and incur the sunk costs I) or to abandon the project altogether.

So, the optimal stopping problem facing the firm is

$$\begin{aligned} F^*(y) &= \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[-c \int_0^\tau e^{-rt} dt + e^{-r\tau} \max\{F(y), 0\} \right] \\ &= -\frac{c}{r} + \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} \max\{F_H(Y_\tau), F_L(Y_\tau)\} \right], \end{aligned}$$

where

$$F_L(y) = \frac{c}{r}, \quad \text{and} \quad F_H(y) = F(y) + \frac{c}{r}.$$

Intuitively, there will be a pair of triggers (Y_L, Y_H), $Y_L < Y_H$, such that abandonment is optimal as soon as Y_L is hit from above and adoption is optimal as soon as Y_H is hit from below. $\triangleleft$

Sufficient conditions for the existence of triggers Y_L and Y_H and a straightforward way of computing them are given in Proposition 7 in Section 3.2.

3 Optimal Stopping Problems with Smooth Payoff Functions

Consider a measurable space $(\Omega, \mathscr{F})$, a state space $E = (a, b) \subset \mathbb{R}$, and a family of probability measures $(\mathsf{P}_y)_{y\in E}$. For each $y \in E$, let $(Y_t)_{t\geq 0}$ be a strongly Markovian, time-homogeneous, càdlàg diffusion with $Y_0 = y$, P_y-a.s. Assume that $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by $(Y_t)_{t\geq 0}$, augmented by the null sets. The process $(Y_t)_{t\geq 0}$ is assumed to solve the stochastic differential equation (SDE)

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t,$$

where $(B_t)_{t\geq 0}$ is a Wiener process. The functions μ and σ are assumed to satisfy the conditions that ensure existence and uniqueness of solutions to this SDE (see, for example, Øksendal, 2000). All payoffs are discounted at a constant rate $r > 0$.

The expectation operator under P_y, $y \in E$, is denoted by E_y. Note that $\mathsf{E}_{Y_t} = \mathsf{E}(\cdot|\mathscr{F}_t)$, P_{Y_t}-a.s., for all $t \geq 0$. The generator of $(Y_t)_{t\geq 0}$ is defined by the operator

$$\mathscr{L}_Y f(y) = \lim_{t\downarrow 0} \frac{\mathsf{E}_y[f(X_t)] - f(y)}{t},$$

whenever the limit exists. It is the equivalent of the derivative of a function in cases where the variable follows a stochastic process. For time-homogeneous diffusions it can be shown that on $C^2(E)$ the generator coincides with the partial differential equation

$$\mathscr{L}_Y g = \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y).$$

The approach in this paper relies on the existence of convex solutions to the differential equation

$$\mathscr{L}_Y \varphi - r\varphi = 0. \tag{3}$$

The following lemma is due to Borodin and Salminen (1996) and Alvarez (2003).

Lemma 1. *If $\mu(\cdot) - ry$ is non-decreasing then there exist convex increasing and convex decreasing solutions to* (3).

Example 4. In this example we consider several often used diffusions. Let $(B_t)_{t\geq 0}$ be a Wiener process.

1. *Arithmetic Brownian motion (ABM).* Suppose that $(Y_t)_{t\geq 0}$ solves the SDE

   $$dY_t = \mu dt + \sigma dB_t,$$

   with $\mu \in \mathbb{R}$ and $\sigma > 0$ constants. Then the quadratic equation

   $$\frac{1}{2}\sigma^2\beta^2 + \mu\beta - r = 0,$$

   has two solutions, $\beta_1 > 0 > \beta_2$. The functions

   $$\hat{\varphi}(y) = e^{\beta_1 y}, \quad \text{and} \quad \check{\varphi}(y) = e^{\beta_2 y},$$

   are convex increasing and convex decreasing solutions to (3), respectively.

2. *Geometric Brownian motion (GBM).* Suppose that $(Y_t)_{t\geq 0}$ solves the SDE

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t,$$

with $\mu < r$ and $\sigma > 0$ constants. Then the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r=0,$$

has two solutions, $\beta_1 > 1 > 0 > \beta_2$. The functions

$$\hat{\varphi}(y)=y^{\beta_1}, \quad \text{and} \quad \check{\varphi}(y)=y^{\beta_2},$$

are convex increasing and convex decreasing solutions to (3), respectively.

3. *Geometric mean reversion (GMR).* Suppose that $(Y_t)_{t\geq 0}$ solves the SDE

$$dY_t = \eta(\bar{Y}-Y_t)Y_t dt + \sigma Y_t dB_t,$$

with $\eta, \bar{Y}, \sigma > 0$. Then the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(r+\eta\bar{Y})\beta-r=0,$$

has two solutions, $\beta_1 > 0 > \beta_2$. The functions

$$\hat{\varphi}(y)=y^{\beta_1}H\left(\frac{2\eta}{\sigma^2}y;\beta_1,b_1\right), \quad \text{and} \quad \check{\varphi}(y)=y^{\beta_2}H\left(\frac{2\eta}{\sigma^2}y;\beta_2,b_2\right),$$

where

$$b_i = 2\beta_i + 2(r+\eta\bar{Y})/\sigma^2, \quad i=1,2,$$

are convex increasing and convex decreasing solutions to (3), respectively. Here

$$H(x;a,b)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)/\Gamma(a)}{\Gamma(b+n)/\Gamma(b)}\frac{x^n}{n!},$$

is the confluent hypergeometric function.

$\triangleleft$

Finally, the *payoff function* is a C^2 function $F : E \to \mathbb{R}$. In investment problems, the payoff can be thought of as the net present value (NPV) of an investment. The problem facing the decision-maker (DM) is to choose a stopping time τ at which it is optimal to invest or, equivalently, at which it is optimal to stop waiting. That is, the DM wishes to solve the *optimal stopping problem*

$$F^*(y) = \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right], \tag{4}$$

where $\mathcal{M}$ is the set of stopping times. It is obvious from this formulation that F^* should (i) be time-independent and (ii) dominate the payoff function F. The solution F^* is called the *value function*. It can also be shown that if F^* solves (4) it must be a *superharmonic* function, i.e. $F^*(y) \geq \mathsf{E}_y \left[e^{-r\tau} F^*(Y_\tau) \right]$, for all stopping times $\tau \in \mathcal{M}$.

In this paper we consider payoff functions F that are (i) monotonically increasing or decreasing, in which case we refer to (4) as a *one-sided* problem, or (ii) decreasing then increasing, in which case we refer to (4) as a *two-sided* problem.

3.1 One-Sided Problems

3.1.1 Problems with Increasing Payoff Functions

First it will be assumed that $F' > 0$, so that an increase in the state variable corresponds with an increase in the payoff. It will be assumed that $F(a) < 0$ and $F(b) > 0$, which implies that there exists a unique $\bar{y} \in E$ such that $F(\bar{y}) = 0$. The assumption is made to to ensure that the problem is not economically vacuous.

The following proposition shows that the problem (4) can be solved by splitting the state space E into a *continuation set* D where $F^* > F$ and a *stopping set* $E \setminus D$, where $F^* = F$. In addition, it shows that (4) has a continuation set of the form $D = (a, y^*)$ for some threshold y^*. That is, the solution is a *trigger policy*: stop as soon as some threshold y^* is reached from below. In order to describe such a policy, denote the *first-hitting time* of some threshold y^* from below (under P_y) by

$$\hat{\tau}_y(y^*) = \inf \{ t \geq 0 \mid Y_t \geq y^* \}.$$

If no confusion is possible, the subscript will be dropped.

Proposition 1. *The continuation set D is a connected set with $D \supset (a, \bar{y}]$.*

Proof. It is well-known (cf. Øksendal, 2000) that the continuation set is time-invariant and, thus, only depends on the state of the process, y, and not explicitly on time, t.

Suppose that problem (4) has a solution. From Peskir and Shiryaev (2006, Theorem 2.4) we know that F^* is the least superharmonic majorant of F on E and that the first exit time of D,

$$\tau_D = \inf \{ t \geq 0 \mid Y_t \notin D \},$$

is the optimal stopping time.

1. We first show that $(a, \bar{y}] \subset D$. Let $y \leq \bar{y}$ and let

$$\tau = \inf \{ t \geq 0 \mid F(Y_t) \geq 0 \} .$$

Note that it is possible that $\mathsf{P}_y(\tau = \infty) > 0$. It holds that

$$\mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right] \geq 0 > F(\bar{y}) > F(y).$$

So, it cannot be optimal to stop at y and, hence, $(a, \bar{y}] \subset D$.

2. We now show that D is connected. Suppose not. Then there exist points

$$y_1 > \bar{Y}, \quad \text{and} \quad y_2 > y_1,$$

such that

$$y_1 \in E \setminus D, \quad \text{and} \quad y_2 \in D.$$

Let $\tau = \inf \{ t \geq 0 \mid Y_t \geq y_2, Y_t \in E \setminus D \}$. Since F^* is a superharmonic majorant of F it holds that

$$F(y_1) = F^*(y_1) \geq \mathsf{E}_{y_1} \left[F^*(Y_\tau) \right] = \mathsf{E}_{y_1} \left[F(Y_\tau) \right] > \mathsf{E}_{y_1} \left[F(y_2) \right] = F(y_2).$$

But this contradicts the fact that F is an increasing function. ■

The problem of finding the optimal stopping time can now be reduced to finding the optimal trigger y^*. Because of the a.s. continuity of the sample paths of $(Y_t)_{t \geq 0}$ this implies that

$$\begin{aligned} F^*(y) &= \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right] \\ &= \sup_{\hat{y} \in E} \mathsf{E}_y \left[e^{-r\hat{\tau}(\hat{y})} F(Y_{\hat{\tau}(\hat{y})}) \right] \\ &= \sup_{\hat{y} \in E} \mathsf{E}_y \left[e^{-r\hat{\tau}(\hat{y})} \right] F(\hat{y}). \end{aligned}$$

In order to find the *expected discount factor* of $\hat{\tau}(\hat{y})$, denoted by

$$\hat{\nu}_y(\hat{y}) := \mathsf{E}_y \left[e^{-r\hat{\tau}(\hat{y})} \right],$$

we use the existence of an increasing function $\hat{\varphi} \in C^2(E)$, such that $\mathscr{L}_Y \hat{\varphi} = r\hat{\varphi}$. In addition, we will assume that $\mu(y) - ry$ is non-decreasing so that this function $\hat{\varphi}$ is convex.

Proposition 2. *For any $\hat{y} \geq y$, it holds that*

$$\hat{\nu}(\hat{y}) = \hat{\varphi}(y)/\hat{\varphi}(\hat{y}).$$

Proof. Define the process $(X_t)_{t\geq 0}$ by

$$X_t = \begin{bmatrix} s+t \\ Y_t \end{bmatrix}.$$

For any function $g \in C^2([0,\infty)\times E)$, the generator of g equals

$$\mathscr{L}_X g = \frac{\partial g(\cdot)}{\partial t}dt + \sum_{i=1}^{d} \mu_i(y)\frac{\partial g(\cdot)}{\partial y_i} + \frac{1}{2}\sum_{i,j}\sigma_i(y)\sigma_j(y)\frac{\partial^2 g(\cdot)}{\partial y_i \partial y_j}.$$

So, in particular, it follows that for $g(t,y) = e^{-rt}F(y)$ we get:

$$\begin{aligned}\mathscr{L}_X g(\cdot) =& e^{-rt}\Big(\sum_{i=1}^{d}\mu_i(y)\frac{\partial g(\cdot)}{\partial y_i} + \frac{1}{2}\sum_{i,j}\sigma_i(y)\sigma_j(y)\frac{\partial^2 g(\cdot)}{\partial y_i\partial y_j} - rg(\cdot)\Big)\\ =& e^{-rt}(\mathscr{L}_Y F(\cdot) - rF(\cdot)).\end{aligned}$$

From Dynkin's formula (see, for example, Øksendal, 2000) it then follows that[3]

$$\begin{aligned}\mathsf{E}_y\left[e^{-r\hat{\tau}(\hat{y})}\hat{\varphi}(Y_{\hat{\tau}(\hat{y})})\right] =& \hat{\varphi}(y) + +\mathsf{E}_y\left[\int_0^{\hat{\tau}(\hat{y})} e^{-rt}\left(\mathscr{L}_Y\hat{\varphi}(Y_t) - r\hat{\varphi}(Y_t)\right)dt\right]\\ =& \hat{\varphi}(y).\end{aligned}$$

The a.s. continuity of sample paths of $(Y_t)_{t\geq 0}$ then implies that

$$\mathsf{E}_y\left[e^{-r\hat{\tau}(\hat{y})}\hat{\varphi}(Y_{\hat{\tau}(\hat{y})})\right] = \mathsf{E}_y\left[e^{-r\hat{\tau}(\hat{y})}\right]\hat{\varphi}(\hat{y}),$$

from which the proposition follows immediately. ■

This proposition implies that the problem (4) can be written as

$$F^*(y) = \hat{\varphi}(y)\sup_{\hat{y}\in E}\frac{F(\hat{y})}{\hat{\varphi}(\hat{y})},$$

so that solving the optimal stopping problem reduces to maximizing the function $\hat{\varphi} := F/\hat{\varphi}$. The main result is summarized in the following proposition.

[3]In most formulations of Dynkin's formula a condition for its application is that $\tau < \infty$, P_y-a.s. to ensure that the integral is finite. Here this condition is irrelevant because the integrand is zero.

Proposition 3. *Suppose that there exists an increasing and convex function $\hat{\varphi} \in C^2$, with $\hat{\varphi}(a) = 0$, which solves $\mathscr{L}_Y \hat{\varphi} = r\hat{\varphi}$. Suppose, in addition, that there exists $y^* \in (\bar{y}, b)$ such that*

$$\hat{\varphi}(y^*)F'(y^*) = \hat{\varphi}'(y^*)F(y^*). \tag{5}$$

Finally, assume that $\hat{\varphi}$ is more convex than F, i.e. that

$$\frac{F''(y)}{F'(y)} < \frac{\hat{\varphi}''(y)}{\hat{\varphi}'(y)}, \quad \textit{on } (a, b). \tag{6}$$

Then the optimal stopping problem (4) *is solved by*

$$F^*(y) = \begin{cases} \frac{\hat{\varphi}(y)}{\hat{\varphi}(y^*)} F(y^*) & \textit{if } y < y^*, \\ F(y) & \textit{if } y \geq y^*, \end{cases} \tag{7}$$

and $\tau^* = \hat{\tau}(y^*)$.

Proof. It first shown that y^* is the unique maximizer of the function $f : E \to \mathbb{R}$, defined by $f(y) = F(y)/\hat{\varphi}(y)$. Since the domain is open, any maximum must occur at an interior location. It is easily checked that

$$f'(y) = 0 \iff \hat{\varphi}(y)F'(y) = \hat{\varphi}'(y)F(y).$$

Let $\hat{y}$ be such that $f'(\hat{y}) = 0$. Then (6) implies that

$$\frac{F''(\hat{y})}{\hat{\varphi}''(\hat{y})} < \frac{F(\hat{y})}{\hat{\varphi}(\hat{y})} = \frac{F'(\hat{y})}{\hat{\varphi}'(\hat{y})},$$

which implies that $f''(\hat{y}) < 0$. So, any solution to $f'(y) = 0$ is a maximum location. But then, y^* must be unique.

The proposition is now proved using Øksendal (2000, Theorem 10.4.1). Note that the only conditions to check are[4]

1. $F^* \geq F$ (F^* dominates F),
2. $F^* \in C^1$ (F^* is smooth),
3. $\mathscr{L}_Y F^* - rF^* = 0$ on (a, y^*) (F^* is minimal),
4. $\mathscr{L}_Y F^* - rF^* \leq 0$ on (y^*, b) (F^* is superharmonic),
5. the family $\{ F^*(Y_\tau) \mid \tau \leq \tau_C \}$ is uniformly integrable with respect to P_y for all $y \in E$, where $\tau_C = \inf \{ t \geq 0 \mid Y_t \notin (a, y^*) \}$.

[4]Øksendal (2000, Theorem 10.4.1) lists more conditions, which are all trivially satisfied.

Typically, only the second and third conditions are checked in applied papers. The other conditions, however, are important to guarantee optimality of a proposed solution.

1. On $(a,\bar{y}]\cup[y^*,b)$ it holds trivially that $F^*\geq F$. Suppose that $F^*\geq F$ does not hold on $(\bar{y},y^*)$. Then there exists $\hat{y}$ such that

$$F^*(\hat{y})=\frac{\hat{\varphi}(\hat{y})}{\hat{\varphi}(y^*)}F(y^*)<F(\hat{y})\iff\frac{F(\hat{y})}{\hat{\varphi}(\hat{y})}>\frac{F(y^*)}{\hat{\varphi}(y^*)}.$$

This contradicts the fact that y^* is the unique maximizer of $F/\hat{\varphi}$.

2. Continuity of F^* is obvious. Differentiability at y^* follows from

$$\lim_{y\uparrow y^*}F^*(y)=\lim_{y\uparrow y^*}\hat{\varphi}'(y)\frac{F(y^*)}{\hat{\varphi}(y^*)}\overset{(*)}{=}\lim_{y\uparrow y^*}\hat{\varphi}'(y)\frac{F'(y^*)}{\hat{\varphi}'(y^*)}=F'(y^*),$$

where $(*)$ follows because y^* solves (5).

3. F^* is minimal, because on (a,y^*)it holds that

$$\mathscr{L}_YF^*-rF^*=\frac{F(y^*)}{\hat{\varphi}(y^*)}\left(\mathscr{L}_Y\hat{\varphi}-r\hat{\varphi}\right)=0.$$

4. F^* is superharmonic on (y^*,b), because

$$\begin{aligned}\mathscr{L}_YF^*-rF^*=&\mathscr{L}_YF-rF\\=&\frac{1}{2}\sigma^2(\cdot)F''+\mu(\cdot)F'-rF\\=&\frac{1}{2}\sigma^2(\cdot)\hat{\varphi}''\frac{F''}{\hat{\varphi}''}+\mu(\cdot)\hat{\varphi}'\frac{F'}{\hat{\varphi}'}-r\hat{\varphi}\frac{F}{\hat{\varphi}}\\=&\frac{1}{2}\sigma^2(\cdot)\hat{\varphi}''\frac{F''}{\hat{\varphi}''}+\frac{F'}{\hat{\varphi}'}\left[r\hat{\varphi}-\frac{1}{2}\sigma^2(\cdot)\hat{\varphi}''\right]-r\hat{\varphi}\frac{F}{\hat{\varphi}}\\=&\frac{1}{2}\sigma^2(\cdot)\hat{\varphi}''\left[\frac{F''}{\hat{\varphi}''}-\frac{F'}{\hat{\varphi}'}\right]+r\hat{\varphi}\left[\frac{F'}{\hat{\varphi}'}-\frac{F}{\hat{\varphi}}\right]\\&<0.\end{aligned}$$

The final inequality follows from (6) and from the fact that, since y^* is the unique maximizer of f it holds that $F'/\hat{\varphi}'<F/\hat{\varphi}$ on $[y^*,b)$.

5. Consider the function $g:[0,\infty)\to[0,\infty)$, defined by $g(x)=x^2$. Then g is increasing and convex on $[0,\infty)$ and $\lim_{x\to\infty}\frac{g(x)}{x}=\infty$, so that g is a *uniform integrability test function.* Since

$$\sup_{\tau\leq\tau_C}\left\{\int g\left(|F^*(Y_\tau)|\right)d\mathsf{P}_y\right\}=\sup_{\tau\leq\tau_C}\mathsf{E}_y\left[F^*(Y_\tau)^2\right]\leq\mathsf{E}_y\left[F^*(y^*)\right]<\infty,$$

the family $\{ F^*(Y_\tau) \mid \tau \leq \tau_C \}$ is uniformly integrable (cf. Øksendal, 2000, Theorem C.3). ■

If (5) has no solution, then it is never optimal to stop. Note that a sufficient condition for (6) is that $F'' \leq 0$. So, a concave NPV function ensures optimality of the optimal stopping trigger, provided it exists in the first place. The proof of the theorem relies on the following result, which is of interest in its own right.

Corollary 1. *The optimal stopping threshold y^* is the unique maximizer of the function $F/\hat{\varphi}$ on $[\bar{y}, y^*)$.*

In most of the literature on investment under uncertainty, following Dixit and Pindyck (1994), it is only checked that $\mathscr{L}_Y F^* = rF^*$ on C and that F^* is C^1. To see that this is not enough consider the following example.

Example 5. Suppose that $(Y_t)_{t\geq 0}$ follows a GBM

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dB_t,$$

on the state space $E = (0, \infty)$ and that the NPV of the project is given by

$$F(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} Y_t^\alpha dt \right] - I = \frac{y^\alpha}{r - \alpha(\mu + .5(\alpha - 1)\sigma^2)} - I,$$

where $\alpha > 0$ and $r > \alpha(\mu + .5(\alpha - 1)\sigma^2)$. Note that when $\alpha = 1$ this model reduces to the basic model discussed in Dixit and Pindyck (1994, Chapter 6).

The solution to $\mathscr{L}_Y \varphi = r\varphi$ is given by

$$\varphi(y) = Ay^{\beta_1} + By^{\beta_2},$$

where $\beta_1 > 0$ and $\beta_2 < 0$ are the roots of the quadratic equation

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \mu\beta - r = 0,$$

and A and B are arbitrary constants. We need the following restrictions on φ:

1. $\varphi(0) = 0$ implies that $B = 0$;

2. $\varphi' > 0$ is satisfied if $A > 0$;

3. $\varphi'' > 0$ is satisfied if $\beta_1 > 1$, i.e. if $r > \mu$.

We now find the optimal stopping trigger by solving (5):

$$y^* = \left(\frac{\beta_1}{\beta_1 - \alpha} [r - \alpha(\mu + .5(\alpha - 1)\sigma^2)] I \right)^{1/\alpha}.$$

This leads to a unique solution of the optimal stopping problem (4)

$$F^*(y) = \begin{cases} \left(\frac{y}{y^*}\right)^{\beta_1} \frac{\alpha}{\beta_1-\alpha} I & \text{if } y < y^*, \\ \frac{y^\alpha}{r-\alpha(\mu+.5(\alpha-1)\sigma^2)} - I & \text{if } y \geq y^*, \end{cases}$$

and $\tau^* = \hat{\tau}(y^*)$.

In order to make sure that y^* is a maximizer and that F^* is superharmonic we need to check (6). This condition is easily verified to hold iff $\alpha < \beta_1$. These conditions show that F^* can only be superharmonic if the NPV does not increase faster than the value of waiting, i.e. if φ is more convex than F. $\triangleleft$

3.1.2 Problems with Decreasing Payoff Functions

Suppose now that $F' < 0$, $F(a) > 0$, and $F(b) < 0$, so that there exists a unique $\bar{y} \in E$ such that $F(\bar{y}) = 0$. This represents a situation where it will be optimal to stop when Y gets small, rather than large. Fix $y \in E$. For any $y^* \leq y$, denote

$$\check{\tau}(y^*) = \inf\{\, t \geq 0 \mid Y_t \leq y^* \,\}, \quad \text{and} \quad \check{\nu}_y(y^*) = \mathsf{E}_y\left[e^{-r\check{\tau}(y^*)}\right].$$

The optimal stopping problem is again

$$F^*(y) = \sup_{\tau \in \mathcal{M}} \mathsf{E}_y\left[e^{-r\tau} F(Y_\tau)\right]. \tag{8}$$

As in the case where $F' > 0$, the optimal policy is to stop as soon as a certain threshold y^* is reached. This time, however, the threshold is reached from *above,* i.e. the continuation region is of the form $(y^*, b) \supset (\bar{y}, b)$.

Proposition 4. *The continuation set D is a connected set with $[\bar{y}, b) \subset D$.*

The proof is identical to that of Proposition 1 and is, therefore, omitted.

So, the optimal stopping problem (8) can be written as

$$F^*(y) = \sup_{\hat{y} \in E} \check{\nu}_y(\hat{y}) F(\hat{y}).$$

Sufficient conditions for the existence of a solution of the optimal stopping problem (8) now follow immediately.

Proposition 5. *Suppose that there exists a decreasing and convex function $\check{\varphi} \in C^2$, with $\check{\varphi}(b) = 0$, which solves $\mathscr{L}_Y \check{\varphi} = r\check{\varphi}$. Suppose, in addition, that there exists $y^* \in (\bar{y}, b)$ such that*

$$\check{\varphi}(y^*) F'(y^*) = \check{\varphi}'(y^*) F(y^*). \tag{9}$$

Finally, assume that $\check{\varphi}$ is more convex than F, i.e.

$$\frac{F''(y)}{F'(y)} > \frac{\check{\varphi}''(y)}{\check{\varphi}'(y)}, \quad \textit{on } (a,b). \tag{10}$$

Then the optimal stopping problem (8) *is solved by*

$$F^*(y) = \begin{cases} F(y) & \textit{if } y \leq y^*, \\ \frac{\check{\varphi}(y)}{\check{\varphi}(y^*)} F(y^*) & \textit{if } y > y^*, \end{cases} \tag{11}$$

and $\tau^* = \check{\tau}(y^*)$.

3.2 Two-Sided Optimal Stopping Problems

Suppose that the payoff function is given by a function F, which is non-increasing on $(a, \bar{y})$ and non-decreasing on $(\bar{y}, b)$, for some unique $\bar{y} \in E$, with $F(a) > F(\bar{y})$ and $F(b) > F(\bar{y})$. Assume that $F \in C^2(E \setminus \{\bar{y}\}) \cap C^1(E \setminus \{\bar{y}\}) \cap C(E)$. The function F can, therefore, be thought of as

$$F(y) = \begin{cases} F_L(y) & \text{if } y \leq \bar{y} \\ F_H(y) & \text{if } y \geq \bar{y}, \end{cases}$$

with $F_L \in C^2(a, \bar{y})$, $F_L' \leq 0$ and $F_H \in C^2(\bar{y}, b)$, $F_H' \geq 0$, with $F_L(\bar{y}) = F_H(\bar{y})$. If the DM stops at a time τ when $Y_\tau < \bar{y}$ ($Y_\tau > \bar{y}$) I refer to this as the *abandonment* (*investment*) decision, in line with the economic situation described in Example 3.

The decision-maker discounts revenues at a constant and deterministic rate $r > 0$ and wishes to find a function F^* and a stopping time τ^* to solve the optimal stopping problem

$$F^*(y) := \mathsf{E}_y \left[e^{-r\tau^*} F(Y_{\tau^*}) \right] = \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right], \tag{12}$$

where $\mathcal{M}$ is the set of stopping times.

If the optimal stopping problem (12) has a solution of the trigger type then one would expect, in analogy with Section 3.1, that (12) can be written as

$$F^*(y) = \sup_{Y_L < \bar{y} < Y_H} \mathsf{E}_y \left[e^{-r\tau^*} \left(1_{(\check{\tau}(Y_L) < \hat{\tau}(Y_H))} F_L(Y_L) + 1_{(\check{\tau}(Y_L) > \hat{t}(Y_H))} F_H(Y_H) \right) \right],$$

where $\tau^* := \inf \{ t \geq 0 \mid Y_t \notin (Y_L, Y_H) \}$. Denoting

$$\check{\nu}_y(Y_L, Y_H) = \mathsf{E}_y \Big[e^{-r\check{\tau}(Y_L)} | \check{\tau}(Y_L) < \hat{\tau}(Y_H) \Big] \mathsf{P}_y \left(\check{\tau}(Y_L) < \hat{\tau}(Y_H) \right), \quad \text{and}$$
$$\hat{\nu}_y(Y_L, Y_H) = \mathsf{E}_y \Big[e^{-r\hat{\tau}(Y_H)} | \hat{\tau}(Y_H) < \check{\tau}(Y_L) \Big] \mathsf{P}_y \left(\hat{\tau}(Y_H) < \check{\tau}(Y_L) \right),$$

this can be rewritten as

$$F^*(y) = \sup_{Y_L < \bar{y} < Y_H} \check{\nu}_y(Y_L, Y_H) F_L(Y_L) + \hat{\nu}_y(Y_L, Y_H) F_H(Y_H). \tag{13}$$

The first task in solving this program is to determine $\hat{\nu}_y(Y_L, Y_H)$ and $\check{\nu}_y(Y_L, Y_H)$. Fix $Y_L, Y_H \in E$, such that $Y_L < Y_H$.

Proposition 6. *Suppose that*

1. *there exists an increasing function $\hat{\varphi} \in C^2$ such that $\mathscr{L}_Y \hat{\varphi} = r\hat{\varphi}$, with $\hat{\varphi}(Y_L) = 0$, and*

2. *there exists a decreasing function $\check{\varphi} \in C^2$ such that $\mathscr{L}_Y \check{\varphi} = r\check{\varphi}$, with $\check{\varphi}(Y_H) = 0$.*

Then for any $y \in (Y_L, Y_H)$

$$\hat{\nu}_y(Y_L, Y_H) = \frac{\hat{\varphi}(y)}{\hat{\varphi}(Y_H)}, \quad \textit{and} \quad \check{\nu}_y(Y_L, Y_H) = \frac{\check{\varphi}(y)}{\check{\varphi}(Y_L)}.$$

Note that $\hat{\varphi}$ will depend explicitly on Y_L due to the boundary condition $\hat{\varphi}(Y_L) = 0$, while $\check{\varphi}$ depends on Y_H.

Proof. Let $\tau = \hat{\tau}(Y_H) \wedge \check{\tau}(Y_L)$ and take any solution $\varphi \in C^2$ to $\mathscr{L}_Y \varphi = r\varphi$. Then

$$\begin{aligned}
\mathsf{E}_y\left[e^{-r\tau}\varphi(Y_\tau)\right] =& \mathsf{E}_y\left[e^{-r\check{\tau}(Y_L)} | \check{\tau}(Y_L) < \hat{\tau}(Y_H)\right] \mathsf{P}_y\left(\check{\tau}(Y_L) < \hat{\tau}(Y_H)\right) \varphi(Y_H) \\
&+ \mathsf{E}_y\left[e^{-r\hat{\tau}(Y_H)} | \hat{\tau}(Y_H) < \check{\tau}(Y_L)\right] \mathsf{P}_y\left(\hat{\tau}(Y_H) < \check{\tau}(Y_L)\right) \varphi(Y_L) \\
=& \hat{\nu}_y(Y_L, Y_H)\varphi(Y_H) + \check{\nu}_y(Y_L, Y_H)\varphi(Y_L).
\end{aligned}$$

An application of Dynkin's formula then gives that

$$\hat{\nu}_y(Y_L, Y_H)\varphi(Y_H) + \check{\nu}_y(Y_L, Y_H)\varphi(Y_L) = \varphi(y).$$

The result now follows immediately by plugging in $\hat{\varphi}$ and $\check{\varphi}$, respectively. ∎

Example 6. Suppose that $(Y_t)_{t \geq 0}$ follows a GBM

$$dY_t = \mu Y_t dt + \sigma Y_t dB.$$

The general solution to $\mathscr{L}_Y \varphi - r\varphi = 0$ is given by

$$\varphi(y) = Ay^{\beta_1} + By^{\beta_2},$$

for constants A and B. Since we need that $\hat{\varphi}(Y_L) = 0$, it follows that

$$B = -AY_L^{\beta_1 - \beta_2}.$$

Therefore,

$$\hat{\varphi}(y) = A\left(y^{\beta_1} - Y_L^{\beta_1-\beta_2} y^{\beta_2}\right),$$

which implies that

$$\hat{\nu}_y(Y_L, Y_H) = \frac{\hat{\varphi}(y)}{\hat{\varphi}(Y_H)} = \frac{y^{\beta_1} - Y_L^{\beta_1-\beta_2} y^{\beta_2}}{Y_H^{\beta_1} - Y_L^{\beta_1-\beta_2} Y_H^{\beta_2}} = \frac{y^{\beta_1} Y_L^{\beta_2} - Y_L^{\beta_1} y^{\beta_2}}{Y_H^{\beta_1} Y_L^{\beta_2} - Y_L^{\beta_1} Y_H^{\beta_2}}.$$

A similar analysis gives that

$$\check{\nu}_y(Y_L, Y_H) = \frac{Y_H^{\beta_1} y^{\beta_2} - y^{\beta_1} Y_H^{\beta_2}}{Y_H^{\beta_1} Y_L^{\beta_2} - Y_L^{\beta_1} Y_H^{\beta_2}}.$$

◁

Note that $\hat{\varphi}$ depends on Y_L and $\check{\varphi}$ depends on Y_H, i.e. $\hat{\varphi}(y; Y_L)$ and $\check{\varphi}(y; Y_H)$, respectively. Under the assumptions of Proposition 6 the first-order condition of the program (13) with respect to Y_H reads

$$\begin{aligned} &-\frac{\hat{\varphi}(y; Y_L)\hat{\varphi}_1'(Y_H; Y_L)}{\hat{\varphi}(Y_H; Y_L)^2} F_H(Y_H) + \frac{\hat{\varphi}(y; Y_L)}{\varphi(Y_H\hat{;} Y_L)} F_H'(Y_H) \\ &+ \frac{\check{\varphi}(Y_L; Y_H)\check{\varphi}_2'(y; Y_H) - \check{\varphi}(y; Y_H)\check{\varphi}_2'(Y_L; Y_H)}{\check{\varphi}(Y_L; Y_H)^2} F_L(Y_L) = 0. \end{aligned}$$

It turns out that this first-order condition only needs to be satisfied at Y_H, i.e.

$$-\frac{\hat{\varphi}_1'(Y_H; Y_L)}{\hat{\varphi}(Y_H; Y_L)} F_H(Y_H) + F_H'(Y_H) + \frac{\check{\varphi}_2'(Y_H; Y_H)}{\check{\varphi}(Y_L; Y_H)} F_L(Y_L) = 0,$$

Since $\check{\varphi}(Y_H) = 0$, it holds that $\check{\varphi}_2'(Y_H; Y_H) = -\check{\varphi}_1'(Y_H; Y_H)$, so that the above equation can be rewritten as

$$-\frac{\hat{\varphi}_1'(Y_H; Y_L)}{\hat{\varphi}(Y_H; Y_L)} F_H(Y_H) + F_H'(Y_H) - \frac{\check{\varphi}_1'(Y_H; Y_H)}{\check{\varphi}(Y_L; Y_H)} F_L(Y_L) = 0, \tag{14}$$

A similar reasoning gives the first-order condition for Y_L:

$$-\frac{\check{\varphi}_1'(Y_L; Y_H)}{\check{\varphi}(Y_L; Y_H)} F_L(Y_L) + F_L'(Y_L) - \frac{\hat{\varphi}_1'(Y_L; Y_L)}{\hat{\varphi}(Y_H; Y_L)} F_H(Y_H) = 0. \tag{15}$$

The following proposition can now be proved.

Proposition 7. *Suppose that there exists a pair* (Y_L, Y_H)*, with* $a < Y_L < \bar{y} < Y_H < b$*, such that* Y_L *and* Y_H *solve* (14) *and* (15)*, and that the assumptions of Proposition 6 hold. Define the function* $\varphi \in C^2$ *by*

$$\varphi(y) = \frac{\check{\varphi}(y)}{\check{\varphi}(Y_L)} F_L(Y_L) + \frac{\hat{\varphi}(y)}{\hat{\varphi}(Y_H)} F_H(Y_H).$$

Let $\hat{y} \in (Y_L, Y_H)$ *be the unique point such that* φ *is decreasing on* $(a, \hat{y})$ *and increasing on* $(\hat{y}, b)$*. Suppose, in addition, that*

1. *φ is strictly convex,*

2. *φ is more convex than F_H on $(\hat{y}, b)$, i.e. $\frac{F''_H(y)}{F'_H(y)} < \frac{\varphi''(y)}{\varphi'(y)}$, $\hat{y} < y < b$, and*

3. *φ is more convex than F_L on $(a, \hat{y})$, i.e. $\frac{F''_L(y)}{F'_L(y)} > \frac{\varphi''(y)}{\varphi'(y)}$, $a < y < \hat{y}$.*

Then a solution to (12) *is given by*

$$F^*(y) = \begin{cases} F_L(y) & \textit{if } y \leq Y_L \\ \varphi(y) & \textit{if } Y_L < y < Y_H \\ F_H(y) & \textit{if } y \geq Y_H, \end{cases}$$

and $\tau^* = \hat{\tau}(Y_H) \wedge \check{\tau}(Y_L)$.

Proof. Note that φ can be written as

$$\varphi = A\hat{\varphi} + B\check{\varphi},$$

where $A = F_H(Y_H)/\hat{\varphi}(Y_H) > 0$ and $B = F_L(Y_L)/\check{\varphi}(Y_L) > 0$ are constants. It will be shown below that $\varphi'(Y_L) = F'_L(Y_L)$ and $\varphi'(Y_H) = F'_H(Y_H)$, which implies that $\varphi'(Y_L) < 0$ and $\varphi'(Y_H) > 0$. Since φ is strictly convex, there exists a unique $\hat{y} \in (Y_L, Y_H)$ such that φ is decreasing on $(a, \hat{y})$ and increasing on $(\hat{y}, b)$.

We first show that Y_L and Y_H maximize the functions

$$f_L = \frac{F_L}{\varphi}, \quad \text{and} \quad f_H = \frac{F_H}{\varphi},$$

on $(a, \hat{y})$ and $(\hat{y}, b)$, respectively. First consider f_L on (a, Y_L). It is easily obtained that

$$\begin{aligned} f'_L &= \frac{\varphi F'_L - F_L \varphi'}{\varphi^2} \\ &\propto [A\hat{\varphi} + B\check{\varphi}]F'_L - F_L[A\hat{\varphi}' + B\check{\varphi}']. \end{aligned}$$

Since $\hat{\varphi}(Y_L) = 0$, it holds that

$$\begin{aligned} f'_l(Y_L) &= B\check{\varphi}(Y_L)\left[F'_L(Y_L) - \frac{A}{B}\frac{F_L(Y_L)}{\check{\varphi}(Y_L)}\hat{\varphi}'(Y_L) - \frac{\check{\varphi}'(Y_L)}{\check{\varphi}(Y_L)}F_L(Y_L)\right] \\ &= B\check{\varphi}(Y_L)\left[F'_L(Y_L) - \frac{\hat{\varphi}'(Y_L)}{\hat{\varphi}(Y_H)}F_H(Y_H) - \frac{\check{\varphi}'(Y_L)}{\check{\varphi}(Y_L)}F_L(Y_L)\right] \\ &= 0. \end{aligned}$$

Suppose that $y_0 \in (a, \hat{y})$ is such that $f'_L(y_0) = 0$. Note that

$$f''_L = \frac{\varphi F''_L - F_L \varphi''}{\varphi^2} - 2\frac{\varphi'}{\varphi}f'_L.$$

Therefore,

$$\begin{aligned} f_L''(y_0) &\propto \varphi(y_0)F_L''(y_0) - F_L(y_0)\varphi''(y_0) \\ &= \varphi(y_0)\varphi''(y_0)\left[\frac{F_L''(y_0)}{\varphi''(y_0)} - \frac{F_L(y_0)}{\varphi(y_0)}\right] = \varphi(y_0)\varphi''(y_0)\left[\frac{F_L''(y_0)}{\varphi''(y_0)} - \frac{F_L'(y_0)}{\varphi'(y_0)}\right] \\ &= \varphi(y_0)F_L'(y_0)\left[\frac{F_L''(y_0)}{F_L'(y_0)} - \frac{\varphi''(y_0)}{\varphi'(y_0)}\right] < 0. \end{aligned}$$

So, any solution to $f_L'(y) = 0$ is a maximum location and, hence, Y_L is the unique maximum location on $(a, \hat{y})$.

A similar procedure establishes that Y_H is the unique maximum location of f_H on $(\hat{y}, b)$.

Following the proof of Proposition 3, we check that

1. $F^* \geq F$,
2. $F^* \in C^1$,
3. $\mathscr{L}_Y F^* - rF^* = 0$ on (Y_L, Y_H),
4. $\mathscr{L}_Y F^* - rF^* \leq 0$ on $(a, Y_L] \cup [Y_H, b)$.

1. On $(Y_L, \hat{y})$ it holds that F^* is decreasing. In addition, φ is more convex than F_L and, as will be established below, $F_L(Y_L) = \varphi(Y_L)$. Therefore, $F^* \geq F_L$ on $(a, \hat{y})$. In case $\hat{y} < \bar{y}$, it obviously holds that $F^* \geq F_L$, because φ is increasing on $[\hat{y}, \bar{y})$. A similar argument establishes that $F^* \geq F_H$ on $(\bar{y}, b)$.

2. It is obvious that $F^* \in C$. To show differentiability, observe that (14) implies that

$$\lim_{y \downarrow Y_L} \varphi'(y) = \frac{\hat{\varphi}'(Y_L)}{\hat{\varphi}(Y_H)} F_H(Y_H) + \frac{\check{\varphi}'(Y_L)}{\check{\varphi}(Y_L)} F_L(Y_L) = F_L'(Y_L),$$

recalling that $\hat{\varphi}(Y_L) = 0$. A similar argument shows that $\varphi'(Y_H) = F_H'(Y_H)$.

3. This follows by construction.

4. In the same way as in the proof of Proposition 3 we can derive that on $(a, Y_L]$ it holds that

$$\mathscr{L}_Y F^* - rF^* = \frac{1}{2}\sigma^2(\cdot)\varphi''\left[\frac{F_L''}{\varphi''} - \frac{F_L'}{\varphi'}\right] + r\varphi\left[\frac{F_L'}{\varphi'} - \frac{F_L}{\varphi}\right].$$

The first term in square brackets is non-positive by assumption. The second term in square brackets is non-positive because Y_L is a maximum location of f_L and, therefore, $f_L' > 0$ on (a, Y_L). A similar argument establishes that $\mathscr{L}_Y F^* - rF^* \leq 0$ on (Y_H, b). ■

4 An Illustration: Bayesian Sequential Testing of Statistical Hypotheses

As an application of Proposition 7, we consider a decision-theoretic approach to the problem of sequential hypothesis testing. The probabilistic set-up follows Shiryaev (1978). That is, we consider a measurable space $(\Omega, \mathscr{F})$ and two probability measures P_0 and P_1 on $(\Omega, \mathscr{F})$. The observed process $(X_t)_{t\geq 0}$ follows an arithmetic Brownian motion

$$dX_t = \sigma dB_t, \quad \text{or} \quad dX_t = \mu dt + \sigma dB_t,$$

under P_0 and P_1, respectively, where $(B_t)_{t\geq 0}$ is a Wiener process. From Girsanov's theorem it follows that P_0 and P_1 are equivalent probability measures, so that the P_0-null sets and the P_1-null sets coincide. The filtration that we use is the one generated by $(X_t)_{t\geq 0}$, augmented with the P_0-null sets, and is denoted by $\left(\mathscr{F}_t^X\right)_{t\geq 0}$.

The problem can be thought of as a sequential hypothesis testing problem $H_0 : \theta = 0$ vs $H_1 : \theta = 1$, where the observed signal follows the SDE

$$dX_t = \theta \mu dt + \sigma dB_t.$$

We use a Bayesian approach and, thus, treat θ as a random variable. Let $p \in (0, 1)$ play the role of *prior probability* for the event $\{\theta = 1\}$. Given $p \in (0, 1)$, define the (equivalent) probability measure

$$\mathsf{P}_p = p\mathsf{P}_1 + (1 - p)\mathsf{P}_0.$$

For $p \in [0, 1]$ the *conditional measure* of P_p at time $t \geq 0$ is denoted by $\mathsf{P}_p|\mathscr{F}_t^X$.

Now define the *likelihood ratio process* $(Y_t)_{t\geq 0}$ as the Radon-Nikodym derivative

$$Y_t = \frac{d\mathsf{P}_1|\mathscr{F}_t^X}{d\mathsf{P}_0|\mathscr{F}_t^X}.$$

The likelihood ratio measures the relative evidence that $(X_t)_{t\geq 0}$ has provided up to time t in favour of H_1 over H_0. It can be shown that $(Y_t)_{t\geq 0}$ follows the geometric Brownian motion

$$dY_t = \frac{\mu}{\sigma} Y_t dB_t. \tag{16}$$

Using Bayes' rule it can be shown (Shiryaev, 1978) that the *posterior probability* of $\{\theta = 1\}$, π_t, follows the diffusion

$$d\pi_t = \frac{\mu}{\sigma} \pi_t (1 - \pi_t) d\bar{B}_t, \tag{17}$$

where $\left(\bar{B}_t\right)_{t\geq 0}$ is a Wiener process.

Suppose that the observations $(X_t)_{t\geq 0}$ follow from research conducted by a firm into the feasibility of, say, marketing a new product. Here $\{\theta = 1\}$ represents the event where the new product is profitable, whereas $\{\theta = 0\}$ represents the event that it is not. The costs of conducting the research are c per unit of time. The firm needs to decide when to stop the research and, at that time, whether to invest in the new product or to abandon the project. Therefore, this is a two-sided optimal stopping problem. Suppose that (i) the profits of investing conditional on $\{\theta = 1\}$ are $P > 0$, (ii) that the losses of investing conditional on $\{\theta = 0\}$ (a Type I error) are $-L$, $L > 0$, and (iii) that the losses associated with abandoning conditional on $\{\theta = 1\}$ (a Type II error) are $-L$ as well. Note that in a standard statistical decision problem, losses are only attached to erroneous decisions. The sunk costs of investment are denoted by $I \geq 0$. The net present value (NPV) of investment, when the posterior belief in $\{\theta = 1\}$ is π then equals

$$F_I(\pi) = \pi P - (1 - \pi)L - I,$$

whereas the NPV of abandonment is equal to

$$F_A(\pi) = -\pi L.$$

Since

$$\pi_t \mapsto Y_t = \frac{\pi_t}{1 - \pi_t} \frac{1 - p}{p},$$

is one-to-one and onto the problem can be formulated both in terms of Y or in terms of π. Since $(Y_t)_{t\geq 0}$ follows a GBM it easier to formulate and solve the problem in terms of Y, leading to an upper and lower trigger, Y_H and Y_L, beyond which investment or abandonment is optimal, respectively. These triggers can then be transformed into triggers for the posterior belief in $\{\theta = 1\}$, which is easier to interpret. The NPVs, formulated as functions of y are given by

$$F_I(y) = \frac{\zeta y}{1 + \zeta y}(P + L) - (I + L), \quad \text{and} \quad F_A(y) = -\frac{\zeta y}{1 + \zeta y} L,$$

where $\zeta = p/(1 - p)$ is the *prior odds ratio*.

The firm's optimal stopping problem then becomes

$$\begin{aligned} F^*(y) &= \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[-c \int_0^\tau e^{-rt} dt + e^{-r\tau} \max(F_I(Y_\tau), F_A(Y_\tau)) \right] \\ &= -\frac{c}{r} + \sup_{\tau \in \mathcal{M}} \mathsf{E}_y \left[e^{-r\tau} F(Y_\tau) \right], \end{aligned}$$

where

$$F(y) = \begin{cases} F_H(y) := F_I(y) + \frac{c}{r} & \text{if } y \geq \bar{y} \\ F_L(y) := F_A(y) + \frac{c}{r} & \text{if } y < \bar{y}, \end{cases}$$

and

$$\bar{y} = \frac{I+L}{\zeta(P+L-I)},$$

is the unique point where $F_H = F_L$. Note that $F_H' > 0$ and $F_L' < 0$. Letting $\beta_1 > 1$ and $\beta_2 < 0$ denote the roots of the equation

$$\mathscr{Q}(\beta) = \frac{1}{2}\frac{\mu^2}{\sigma^2}\beta(\beta-1) - r = 0,$$

it follows that the increasing and decreasing solutions to $\mathscr{L}_Y\varphi = r\varphi$ are

$$\hat{\varphi}(y) = y^{\beta_1} - Y_L^{\beta_1-\beta_2}y^{\beta_2}, \quad \text{and} \quad \check{\varphi}(y) = y^{\beta_2} - Y_H^{\beta_2-\beta_1}y^{\beta_1}.$$

If the first-order conditions (14) and (15) have a solution (Y_L, Y_H), then the function φ can be written as

$$\varphi(y) = Ay^{\beta_1} + By^{\beta_2},$$

where

$$A = \frac{Y_L^{\beta_2}F_H(Y_H) - Y_H^{\beta_2}F_L(Y_L)}{Y_H^{\beta_1}Y_L^{\beta_2} - Y_L^{\beta_1}Y_H^{\beta_2}}, \quad \text{and} \quad B = \frac{Y_H^{\beta_1}F_L(Y_L) - Y_L^{\beta_1}F_H(Y_H)}{Y_H^{\beta_1}Y_L^{\beta_2} - Y_L^{\beta_1}Y_H^{\beta_2}}.$$

If φ is a convex function, then, since $F_H'' < 0$, it always holds that φ is more convex than F_H. However, since $F_L'' > 0$ there may be situations where F_L is more convex than φ. In such cases the value function is no longer superharmonic. This means that then the abandonment option has no value and that the firm should never exercise it. In such cases the investment problem can be solved using the theory presented in Section 3.1. If a solution exists, then the optimal decision time is $\tau^* = \inf\{t \geq 0 | Y_t \notin (Y_L, Y_H)\}$.

As a numerical illustration, consider the model with the parameter values $P = 10$, $I = 5$, $L = 8$, $c = 1$, $r = \mu = .1$, and $\sigma = .1$. Figure 1 shows the value functions and triggers (in terms of the likelihood ratio) for the prior $p = .5$. Note that $Y_0 = 1$, because at time $t = 0$ no evidence has been gathered yet so that $X_0 = 0$. In terms of the posterior probability of the event $\{\theta = 1\}$, the triggers can be found to be equal to $(.3279, ..6142)$. These triggers are the same for every value of the prior p, due to the Markovian structure of the problem.

What does change with the prior are the inferential properties of the optimal stopping rule. For example, it can be found that (Poor and Hadjiliadis, 2009) the (implied) probabilities of Type I and Type II errors are

$$\mathsf{P}_0(Y_{\tau^*} = Y_H) = \frac{1-Y_L}{Y_H - Y_L}, \quad \text{and} \quad \mathsf{P}_1(Y_{\tau^*} = Y_L) = Y_L\frac{Y_H - 1}{Y_H - Y_L},$$

respectively. Obviously, these probabilities are only meaningful when $p \in (\pi_L, \pi_H)$. For $p = .5$, the probabilities are .4637 and .2617, respectively. One has to be careful in interpreting these probabilities, because they reflect probabilities of reaching thresholds under different measures *at* $t = 0$ *only*. These

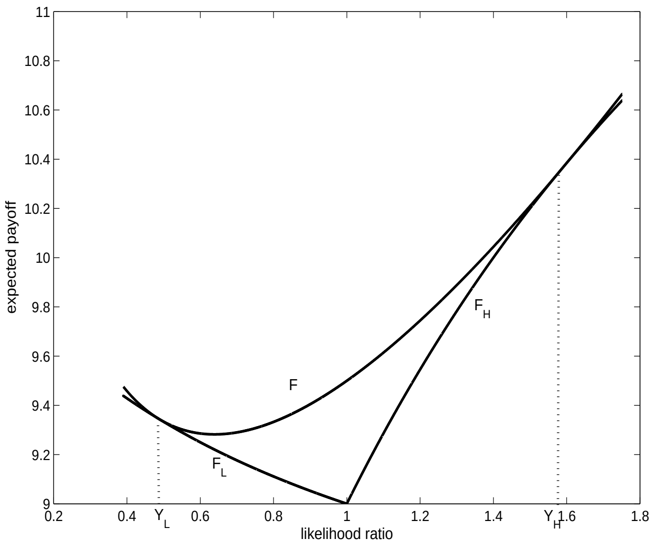

Figure 1: Value functions for different priors.

probabilities change when the posterior belief in $\{\theta = 1\}$ change, i.e. when evidence accrues. This is due to the Bayesian nature of the procedure. A frequentist error probabilities approach is possible but requires a slightly different set-up (see, for example, Shiryaev, 1978 or Poor and Hadjiliadis, 2009).

5 Concluding Remarks

Ever since the seminal contribution of Dixit and Pindyck (1994) the literature on investment under uncertainty solves optimal stopping problems by solving a Bellman equation and then to find a threshold that satisfies the value-matching and smooth-pasting conditions. This procedure, however, does not check the necessary conditions that the value function should dominate the payoff function and should be superharmonic.

This paper introduces easy-to-check sufficient conditions that allow for the solution to a wide variety of optimal stopping problems to be obtained in a straightforward way. Letting F denote the payoff function and φ a solution to the Bellman equation, it is shown that the solution F^* dominates F and is superharmonic if φ is more convex than F. In addition, the threshold beyond which stopping is optimal is obtained by solving the first-order condition of optimizing the function F/φ. Under the condition that φ is more convex than F this optimization problem has at most one solution, which is a maximum.

The approach presented here also brings some computational advantages because the number of equations to be solved is halved when compared to the standard value-matching/smooth-pasting approach. An example based on a Bayesian sequential hypothesis testing problem shows the applicability of the approach to a wide variety of problems.

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