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# The US Economy, the Treasury Bond Market and the Specification of Macro-Finance Models

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# The US Economy, the Treasury Bond Market and the Specification of Macro-Finance Models. *

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**Abstract**

This paper addresses questions about the structure of the economy and financial markets raised by recent research on the term structure. The work of Duffee (2012) and Joslin, Preibsch and Singleton (2012) suggests that macroeconomic variables affect risk premia rather than bond yields, which are driven by just three factors as in the traditional model. This is consistent with the observation that the real world macro-dynamics appear to be much richer than the risk neutral dynamics underpinning the term structure. On the other hand, Cochrane and Piazzesi (2005) and (2010) suggest that premia are much simpler, depending upon a single return forecasting factor but not macro variables. This paper suggests that the traditional model is too restrictive, performing poorly at the long end. A model with two return-forecasting factors works remarkably well.

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# 1 Introduction

The affine model of the term structure of interest rates (due to Vasicek (1977), Cox, Ingersoll, and Ross (1985) and Duffie and Kan (1996)) has greatly enriched our understanding of the behavior of bond markets. This model eliminates arbitrage opportunities, allowing differential expected excess returns to occur only as a compensation for risk. It shows that risk premia are highly significant and tend to vary over time. However recent studies raise serious questions about the nature of the US treasury bond market and in particular its links with the macroeconomy. This paper develops a macro-finance framework for the US economy and Treasury bond market that is designed to address these issues.

The early work on the affine term structure model (ATSM) suggested that bond yields were determined by low-dimensional processes: most of the variation of the yield curve could be explained by three explanatory variables such as principal components of yields. Recent papers such as Duffee (2012) and Adrian, Crump, and Moench (2013) find that although the fourth and fifth factors have little effect on bond yields they have a powerful effect on risk premia and returns. Ang and Piazzesi (2003) expanded the traditional framework by including macroeconomic variables reflecting inflation and the business cycle as additional factors. This macro term structure approach takes advantage of the interplay between the macroeconomy, the policy interest rate and the bond market to provide a better representation of the real world dynamics. However, as Duffee (2012), Joslin, Priebsch, and Singleton (2012) and others have noted, the real-world dynamics implied by macro models are much richer than the risk-neutral dynamics that seem to explain the cross section of bond yields in an ATSM. Macro term structure models (MTSMs) include more variables in the state vector and typically require longer lags than the first order autoregressive model normally employed in a term structure model. The high dimension of the dynamic interaction between bond yields and macroeconomic variables under the real world probability measure stands in marked contrast to the low dimension of yields apparent under the risk-neutral measure used to model the cross-section.

Theoretically, any such difference between the risk-neutral and real-world yield dynamics must be due to fluctuations in the price of risk. This observation has led Joslin, Preibsch and Singleton (2012) to conclude that these additional macroeconomic variables must affect risk premia rather than the yield structure. These macro variables represent 'unspanned risks' or 'hidden factors' that affect the future evolution of the economy and yields but not the current interest rate structure. As such, they cannot be hedged by investors in the bond market. Duffee (2012) shows that to have neutral effects on the current cross-section of yields, such factors must affect risk premia and expected interest rates in opposite ways, with their effects broadly cancelling out. Chernov and Mueller (2012) find that although inflation itself is not hidden, a latent factor that influences inflationary expectations is hidden. The implication is that because the real world dynamics are of higher dimension than the risk neutral dynamics, risk premia are relatively complicated, high-dimension phenomena.

To set against these findings, empirical studies of expected returns (using ex post returns as proxies for expectations) suggest that the structure of risk premia is relatively simple. Cochrane and Piazzesi (2005) and (2010) observe that excess returns of all maturities move together and can be explained by a single 'return forecasting factor' that can be represented by a linear combination of bond yields or forward rates. Dewachter and Iania (2011) identify a similar forecasting factor using a model with Kalman filters. Duffee (2012) and Adrian et al (2013) find that two return forecasting factors are significant. In these models macro factors (other than the spot interest rate) can only affect risk premia indirectly, through their correlation with the return forecasting factor. The role of macroeconomic variables in helping to explain the real world dynamics of the yield curve means that theoretically they must affect the cross section of yields or the risk premia or both. Yet there seems to be scant evidence of either effect.

Another strong theoretical implication of the affine pricing model is that if macro or other observable variables affect the bond market then they must be spanned by it: in principle it should be possible to represent such variables perfectly as linear combinations of bond yields using regression techniques. Similarly, it should be possible to replace one of the factors in the model of the cross-section of yields by an appropriate combination or 'portfolio' of macro variables. This is the basic approach taken by studies such as Joslin, Le, and Singleton (2013) who substitute macro variables for market-based factors on a one-for-one basis. However, there are good reasons why such simple substitution strategies may not work well in practice.

First, it is not clear what combinations of macro variables are appropriate to represent the shocks hitting the system. The effect of supply or demand shocks for example would have to be modelled by combinations of inflation and output variables, probably involving lagged values of both. In this case, we would not expect to find a good fit by regressing individual inflation or output variables on yield variables. It should still be possible to replace a single market based factor in an ATSM by a combination of macro variables, but realistically this would involve a search (using canonical regression or other multivariate techniques) to find the appropriate combination. The use of market-based factors such as principal components of yields would be much more reliable in this situation. Second, macro-finance relationships could be disturbed by structural change in the system. Kalman filtering techniques are more robust to such problems, extracting information optimally from innovations in both market and macro data. However, because these filters are not linear combinations of the macro and yield data the model is no longer affine in this data.

In view of these problems, my research strategy is to add macro variables to the model rather than to attempt to replace market-based yield factors on a one-for-one basis. This approach has the effect of increasing the total number of factors, running the risk of over-parameterization. However, since it is hard to be sure what the appropriate number of factors is in the first place, this makes it appropriate to employ a general to specific research strategy, starting with a general model which is then refined to exclude insignificant factors and parameters. I use this approach to investigate the effect of macro variables on yields and risk premia in the Treasury bond market.

I begin by setting out a standard macro model which determines the policy interest rate. Following Dewachter et al (2006) and Spencer (2008), this handles the unit root seen empirically in nominal variables like inflation and interest rates using a non-stationary latent variable to represent the inflation asymptote. A second latent variable models fluctuations in the equilibrium real interest rate. The latent variables are represented by the Kalman filter in a Kalman Vector Autoregression (KVAR) model, which is specified and estimated under the real world or historical probability measure. The first specification (model $M_0$) complements this with a separate 'yields-only' model which explains the term structure of yields using five latent variables as in Duffee (2012). Macro variables do not affect bond yields (or vice versa) in this model, which is used to establish a performance benchmark for models in which they do.

I then set up a MTSM framework to explore the links between the bond market and the macroeconomy. In this framework, the macro KVAR generates the interest rate expectations that feed into the model of the yield curve. The latter models the risk premia in bond yields in terms of the macro variables and two return forecasting factors that affect the risk premia without affecting the macro model or interest rate expectations. This general macro-finance specification allows the macro variables to affect both interest rate expectations and risk premia. I find that excluding them from the process driving the yield curve as suggested by the work of Joslin et al (2012) leads to a marked deterioration in explanatory power, which is particularly pronounced at the long end of the maturity spectrum (10 to 15 years). Excluding the macro variables from the process determining the risk premia as suggested by the single return-forecasting factor models of Cochrane and Piazzesi (2010) and Dewachter and Iania (2011) is also rejected by the data. However excluding them from a model with two return-forecasting factors is acceptable, indeed, these exclusions have practically no effect on the fit of the model.

The paper is structured as follows. The next section sets out the general model framework. Section three reports the empirical results, starting with the model selection tests and then going on to discuss the implications of the final model for the macroeconomy and the yield curve in some detail. The final section provides some further reflections on these results and looks at the implications for the monetary transmission mechanism and unconventional financial policies.

## 2 The econometric model framework

This section follows Cochrane and Piazzesi (2005, 2010) in developing a model in which financial but not macro factors influence the risk premia. Rather than using an amalgam of forward rates as they do, these are represented using latent variables modeled by Kalman filters as in Dewachter and Iania (2011) and Duffee (2012). The first part of this section describes the macromodel used to generate interest rate expectations and the second part describes the model of the risk premia.

### 2.1 Modeling the macroeconomy

The macroeconomy is represented by a Kalman vector autoregression or KVAR. This is an $L-$th order difference equation system that models the dynamic behavior of a vector $\boldsymbol{z}_t = \{g_t, \pi_t, r_t\}'$ of three observable macroeconomic variables, respectively the output gap, inflation and the spot interest rate.

$$\boldsymbol{z}_t = \boldsymbol{K} + \boldsymbol{\Theta}_{\boldsymbol{y}} \boldsymbol{y}_t + \sum\nolimits_{l=1}^{L} \boldsymbol{\Phi}_l \boldsymbol{z}_{t-l} + \boldsymbol{w}_{\boldsymbol{z},t}, \tag{1}$$

where: $\boldsymbol{w}_{\boldsymbol{z},t}$ is a $3 \times 1$ vector of *i.i.d.N.* (independent normally distributed) orthogonal errors. This model includes two unobservable or latent macro factors $\boldsymbol{y}_t = \{\boldsymbol{y}_{\pi^*,t}, \boldsymbol{y}_{\boldsymbol{\rho},t}\}'$ that are designed to pick up the effect of any missing variables on the macroeconomy and follow the first order process:

$$\boldsymbol{y}_t = \boldsymbol{\Gamma} \boldsymbol{y}_{t-1} + \boldsymbol{w}_{\boldsymbol{y},t}, \tag{2}$$

where $\boldsymbol{\Gamma} = Diag\{\xi_{\pi^*}, \xi_{\boldsymbol{\rho}}\}$;[1] and $\boldsymbol{w}_{\boldsymbol{y},t}$ is a $2 \times 1$ vector of *i.i.d.N.* errors. I follow Dewachter, Lyrio and Maes (2006) and Dewachter and Iania (2011) in interpreting the latent macro factors as the inflation and real interest asymptotes and using the restrictions:

$$\boldsymbol{\Theta}_{\boldsymbol{y}} = (\boldsymbol{I}_3 - \sum\nolimits_{l=1}^{L} \boldsymbol{\Phi}_l)\boldsymbol{R}, \quad \text{where: } \boldsymbol{R} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{3}$$

This ensures that the macro vector converges on the asymptote $\boldsymbol{z}_t^*$, where:

$$\begin{aligned} \boldsymbol{z}_t^* = \{g_t^*, \pi_t^*, r_t^*\}' &= (\boldsymbol{I} - \sum\nolimits_{l=1}^{L} \boldsymbol{\Phi}_l)^{-1}[\boldsymbol{K} + \boldsymbol{\Theta}_{\boldsymbol{y}} \boldsymbol{y}_t] \\ &= (\boldsymbol{I} - \sum\nolimits_{l=1}^{L} \boldsymbol{\Phi}_l)^{-1}\boldsymbol{K} + \boldsymbol{R}\boldsymbol{y}_t. \end{aligned}$$

The spot rate $r_t$ plays a crucial role in a MTSM since it is assumed to be default free and measured without error. Its maturity equals the periodicity of the model.[2] This makes it the return on the one-period bond denoted in this paper by $r_{1,t}$, the conduit through which the macro variables affect future real world interest rate expectations. This makes it important to distinguish it from the other macro variables $\boldsymbol{m}_t = \{g_t, \pi_t\}'$. Partitioning $\boldsymbol{\Theta}_{\boldsymbol{y}}$ and $\boldsymbol{\Phi}_1$ in (1) conformably gives:

$$\begin{bmatrix} \boldsymbol{m}_t \\ r_t \end{bmatrix} = \boldsymbol{K} + \begin{bmatrix} \boldsymbol{\Theta}_{\boldsymbol{my}} \\ \boldsymbol{\Theta}_{r\boldsymbol{y}} \end{bmatrix} \boldsymbol{y}_t + \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{mm}} & \boldsymbol{\Phi}_{\boldsymbol{m}r} \\ \boldsymbol{\Phi}_{r\boldsymbol{m}} & \boldsymbol{\Phi}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_{t-1} \\ r_{t-1} \end{bmatrix} + \sum\nolimits_{l=2}^{L} \boldsymbol{\Phi}_l \boldsymbol{z}_{t-l} + \boldsymbol{w}_{\boldsymbol{z},t}. \tag{4}$$

[1] In this paper, $Diag\{\delta\}$ represents a matrix with the elements of the row vector $\delta$ in the main diagonal and zeros elsewhere. $0_a$ is the $(a \times 1)$ zero vector, $1_a$ is the $(a \times 1)$ summation vector, $0_{a,b}$ the $(a \times b)$ zero matrix; and $I_a$ the $a^2$ identity matrix.

[2] So for example a 3 month Treasury bill yield or swap rate is the usual choice in a quarterly time series framework such as the one used in this paper.

The shape of the factor loadings (which show the effect of any factor on different yield maturities) on $r_{1,t} = r_t$ then differs from those of other factors. Since it is the return on the one-period maturity, it is appropriate to set its initial loading to unity and those on all the other factors to zero. This is ensured by using a selection vector ($\mathbf{J}_r$ in (28) below) to pick the spot rate $r_t$ from the state vector. Depending upon its persistence, the spot rate loadings then decline with maturity as the loadings on the other factors depart from zero. Traditional term structure modelers have however been more eclectic about the one period yield, assuming that it is a weighted average of all the factors, not just an observed short term interest rate. That would be appropriate if the short term interest rate were affected by default risk or measurement error for example. In this specification other factors can have an initial impact on the yield curve similar to that of the so-called spot rate. Moreover, the dynamics of $r_{1,t}$ could be simpler than those observed in a macro model for $r_t$. I will explore this more general specification in this paper as well as the restricted MTSM specification.

## 2.2 The risk-neutral pricing framework

The KVAR model described by (5) and (4) is specified under the real world measure $\mathcal{P}$. Taking expectations and running this forward generates future interest rate expectations under this measure. Forward interest rates (and hence the yield curve) are determined by a similar system defined under the risk neutral measure $\mathcal{Q}$. Following Cochrane and Piazzesi (2010) I introduce a vector of latent financial factors $\boldsymbol{f}_t$, which pick up the effects of any influences that affect the KVAR under $\mathcal{Q}$, but not $\mathcal{P}$. This vector follows the processes:

$$\boldsymbol{f}_t = \Xi \boldsymbol{f}_{t-1} + \Xi_r r_{t-1} + \boldsymbol{w}_{\boldsymbol{f},t} \tag{5}$$

$$= \boldsymbol{k}_{\boldsymbol{f}}^Q + \Xi^Q \boldsymbol{f}_{t-1} + \Xi_{\boldsymbol{y}}^Q \boldsymbol{y}_{t-1} + \Xi_{\boldsymbol{m}}^Q \boldsymbol{m}_{t-1} + \Xi_r^Q r_{t-1} + \boldsymbol{u}_{\boldsymbol{f},t} \tag{6}$$

In the benchmark yield-only term structure model $M_0$ this is independent of the macro KVAR and the parameters of $\Xi_{\boldsymbol{y}}^Q, \Xi_{\boldsymbol{m}}^Q, \Xi_r^Q$ and $\Xi_r$ are set to zero. As usual, the change of measure shifts the the error terms and the dynamic parameters, while leaving the variance structure unchanged. I use the symbol $\boldsymbol{u}$ to represent the value under $\mathcal{Q}$ of an error vector represented as $\boldsymbol{w}$ under $\mathcal{P}$ and use superscripts to denote the parameters under $\mathcal{Q}$. Re-specifying (2) and (4) under $\mathcal{Q}$ gives:

$$\boldsymbol{y}_t = \boldsymbol{k}_{\boldsymbol{y}}^Q + \boldsymbol{\Gamma}_{\boldsymbol{f}}^Q \boldsymbol{f}_{t-1} + \boldsymbol{\Gamma}^Q \boldsymbol{y}_{t-1} + \boldsymbol{\Gamma}_{\boldsymbol{m}}^Q \boldsymbol{m}_{t-1} + \boldsymbol{\Gamma}_r^Q r_{t-1} + \boldsymbol{u}_{\boldsymbol{y},t}, \tag{7}$$

and:

$$\begin{bmatrix} \boldsymbol{m}_t \\ r_t \end{bmatrix} = \boldsymbol{K} + \begin{bmatrix} \boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \\ \boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \end{bmatrix} \boldsymbol{f}_t + \begin{bmatrix} \boldsymbol{\Theta}_{\boldsymbol{my}}^Q \\ \boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \end{bmatrix} \boldsymbol{y}_t + \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{mm}}^Q & \boldsymbol{\Phi}_{\boldsymbol{m}r}^Q \\ \boldsymbol{\Phi}_{r\boldsymbol{m}}^Q & \boldsymbol{\Phi}_{rr}^Q \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_{t-1} \\ r_{t-1} \end{bmatrix} + \sum\nolimits_{l=2}^{L} \boldsymbol{\Phi}_l \boldsymbol{z}_{t-l} + \boldsymbol{u}_{\boldsymbol{z},t}. \tag{8}$$

Substituting (6) and (7) gives:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{m}_t \\ r_t \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_{\boldsymbol{mf}}^Q \\ \mathbf{F}_{r\boldsymbol{f}}^Q \end{bmatrix} \boldsymbol{f}_{t-1} + \begin{bmatrix} \mathbf{F}_{\boldsymbol{my}}^Q \\ \mathbf{F}_{r\boldsymbol{y}}^Q \end{bmatrix} \boldsymbol{y}_{t-1} + \begin{bmatrix} \boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \\ \boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \end{bmatrix} \boldsymbol{k}_{\boldsymbol{f}} + \begin{bmatrix} \boldsymbol{\Theta}_{\boldsymbol{my}}^Q \\ \boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \end{bmatrix} \boldsymbol{k}_{\boldsymbol{y}} \\ &+ \begin{bmatrix} \mathbf{F}_{\boldsymbol{mm}}^Q & \mathbf{F}_{\boldsymbol{m}r}^Q \\ \mathbf{F}_{r\boldsymbol{m}}^Q & \mathbf{F}_{rr}^Q \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_{t-1} \\ r_{t-1} \end{bmatrix} + \boldsymbol{K}^Q + \sum\nolimits_{l=2}^{L} \boldsymbol{\Phi}_l^Q \boldsymbol{z}_{t-l} + \boldsymbol{v}_{z,t} \end{aligned} \tag{9}$$

where:

$$\boldsymbol{v}_{z,t} = \boldsymbol{\Theta}_{\boldsymbol{y}} \boldsymbol{\Xi} \boldsymbol{u}_{\boldsymbol{y},t} + \boldsymbol{\Phi}_1^Q \boldsymbol{u}_{\boldsymbol{z},t} + \boldsymbol{\Theta}_{\boldsymbol{zf}}^Q \boldsymbol{u}_t \tag{10}$$

$$\mathbf{F}_{\boldsymbol{mf}}^Q = (\boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \boldsymbol{\Xi}^Q + \boldsymbol{\Theta}_{\boldsymbol{my}}^Q \boldsymbol{\Gamma}_{\boldsymbol{f}}^Q), \quad \mathbf{F}_{\boldsymbol{my}}^Q = (\boldsymbol{\Theta}_{\boldsymbol{my}}^Q \boldsymbol{\Gamma}^Q + \boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \boldsymbol{\Xi}_{\boldsymbol{y}}^Q), \tag{11}$$

$$\mathbf{F}_{\boldsymbol{mm}}^Q = (\boldsymbol{\Phi}_{\boldsymbol{mm}}^Q + \boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \boldsymbol{\Xi}_{\boldsymbol{m}}^Q + \boldsymbol{\Theta}_{\boldsymbol{my}}^Q \boldsymbol{\Gamma}_{\boldsymbol{m}}^Q), \quad \mathbf{F}_{\boldsymbol{m}r}^Q = (\boldsymbol{\Phi}_{\boldsymbol{m}r}^Q + \boldsymbol{\Theta}_{\boldsymbol{mf}}^Q \boldsymbol{\Xi}_r^Q + \boldsymbol{\Theta}_{\boldsymbol{my}}^Q \boldsymbol{\Gamma}_r^Q) \tag{12}$$

$$\mathbf{F}_{r\boldsymbol{f}}^Q = (\boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \boldsymbol{\Xi}^Q + \boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \boldsymbol{\Gamma}_{\boldsymbol{f}\_}^Q, \quad \mathbf{F}_{r\boldsymbol{y}}^Q = (\boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \boldsymbol{\Gamma}^Q + \boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \boldsymbol{\Xi}_{\boldsymbol{y}}^Q), \tag{13}$$

$$\mathbf{F}_{r\boldsymbol{m}}^Q = (\boldsymbol{\Phi}_{rr}^Q + \boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \boldsymbol{\Xi}_{\boldsymbol{m}}^Q + \boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \boldsymbol{\Gamma}_{\boldsymbol{m}}^Q), \quad \mathbf{F}_{rr}^Q = (\boldsymbol{\Phi}_{rr}^Q + \boldsymbol{\Theta}_{r\boldsymbol{f}}^Q \boldsymbol{\Xi}_r^Q + \boldsymbol{\Theta}_{r\boldsymbol{y}}^Q \boldsymbol{\Gamma}_r^Q). \tag{14}$$

Table 1 shows that because the financial factors do not affect the spot rate or the macro system under $\mathcal{P}$, they are reflected in the yield curve through the risk premia, not through expectations.

The current state variables of this system are represented by the vector $\boldsymbol{x}_t = \{\boldsymbol{f}'_t, \boldsymbol{y}'_t, \boldsymbol{m}'_t, r_t\}'$. Stacking equations (2) to (9) conformably gives:

$$
\begin{aligned}
\boldsymbol{x}_t &= \boldsymbol{H} + \mathbf{F}_1\boldsymbol{x}_{t-1} + \sum\nolimits_{l=2}^{L} \mathbf{F}_l\boldsymbol{x}_{t-l} + \boldsymbol{w}_t, \\
&= \boldsymbol{H}^Q + \mathbf{F}_1^Q\boldsymbol{x}_{t-1} + \sum\nolimits_{l=2}^{L} \mathbf{F}_l\boldsymbol{x}_{t-l} + \boldsymbol{u}_t,
\end{aligned} \tag{15}
$$

where:

$$
\boldsymbol{w}_t, \boldsymbol{u}_t \sim N[0_7, \boldsymbol{G}.\boldsymbol{D}.\boldsymbol{G}'] \tag{16}
$$

$$
\mathbf{F}_1 = \begin{bmatrix} \boldsymbol{\Xi} & & & \boldsymbol{\Xi}_r \\ 0 & \boldsymbol{\Gamma} & 0 & 0 \\ 0 & \boldsymbol{\Theta}_{\boldsymbol{my}}\boldsymbol{\Gamma} & \boldsymbol{\Phi}_{\boldsymbol{mm}} & \boldsymbol{\Phi}_{\boldsymbol{m}r} \\ 0 & \boldsymbol{\Theta}_{r\boldsymbol{y}}\boldsymbol{\Gamma} & \boldsymbol{\Phi}_{r\boldsymbol{m}} & \boldsymbol{\Phi}_{rr} \end{bmatrix}; \quad \mathbf{F}_1^Q = \begin{bmatrix} \boldsymbol{\Xi}^Q & \boldsymbol{\Xi}_{\boldsymbol{y}}^Q & \boldsymbol{\Xi}_{\boldsymbol{m}}^Q & \boldsymbol{\Xi}_r^Q \\ \boldsymbol{\Gamma}_{\boldsymbol{f}}^Q & \boldsymbol{\Gamma}^Q & \boldsymbol{\Gamma}_{\boldsymbol{m}}^Q & \boldsymbol{\Gamma}_r^Q \\ \mathbf{F}_{\boldsymbol{mf}}^Q & \mathbf{F}_{\boldsymbol{my}}^Q & \mathbf{F}_{\boldsymbol{mm}}^Q & \mathbf{F}_{\boldsymbol{m}r}^Q \\ \mathbf{F}_{r\boldsymbol{f}}^Q & \mathbf{F}_{r\boldsymbol{y}}^Q & \mathbf{F}_{r\boldsymbol{m}}^Q & \mathbf{F}_{rr}^Q \end{bmatrix} \tag{17}
$$

$$
\boldsymbol{G} = \begin{bmatrix} \boldsymbol{I}_2 & 0_2' & 0_2' \\ 0_2 & \boldsymbol{G}_1 & 0_{4,3} \\ 0_2 & \boldsymbol{\Gamma}\boldsymbol{G}_1 & \boldsymbol{G}_2 \end{bmatrix}; \quad \boldsymbol{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ g_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & g_2 & 1 \end{bmatrix} \boldsymbol{G}_2 = \begin{bmatrix} 1 & 0 & 0 \\ g_3 & 1 & 0 \\ g_4 & g_5 & 1 \end{bmatrix} \tag{18}
$$

and $\boldsymbol{D}$ is a 7×7 diagonal variance matrix. $\boldsymbol{H}, \boldsymbol{H}^Q$ and $\mathbf{F}_l, \mathbf{F}_l^Q, l = 2, ..., L$ are defined in the Appendix . The difference between the response matrices in (17) is given by the matrix $\boldsymbol{\Lambda}_1$ where:

$$
\mathbf{F}_1 - \mathbf{F}_1^Q = \boldsymbol{\Lambda}_1 = \begin{bmatrix} (\boldsymbol{\Xi} - \boldsymbol{\Xi}^Q) & -\boldsymbol{\Xi}_{\boldsymbol{y}}^Q & -\boldsymbol{\Xi}_{\boldsymbol{m}}^Q & (\boldsymbol{\Xi}_r - \boldsymbol{\Xi}_r^Q) \\ -\boldsymbol{\Gamma}_{\boldsymbol{f}}^Q & \boldsymbol{\Gamma} - \boldsymbol{\Gamma}^Q & -\boldsymbol{\Gamma}_{\boldsymbol{m}}^Q & -\boldsymbol{\Gamma}_r^Q \\ -\mathbf{F}_{\boldsymbol{mf}}^Q & (\boldsymbol{\Theta}_{\boldsymbol{my}}\boldsymbol{\Gamma} - \mathbf{F}_{\boldsymbol{my}}^Q) & (\boldsymbol{\Phi}_{\boldsymbol{mm}} - \mathbf{F}_{\boldsymbol{mm}}^Q) & (\boldsymbol{\Phi}_{\boldsymbol{m}r} - \mathbf{F}_{\boldsymbol{m}r}^Q) \\ -\mathbf{F}_{r\boldsymbol{f}}^Q & (\boldsymbol{\Theta}_{r\boldsymbol{y}}\boldsymbol{\Gamma} - \mathbf{F}_{r\boldsymbol{y}}^Q) & (\boldsymbol{\Phi}_{r\boldsymbol{m}} - \mathbf{F}_{r\boldsymbol{m}}^Q) & (\boldsymbol{\Phi}_{rr} - \mathbf{F}_{rr}^Q) \end{bmatrix} \tag{19}
$$

The columns of this matrix show the effect of the state variables on their prices of risk and hence the risk premia (Duffee (2002)).

The specification represented by (15) to (17) can be specialized to address the issues raised in the introduction and in particular the effect of different state variables on the risk premia and the yield curve.

This concerns the associated rows and columns of the matrices $\mathbf{F}_l$ and $\mathbf{F}_l^Q = \mathbf{F}_l - \mathbf{\Lambda}_l, l = 1, ..., L$. Table 1 summarizes two important special cases. First, if we set all but the first two columns of $\mathbf{\Lambda}_l$ to zero (i.e. equate the associated rows and columns of $\mathbf{F}_l$ and $\mathbf{F}_l^Q$) then only the financial factors affect the price of risk[3] as shown in Table 1:

$$\mathbf{F}_1^Q = \begin{bmatrix} \mathbf{\Xi}^Q & 0 & 0 & 0 \\ \mathbf{F}_{\boldsymbol{yf}}^Q & \mathbf{\Gamma} & 0 & 0 \\ \mathbf{F}_{\boldsymbol{mf}}^Q & \mathbf{\Theta}_{\boldsymbol{my}}\mathbf{\Gamma} & \mathbf{\Phi}_{\boldsymbol{mm}} & \mathbf{\Phi}_{\boldsymbol{m}r} \\ \mathbf{F}_{r\boldsymbol{f}}^Q & \mathbf{\Theta}_{r\boldsymbol{y}}\mathbf{\Gamma} & \mathbf{\Phi}_{r\boldsymbol{m}} & \mathbf{\Phi}_{rr} \end{bmatrix}; \quad \mathbf{\Lambda}_1 = \begin{bmatrix} \mathbf{\Xi} - \mathbf{\Xi}^Q \, 0\,0\,0 \\ -\mathbf{F}_{\boldsymbol{yf}}^Q \;\; 0\,0\,0 \\ -\mathbf{F}_{\boldsymbol{mf}}^Q \;\; 0\,0\,0 \\ -\mathbf{F}_{r\boldsymbol{f}}^Q \;\; 0\,0\,0 \end{bmatrix} \tag{20}$$

$$\mathbf{F}_l = \mathbf{F}_l^Q, l = 2, ..., L. \tag{21}$$

where $\mathbf{F}_1$ is specified in (17) and: $\mathbf{F}_{\boldsymbol{yf}}^Q = \mathbf{\Gamma}_{\boldsymbol{f}}^Q$. Macro shocks can only have an indirect effect, through the revisions to the Kalman filters used to model the financial factors.

The second special case results if we set the rows and columns of $\mathbf{\Phi}_l^Q$ associated with $(\boldsymbol{y}_t, \boldsymbol{m}_t, z_{t-l})$ to zero so that these variables can affect the risk premia but have no effect on the current term structure. They are 'hidden' and cannot be hedged in the bond market Duffee (2012). We can follow Joslin et al (2012) and model the yields under $\mathcal{Q}$, using a traditional ATSM with say three factors $(r_t, \boldsymbol{f}_t)$ using:

$$\mathbf{F}_1^Q = \begin{bmatrix} \mathbf{\Xi}^Q \; 0\,0\,0 \; \mathbf{\Xi}_r^Q \\ 0 \;\; 0\,0\,0 \;\; 0 \\ 0 \;\; 0\,0\,0 \;\; 0 \\ \mathbf{F}_{r\boldsymbol{f}}^Q \; 0\,0\,0 \; \mathbf{\Phi}_{rr}^Q \end{bmatrix}, \mathbf{\Lambda}_1 = \begin{bmatrix} (\mathbf{\Xi} - \mathbf{\Xi}^Q) & 0 & 0 & -\mathbf{\Xi}_r^Q \\ 0 & -\mathbf{\Gamma} & 0 & 0 \\ 0 & -\mathbf{\Theta}_{\boldsymbol{my}}\mathbf{\Gamma} & -\mathbf{\Phi}_{\boldsymbol{mm}} & -\mathbf{\Phi}_{\boldsymbol{m}r} \\ -\mathbf{\Theta}_{r\boldsymbol{f}}^Q\mathbf{\Xi}^Q & -\mathbf{\Theta}_{r\boldsymbol{y}}\mathbf{\Gamma} & -\mathbf{\Phi}_{r\boldsymbol{m}} & (\mathbf{\Phi}_{rr} - \mathbf{\Phi}_{rr}^Q) \end{bmatrix} \tag{22}$$

$$\mathbf{F}_l^Q = \mathbf{0}, l = 2, ..., L. \tag{23}$$

[3] The zeros in the top super-rows of $F_1$ and $F_1^Q$ indicate that the financial factors are not affected by lagged macroeconomic effects. This hypothesis was tested successfuly against the alternative. These results are not reported but available from the author.

with $\mathbf{F}_1$ still specified in (17).

The Appendix shows that this framework system can be arranged in the form of a first order VAR known as the companion form:

$$\boldsymbol{X}_t = \boldsymbol{\Theta} + \boldsymbol{\Phi}\boldsymbol{X}_{t-1} + \boldsymbol{W}_t, \tag{24}$$

$$= \boldsymbol{\Theta}^Q + \boldsymbol{\Phi}^Q \mathbf{X}_{t-1} + \boldsymbol{U}_t, \tag{25}$$

where the state vector is $\boldsymbol{X}_t = \{\boldsymbol{x}_t', \boldsymbol{z}_{t-1}', ..., \boldsymbol{z}_{t-l}'\}'$, $\boldsymbol{W}_t = \{\boldsymbol{w}_t', 0_6'\}'$, $\boldsymbol{U}_t = \{\boldsymbol{u}_t', 0_6'\}'$ and $\boldsymbol{\Theta}, \boldsymbol{\Theta}^Q, \boldsymbol{\Phi}, \boldsymbol{\Phi}^Q$ and $\boldsymbol{C}$ are defined in the Appendix .

## 2.3 Modeling the yield curve

As the name indicates, an ATSM makes discount bond prices $P_{\tau,t}$ loglinear in the state variables. At time $t$, the negative of the log price can be represented as

$$-p_{\tau,t} = \gamma_\tau + \boldsymbol{\Psi}_\tau' \ \mathbf{X}_t, \quad \tau = 1, ..., M. \tag{26}$$

where $\tau$ denotes maturity. Dividing by maturity $\tau$ gives the discount yields: $r_{\tau,t} = -p_{\tau,t}/\tau = \alpha_\tau + \boldsymbol{\beta}_\tau' \mathbf{X}_t$, where: $\alpha_\tau = \gamma_\tau/\tau$, $\boldsymbol{\beta}_\tau = \boldsymbol{\Psi}_\tau/\tau$ (using (26)) and these slope coefficients $\boldsymbol{\beta}_\tau$ are known as the 'factor loadings'. The parameters of (26) are then given by well-known recursion relations (see for example Ang and Piazzesi (2003)).

$$\boldsymbol{\Psi}_\tau = (\boldsymbol{\Phi}^Q)' \boldsymbol{\Psi}_{\tau-1} + \mathbf{J}_r \tag{27}$$

$$\gamma_\tau = \gamma_{\tau-1} + (\boldsymbol{\Theta}^Q)' \boldsymbol{\Psi}_{\tau-1} - \frac{1}{2} \boldsymbol{\Psi}_{\tau-1}' \quad \boldsymbol{\Psi}_{\tau-1}. \tag{28}$$

where $\mathbf{J}_r$ is a selection vector that defines the return on the shortest maturity: $r_{1,t} = \mathbf{J}_r' \mathbf{X}_t$. (Recall that just selects the spot rate $r_t$ in a MTSM.) Similarly it can be shown that the implied risk premia, defined as the expected one-period expected returns relative to the spot rate are given by:

$$\rho_{\tau,t} = -\boldsymbol{\Psi}'_{\tau-1}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1 \mathbf{X}_t). \tag{29}$$

The empirical models reported in the next section stack the yield equations for $\tau = 4, 8, 12, 20, 28, 40$ and 60 quarters, to get the vector $\mathbf{r}_t = \{r_{4,t}, r_{8,t}, r_{12,t}, r_{20,t}, r_{28,t}, r_{40,t}, r_{60,t}\}'$. Adding a conformable *i.i.d.* Gaussian error vector $\boldsymbol{e}_t$ gives the multivariate linear-in-variables regression model describing the cross-section of yields:

$$\begin{aligned} \mathbf{r}_t &= \boldsymbol{\alpha} + \boldsymbol{B}'\mathbf{X}_t + \boldsymbol{e}_t = \boldsymbol{\alpha} + \boldsymbol{B}_0'\boldsymbol{y}_t + \Sigma_{l-1}^{L}\boldsymbol{B}_l'\boldsymbol{z}_{t+1-l} + \boldsymbol{e}_t, \\ \text{where : } & \boldsymbol{e}_t \sim N(0, \boldsymbol{\Sigma}); \boldsymbol{\Sigma} = Diag\{\sigma_1^2, \sigma_2^2, ..., \sigma_7^2\}, \\ \boldsymbol{v}_{r,t} &= \boldsymbol{B}'\boldsymbol{W}_t + \boldsymbol{e}_t. \end{aligned} \tag{30}$$

The error vector $\boldsymbol{e}_t$ is assumed to be independent of the error vector $\boldsymbol{W}_t$ in the KVAR (24). It is normally viewed as reflecting pricing and measurement error but it also reflects mis-specification error in the model of the cross section. These errors affect the likelihood statistics, which are used to evaluate the overall performance of different specifications of the cross section in the next section. Their performance in different maturity areas can then be analyzed using estimates of the $\sigma_i$ parameters or equivalently the Root Mean Square Error of the equation residuals for different maturities.

## 3 The empirical models

My data set is similar to that used in previous US MTSM studies such as Dewachter, Lyrio, and Maes (2006). The series $g_t$ is the quarterly OECD output gap measure. $\pi_t$ is the annual inflation rate and $r_t$ the 3 month Treasury Bill rate, both provided by Datastream. The yield data were taken from McCulloch and Kwon (1991), updated by the New York Federal Reserve Bank. The macroeconomic data are available on a quarterly basis and the sample (1961Q4-2011Q2) gave a total of 199 observations. Table 2 displays the summary statistics. As in previous studies in this area, the inflation and interest rates all exhibit a high degree of persistence and the ADF tests show that the null hypothesis of non-stationarity for these variables cannot be rejected at the 5% level. Johansen-Juselius tests show that these nominal data are cointegrated and I follow Dewachter et al (2006) in analyzing a KVAR model with an inflation trend characterized by a unit root. Preliminary VAR tests using the BIC criterion suggested that a third order lag was appropriate ($L = 3$). The general macro-finance specification contains two financial factors, with a view to testing the Cochrane-Piazzesi single factor model of the risk premium against a more general specification.

The empirical framework consists of three equations describing the macroeconomic variables (1) and (2) plus seven equations describing the representative yields (30).[4] $M_0$ comprises two separate yield and macro models. Its stand-alone yield model (30) just uses latent variables, which are represented using the Kalman filter. This is based on based on the five factor specification of Duffee (2012). Thus the spot rate driving the yield curve in $M_0$ is specified independently of the policy rate modelled in the macro KVAR and is as a weighted average of the five latent variables (see section 2.3). (Experiments with a variant in which the first three factors drove yields and the fourth and fifth drove risk premia led to a serious reduction in the explanatory power of the model, suggesting that although the effect of the last two factors on yields is numerically small these influences are highly significant statistically.) Following Duffee (2012), term structure slope and level risks are both priced in $M_0$, the former being constant and the latter depending upon all five latent factors. The macro model is the KVAR described in section 2.1 and is common to all of the models used in this paper. $M_0$ forms a benchmark in which macro variables do not affect the yield curve in any way. Similarly, the Kalman filters driving the long run inflation and policy rate asymptotes in the macro KVAR are informed by shocks to the macro variables but not yield curve shocks.

The remaining models relax this separation property and are compared with the benchmark model. Table 3 shows the results of these tests. It reports the number of parameters ($\boldsymbol{k}_i$) used in each model ($M_i$), its loglikelihood value ($\ln L_i$) and (where appropriate) the value of the likelihood ratio test $\chi^2 = 2(lnL_1 - lnL_i)$ against $M_1$ (or $M_3$). The number $\chi^2_{0.02}$ shows the critical value of this test for $p = 0.02$, which the analysis of Hendry (1995) suggests will give a null-rejection frequency comparable to that of the conventional 95% value ($p$ =0.05) used in a small sample. (With 199 quarterly observations on ten variables the sample size is: $N =$

[4] Spencer (2008) describes the Kalman learning model and the resulting likelihood function for this type of model as well as the quasi-maximum likelihood method used to estimate it. The models presented in this paper were estimated and tested using the Nelder-Mead Simplex and numerical gradient algorithms *fminsearch* and *fminunc* on *Matlab*.

$1,990$). Similarly, the Schwarz statistic or Bayesian Information Criterion $BIC_i = (lnL_i - 0.5 \times k_i \times \ln(N))$ provides an asymptotically consistent selection criterion (Canova (2007)). This is the basic model selection criterion used in this section.

The BIC is supplemented by the $\chi^2$ criterion for the models that are nested in $M_1$, which is a general MTSM structure defined by (17), (24), (25) and hence (27) and (28). In order to keep the number of parameters in this model manageable I assume that $\mathbf{\Phi}_l^Q = \mathbf{\Phi}_l, l = 2, ..., L$. Table 1 shows that this means these lagged state variables only affect the yield curve through interest rate expectations, not the risk premia. Nevertheless, $M_1$ includes four latent variables and 3 macro variables and with $L = 3$, it contains a large number (100) of parameters. Despite the large number of parameters, the BIC criterion indicates that this model is preferable to $M_0$.

$M_1$ is then specialized to assess the effect of the macro variables on the risk premia and the yield curve. Model $M_2$ simplifies $M_1$ by replacing (17) by (22) so that the output and inflation variables only affect the risk premia, not the yield curve. It is a conventional model with three factors (the two financial factors and the policy rate). The zero entries in the rows of the selection vector $\mathbf{J}$ corresponding to $f_{1,t}$ and $f_{2,t}$ were replaced by estimated parameters (as discussed in section 2.1). Optimizing these parameters gave the spot rate model: $r_{1,t} = 1.0r_t - 0.650f_{1,t} + 0.28f_{2,t}$. This model saves 35 degrees of freedom compared to $M_1$, but the large reduction in the loglikelihood outweighs this and reduces the value of the BIC (see Table 3). In other words a better model is obtained by allowing the innovations in the output gap and inflation to affect the yield curve. This observation can be checked by regressing the measurement errors from the yield model on the macro innovations, which gives the result shown in Table 4. The correlations shown in the table indicate mis-specification. They show that the three yield factors that are allowed for in $M_1$ do not span the yield structure properly and that macro innovations also have a significant impact[5].

Alternatively, replacing (17) by (20) and thus using the financial factors to influence the risk premia gives model $M_3$. This is nested within $M_1$ and saves 34 degrees of freedom. These restrictions have remarkably little effect on the likelihood and have the effect of increasing the BIC. This model is much simpler than $M_0$,

[5] Gurkaynak, Sack, and Swanson (2005) and Swanson and Williams (2012) analyze the effect of surprises in macroeconomic data announcements (like the monthly non-farm payroll release) on the yield curve. However their effects could work through the factors spanning the yield curve, while the tests shown in table 4 condition for changes in the three $M_1$ yield factors.

eliminating the columns of $\mathbf{\Lambda}_1$ associated with the macro variables and factors in (20). However, the macro variables do affect the risk premia through their effect on the two financial factors. These are modelled by Kalman filters and are progressively revised in line with shocks in both yields and macro variables. The restrictions in (20) eliminate additional macro effects, and the ease with which the data accepts this shows that the two Kalman filters $f_{1,t}$ and $f_{2,t}$ act as summary statistics, showing the effect of shocks on the risk premia. It appears that both of these risk factors are necessary: elimination of one of these gives model $M_4$ in which $\mathbf{\Lambda}_1$ depends upon a single price of risk factor as in Dewachter and Iania (2011), but this model has a much lower BIC. Model $M_5$ shows the effect of eliminating one of the financial factors from model $M_1$. This specification allows the macro variables to have a direct effect on the risk premia. This is inferior to model $M_3$ but is an improvement upon $M_4$. Both of these models are rejected against $M_1$ on the basis of the $\chi^2$ criterion.

Using the BIC, the stylized Cochrane Piazzesi specification $M_3$ was selected as the best model. These tests are discriminating because the likelihood of the cross section depends upon measurement errors that are as Cochrane and Piazzesi say in their (2010) paper 'tiny' empirically, so that restrictions that generate small perturbations in the yield curve estimates are often rejected. Analysis of the cross section equation residuals defined in (30) for different maturities indicates that these models all fit the short maturities quite well, with Root Mean Square Error values of less than 10 basis points for the 2-7 year maturities, consistent with the view that these are measurement or pricing errors rather than indicating mis-specification. Table 5 reports these statistics for the $M_0$, $M_2$, $M_3$ and $M_4$ models. This reveals that the performance of the $M_2$ and $M_4$ models deteriorates badly for the longer maturities, generating errors that are economically as well as statistically significant. For example the yield error of 27bp for the 15 year residual in $M_2$ implies a pricing error of \$1.34 per \$100 face value at the mean yield of 7.22%. However, the table suggests that $M_4$ performs reasonably well in the 1 to 10 year maturities. Adrian et al (2012) and Duffee (2012) also report problems in fitting 10 year yields, but the table shows that $M_3$ performs well right across the curve.

On the basis of these tests I conclude that macro variables do affect the yield curve. As the next section shows, they work largely through their effect on the macro asymptotes $y_{1,t}$ and $y_{2,t}$. They may affect the risk premia through their effect on the two latent financial factors $f_{1,t}$ and $f_{2,t}$. $M_3$ includes a lot of insignificant price of risk parameters. Eliminating these sequentially (on the basis of the lowest t-statistic) gave model $M_6$. This has 69 parameters, which are shown in Tables 6, 7 and 8. The estimated and actual values for the three macro variables and three representative yields are shown in Figure 1. The remainder of this section describes this model in detail.

### 3.1 The empirical macro-model

The basic macro specification and data used in this paper are similar to that used in previous studies such as Dewachter et al (2006) and consequently the macro dynamics are comparable. They are defined by (1) and (2) with the parameters shown in table 6. The latent variables representing the central tendencies of inflation and the real interest rate are shown in the lower panel of Figure 3. The first reflects the inflation trends seen over the period which saw an increase during the 1960 and 1970s, followed by a subsequent decline. The second clearly reflects developments in monetary policy and in particular the low level of real interest rates seen in the early1990s and more recently.

The impulse responses are shown in Figure 2 (a) and give a plausible description of the dynamics. In particular the bottom row of the upper panel shows that as we would expect following the work of Bikbov and Chernov (2010) and many others, positive inflation and output shocks impact the policy interest rate without a lag. Indeed the impact effect if inflation is very significant, shown by the coefficient $g_5$ in table 7. This underlines a key difference between the MTSM and the conventional ATSM represented here by $M_6$ and $M_2$: because macro variables like output and inflation are unspanned in such models they can only affect the spot rate with a lag. This restriction is particularly troublesome in $M_2$ since this specification equates the spot rate with the policy rate, and has the implication that the Federal Reserve reacts to inflation and output shocks with a lag of three months or more.

### 3.2 The empirical yield model

The behavior of the yield curve is dictated by the factor loadings ($\boldsymbol{\beta}_\tau$). These are depicted in Figure 4 and show the effect on different maturities (expressed in quarters) of increasing the each of the factors in turn by one percentage point compared to its historical value. The top panel shows the loadings on the financial factors $f_1$ and $f_2$, the central panel those on the central tendencies $y_\pi$ and $y_r$ while the lower panel shows those on $\pi$, $g$ and $r$. As noted in section 2.2, these depend critically upon the type of shock and its persistence. Since the spot rate is the 3 month yield in model $M_6$, this has a unit coefficient at a maturity of one quarter while other factors have a zero loading. The loading on this factor then fades as it mean reverts while the effects of other factors move away from the zero line. The loadings on the stationary factors $f_2$ and $y_r$ peak respectively in the 4 and 10 year maturities, while those on the non-stationary factors $f_1$ and $y_\pi$ are much more persistent. Although the residual effects of output and inflation are small, this gives a misleading impression of their overall effect on the yield curve since shocks to these variables largely work through their impact on $y_\pi$ and $y_r$. Figure 2 (b) shows the dynamic response of the yield curve to shocks in these variables. The financial factors work through the risk premia: $f_1$ has a persistent effect on the yield structure while shocks to $f_2$ decay quickly. The remaining factors work through investor expectations and thus have a more complex dynamic structure. The effect of shocks to $y_\pi$ tend to increase over time as well as maturity since they are persistent. The effect of $y_{\boldsymbol{\rho}}$ builds up initially but then decays due to mean reversion.

Perhaps the most interesting aspect of this model given its motivation is its description of the risk premia. These are determined by the two financial factors $f_1$ and $f_2$. Table 8 shows the parameters of the price of risk equation (29) and (20)) that remain significant in model $M_6$. The (-) entries (which indicate that a parameter is not significant.)in the first row indicate that the risk associated with the first financial factor is not priced, but the non-zero entries in the central column indicate that this factor does affect the price of risk associated with the inflation trend, the output gap and the interest rate. The importance of the two financial factors can be described in terms of their effect over time on (a) expected one period ahead returns and (b) yields to maturity. Figure 5 shows the effects of variations in the factors shown in figure 3 upon the one period return required by investors on the 15 year bond, computed using (29). Although the effect of variations in the second factor is much smaller for the 15 year than for shorter maturities, the single period effect is nevertheless on a similar scale to that of the first factor. In contrast, figure 6 shows that the effect of $f_1$ on the required 15 year yield to maturity is much greater than the effect of variations in $f_2$. The construction of the model means that these effects can be computed by applying the factor loadings $\beta_{1,60}$ and $\beta_{2,60}$ to the financial factors (shown in Figure 3). These components of the yield are shown in the first two panels of Figure 6. The lower panel shows the 15 year average interest rate expected under the risk neutral measure, computed using the remaining factors and loadings.

As usual, the financial factors are hard to interpret and to align with macroeconomic variables. Being latent variables, they reflect factors such as market sentiment, liquidity and flight to quality effects that are not picked up my the macro trends and variables. Nevertheless, $f_1$ is weakly (negatively) correlated with the output gap (with a t-statistic of (-)1.70) and $f_2$ with the yield gap $r_t - r_{60,t}$ (t=1.61). The negative association between $g$ and $f_1$ can be rationalized in terms of the macroeconomic theory of the stochastic discount factor (SDF). This suggests that investors will seek a high (low) term premium to induce them to defer consumption when they expect the economy to improve (deteriorate), since they attach relatively low value to consumption in good states. The peaks seen in $f_1$ in 1975 and 1982 in figure 5 align quite well with recessional troughs in the output gap (shown in the top left panel of figure 1). This factor fell during the late 1980s (which was a period of strong GDP growth when the output gap was high) and then increased during the subsequent recession and recovery phases. The peak seen in 1999 coincided with the dot-com boom and was clearly reflected in bond yields, while the peak in 2003 was aligned with the beginning of the recovery from the subsequent recession.

It is probably too early to interpret the behavior of $f_1$ since the onset of the 2009 recession. The risk premium fell during the initial phase of the recession in 2009 as the macro SDF theory would predict, but then remained at this very low level rather than recovering. This could be because investors were worried about further falls in output. It might also be due to the introduction of the Fed's quantitative easing programme in March 2009, which was designed to reduce bond yields by bearing down on the term premium. Flight to quality effects could also have held down the premium since the onset of the crisis.

## 4 Conclusion

This paper uses a set of MTSMs to explore some of the issues raised by recent research on the term structure of interest rates. These models are highly discriminating because as Cochrane and Piazzesi (2010) note, the likelihood of the cross section depends upon measurement errors that are small empirically, so that restrictions that increases in yield error variances that are numerically small are often rejected. The cross sectional errors reported in this paper for the 2-7 year maturities are less than 10 basis points, consistent with the view that they are pricing or measurement errors and economically insignificant. The models that allow for macro effects on yield and have two return-forecasting-factors $M_3$ and $M_6$ generate similar errors for the longer maturities, but the 15 year errors are much larger for the single return-forecasting-factor model $M_4$ and the 10 and 15 year errors are larger still for the conventional three factor model $M_2$. These errors are economically as well as statistically significant. Although the traditional three factor model explains a high proportion of the variance in the cross section of yields, the addition of macro variables significantly improves performance, particularly at the long end.

These results suggest that inflation and output do affect bond yields, even after the effect of conventional yield factors are allowed for. They also suggest that two return forecasting factors are necessary to model the behavior of the yield curve in the very long maturities but that the single factor model may prove adequate up to 10 years. In these specifications, macro variables affect the yield curve through interest rate expectations but only affect the risk premia indirectly, through their effect on the return forecasting factors. These findings are important given the recent initiatives of monetary authorities in the US and elsewhere to push long yields towards the lower bound to help stimulate the economy. Central bank researchers have in the main followed academic researchers in analyzing the 1-5 (or at most 10) year maturities using a first-order three-factor model of the risk neutral dynamics. However this paper would suggest that more factors are needed, ideally including inflation and output.

These results also throw light on the way that the transmission mechanism of monetary policy works. First, in line with previous work, it is apparent that the inflation asymptote $f_1$ drifts over time in response to inflationary supply-side shocks, as noted in the context of a New Keynesian macro model by Ireland (2007). He also models this asymptote using a Kalman filter and interprets it as the authorities long run inflation target. In my framework, macro innovations also affect bond yields through the real interest rate asymptote. These appear to be the main channels through which the macroeconomy influences the bond market: after allowing for these influences the residual effects of inflation and the output gap on the yield curve are very small. These inflation and interest rate asymptotes are also informed by yield curve surprises in the MTSM, which should lead to a better calibration of the long run economic trends than in a macro model such as Ireland (2007).

The macro-finance literature has so far been silent on the effects of unconventional monetary policies. However, the distinction that MTSMs make between the effects of interest rate expectations and risk premia makes them potentially very useful in this role, since forward guidance is designed to affect the former and quantitative easing (open market operations) to influence the latter. This paper can say very little about these effects. Quantitative easing was introduced in March 2009, but my data sample ends just two years later, making it difficult to formally identify its effects. Forward guidance was introduced afterwards, in August 2011. Swanson and Williams (2012) analyze the effect of macroeconomic data announcements on the yield curve as a way of assessing the effect of the lower interest rate bound and conclude that interest rates expectations were not affected before that date. However I am currently extending the estimation period with a view to exploring the effects of unconventional monetary policies. More recent data may also allow researchers to use MTSMs to investigate the effect of fiscal variables like deficit and debt ratios on the economy and financial markets[6].

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[6] Fiscal ratios should also to influence bond yields through interest rate expectations and risk channels but these influences are notoriously difficult to pin down empirically and fiscal variables are not used in macro-finance models. However, it is possible that these effects become significant once they pass critical thresholds, in which case they may become significant in future research

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## Appendix: The state-space representation

Stacking (15) and using identities to handle the longer lags with $L = 3$, puts the system into state space form (24), where $\boldsymbol{\Theta} = \{\boldsymbol{H}', 0_{1,9}\}'$ and:

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 & \mathbf{F}_4 \\ \mathbf{F}_5 & 0_{3,3} & 0_{3,3} & 0_{3,3} \\ 0_{7,3} & \boldsymbol{I}_3 & 0_{3,3} & 0_{3,3} \\ 0_{7,3} & 0_{3,3} & \boldsymbol{I}_3 & 0_{3,3} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \boldsymbol{G} & 0_{7,9} \\ 0_{9,7} & 0_{7,7} \end{bmatrix} \tag{31}$$

where:

$$\mathbf{F}_l^Q = \mathbf{F}_l = \begin{bmatrix} 0_{4,3} \\ \boldsymbol{\Phi}_l \end{bmatrix}, l = 1, ..., 3, \quad \mathbf{F}_5^Q = \mathbf{F}_5 = \begin{bmatrix} 0_{3,4} \, \boldsymbol{I}_3 \end{bmatrix}, \tag{32}$$

and $\boldsymbol{\Theta}^Q$ and $\boldsymbol{\Phi}^Q$ follow by replacing $\boldsymbol{H} = \{0_{1,4}, \boldsymbol{K}', 0_{1,9}\}'$ and $\mathbf{F}_1$ and by $\boldsymbol{H}^Q = \{\boldsymbol{k}_{\boldsymbol{f}}^Q, \boldsymbol{k}_{\boldsymbol{y}}^Q, \boldsymbol{K}^{Q\prime}, 0_{1,9}\}' = \boldsymbol{H}' + \boldsymbol{\Lambda}_0'$ and $\mathbf{F}_1^Q$.

# 5 Tables

Table 1: Alternative yield model structures

| Model | Type | Effect of: on: | $\boldsymbol{f}_t$ | $r_t$ | $\boldsymbol{y}_t$ | $\boldsymbol{m}_t$ | $\boldsymbol{z}_{t-l}$ |
|---|---|---|---|---|---|---|---|
| $M_0$ | Separate macro and yield models | Interest rate expectations | ✓ | × | × | × | × |
| | | Risk premia | ✓ | × | × | × | × |
| | | Yield curve | ✓✓ | × | × | × | × |
| $M_1$ | Encompassing macro-yield model | Interest rate expectations | × | ✓ | ✓ | ✓ | ✓ |
| | | Risk premia | ✓ | ✓ | ✓ | ✓ | × |
| | | Yield curve | ✓ | ✓✓ | ✓✓ | ✓✓ | ✓ |
| $M_2$ | Hidden macro factor model | Interest rate expectations | × | ✓ | ✓ | ✓ | ✓ |
| | | Risk premia | ✓ | ✓ | ✓ | ✓ | ✓ |
| | | Yield curve | ✓ | ✓✓ | × | × | × |
| $M_3$ | Latent factor model of price of risk | Interest rate expectations | × | ✓ | ✓ | ✓ | ✓ |
| | | Risk premia | ✓ | × | × | × | × |
| | | Yield curve | ✓ | ✓ | ✓ | ✓ | ✓ |

The ticks and crosses in this table show whether or not the various factors affect interest rate expectations, premia and hence the yield curve in the four representative models. Factors affect expectations through the real world dynamics and the yield curve through the risk-neutral dynamics, with the risk premia accounting for the difference. A double tick for any factor indicates that it affects yields through both expectations and risk premia. By construction, the financial factors $\mathbf{f}_t$ do not affect expectations in any MTSM, they can only influence yields through the risk premia. $M_0$ comprises a 5-factor yield-only model based on Duffee (2012) and a separate macro KVAR. $M_1$ is an encompassing macro-finance model. In $M_2$ the effects of the macro factors $(\mathbf{y}_t, \mathbf{m}_t, \mathbf{z}_{t-l})$ on expectations are offset in the premia as in Joslin et al (2012), so yields are only affected by the spot rate $r_t$ and the financial factors. The $M_3$ model follows Cochrane and Piazzesi (2005) in suppressing the effect of these macro variables (and $r_t$) on the risk premia, which are only affected by the financial factors. Table 3 reports the likelihood and BIC statistics for these models.

Table 2: Data summary statistics: 1961Q4-2011Q2

| | $g$ | $\pi$ | $r_1$ | $r_4$ | $r_8$ | $r_{12}$ | $r_{20}$ | $r_{28}$ | $r_{40}$ | $r_{60}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Mean$ | 0.133 | 4.133 | 5.337 | 5.797 | 6.023 | 6.191 | 6.440 | 6.618 | 6.775 | 7.022 |
| $Std.$ | 2.263 | 2.907 | 2.983 | 3.052 | 2.977 | 2.882 | 2.738 | 2.633 | 2.523 | 2.435 |
| $Skew.$ | -0.481 | 1.450 | 0.873 | 0.630 | 0.628 | 0.652 | 0.783 | 0.861 | 0.902 | 0.882 |
| $Kurt.$ | 0.334 | 2.020 | 1.645 | 0.855 | 0.741 | 0.668 | 0.689 | 0.613 | 0.553 | 0.446 |
| $Auto.$ | 0.466 | 0.993 | 0.982 | 0.989 | 0.994 | 0.995 | 0.996 | 0.997 | 0.997 | 0.997 |
| $ADF$ | -4.415 | -2.2 | -2.206 | -2.02 | -1.864 | -1.756 | -1.735 | -1.69 | -1.712 | -1.711 |
| $KPSS$ | -4.155 | -2.454 | -2.111 | -2.213 | -2,067 | -2.043 | -2.043 | -2.010 | -1.995 | 2.033 |

Output gap ($g$) is from OECD. Inflation ($\pi$) and T-bill rate ($r_1$) are from Datastream. Yield data are US Treasury discount bond equivalent data compliled by McCulloch and Kwon (1990) updated by the New York Federal Reserve Bank. Mean denotes sample arithmetic mean expressed as percentage p.a.; $Std.$ standard deviation and $Auto.$ the first order quarterly autocorrelation coefficient. $Skew.$ & $Kurt.$ are standard measures of skewness (the third moment) and kurtosis (the fourth moment). $KPSS$ is the Kwiatowski et al (1992) statistic testing the null hypothesis of level stationarity. The 10% and 5% significance levels are 0.347 and 0.463 respectively. $ADF$ is the Adjusted Dickey-Fuller statistic testing the null hypothesis of non-stationarity. The 10% and 5% significance levels are 2.575 and 2.877 respectively.

Table 3: Model selection criteria

| Model: | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---|---|---|---|---|---|---|---|
| Statistic | | | | | | | |
| $lnL_i$ | 13758.1 | 14000.3 | 13561.7 | 13991.1 | 13862.0 | 13968.1 | 13987.0 |
| $k_i$ | 89 | 100 | 65 | 66 | 50 | 80 | 55 |
| $\chi^2$ test against | | | | $M_1$ | $M_3$ | $M_1$ | $M_3$ |
| $\chi^2$ | (-) | (-) | (-) | 18.4 | 258.2 | 64.6 | 8.2 |
| $\chi^2_{0.02}$ | (-) | (-) | | 53.0 | 72.6 | 35.0 | 22.6 |
| $p(\chi^2)$ | (-) | (-) | (-) | 1 | 0 | 0 | 0.70 |
| $BIC_i$ | 26839.6 | 27240.6 | 26629.4 | 27480.6 | 27344.0 | 27328.6 | 27556.0 |

Table 1 provides a basic description of these models. The model selection tests depend upon the number of parameters ($k_i$) used in each model ($M_i$) and its loglikelihood value ($\ln L_i$). $\chi^2$ shows the value of a loglikelihood ratio test against the general macro-finance model $M_1$ or $M_3$ as appropriate. The next row shows the 2% critical value $\chi^2_{0.02}$ which is appropriate for this sample size. The BIC also provides an asymptotically consistent criterion. In $M_2$ the effects of the macro factors $(y_t, m_t, z_{t-l})$ on expectations are offset in the premia as in Joslin et al (2012), so yields are only affected by the spot rate and the financial factors (see table 1). The $M_3$ model follows Cochrane and Piazzesi (2005) in suppressing the effect of these macro variables (and $r_t$) on the risk premia, which are only affected by the two financial (price of risk) factors (table 1). These restrictions are easily accepted. $M_4$ eliminates one of these factors from $M_3$, but is strongly rejected. $M_5$ eliminates one of these factors from $M_1$, and is also rejected. The final model $M_6$ sequentially eliminates insignificant price of risk parameters from $M_3$.

Table 4: Model M1: Regressions of the measurement errors from the yield equations on macro innovations

| Regressand: | $r_4$ | $r_8$ | $r_{12}$ | $r_{20}$ | $r_{28}$ | $r_{40}$ | $r_{60}$ |
|---|---|---|---|---|---|---|---|
| | | | | Regression 1 | | | |
| Regressor parameters: | | | | | | | |
| $g$ | 0.003 | 0.001 | 0.011 | 0.020 | 0.001 | -0.014 | -0.044 |
| | (0.17) | (2.28) | (1.44) | (2.95) | (1.04) | (1.24 | (1.57) |
| $\pi$ | 0.071 | 0.001 | -0.024 | -0.020 | 0.001 | 0.035 | 0.076 |
| | (3.64) | (1.53) | (2.87) | (2.71) | (0.85) | (2.76) | (2.53) |
| $r_1$ | 0.060 | 0.000 | -0.016 | -0.008 | 0.002 | 0.001 | -0.003 |
| | (3.63) | (0.03) | (2.27) | (1.22) | (1.87) | (0.12) | (0.11) |
| Summary statistics: | | | | | | | |
| $R^2$ | 0.141 | 0.033 | 0.073 | 0.072 | 0.027 | 0.044 | 0.039 |
| $F(199, 3)$ | 16.654 | 3.690 | 7.836 | 7.593 | 6.093 | 10.086 | 6.319 |
| $p$ | 0.000 | 0.027 | 0.001 | 0.001 | 0.003 | 0.000 | 0.002 |
| | | | | Regression 2 | | | |
| Regressor parameters: | | | | | | | |
| $g$ | 0.012 | 0.001 | 0.009 | 0.019 | 0.001 | -0.014 | -0.044 |
| | (0.65) | (2.30) | (1.13) | (2.80) | (0.79) | (1.24) | (1.60) |
| $\pi$ | 0.078 | 0.001 | -0.026 | -0.021 | 0.001 | 0.035 | 0.076 |
| | (3.94) | (1.55) | (3.10) | (2.85) | (1.05) | (2.80) | (2.54) |
| Summary statistics: | | | | | | | |
| $R^2$ | 0.082 | 0.033 | 0.048 | 0.064 | 0.007 | 0.044 | 0.039 |
| $F(199, 2)$ | 18.689 | 7.419 | 10.227 | 13.632 | 8.262 | 20.287 | 12.706 |
| $p$ | 0.000 | 0.007 | 0.002 | 0.000 | 0.004 | 0.000 | 0.000 |

In $M_1$, the macro model (9).and the yield model (30) are independent. To test this specification, I first regress the measurement errors from the yield equations $e_t$ from (30) upon the macro innovations $v_{z,t}$ from (9). The first regression shows the effect of the output gap and inflation, while the second regression shows the effect of allowing in addition for spot rate effects missing from (30). (Parameter t-values in parentheses)

Table 5: Cross section errors by maturity for representative models

| | Model: | $M_0$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|---|
| Maturity: | | | | | |
| 1 | | 15.482 | 18.166 | 12.462 | 13.196 |
| 2 | | 0.090 | 0.100 | 0.201 | 0.101 |
| 3 | | 5.578 | 7.680 | 4.233 | 4.757 |
| 5 | | 4.598 | 6.864 | 2.697 | 2.120 |
| 7 | | 0.911 | 1.160 | 3.561 | 4.312 |
| 10 | | 5.450 | 11.317 | 3.987 | 3.636 |
| 15 | | 14.385 | 27.189 | 3.710 | 15.569 |

Root Mean Square Error values for the cross section equation residuals defined in (30) for different maturities, expressed in basis points. These models (see notes to table 1) all fit the short maturities very well, but $M_3$ outperforms in the longer maturities.

Table 6: Dynamic parameters in model M6

| $\mathbf{\Phi}_1$ | Parameter | t-statistic | $\mathbf{\Phi}_2$ | Parameter | t-statistic | $\mathbf{\Phi}_3$ | Parameter | t-statistic |
|---|---|---|---|---|---|---|---|---|
| $\phi_{1,gg}$ | 1.0939 | 15.62 | $\phi_{2,gg}$ | 0.0018 | 0.02 | $\phi_{3,gg}$ | -0.1182 | -1.94 |
| $\phi_{1,\pi g}$ | -0.0017 | -0.02 | $\phi_{2,\pi g}$ | -0.1830 | -1.61 | $\phi_{3,\pi g}$ | 0.1621 | 2.44 |
| $\phi_{1,rg}$ | -0.0040 | -0.07 | $\phi_{2,rg}$ | -0.2995 | -3.83 | $\phi_{3,rg}$ | 0.1191 | 2.14 |
| $\phi_{1,g\pi}$ | 0.0403 | 0.61 | $\phi_{2,g\pi}$ | 0.0068 | 0.07 | $\phi_{3,g\pi}$ | -0.0056 | -0.13 |
| $\phi_{1,\pi\pi}$ | 1.2336 | 17.32 | $\phi_{2,\pi\pi}$ | -0.2637 | -2.52 | $\phi_{3,\pi\pi}$ | 0.0000 | 0.00 |
| $\phi_{1,r\pi}$ | 0.1135 | 2.10 | $\phi_{2,r\pi}$ | -0.0980 | -1.29 | $\phi_{3,r\pi}$ | 0.0136 | 0.26 |
| $\phi_{1,gr}$ | 0.0495 | 0.87 | $\phi_{2,gr}$ | -0.0764 | -0.86 | $\phi_{3,gr}$ | 0.0092 | 0.22 |
| $\phi_{1,\pi r}$ | 0.0289 | 0.49 | $\phi_{2,\pi r}$ | 0.0567 | 0.58 | $\phi_{3,\pi r}$ | -0.0967 | -1.73 |
| $\phi_{1,rr}$ | 0.9803 | 18.86 | $\phi_{2,rr}$ | -0.4488 | -6.48 | $\phi_{3,rr}$ | 0.2806 | 7.09 |
| $\xi_{\boldsymbol{f}_2}$ | 0.4597 | 7.04 | $\boldsymbol{k}_g$ | 0.0016 | 4.02 | $\boldsymbol{k}_r$ | 0.0023 | 4.56 |
| $\xi_{\boldsymbol{\rho}}$ | 0.9298 | 126.7 | $\boldsymbol{k}_\pi$ | 0.0002 | 0.76 | | | |

These are the parameters of the dynamic equations ((1) and (2)) of the of the preferred model $M_6$. This is a development of the Cochrane and Piazzesi (2005) and Dewachter and Iania (2011) models with two financial factors driving the risk premia. The first of these factors (and the inflation variable) follow stochastic trends so that: $\xi_{f_1} = \xi_\pi = 1$.

Table 7: Variance structure of Model M6

| $\boldsymbol{G}$ | Parameter | t-statistic | $\boldsymbol{\Delta}$ | Parameter | t-statistic |
|---|---|---|---|---|---|
| $g_1$ | (-) | | $\delta_{01}$ | 0.1273 | 0.96 |
| $g_2$ | -0.6307 | -3.12 | $\delta_{02}$ | 0.0912 | 0.66 |
| $g_3$ | 0.1769 | 3.52 | $\delta_{03}$ | 0.0017 | 7.77 |
| $g_4$ | 0.0643 | 0.71 | $\delta_{04}$ | 0.0025 | 9.51 |
| $g_5$ | 0.2108 | 3.34 | $\delta_{05}$ | 0.0017 | 19.57 |
| | | | $\delta_{06}$ | 0.0016 | 19.94 |
| | | | $\delta_{07}$ | 0.0020 | 19.08 |

These are the variance parameters of the preferred model (equation (16)). See notes to table 4.

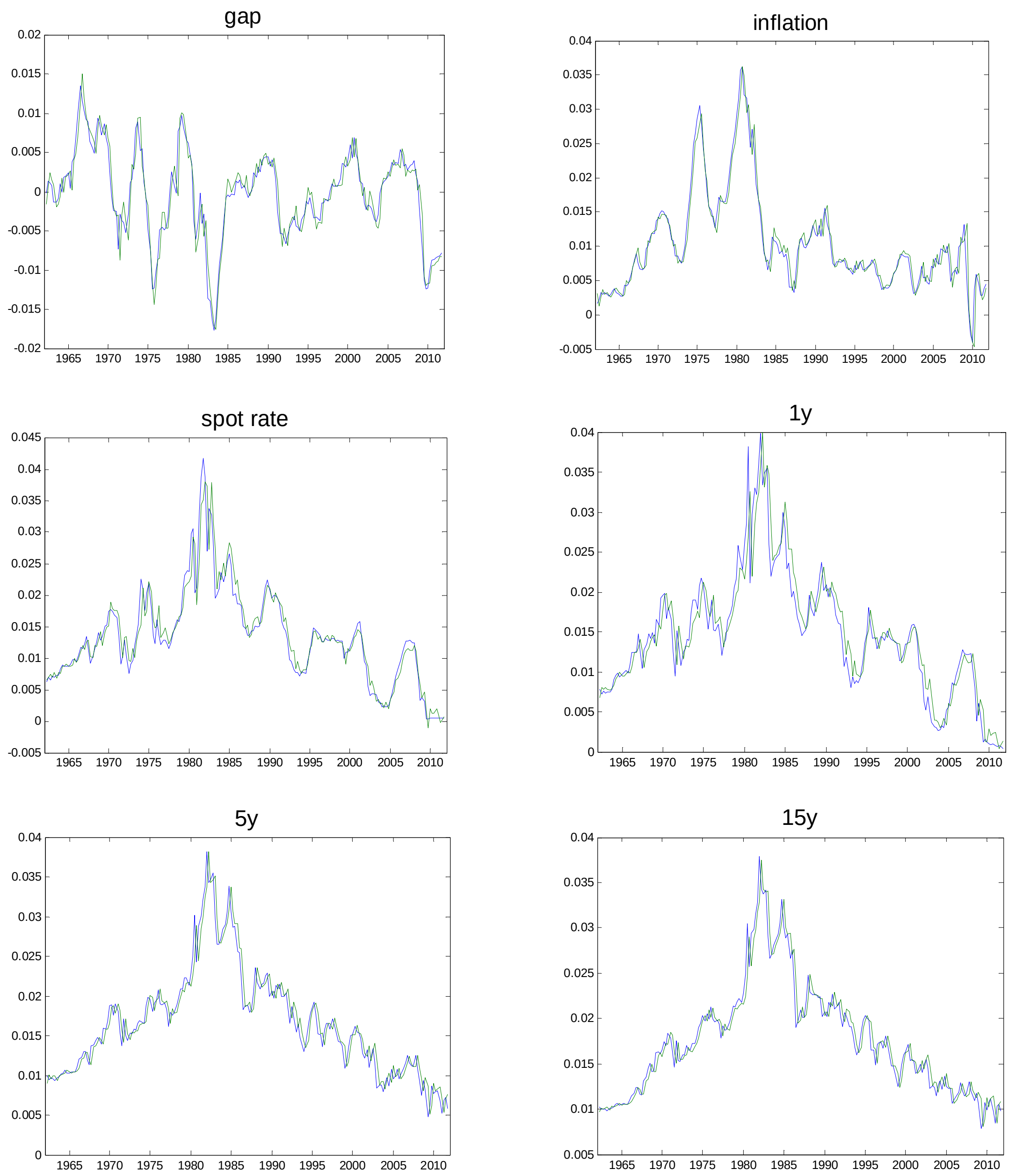
Table 8: Risk parameters in Model M6

| $\mathbf{\Lambda}_0$ | Parameter | t-statistic | $\mathbf{\Lambda}_{\boldsymbol{f}_1}$ | Parameter | t-statistic | $\mathbf{\Lambda}_{\boldsymbol{f}_2}$ | Parameter | t-statistic |
|---|---|---|---|---|---|---|---|---|
| $\lambda_{\boldsymbol{f}_1}$ | (-) | | $\lambda_{\boldsymbol{f}_1\boldsymbol{f}_1}$ | (-) | | $\lambda_{\boldsymbol{f}_2\boldsymbol{f}1}$ | (-) | |
| $\lambda_{\boldsymbol{f}_2}$ | 0.0402 | 2.03 | $\lambda_{\boldsymbol{f}_1\boldsymbol{f}_2}$ | (-) | | $\lambda_{\boldsymbol{f}_2\boldsymbol{f}_2}$ | -0.5756 | 3.29 |
| $\lambda_{\pi^*}$ | -0.0002 | 2.84 | $\lambda_{\boldsymbol{f}_1\pi^*}$ | -0.0033 | 3.88 | $\lambda_{\boldsymbol{f}_2\pi^*}$ | (-) | |
| $\lambda_{\boldsymbol{\rho}}$ | (-) | | $\lambda_{\boldsymbol{f}_1\boldsymbol{\rho}}$ | 0.0023 | 2.82 | $\lambda_{\boldsymbol{f}_2\boldsymbol{\rho}}$ | 0.0116 | 2.78 |
| $\lambda_g$ | (-) | | $\lambda_{\boldsymbol{f}_1 g}$ | -0.0211 | 2.06 | $\lambda_{\boldsymbol{f}_2 g}$ | (-) | |
| $\lambda_\pi$ | (-) | | $\lambda_{\boldsymbol{f}_1\pi}$ | (-) | | $\lambda_{\boldsymbol{f}_2\pi}$ | 0.0783 | 2.78 |
| $\lambda_r$ | -0.0014 | 2.90 | $\lambda_{\boldsymbol{f}_1 r}$ | (-) | | $\lambda_{\boldsymbol{f}_2 r}$ | -0.0165 | 2.64 |

This table shows the parameters of the price of risk in the preferred model (equations (15) and (20)). Their effect on the risk premia is shown by (29). See notes to table 4. The entry (-) indicates that the parameter was not significant in $M_3$ and deleted from $M_6$. The entries in the first row indicate that the risk of variations in the first financial factor is not priced, but the non-zero entries in the central column indicate that this factor does affect the price of risk associated with the inflation trend, the output gap and the interest rate.

## 6 Figures

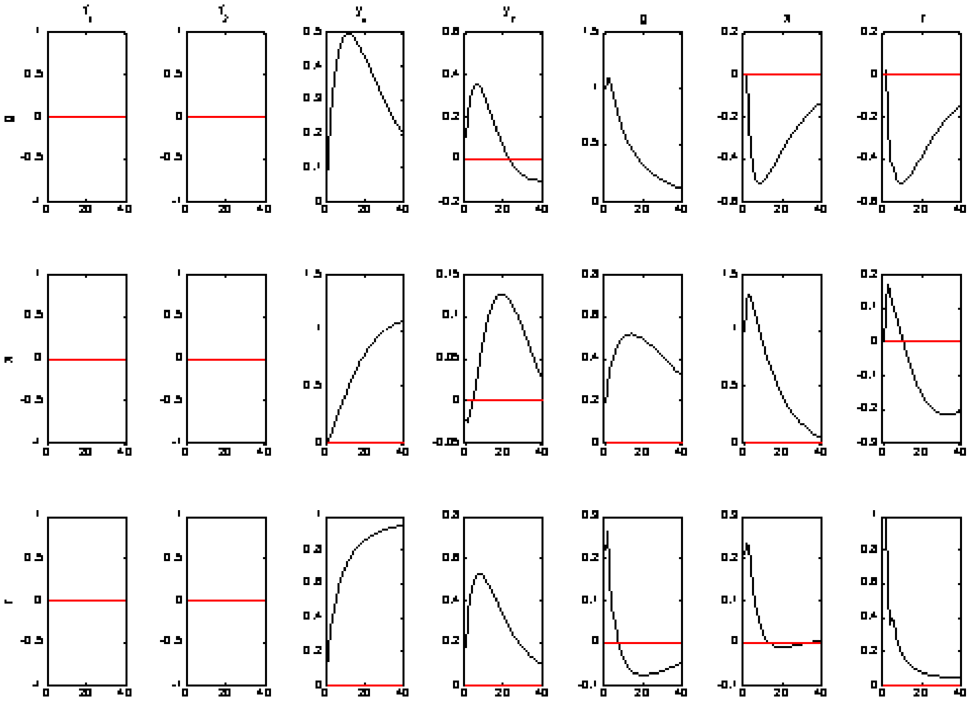
Figure 1: Model M6 estimates and actual values



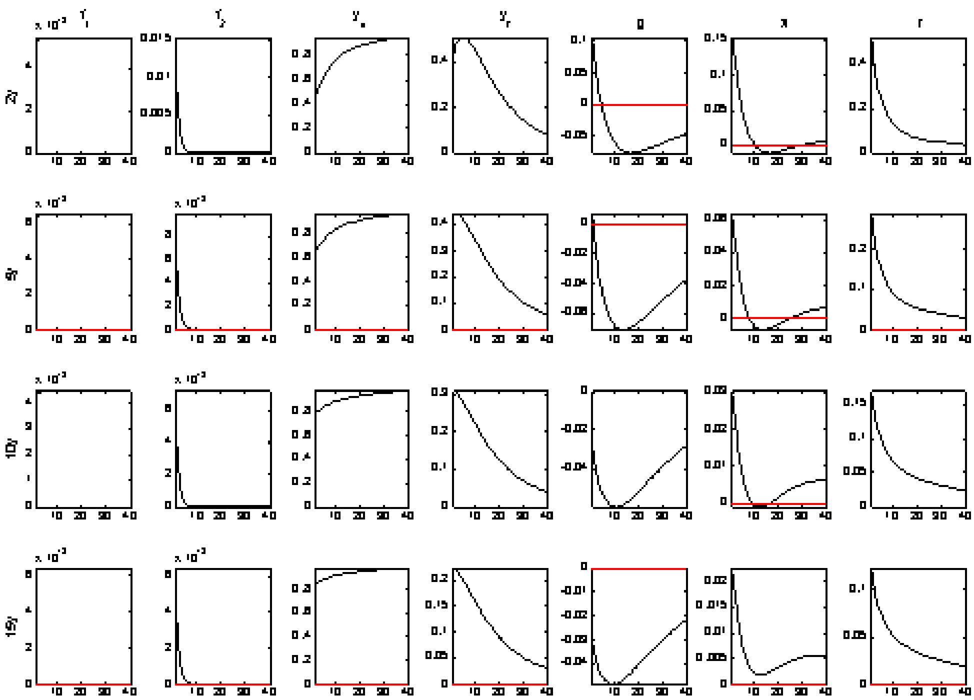

The blue lines in these figures show the actual values for the three macro variables and three representative yields, while the green lines show the one-step ahead estimated values from model $M_6$. The data sources are listed in the footnote to table 1.

Figure 2: Impulse response functions

(a) Macro variables



(b) Yields

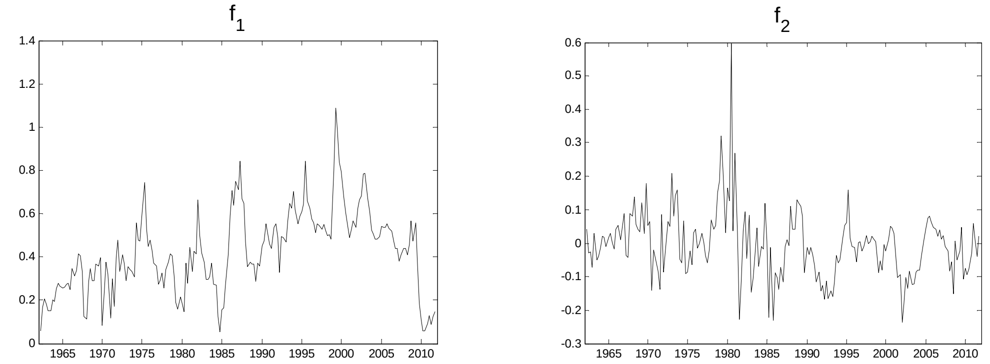




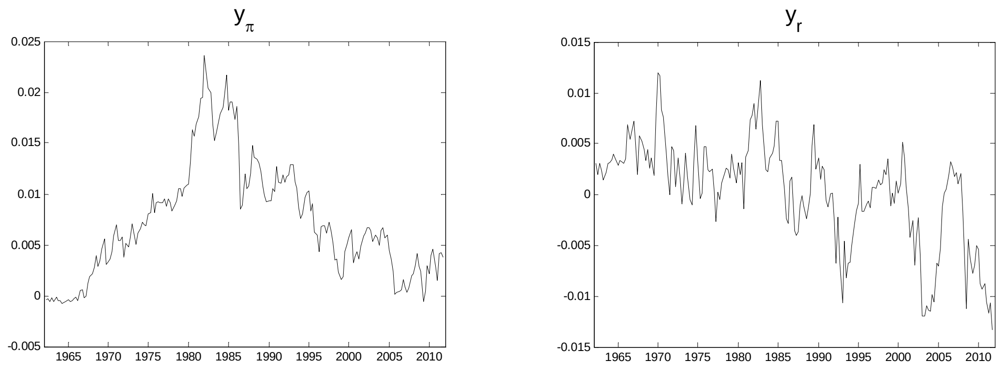
Each sub-plot shows the effect in model $M_6$ of a unit one period shock to one the innovations $(W_t)$ shown in (17). The first panel shows the macroeconomic effects and the second the effect on the 2,5,10 and 15 year yields. Since $f_1$ and $y_\pi$ are martingales, their shocks have permanent effects, while other shocks are transient. Time is measured in quarters.

Figure 3: Estimated financial and macro factors in model M6

(a) Financial factors

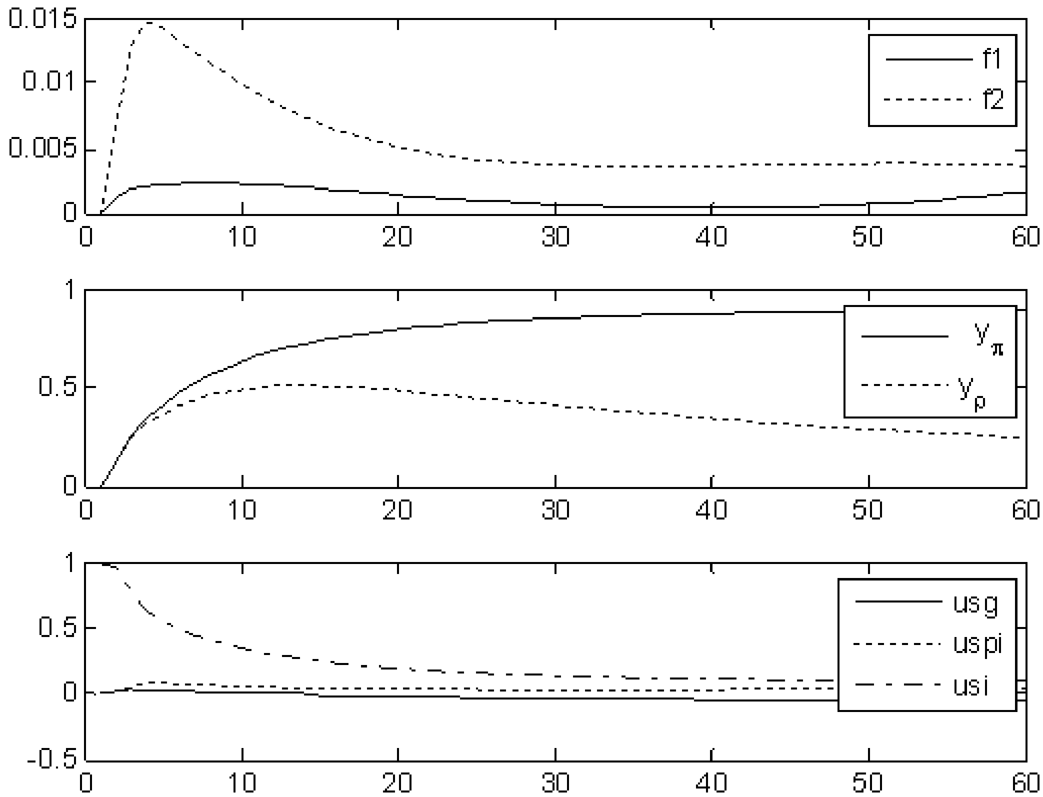



(b) Central tendencies



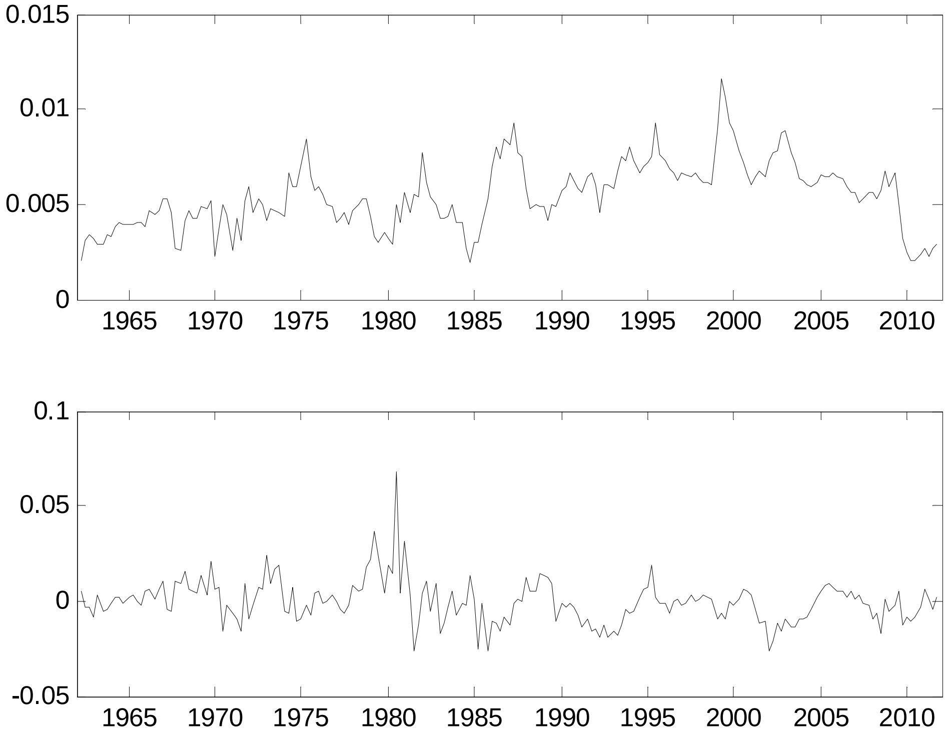

The top panel shows the financial factors $f_1$ and $f_2$ that drive the risk premium in model $M_6$ while the lower panel shows the factors representing the central tendencies of inflation and real interest rates. These factors are modeled using the Extended Kalman Filter

Figure 4: Model M6 factor loadings



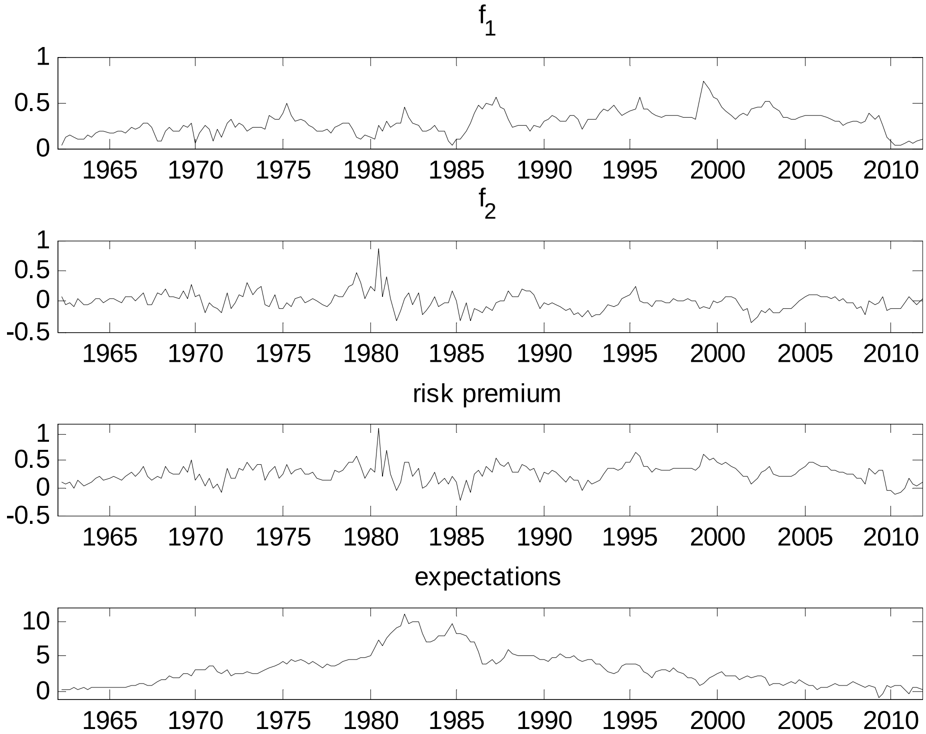

The top panel shows the loadings on the financial factors $f_1$ and $f_2$ that drive the risk premium in model $M_6$, the central panel those on the central tendencies $y_{\pi,t}$ and $y_{r,t}$ while the lower panel shows those on $\pi$, $g$ and $r$. As explained in section 2.2, these loadings depend critically upon the type of shock and its persistence.

Figure 5: Contribution of the financial factors to the required one period return on the 15 year bond in model M6




The risk premia in model $M_6$ are driven by the financial factors $f_1$ and $f_2$. This figure shows the effects of variations in these factors upon the single period returns requited by investors in the 15 year maturity while figure 5 shows their effect on the 15 year yield to maturity. Although the effect of variations in the price of risk is much smaller here than for shorter maturities the single period effect is nevertheless on a similar scale to that of the volatility factor. In contrast, figure 5 shows that the effect of variations in volatility on the required 15 year yield to maturity were much greater than the effect of variations in the price of risk.

Figure 6: Contribution of financial and macrofactors to the required 15 year yield in model M6




See footnote to table 4. The first two panels show the effect of the price of risk factors $f_1$ and $f_2$on the required rate of return on 15 year bonds in model $M_6$. The third panel shows their combined effect, which is the risk premium. The bottom panel shows the expectations component, essentially the average interest rate investors expect over the following 15 years. This is determined by the macromodel set out in (1) and (2) under the real world probability measure. Adding this to the risk premium would give the estimated 15 year yield.