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# The behavior of the hazard rate in the Gaussian structural default model under asymmetric information

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**Abstract**

This paper shows that the standard and deferred filtration structural models of corporate default are isomorphic, allowing the insights of the standard full information setting to be carried over to the more complex case of asymmetric information. It shows that the accounting lag, which provides a general indicator of uncertainty and opacity in the deferred filtration model, plays a role analogous to that of forward maturity in the standard model. The comparative static properties of the standard model carry over *mutatis mutandis* and can also be used to sign the effect of signals upon the effective accounting lag and drift parameters.

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## Part I

# Introduction

The structural approach to corporate default models the decision in terms of accounting information: the firm's management is assumed to observe the value of its assets and liabilities and to default when the net asset value attains a critical value. The basic Gaussian model was developed by Black and Cox (1976), and has been extensively used in studying optimal capital structure, default and the pricing of corporate securities. Important extensions include Leland and Toft (1996) who allow for strategic default behavior and Hackbarth and Morellec (2006) who allow the dynamics of the firm to depend upon macroeconomic conditions that can change regime. These models have a similar Gaussian structure which lends itself to closed form solutions that are easily understood in terms of their structural parameters. However, the firm's asset value follows a Brownian motion and is observed by investors, so default never comes as a surprise. The continuity of this process means that short spreads remain negligible until the asset value approaches the default boundary. For this reason, the reduced form hazard rate model is typically used in empirical research (Duffie and Singleton (2003)). Alternatively, jump processes can be added to the Brownian motion to make the default intensity and short spreads significant when the asset value is within range of the boundary. The Levy distribution can be used to analyze default intensity in this situation and has been used to develop structural default models (Baxter (2007)). Unfortunately, solutions for these non-Gaussian models are not available in closed form and so numerical approximations have to be employed when using them.

However, Duffie and Lando (2001) deal with these problems by assuming instead that the investment decision is conditioned by a 'deferred filtration'. Heuristically, investors observe a lagged information set, reflecting delays in financial reporting. This lag damps the effect of accounting information on market prices in the same way that maturity does in the forward market and allows room for other indicators to affect them. Investors do not know precisely how close the firm is to default, which therefore can come as a surprise. The Duffie and Lando (2001) model potentially provides a more realistic representation of the short risk spreads while preserving the tractability of the basic Gaussian framework.

Despite these advantages, the deferred filtration model has received very little attention in the literature. This may reflect the complicated mathematical structure of the conditional distributions used by Duffie and Lando to model prices. However, this paper shows that analyzing the model in terms of its default intensity makes it much easier to manipulate and understand. Indeed, it shows that the deferred filtration model and the standard full information model are isomorphic, sharing the same default mechanism and essentially the same default probability structure. That is because in the risk-neutral world used for asset pricing, spot prices evolve in line with the structure of forward prices if no 'news' arrives to perturb the system. Additional value signal can perturb the system, but this effect can be allowed for by linear change of variable techniques which preserve the default probability structure of the standard model.

The paper is set out along the following lines. The next section sets out the basic Black Cox (1976) structure, which is a simplified version of the Leland and Toft (1996) full information model employed by Duffie and Lando (2001) as their baseline. Section 3 establishes the basic isomorphism between the Black & Cox and Duffie & Lando models. I look at the simple case in which investors observe a lagged asset value and no additional value signals. I then show how additional signals affect the estimate of the distance to default and its variance, preserving the isomorphism. This updating mechanism is illustrated using data for the effect of the Lehman default on the survivorship values of other US investment banks. Finally I show how the general model encompasses a variety of models used in the default literature.

## 1 The model of Black and Cox (1976)

Gaussian structural default models assume that the logarithm $v = \ln V(t)$ of the value of the firm $V(t)$ follows a Brownian Motion under the risk neutral measure:

$$dv = \mu dt + \sigma dw. \tag{1}$$

where: $\mu = (\theta - \kappa)$; $\theta$ is the expected total logarithmic return on assets and $\kappa$ the percentage cash flow return to the equity owners. This makes $V(t)$ lognormal. The firm has perpetual debt with a face value of $L$. The full information variant of the model then follows from:

Assumption 1: All agents observe the value of the firm $V(t)$ at time $t$. Formally: Agents have the information filtration $\mathcal{F}_t$ generated by $V(t)$, where for each $t > 0$, $\mathcal{F}_t$ is the $\sigma-$algebra generated by $\{V(s) : 0 \leq s \leq t\}$.

The original model of Black and Cox (1976) is simpler than the model of Leland and Toft (1996) because they assume that there are no taxes and that legal restrictions prevent the firm trading with a negative net asset value. In this case, default occurs at time $\tau$ which is the first time that the net asset value $V(t)$ reaches $V_B = L$, or equivalently when the logarithm of the distance to default ratio $x(t) = \ln(V(t)/L)$ first reaches zero. The critical default value is a negative constant in the Leland and Toft (1996) model. Apart from that, the mathematical structure of their model is identical to that of Black and Cox.

In these Gaussian models, the probability $p$ of default during an investment period of length $t$ and a starting value of $x(0) = z$ is the probability of a first passage from $z$ to default at $x = 0$ during the period. The probability $s = (1 - p)$ of survival is:

$$s(z/\sigma, \mu/\sigma, t) = \Phi\left[\frac{z + \mu t}{\sigma\sqrt{t}}\right] - \exp\left[-\frac{2\mu z}{\sigma^2}\right]\Phi\left[\frac{-z + \mu t}{\sigma\sqrt{t}}\right] \geq 0; \quad t > 0. \qquad (2)$$

(Duffie and Singleton (2003)) where $\Phi[.]$ is the standard normal distribution function and $\phi[.]$ its density function:

$$\phi[x] = \left(\frac{1}{\sqrt{2\pi}}\right)\exp\left[-\frac{x^2}{2}\right].$$

The Gaussian structure allows the state and drift values to be standardized as $z/\sigma$ and $\mu/\sigma$. The forward default intensity, which plays the same role in the defaultable debt valuation as the forward rate does in non-defaultable valuation, follows by substituting the derivative of (2) into the definition (Duffie and Singleton (2003)):

$$\begin{aligned} f(z/\sigma, \mu/\sigma, t) &= -\frac{1}{s(z/\sigma, \mu/\sigma, t)}\frac{\partial s(z/\sigma, \mu/\sigma, t)}{\partial t} && (3) \\ &= \left(\frac{z}{\sigma t^{3/2} s(z/\sigma, \mu/\sigma, t))}\right)\phi\left[\frac{z + \mu t}{\sigma\sqrt{t}}\right] \quad t > 0; z \geq 0. && (4) \end{aligned}$$

The default arrival intensity or hazard rate $h(z/\sigma, \mu/\sigma) = f(z/\sigma, \mu/\sigma, 0)$ is the limit of the forward default intensity as the forward maturity goes to zero and is equivalent to the spot interest rate in the non-defaultable market. Duffie and Lando (2001) show that although this is identically zero for $z > 0$ in the standard model this is not the case in the deferred filtration model.

As we would expect, increases in the firm's distance to default ($z$) and its growth rate ($\mu$) reduce the forward default intensity and hence the value of redeemable debt:

$$\frac{\partial f}{\partial z} < 0; \quad \frac{\partial f}{\partial \mu} < 0. \tag{5}$$

(Spencer (2013a)). Note that the Gaussian structure means that the comparative statics of the model can be analyzed in terms of the risk and maturity-adjusted distance to default and growth variables $z/\sigma\sqrt{t}$ and $\mu\sqrt{t}/\sigma$. In other words, the effect of shocks in $z$ on the forward structure falls with forward maturity at the rate $t^{-\frac{1}{2}}$, while the effect of $\mu$ increases at the rate $t^{\frac{1}{2}}$. Black and Cox (1976) show that the effect of an increase in the forward maturity on the forward default intensity is ambiguous, depending critically upon the initial distance to default. Lengthening the forward maturity normally increases the forward default rate but reduces it if the initial distance to default is low. In that case the passage of time without default makes it likely that the firm has been able to rebuild its asset value, reducing the forward default intensity as maturity increases.

## 2 The Duffie and Lando (2001) deferred filtration model

Duffie and Lando (2001) maintain assumption 1 for the firm's manager, who declares bankruptcy when $x(t)$ attains zero ($V(t)$ reaches $V_B = L$), as in the standard model. They employ the event indicator $1_{\{\tau<s\}}$ which takes the value 1 at time $s$ if the firm has not defaulted, zero otherwise. However they assume:

Assumption 1′: At time $t$, investors observe a lagged accounting value and may receive subsequent value signals. Formally: investors have the deferred information filtration $\mathcal{H}_t$ defined in Duffie and Lando (2001) equation (14):

$$\mathcal{H}_t = \sigma(\{Y(t_1), ..., Y(t_n), 1_{\{\tau<s\}}\colon\ 0 \leq s \leq t\}).$$

where $Y(t_1), ..., Y(t_n)$ are value signals published at times $0 \leq t_1, ..., t_n \leq t$. These allow an estimate of the distance to default to be inferred. Importantly investors know whether the firm has defaulted or not ($1_{\{\tau<s\}}$), which is also informative.

Duffie and Lando show that a succession of informative signals allows investors to update the parameters describing the conditional distribution of the distance to default. They illustrate this using a basic example in which investors see a precise but lagged value observation $A(t) = V(0)$ followed by a second noisy value signal $y(t) = \ln Y(t)$. This is normally distributed and related to the true value $x(t)$ by:

$$y(t) = x(t) + u = z + \mu t + w(t) + u; \tag{6}$$
$$\text{where:} \qquad u \sim N(0, \alpha^2).$$

(given (1), $w(t) \sim N(0, \sigma^2 t)$). Appendix 1 establishes the following proposition:

*Proposition 1: In the Duffie and Lando model, the initial indicator $z$ and the signal $y$ can be combined into a composite indicator $\upsilon$ with a variance of $\sigma^2 l < \sigma^2 t$, allowing the hazard rate to be represented as:*

$$h(\upsilon/\sigma, \rho/\sigma, t) = \left(\frac{\upsilon}{\sigma l^{3/2} s(\upsilon/\sigma, \rho/\sigma, l)}\right) \phi\left[\frac{\upsilon + \rho l}{\sigma\sqrt{l}}\right] \quad ; \quad l > 0; z \geq 0 \tag{7}$$

*where $\upsilon$, $l$ and $\rho$ are signal-adjusted location, time and drift values and is $\chi$ a scale parameter, defined respectively as:*

$$\upsilon = \chi z \leq z; \quad l = \chi t \leq t. \tag{8}$$
$$\rho = \mu + \frac{\sigma^2}{\alpha^2} y, \quad \chi = \frac{\alpha^2}{\alpha^2 + \sigma^2 t} \leq 1.$$

This expression is isomorphic with the expression (4) for the forward default intensity in the standard model (7). The reason for this is apparent if the second signal is uninformative (the limit in which $\alpha$ tends to infinity). In this case $\chi = 1$, $\rho = \mu$, $\upsilon = z$; $l = t$ and:

$$h(z/\sigma, \mu/\sigma, t) = \left( \frac{z}{\sigma t^{3/2} s(z/\sigma, \mu/\sigma, t))} \right) \phi \left[ \frac{z + \mu t}{\sigma \sqrt{t}} \right] \quad t > 0; z \geq 0, \tag{9}$$

This formula is the same as that for the forward default intensity in (4), but in this context $t$ represents the length of the accounting lag rather than the forward maturity and $z$ the lagged rather than the current accounting value indicator. Heuristically, as noted in the introduction, in this case the deferred filtration has the effect of stopping the flow of news to investors and the hazard rate simply evolves in line with the forward rates established when the flow of new information ceased. I call this signal-free specification the basic deferred filtration (BDF) model. Obviously, in the limit as the information lag $t$ tends to zero, (9) converges on the hazard rate in the full information model: $h(z/\sigma, \mu/\sigma) = f(z/\sigma, \mu/\sigma, 0)$.

If on the other hand, the signal $Y(t)$ is informative, $z$ is simply replaced in (4) or (9) by the rescaled capital indicator: $\upsilon = \chi z$; $t$ by the rescaled accounting lag $l = \chi t$ and $\mu$ by the shifted drift parameter $\rho$. Since $\chi < 1$ this has the effect of shrinking both the distance to default and the accounting lag. Apart from these linear changes of variable the basic structure of the Gaussian default model is not affected.

## 2.1 Comparative static properties

The isomorphism between equations like (4) and (7) immediately allows us to analyze the comparative static properties of the model using those of the standard model. For example, (5) implies that the post-signal capital $\upsilon$ and drift $\rho$ parameters (like the initial capital and drift parameters $z$ and $\mu$ in the standard model) reduce the hazard rate in the deferred filtration model. Although the effect of the rescaled lag $l$ on the hazard rate (like $t$ in the standard model) remains ambiguous, (depending as we have seen upon the adjusted distance to default $\upsilon$), appendix 2 shows that

*Proposition 2: The the overall effect of a value signal on the scale parameter $\chi$ is negative:*

$$\frac{dh}{d\chi} < 0 \tag{10}$$

Since $\chi < 1$, this means that the scale effect of a signal increases the hazard rate. The updating equation (8) shows that the magnitude of this increase depends upon the share of $\alpha^2$ (the variance of of $x$ conditional upon both $z$ and $y$) in the total variance: $\alpha^2 + \sigma^2 t$. However, the scale effect could be offset by the shift effect of 'good' signal (defined as. a value $y > 0$) on $\rho$, which is positive. Using (5) and (8):

$$\frac{dh}{dy} = \frac{\partial h}{\partial \rho}\frac{\sigma^2}{\alpha^2} = \frac{\partial h}{\partial \mu}\frac{\sigma^2}{\alpha^2} < 0.$$

The magnitude of the shift effect depends upon $\sigma^2/\alpha^2$ : the precision of this signal relative to that of $z$. If the initial accounting information is relatively uninformative, a good signal will have a powerful effect, tending to reduce the hazard rate.

## 2.2 The Lehman default as an illustrative example

The recent financial crisis provides a good example of an additional value signal: the effect of the Lehman bankruptcy on 15 September 2008 on the position of similar financial firms. The deferred filtration model can be used to analyze the effect of this on hazard rates. Let $y$ represent the Lehman log distance to default and $z$ that of a similar firm and suppose that $\alpha$ reflects the correlation between their asset value diffusions. Because default occurs in this model when $y = 0$, there is no 'good signal' shift effect, only a scale effect which unambiguously increases the hazard rate of the other bank.

The two tables show the effect of this on default swap (CDS) spreads for the subordinated debt of the two surviving investment banks, Goldman Sachs and Morgan Stanley. It is convenient to use spreads from the CDS swap market because this circumvents the problem of specifying the tax regime (Houweling and Ton Vorst (2005)), consistent with the use of the Black-Cox rather than the more general Leland-Toft model. The CDS market was also very active over this period. The spread data is provided by Credit Market Analysis Ltd. and taken from Datastream. I use the difference between the end August and September observations to calibrate the spillover effect. I then back out the implied default probabilities $(1 - p_{t,t+m})$ and forward default probabilities $(p_{t,t+m} - p_{t,t+m-1})$ using standard recursion formulae (Hull (2003)).

What light does the deferred filtration model throw on these figures? Recall from the basic model of section 2 that the effect of the distance to default is negative and tends to weaken with forward maturity while that of the drift or growth factor is also negative but tends to become more powerful with maturity. Looking first at the structure of rates before the Lehman default, we see from the first line of the lower panel of these tables that the forward default probabilities are relatively flat for Goldmans and decline gradually for Morgan Stanley, suggesting that these influences broadly offset each other initially.

Table 1: The effect of the Lehman default on the market's assessment of the risk of a default by Goldman Sachs

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | CDS spread (% p.a.) | | | | | | |
| Date | 1 year | 2 year | 3 year | 4 year | 5 year | 6 year | 7 year |
| 31/8/2008 | 1.675 | 1.69 | 1.725 | 1.765 | 1.8 | 1.8 | 1.8 |
| 30/9/2008 | 5.935 | 5.24 | 5.075 | 4.84 | 4.775 | 4.712 | 4.65 |
| Increase | 4.26 | 3.55 | 3.35 | 3.075 | 2.975 | 2.912 | 2.85 |
| | Default probability | | | | | | |
| 31/8/2008 | 0.016 | 0.033 | 0.050 | 0.068 | 0.086 | 0.102 | 0.118 |
| 30/9/2008 | 0.056 | 0.097 | 0.137 | 0.170 | 0.205 | 0.238 | 0.268 |
| Increase | 0.040 | 0.064 | 0.087 | 0.102 | 0.120 | 0.136 | 0.150 |
| | Forward default probability | | | | | | |
| 31/8/2008 | 0.016 | 0.016 | 0.017 | 0.018 | 0.018 | 0.016 | 0.016 |
| 30/9/2008 | 0.056 | 0.041 | 0.041 | 0.033 | 0.035 | 0.033 | 0.030 |
| Increase | 0.040 | 0.024 | 0.023 | 0.015 | 0.017 | 0.016 | 0.014 |

Notes: CDS spreads are provided by Credit Market Analysis Ltd. Default probabilities are backed out using standard recursion formulae (Hull (2003)).

Looking at the effect of the Lehman default, the bottom line of table 1 confirms that the shock to the forward default probability structure for Goldman Sachs was positive and declined gradually across forward maturities. This is consistent with the idea that we are observing the effect of a downward shift in the shock-adjusted distance to default $\upsilon$ which is not offset by an increase in the effective drift rate $\rho$, which would have the opposite effect (of reducing the forward default probabilities at a rate increasing with maturity). The slow rate of decay indicates that the effect is damped by a relatively large shock-adjusted accounting lag $l$. In contrast, the shock to the forward default probability structure for Morgan Stanley (the bottom line of table 2) is much larger and declines at a faster rate. This suggests that the signal was more informative for Morgan Stanley that for Goldman Sachs, resulting in a much bigger percentage decrease in the shock-adjusted distance to default. The rapid rate of decay suggest that the shock also had the effect of reducing the adjusted accounting lag $l$, which seems much smaller than for Goldmans.

Table 2: The effect of the Lehman default on the market's assessment of the risk of a default by Morgan Stanley

| Date | CDS spread (% p.a.) | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 year | 2 year | 3 year | 4 year | 5 year | 6 year | 7 year |
| 31/8/2008 | 3.050 | 2.900 | 2.775 | 2.640 | 2.500 | 2.473 | 2.445 |
| 30/9/2008 | 16.519 | 13.774 | 11.817 | 10.593 | 9.995 | 9.609 | 9.223 |
| Increase | 13.469 | 10.874 | 9.042 | 7.953 | 7.495 | 7.137 | 6.778 |
| | Default probability | | | | | | |
| 31/8/2008 | 0.030 | 0.055 | 0.079 | 0.098 | 0.114 | 0.134 | 0.153 |
| 30/9/2008 | 0.142 | 0.224 | 0.276 | 0.316 | 0.359 | 0.399 | 0.430 |
| Increase | 0.112 | 0.169 | 0.197 | 0.218 | 0.245 | 0.265 | 0.277 |
| | Forward default probability | | | | | | |
| 31/8/2008 | 0.030 | 0.026 | 0.023 | 0.020 | 0.016 | 0.020 | 0.019 |
| 30/9/2008 | 0.142 | 0.083 | 0.051 | 0.040 | 0.043 | 0.040 | 0.031 |
| Increase | 0.112 | 0.057 | 0.028 | 0.021 | 0.027 | 0.020 | 0.012 |

## 2.3 The deferred filtration model as an encompassing model

Finally it is worth noting that the deferred filtration model encompasses variety of other Gaussian default models. These are shown as limiting cases in the table below. As we have seen, the standard structural default model (i) is obtained as the limit in which the accounting lag $t \to 0$. In case (ii), $\alpha \to 0$ so that $y > 0$ becomes perfectly informative, acting just like the observed excess capital value does in the standard model and making earlier accounts and signals redundant. As we have also seen, the BDF model (in which there are no value signals other than $z$) results in case (iii) as: $\alpha \to \infty$ with $t > 0$ so that $\chi \to 1, \rho \to \mu$. At the opposite extreme, case (iv) describes the limit in which $t \to \infty$ with $\alpha > 0$. This is analyzed in appendix 2 and is a 'worst case scenario' in the sense that the adjusted excess capital value is negligibly small (since $\chi \to 0$ and $l \to 1$).

The final row of the table shows the limit $\alpha \to \infty$; $t \to \infty$ in which both signals are uninformative and the depositor assumes that the diffusion is in a steady state. The hazard rate is constant in this case, in line with the steady state.

**Table 2: Limiting cases of the Deferred Filtration (DF) model**

| Model resulting as special case | Parameter limit | Informativeness of value signals | |
|---|---|---|---|
| | | $z$ | $y$ |
| (i) Standard | $t \to 0$ | precise | uninformative |
| (ii) Standard (with $y$ replacing $z$) | $\alpha \to 0$ with $t > 0$ | uninformative | precise |
| (iii) Basic deferred filtration | $\alpha \to \infty$ with $t > 0$ | informative | uninformative |
| (iv) Worst case scenario | $t \to \infty$ with $\alpha > 0$ | uninformative | informative |
| (v) Static hazard rate model | $t \to \infty$ and $\alpha \to \infty$ | uninformative | uninformative |

## 3 Conclusion

The isomorphism between the standard and deferred filtration models allows the insights of the full information setting to be carried over to the more complex case of asymmetric information. For example, it shows that the accounting lag, which provides a general indicator of uncertainty and opacity in the deferred filtration model, plays a similar role to that of forward maturity in the standard model. The comparative static properties of the standard model carry over *mutatis mutandis* and can also be used to sign the effect of signals upon the effective accounting lag and drift parameters. It shows how standard pricing formulae can be adapted to allow for asymmetric information and facilitates empirical application. A companion paper (Spencer (2013b)) shows how these formulae can be used to analyze the behavior of US bank CDS spreads during the recent crisis and shows that the performance of the structural model compares favorably with that of the reduced form pricing model normally used in such studies.

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# 4 Appendix 1: Proof of proposition 1

Duffie and Lando (2001) derive the conditional distribution of the firm's asset value given incomplete accounting information and then use this to derive the default arrival intensity and credit spreads. They use a tilde notation $(\tilde{x}, \tilde{z}, \tilde{y})$ to denote deviations from bankruptcy trigger values, but we can drop this since our notation is based on the zero trigger value of the Black Cox model. With this modification, their E(20) then shows that the density of the excess value $x(t) = x > 0$ at time $t$ conditional upon survivorship until then; the lagged value $z = x(0)$ and the noisy signal $y(t) = y$ is[1]:

$$g(x|z,y,t) = \frac{\sqrt{\frac{\beta_0}{\pi}}\left[1 - \exp\left[\frac{-2xz}{\sigma^2}\right]\right]\exp\left[-J(z,x,y)\right]}{\left(\exp\left[\frac{\beta_1^2}{4\beta_0} - \beta_3\right]\Phi\left[\frac{\beta_1}{\sqrt{2\beta_0}}\right] - \exp\left[\frac{\beta_2^2}{4\beta_0} - \beta_3\right]\Phi\left[-\frac{\beta_2}{\sqrt{2\beta_0}}\right]\right)} \tag{11}$$

(Duffie and Lando (2001), equation (20)).where:

$$\begin{aligned}
J(z,x,y) &= \frac{(x-y)^2}{2\alpha^2} + \frac{(z+\mu t - x)^2}{2\sigma^2 t}, \\
\beta_0 &= \frac{\alpha^2 + \sigma^2 t}{2\alpha^2\sigma^2 t}, \quad \beta_2 = \frac{(z-\mu t)}{\sigma^2 t} - \frac{y}{\alpha^2}, \\
\beta_1 &= \frac{(z+\mu t)}{\sigma^2 t} + \frac{y}{\alpha^2}, \quad \beta_3 = \frac{y^2}{2\alpha^2} + \frac{(z+\mu t)^2}{2\sigma^2 t}.
\end{aligned}$$

The numerator and denominator terms can be arranged as:

$$g(x|z,y,t) = \frac{\sqrt{\frac{\beta_0}{\pi}}\left[1 - \exp\left[\frac{-2xz}{\sigma^2}\right]\right]\exp\left[-I(z,x,y)\right]\exp\left[\frac{\beta_1^2}{4\beta_0} - \beta_3\right]}{\left(\Phi\left[\frac{\beta_1}{\sqrt{2\beta_0}}\right] - \exp\left[-\frac{2z(\sigma^2 y + \alpha^2\mu)}{(\sigma^2 t + \alpha^2)\sigma^2}\right]\Phi\left[-\frac{\beta_2}{\sqrt{2\beta_0}}\right]\right)\exp\left[\frac{\beta_1^2}{4\beta_0} - \beta_3\right]} \tag{12}$$

[1] Their Equation (20), with $t = l$.

where:

$$I(z,x,y) = \frac{(\alpha^2(x-z) + (\sigma^2(x-y) - \alpha^2\mu)t)^2}{2\alpha^2\sigma^2 t(\alpha^2+\sigma^2 t)}$$

[To see this, first consider the difference:

$$\begin{aligned}
J(z,x,y) - I(z,x,y) &= \frac{\sigma^2 t(x-y)^2 + \alpha^2(z+\mu t - x)^2}{2\alpha^2\sigma^2 t} - \frac{(\alpha^2(x-z) + (\sigma^2(x-y) - \alpha^2\mu)t)^2}{2\alpha^2\sigma^2 t(\alpha^2+\sigma^2 t)}, \\
&= \frac{(\alpha^2+\sigma^2 t)[\sigma^2 t(x-y)^2 + \alpha^2(z+\mu t-x)^2] - (\alpha^2(x-z) + (\sigma^2(x-y) - \alpha^2\mu)t))^2}{2\alpha^2\sigma^2 t(\alpha^2+\sigma^2 t).}
\end{aligned}$$

Expanding the quadratic terms in the numerator in this expression and cancelling common terms simplifies this numerator to:

$$\begin{aligned}
&t^3\alpha^2\sigma^2\mu^2 + t^2x^2\sigma^4 - 2t^2xy\sigma^4 - 2t^2x\alpha^2\sigma^2\mu + t^2y^2\sigma^4 + 2t^2z\alpha^2\sigma^2\mu + t^2\alpha^4\mu^2 + 2tx^2\alpha^2\sigma^2 \\
&-2txy\alpha^2\sigma^2 - 2txz\alpha^2\sigma^2 - 2tx\alpha^4\mu + ty^2\alpha^2\sigma^2 + tz^2\alpha^2\sigma^2 + 2tz\alpha^4\mu + x^2\alpha^4 - 2xz\alpha^4 + z^2\alpha^4 \\
&-(t^2x^2\sigma^4 - 2t^2xy\sigma^4 - 2t^2x\alpha^2\sigma^2\mu + t^2y^2\sigma^4 + 2t^2y\alpha^2\sigma^2\mu + t^2\alpha^4\mu^2 + 2tx^2\alpha^2\sigma^2 - 2txy\alpha^2\sigma^2 \\
&-2txz\alpha^2\sigma^2 - 2tx\alpha^4\mu + 2tyz\alpha^2\sigma^2 + 2tz\alpha^4\mu + x^2\alpha^4 - 2xz\alpha^4 + z^2\alpha^4) \\
&= -t^3\alpha^2\sigma^2\mu^2 + 2t^2y\alpha^2\sigma^2\mu - 2t^2z\alpha^2\sigma^2\mu - ty^2\alpha^2\sigma^2 + 2tyz\alpha^2\sigma^2 - tz^2\alpha^2\sigma^2 \\
&= -t\alpha^2\sigma^2\left(z-y+t\mu\right)^2
\end{aligned}$$

allowing the difference to be related to $\phi_Y[Y_t]$:

$$J(z,x,y) - I(z,x,y) = \frac{\left(z-y+t\mu\right)^2}{2(\alpha^2+\sigma^2 t)}.$$

Also note that:

$$\beta_3 - \beta_1^2/4\beta_0 = \frac{\sigma^2 ty^2 + \alpha^2(z+\mu t)^2}{2\alpha^2\sigma^2 t} - \frac{(\alpha^2(z+\mu t)+\sigma^2 ty)^2}{2\alpha^2\sigma^2 t(\alpha^2+\sigma^2 t)},$$
$$= \frac{(\alpha^2+\sigma^2 t)(\sigma^2 ty^2 + \alpha^2(z+\mu t)^2) - (\alpha^2(z+\mu t)+\sigma^2 ty)^2}{2\alpha^2\sigma^2 t(\alpha^2+\sigma^2 t)},$$
$$= \frac{-(z-y+t\mu)^2}{2(\alpha^2+\sigma^2 t)},$$

allowing exponent in the numerator of (11) to be factorized using:.

$$J(z,x,y) = I(z,x,y) - \frac{\beta_1^2}{4\beta_0} + \beta_3.$$

To factorize the denominator of (11) first note that:

$$\beta_1^2 - \beta_2^2 = (\beta_1+\beta_2)(\beta_1-\beta_2)$$
$$= \frac{2z}{\sigma^2 t}\left(\frac{2(\alpha^2\mu t + \sigma^2 ty)}{\alpha^2\sigma^2 t}\right)$$

Thus:

$$\frac{\beta_2^2}{4\beta_0} = \frac{\beta_1^2}{4\beta_0} - \frac{2z(\sigma^2 y + \alpha^2\mu)}{(\sigma^2 t+\alpha^2)\sigma^2}.$$

Substituting this into the denominator of (11) allows this to be arranges as in (12).]

Equation (11) shows the common factor:

$$\phi_Y[Y_t] = \left(\frac{1}{\sqrt{2\pi}}\right)\exp\left[-\frac{(z-y+t\mu)^2}{2(\alpha^2+\sigma^2 t)}\right] = \left(\frac{1}{\sqrt{2\pi}}\right)\exp\left[\frac{\beta_1^2}{4\beta_0} - \beta_3\right].$$

Cancelling this factor and using (8) allows (11) to be arranged as:

$$g(x\chi/\sigma, \upsilon/\sigma, \rho/\sigma, l) = \frac{\sqrt{\frac{1}{2\pi\sigma^2 l}}\left(\exp\left[-\frac{(x-(\upsilon+\rho l))^2}{2\sigma^2 l}\right] - \exp\left[-\frac{2\upsilon\rho}{\sigma^2}\right]\exp\left[-\frac{(x-(-\upsilon+\rho l))^2}{2\sigma^2 l}\right]\right)}{s(\upsilon/\sigma, \rho/\sigma, l)} \tag{13}$$

$$\text{where:} \qquad s(\upsilon/\sigma, \rho/\sigma, l) = \left(\Phi\left[\frac{(\upsilon+\rho l)}{\sigma\sqrt{l}}\right] - \exp\left[-\frac{2\upsilon\rho}{\sigma^2}\right]\Phi\left[\frac{(-\upsilon+\rho l)}{\sigma\sqrt{l}}\right]\right) \tag{14}$$

[To see this, note that the denominator of (13) follows directly by substituting (8). The first term in the numerator of (13) follows by re-arranging the the exponent of $I(z, x, y)$ and substituting (8):

$$\begin{aligned} I(z,x,y) &= \frac{\{x(\sigma^2 t + \alpha^2) - z\alpha^2 - (\alpha^2\mu + \sigma^2 y)t\}^2}{2\alpha^2\sigma^2 t(\alpha^2 + \sigma^2 t)} \\ &= \frac{\{(\alpha^2 + \sigma^2 t)(x - z\chi - (\mu + \frac{\sigma^2}{\alpha^2}y)\chi t)\}^2}{2\alpha^2\sigma^2 t(\alpha^2 + \sigma^2 t)} \\ &= \frac{(x - (\upsilon + \rho l))^2}{2\sigma^2 l} \end{aligned}$$

The second term in the numerator is the product of two exponential terms and follows by adding and rearranging the exponents to obtain.-$\frac{(x+(\upsilon+\rho l))^2}{2\sigma^2 l}$.]

The conditional density in (11) and (13) is central to the behavior of the default arrival intensity and credit spreads in the Duffie and Lando (2001) model. They use the dominated convergence theorem to show that the default arrival intensity is:

$$\frac{1}{2}\sigma^2\left[\frac{g(x\chi/\sigma, z\chi/\sigma, \rho/\sigma, l)}{\partial x}\right]_{x=0} \tag{15}$$

Evaluating this using (13) gives:

$$h(z\chi/\sigma,\rho/\sigma,l)=\frac{\left(\frac{\upsilon}{\sigma l^{3/2}\sqrt{2\pi}}\right)\exp\left[-\frac{(\upsilon+\rho l)^2}{2\sigma^2 l}\right]}{s(\upsilon/\sigma,\rho/\sigma,l)} \tag{16}$$

which may also be expressed as (7). Proposition 1 follows immediately.

# 5 Appendix 2. Proof of proposition 2

The hazard rate is difficult to analyze because it involves the ratio of the Gaussian density and distribution functions:

$$\Psi[q]=\frac{\Phi\left[q\right]}{\phi\left[q\right]}. \tag{17}$$

We can use this together with the well-known relationship:

$$\exp\left[-\frac{2\mu z}{\sigma^2}\right]\phi\left[\frac{\upsilon-\rho l}{\sigma\sqrt{l}}\right]=\phi\left[\frac{\upsilon+\rho l}{\sigma\sqrt{l}}\right] \tag{18}$$

to express the post-signal survivorship function (14) as:

$$s(z\chi/\sigma,\rho/\sigma,l)=\phi\left[\frac{\upsilon+\rho l}{\sigma\sqrt{l}}\right]\left\{\Psi\left[\frac{\upsilon+\rho l}{\sigma\sqrt{l}}\right]-\Psi\left[\frac{\upsilon+\rho l}{\sigma\sqrt{l}}\right]\right\}$$

and thus the post-signal hazard rate (16) as:

$$h(z\chi/\sigma,\rho/\sigma,\chi t)=\left(\frac{z}{\sigma t^{3/2}}\right)\left\{\begin{array}{c}\chi\Psi\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l+z)\sqrt{\chi}\right]\\ -\chi\Psi\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l-z)\sqrt{\chi}\right]\end{array}\right\}^{-1}. \tag{19}$$

The ratio (17) is closely related to the Mills Ratio (Mills (1926)):

$$R[x]=(1-\Phi\left[x\right])/\phi\left[x\right]. \tag{20}$$

Gordon (1941) shows that for $x > 0$:

$$\frac{x}{1+x^2} \leq R[x] \leq \frac{1}{x}. \tag{21}$$

The ratio (17) has the derivatives:

$$\Psi'[q] = \frac{\partial \Psi}{\partial q} = q\Psi + 1 \geq 0 \tag{22}$$

$$\Psi''[q] = \frac{\partial^2 \Psi}{\partial q^2} = (1+q^2)\Psi + q \geq 0. \tag{23}$$

The signs of these derivative follow immediately for $q \geq 0$. In the case $q < 0$, the change of variable $x = -q > 0$ and the transform:

$$\Psi[q] = \frac{\Phi[q]}{\phi[q]} = \frac{(1 - \Phi[-q])}{\phi[q]} = \frac{(1 - \Phi[x])}{\phi[x]} = R[x]$$

allows us to transform (21) and get the bounds:

$$\frac{-q}{(1+q^2)} \leq \Psi[q] \leq \frac{-1}{q}; \quad q < 0$$

which establishes (22) and (23) for $q < 0$. It follows immediately from (23) that $q\Psi[q]$ is also increasing in $q$ :

$$\frac{\partial q\Psi[q]}{\partial q} = (1+q^2)\Psi[q] + q \geq 0. \tag{24}$$

These inequalities allow us to show that the effect of the scale variable $\chi$ on the hazard rate (19) is positive (Proposition 2). Differentiating the term in curly brackets with respect to $\chi$ gives the positive expression:

$$\Psi\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l+z)\sqrt{\chi}\right]-\Psi\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l-z)\sqrt{\chi}\right] \tag{25}$$

$$+\frac{1}{2}\left\{\begin{array}{c}\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l+z)\sqrt{\chi}\Psi'\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l+z)\sqrt{\chi}\right]\\ -\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l-z)\sqrt{\chi}\Psi'\left[\left(\frac{1}{\sigma\sqrt{t}}\right)(\rho l-z)\sqrt{\chi}\right]\end{array}\right\}\geq 0. \tag{26}$$

The positive sign of the first line follows directly from (22) and the fact that with $\dot{z}>0$ the argument of the first component is larger than that of the second. The positive sign of the second line follows from a similar argument, noting that it is comprises two components of the form $q\Psi[q]$ and using (23) and the fact that $\chi>0$. Since the term in curly brackets is on the denominator of (19) it follows that an increase in the scale variable $\chi$ reduces the hazard rate: (10).

# 6 Appendix 3: Worst case scenarios

Now consider how the overnight forward default intensity behaves in a 'worst case' scenario - the limit in which $z\chi$ tends to zero. First approximate the numerator of (16) by taking a first order approximation around $z\chi=0$:

$$0+z\chi\left\{\left(\frac{1}{\sigma l^{3/2}\sqrt{2\pi}}\right)\exp\left[-\frac{(\rho l)^2}{2\sigma^2 l}\right]\right\}+O((z\chi)^2$$

Similarly for the denominator:

$$0+z\chi\left\{\frac{2}{\sigma\sqrt{l}}\phi\left[\frac{\rho l}{\sigma\sqrt{l}}\right]+\frac{2\rho}{\sigma^2}\Phi\left[\frac{\rho l}{\sigma\sqrt{l}}\right]\right\}+O((z\chi)^2$$

Taking the ratio of these approximations of numerator and denominator cancelling the common factor $z\chi/\sigma\sqrt{l}$ gives:

$$\frac{\left(\frac{1}{\sqrt{2\pi}}\right)\exp\left[-\frac{(\rho l)^2}{2\sigma^2 l}\right]}{\frac{2}{\sqrt{l}}\phi\left[\frac{\rho l}{\sigma\sqrt{l}}\right]+\frac{2\rho}{\sigma}\Phi\left[\frac{\rho l}{\sigma\sqrt{l}}\right]}=\frac{\frac{1}{2l}\phi\left[\frac{\rho\sqrt{l}}{\sigma}\right]}{\phi\left[\frac{\rho\sqrt{l}}{\sigma}\right]+\frac{\rho\sqrt{l}}{\sigma}\Phi\left[\frac{\rho\sqrt{l}}{\sigma}\right]} \tag{27}$$

Note that if the drift term $\rho = 0$ then this simplifies to $\left(\frac{1}{2l}\right)$.