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The Long and Winding Road: Valuing Investment under Construction Uncertainty

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Abstract

This paper presents a model of investment in projects that are characterized by (i) uncertainty over both the construction costs and revenues, and (ii) revenues that accrue only after construction is completed. Both processes are modeled as spectrally negative Lévy jump-diffusions. The optimal stopping problem that determines the value of the project is solved under fairly general assumptions. It is found that the threshold for the benefit-to-cost ratio (BCR) beyond which investment is optimal is higher than when investment costs are sunk and upfront. In addition, the current value of the BCR decreases sharply in the frequency of negative shocks to the construction process. This implies that the cost overruns that can be expected if one ignores such shocks are sharply increasing in their frequency. Based on calibrated data, the model is applied to the construction of high-speed rail in the UK and it is found that the economic case for the first phase of High Speed 2 cannot be made and is unlikely to be met in the next 10 years.

Keywords: Investment under Uncertainty, Infrastructure investment, Optimal stopping

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1 Introduction

The theory of investment under uncertainty has been successful in the past few decades to help decisionmakers understand how uncertainty over future payoffs influences the optimal timing of investment projects.¹ Many investment projects, however, are not only characterized by uncertainty over future payoffs, but also over the costs of construction. For example, large-scale infrastructure projects are often plagued by substantial uncertainty over the time it takes to construct them. Moreover, the benefits of such projects do not accrue until the construction process is finalized. As a result the present value of the revenues has to be discounted more than the present value of the costs.

This paper presents a model for projects that are influenced by two sources of uncertainty: one over future revenues and one over the construction period. These two factors are modeled as two (possibly correlated) Lévy jump-diffusions. The main result presents a way to compute the optimal time to invest when revenues do not accrue until some pre-specified level of the process influencing the construction is reached. For example, if the project under consideration is a railway line, the process underlying the construction could represent the mileage of track that has been constructed up to a certain point in time. Revenues can only accrue when the distance between two cities has been covered.

It is common practice in project evaluation to base decisions on the *benefit-to-cost ratio* (BCR).² This is the ratio of the (estimated) present value of future revenues and the (estimated) present value of the construction costs. Orthodox theory teaches that a project is worthwhile if the BCR exceeds unity. Standard real options theory shows that this threshold should be increased in order to take into account revenue uncertainty. This paper argues that the threshold should be even higher to account for (i) uncertainty over total construction costs and (ii) the time difference between incurring costs and accruing revenues.

The importance of the development of techniques dealing with construction uncertainty is well-established empirically. For example, Pohl and Mihaljek (1992) show that there tends to be a divergence between *ex ante* and *ex post* project evaluations, especially when construction times are long and uncertain. In particular, appraisal estimates tend to be too optimistic (i.e. the reported BCR is too high). A study by Flyvbjerg et al. (2002), using data on 258 transportation infrastructure projects worth US\$90 billion, shows that almost 9 out of 10 projects have higher costs than estimated and that the average cost overrun is 28%. For rail projects this increases to 45%. The same authors, in Flyvbjerg et al. (2004), expand on these results

¹See, for example, McDonald and Siegel (1986), Brennan and Schwartz (1985), Dixit and Pindyck (1994), Cortazar et al. (1998).

²See, for example, Vickerman (2007) for an overview.

and find evidence that cost overruns are more prominent the longer the implementation phase of the project. Even though the engineering profession continues to work on improving the methods used for cost-benefit analysis, none of these models is explicitly dynamic.³

The model developed here is illustrated for a project where the construction process follows a spectrally negative geometric Lévy process and the revenue process follows a geometric Brownian motion. A 2013 report into the viability of a high-speed rail link between the UK cities of London and Birmingham (HS2) serves as a basis for a numerical illustration to estimate the current BCR estimate and its threshold of this project. It is found that the report overestimates the current BCR an that it does not meet the threshold that arises from the methodology advocated in this paper. In fact, it is argued that the probability that the economic case for HS2 is, based on the figures used, very unlikely to be met in the next 10 years. In its focus on high-speed rail investment as a case study, the paper is related to Pimantel et al. (2012). That paper does identify time-to-build as an important factor in high-speed rail construction, but does not take it specifically into account.

The contribution of the paper is two-fold. On the mathematical side, a new optimal stopping problem is solved under fairly general conditions that can be used to model many real-life situations. Second, the paper provides a methodology that can help policy-makers in assessing large-scale investment projects where the time (and, hence, the cost) of construction are uncertain. Current practice often produces present values of costs and benefits based on predictions far into the future with some prediction interval. In order to apply the methodology advocated here, one needs estimates for benefits and costs *in the current period only*, together with estimates of the growth rate and volatility. The chosen stochastic process then automatically delivers the correct estimates and prediction intervals for any time in the future.

In the existing literature on real options time to build is largely ignored. A notable exception is Alvarez and Keppo (2002), who consider a model of investment under uncertainty where the time to build is deterministic. This paper builds on theirs by allowing the time to build to be stochastic. In fact, a stochastic time to build is easier to deal with in some sense. Mathematically, the result presented here relies heavily on the strong Markovian nature of Lévy processes and applications of Dynkin's formula.

The paper is organized as follows. In Section 2 the issues surrounding appraisal of investment projects under uncertainty are introduced. Section 3 presents the model and the main results. In Section 4 the optimal threshold for the BCR is computed and it is shown that uncertain time to build implies a higher threshold BCR over and above the threshold when only revenues are uncertain. Section 5 provides a particular ex-

³See, for example, Mills (2001), Molenaar (2005), and Touran and Lopez (2006).

ample, where the optimal BCR threshold is computed analytically when the stochastic processes follow spectrally negative geometric Lévy processes. A case study of high-speed rail in the UK is presented in Section 6 and Section 7 provides some concluding remarks.

2 An Introduction to the Issues at Stake

The standard way of appraising investment projects is by conducting a cost-benefit analysis, resulting in a benefit-to-cost ratio (BCR). Typically, such an exercise consists of estimating the present value of the benefits, PV, and an estimate of the sunk costs, I, resulting in BCR = PV/I. Investment should take place if, and only if, BCR > 1.

It has been recognized for several decades now that this approach ignores the irreversibility of the decision and the uncertainty surrounding benefits and/or costs. These give the decision-maker an *option value of waiting*: by delaying investment one can see how the probability of future losses evolves. A decision to invest should be made only when that probability is low enough. For example, consider the construction of a railway line. The future benefits of the line depend crucially on passenger numbers, Y. Suppose that the process $(Y_t)_{t>0}$ follows a geometric Brownian motion, i.e.

$$\frac{dY}{Y} = \mu dt + \sigma dB_t, \quad Y_0 = y,$$

where $(B_t)_{t\geq 0}$ is a standard Wiener process, i.e. $B_t \sim N(0, t)$. Then a railway line that runs forever at constant operating costs, *oc*, and a constant ticket price, *p*, has a present value, discounted at the constant rate $r > \mu$, of

$$PV(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} (pY_t - oc) dt \right] = \frac{py}{r - \mu} - \frac{oc}{r}$$

The optimal time of investment is determined by the solution to the optimal stopping problem

$$F^*(y) = \sup_{\tau} \mathsf{E}_y \left[e^{-r\tau} \left(PV(Y_{\tau}) - I \right) \right],$$

over the set of all stopping times.⁴ Note that the present value is computed at time τ , i.e. when the railway becomes operational. So, one has to find the optimal stopping time τ at which to exchange the sunk costs I for the then current estimate of the present value of life-long benefits. It is well-known (see, for example, Dixit and Pindyck, 1994) that the solution to this problem prescribes that one should invest as soon as passenger numbers exceed the threshold

$$Y^* = \frac{\beta_1}{\beta_1 - 1} \frac{r - \mu}{p} \left(\frac{oc}{r} + I\right),$$

⁴Intuitively, a stopping time is a random time of which you can ascertain at any point in time whether it has passed or not.

where $\beta_1 > 1$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0.$$

In terms of the BCR that means that investment should take place as soon as

$$BCR(y) = \frac{PV(y)}{I} \ge \frac{PV(Y^*)}{I} = 1 + \frac{1}{\beta_1 - 1} \left[1 + \frac{oc/r}{I} \right] > 1.$$

This is the familiar threshold that is determined by a balance of the cost of foregoing revenues and the benefits of waiting for more information and reducing the risk of encountering future losses.

This approach, however, ignores that the construction process takes time and is, in general, of an uncertain duration. Typically one estimates the construction time, say T years, and then projects passenger numbers T years from now, on the basis of which the present value of revenues is computed and then discounted back to the present time. However, if at some stage the construction process runs into unexpected delays (due to unexpected geological or environmental issues, strikes, problems in the supply chain, etc.) then the projected time at which revenues start being accrued has to be pushed back. This has three consequences:

- 1. the construction costs are incurred for longer,
- 2. the present value of revenues has to be discounted over a longer period, and
- 3. the projected passenger numbers on which the present value of the project is based are incorrect.

The main idea of this paper is to model the construction process and the revenue process as two, possibly correlated, stochastic processes. Any delays that may occur during construction are automatically tracked and taken into account in the cost and benefit estimates. As a consequence, the model inputs are less demanding than those used in current practice. First, no estimate for the time of completion is necessary. Rather, an expected rate of construction together with an estimate of volatility suffice. The distributional properties of construction progress then follow automatically from the model of the evolution of construction. One consequence is that the expected time of completion can then be computed (or simulated). Secondly, no projected passenger numbers far into the future are needed. Rather, one needs an estimate of *current* passenger demand together with a growth rate and a volatility rate. Again, together with the specific form that is assumed for the stochastic process driving passenger numbers this gives all the distributional information needed. In particular, this would lead to a point estimate of passenger numbers at the expected time of completion interval can be computed.

3 The Model and Main Results

Consider a decision-maker (DM) who can invest in a project that requires costs to be incurred over an uncertain construction time and leads to an uncertain stream of payoffs once construction is completed. The two sources of uncertainty are represented by a stochastic process $(X_t)_{t\geq 0}$, taking values in $\mathcal{X} = (a_1, b_1) \subset \mathbb{R}$, for the construction process, and a stochastic process $(Y_t)_{t\geq 0}$, taking values in $\mathcal{Y} = (a_2, b_2) \subset \mathbb{R}$, for the profit stream. So, the state variable is Z = (X, Y), taking values in $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$. Uncertainty is modeled by a family of probability measures $(\mathsf{P}_z)_{z\in \mathcal{Z}}$ on a measurable space (Ω, \mathscr{F}) , endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$. The restrictions of P_z to \mathcal{X} and \mathcal{Y} are denoted by P_x and P_y , respectively.⁵

The processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are assumed to be adapted to $(\mathscr{F}_t)_{t\geq 0}$ and to follow the time homogeneous Lévy processes

$$dX_{t} = \mu_{1}(X_{t-})dt + \sigma_{1}(X_{t-})dB_{1t} + \int_{\mathbb{R}} \kappa_{1}(u, X_{t-})\tilde{N}_{1}(dt, du), \text{ and } dY_{t} = \mu_{2}(Y_{t-})dt + \sigma_{2}(Y_{t-})dB_{2t} + \int_{\mathbb{R}} \kappa_{2}(u, Y_{t-})\tilde{N}_{2}(dt, du),$$

respectively, where $(B_{i,t})_{t\geq 0}$ are standard Brownian motions with $\mathsf{E}_{(x,y)}(dB_{1t}dB_{2t}) = \rho dt$, \tilde{N}_1 and \tilde{N}_2 are independent compensated Poisson random measures, with Lévy measures m_1 and m_2 , respectively, and $(X_0, Y_0) = (x, y)$, $\mathsf{P}_x \otimes \mathsf{P}_y$ -a.s. It is assumed that both $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are spectrally negative, i.e. that $\kappa_1(\cdot) \leq 0$, P_x -a.s. and $\kappa_2(\cdot) \leq 0$, P_y -a.s.

For any $x, X^* \in \mathcal{X}$ and $y, Y^* \in \mathcal{Y}$, let

$$\tau_x(X^*) = \inf\{t \ge 0 | X_t \ge X^*\} \quad \text{and} \quad \tau_y(Y^*) = \inf\{t \ge 0 | Y_t \ge Y^*\},$$

under P_x and P_y , respectively, be the first hitting times of X^* and Y^* . The generators of $(X_t)_{t\geq 0}$ (on $C^2(\mathcal{X})$), $(Y_t)_{t\geq 0}$ (on $C^2(\mathcal{Y})$) and $(Z_t)_{t\geq 0}$, (on $C^2(\mathcal{Z})$) are given by the partial integro-differential equations

$$\mathscr{L}_X g = \frac{1}{2} \sigma_1^2(x) \frac{\partial^2 g(\cdot)}{\partial x^2} + \mu_1(x) \frac{\partial g(\cdot)}{\partial x} + \int_{\mathbb{R}} [g(x + \kappa_1(u)) - g(x) - \frac{\partial g(\cdot)}{\partial x} \kappa_1(u)] m_1(du),$$

$$\mathscr{L}_Y g = \frac{1}{2} \sigma_2^2(y) \frac{\partial^2 g(\cdot)}{\partial y^2} + \mu_2(y) \frac{\partial g(\cdot)}{\partial y} + \int_{\mathbb{R}} [g(y + \kappa_2(u)) - g(y) - \frac{\partial g(\cdot)}{\partial y} \kappa_2(u)] m_2(du),$$

⁵That is, P_z is the product measure of P_x and P_y .

and

$$\begin{aligned} \mathscr{L}_{Z}g &= \frac{1}{2}\sigma_{1}^{2}(x)\frac{\partial^{2}g(\cdot)}{\partial x^{2}} + \frac{1}{2}\sigma_{2}^{2}(y)\frac{\partial^{2}g(\cdot)}{\partial y^{2}} + \rho(x,y)\sigma(x)\sigma(y)\frac{\partial^{2}g(\cdot)}{\partial x\partial y} \\ &+ \mu_{1}(x)\frac{\partial g(\cdot)}{\partial x} + \mu_{2}(y)\frac{\partial g(\cdot)}{\partial y} \\ &+ \int_{\mathbb{R}}[g(x+\kappa_{1}(u)) - g(x) - \frac{\partial g(\cdot)}{\partial x}\kappa_{1}(u)]m_{1}(du) \\ &+ \int_{\mathbb{R}}[g(y+\kappa_{2}(u)) - g(y) - \frac{\partial g(\cdot)}{\partial y}\kappa_{2}(u)]m_{2}(du) \\ &= \mathscr{L}_{X}g + \mathscr{L}_{Y}g + + \rho(x,y)\sigma(x)\sigma(y)\frac{\partial^{2}g(\cdot)}{\partial x\partial y}, \end{aligned}$$

respectively.

The process $(X_t)_{t\geq 0}$ represents the progress of construction. It starts at some \hat{x} and is finished as soon as some exogenously given $x^* > \hat{x}$ is reached. It is assumed that $[\hat{x}, x^*] \subset \mathcal{X}$, and that $\tau_{\hat{x}}(x^*) < \infty$, $\mathsf{P}_{\hat{x}}$ -a.s. The latter assumption ensures that construction is completed in finite time a.s. The construction costs are given by a measurable function $c : \mathcal{X} \to \mathbb{R}_+$, where it is assumed that

$$\mathsf{E}_{\hat{x}}\left[\int_0^\infty e^{-rt}|c(X_t)|dt\right] < \infty.$$

This assumption ensures that discounted construction costs are, in expectation, finite. Denote these by

$$I(x) = \mathsf{E}_x \left[\int_0^\infty e^{-rt} c(X_t) dt \right] > 0, \quad \hat{x} \le x \le x^*.$$

On the revenue side it is assumed that, once construction is finished, the profit flow accruing from the project is given by some measurable function $f : \mathcal{Y} \to \mathbb{R}$, $f \in C^2(\mathcal{Y})$, with f' > 0 and $f'' \leq 0$, where it is assumed that

$$\mathsf{E}_{y}\left[\int_{0}^{\infty}e^{-rt}|f(Y_{t})|dt
ight]<\infty, \quad \text{all } y\in\mathcal{X}.$$

Denote the present value of revenues by

$$D(y) = \mathsf{E}_y\left[\int_0^\infty e^{-rt} f(Y_t) dt\right], \quad y \in \mathcal{Y}.$$

The net present value of the project, under P_z , $z = (x, y) \in [\hat{x}, x^*] \times \mathcal{Y}$, then equals

$$F(x,y) = \mathsf{E}_{z} \left[-\int_{0}^{\tau_{x}(x^{*})} e^{-rt} c(X_{t}) dt + \int_{\tau_{x}(x^{*})}^{\infty} e^{-rt} f(Y_{t}) dt \right].$$

Note that, for $x \ge x^*$ it holds that $\tau_x(x^*) = 0$ and, thus, that

$$F(x,y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} f(Y_t) dt \right] = D(y), \quad x \ge x^*.$$

For any project that has not started yet, the NPV of commencing when the current value of the process $(Y_t)_{t>0}$ is y equals $F(\hat{x}, y)$.

The DM wishes to choose the investment time to maximize the project's value, i.e. to solve the optimal stopping problem

$$F^*(y) = \sup_{\tau \in \mathscr{M}} \mathsf{E}_y \left[e^{-r\tau} F(\hat{x}, Y_\tau) \right], \tag{1}$$

where \mathcal{M} is the set of stopping times.

Sufficient conditions for a solution to this problem will be given below. First, however, the net present value of the project is determined. The NPV can be used to compute the current estimate of the benefit-to-cost ratio of the project.

Proposition 1. Suppose that

- (i) there exists an increasing solution $\zeta \in C^2(\mathcal{X})$ to the equation $\mathscr{L}_X \zeta = r\zeta$, such that $\zeta(a_1) = 0$;
- (ii) there exists a solution $\varphi \in C^2(\mathbb{Z})$ to the equation $\mathscr{L}_{(X,Y)}\varphi = r\varphi$, such that $\varphi(a_1, y) = \varphi(x, a_2) = 0$, all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and $\varphi(x^*, y) = D(y)$, all $y \in \mathcal{Y}$;

Then the net present value of the investment project is given by,

$$F(\hat{x}, y) = \varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x}).$$
⁽²⁾

The proof of this proposition can be found in Appendix A.

Let the expected construction costs (under P_x) be denoted by

$$\Delta I(x) = \left(1 - \frac{\zeta(x)}{\zeta(x^*)}\right)I(x) > 0,$$

and let \bar{y} denote the traditional NPV threshold of the project, i.e. the smallest value that solves $\varphi(\hat{x}, \bar{y}) = \Delta I(\hat{x})$ (provided it exists). Sufficient conditions for the existence of a solution to the optimal stopping problem (1) can now be established. The solution to this problem will provide the threshold benefit-to-cost ratio against which any current estimate should be compared.

Proposition 2. Suppose that, in addition to the assumptions of Proposition 1,

- (i) the function φ is such that $\varphi'_y > 0$ and $\varphi''_{yy} \le 0$;
- (ii) there exists an increasing and convex solution $\psi \in C^2(\mathcal{Y})$ to the equation $\mathscr{L}_Y \psi = r\psi$, such that $\psi(a_2) = 0$;
- (iii) $\lim_{y\uparrow b_2} \varphi(\hat{x}, y) > \Delta I(\hat{x})$; and

(iv) the function

$$\frac{1}{\psi(y)} \left[\varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)} \right) I(\hat{x}) \right],\tag{3}$$

has a stationary point $y^* \in \mathcal{Y}$.

Then y^* is unique, \bar{y} is unique, $y^* > \bar{y}$, and $\tau(y^*)$ is a solution to the optimal stopping problem (1) with

$$F^*(y) = \begin{cases} \frac{\psi(y)}{\psi(y^*)} \left[\varphi(\hat{x}, y^*) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x}) \right] & \text{if } y < y^* \\ \varphi(\hat{x}, y) - \left(1 - \frac{\zeta(\hat{x})}{\zeta(x^*)}\right) I(\hat{x}) & \text{if } y \ge y^*. \end{cases}$$

If (3) has no stationary point, then the optimal stopping problem has no solution and investment is never optimal.

The proof of this proposition can be found in Appendix B.

A question that remains is whether functions $\zeta(\cdot)$, $\varphi(\cdot)$, and $\psi(\cdot)$ as described in the propositions actually exist. Based on known results in the literature it can be shown that increasing functions $\zeta(\cdot)$ and $\psi(\cdot)$ always exist for any diffusion (cf. Borodin and Salminen, 1996). In addition, if $\mu_y(y) - ry$ is non-increasing, the increasing function $\psi(\cdot)$ is also convex (cf. Alvarez, 2003). The conditions on $\varphi(\cdot)$ are more difficult to establish in any generality. As will be seen in Section 5, for $\varphi(\cdot)$ to be concave in y, the expected growth of the revenues should not be higher than the rate at which revenues are discounted. This makes intuitive sense, for if this is not the case, then the expected discounted revenues will explode. At the same time, the growth rate of construction should exceed the discount rate, because otherwise the revenues will not be positively valued and only the construction costs matter.

4 Evaluating Projects: the Dynamic BCR

A standard way to evaluate a project is to compute its *benefit-to-cost ratio* (BCR). This is simply the ratio of the project's present value and its sunk costs of investment. In the context of this paper the BCR is easily computed as

$$BCR(x,y) = \frac{\varphi(x,y)}{[1-\zeta(x)/\zeta(x^*)]I(x)}.$$

For a project that has not started yet this can be reduced to

$$BCR(y) = \frac{\varphi(\hat{x}, y)}{[1 - \zeta(\hat{x})/\zeta(x^*)]I(\hat{x})}.$$

Standard practice prescribes that an investment should be undertaken when the BCR exceeds unity. Proposition 2, however, prescribes another threshold. The optimal stopping time (i.e. the optimal time of investment) is $\tau(y^*) = \inf\{t \ge 0 | Y_t \ge y^*\}$. Since y^* is a stationary point of (3), i.e.

$$\psi(y^*)\varphi'_y(\hat{x}, y^*) = [\varphi(\hat{x}, y^*) - (1 - \zeta(\hat{x})/\zeta(x^*))I(\hat{x})]\,\psi'(y^*),$$

it follows that investment is optimal if the BCR exceeds the threshold

$$BCR(y^*) = 1 + \frac{\psi(y^*)}{\psi'(y^*)} \frac{\varphi'_y(\hat{x}, y^*)}{(1 - \zeta(\hat{x})/\zeta(x^*))I(\hat{x})} \equiv BCR^*.$$

So, Proposition 2 shows that investment only should take place when the *current* (estimate of) benefits of the investment, y, is such that $BCR(y) \ge BCR^*$. From the assumptions it is obvious that $BCR^* > 1$. Policy makers should, therefore, increase the hurdle rate of investment, a result that is well-known and standard in the literature on real options (see, for example, Dixit and Pindyck, 1994).

5 An Illustration: Building a High-Speed Rail Link

To illustrate how Propositions 1 and 2 can be used, consider the construction of a new high-speed rail link. We model the revenues as a geometric Brownian motion (GBM) on $\mathcal{Y} = \mathbb{R}_+$, and assume $(Y_t)_{t\geq 0}$ follows the stochastic differential equation

$$\frac{dY}{Y} = \mu_2 dt + \sigma_2 dB_2. \tag{4}$$

The construction progress is modeled as a jump-diffusion on $\mathcal{X} = \mathbb{R}_+$, solving the stochastic differential equation

$$\frac{dX_t}{X_{t-}} = \mu_1 dt + \sigma_1 dB_2 - \int_{\mathbb{R}} u \tilde{N}(dt, du), \tag{5}$$

where 0 < u < 1, P_x -a.s. We allow for possible correlation between the two processes: $E[dB_1dB_2] = \rho dt$, where $\rho \in (-1, 1)$. The stream of construction costs is assumed to be constant at c > 0. The costs of operating the rail line are assumed to be constant and equal to oc > 0 per period. The jumps in $(X_t)_{t\geq 0}$ are assumed to be Beta distributed with parameters a and b, i.e.

$$m'(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1} (1-u)^{b-1}.$$
(6)

The increasing solution to $\mathscr{L}_Y \psi = r \psi$ with $\psi(0) = 0$ is easily obtained as

$$\psi(y) = A_2 y^{\beta_1},$$

where $\beta_1 > 0$ is the positive root of the quadratic equation

$$\mathscr{Q}_y(\beta) \equiv \frac{1}{2}\sigma_y^2\beta(\beta-1) + \mu_y\beta - r = 0,$$

and A_2 is a positive constant. The function ψ is convex only if $\beta_1 > 1$, i.e. if $r > \mu_2$.

The increasing solution to $\mathscr{L}_X \zeta = r\zeta$ with $\zeta(0) = 0$ is solved by (cf. Alvarez and Rakkolainen, 2010) $\zeta(x) = A_1 x^{\gamma_1}$, where $\gamma_1 > 0$ is the positive root of the equation

$$\frac{1}{2}\sigma_1^2\gamma(\gamma-1)x^\gamma + \mu_1\gamma x^\gamma - r + \int_{\mathbb{R}} [(x-xu)^\gamma - x^\gamma + \gamma u]m_1(du) = 0$$

$$\iff \frac{1}{2}\sigma_1^2\gamma(\gamma-1) + (\mu_1 + \lambda \mathsf{E}(U))\gamma - r - \lambda + \lambda \mathsf{E}[(1-U)^\gamma] = 0$$

$$\iff \frac{1}{2}\sigma_1^2\gamma(\gamma-1) + \left(\mu_1 + \lambda \frac{a}{a+b}\right)\gamma - (r+\lambda) + \lambda \frac{\Gamma(a+b)\Gamma(b+\gamma)}{\Gamma(b)\Gamma(a+b+\gamma)} = 0,$$

and A_1 is a positive constant.

In order to obtain φ , first note that

$$\varphi(v,y) = Bx^{\alpha}y^{1-\alpha}$$

solves the differential equation $\mathscr{L}_{(X,Y)}\varphi - r\varphi = 0$ only if α solves the equation

$$\mathcal{Q}(\alpha) \equiv \frac{1}{2} [(\sigma_1 - \sigma_2)^2 + 2(1 - \rho)\sigma_1\sigma_2]\alpha(\alpha - 1) + \left(\mu_1 - \mu_2 + \lambda \frac{a}{a + b}\right)\alpha + \mu_2 - (r + \lambda) + \lambda \frac{\Gamma(a + b)\Gamma(b + \alpha)}{\Gamma(b)\Gamma(a + b + \alpha)} = 0.$$

If $\rho < 1$ this equation has two roots, α_1 and α_2 , and, since $\mathscr{Q}(0) < 0$, it holds that $\alpha_1 > 0$ and $\alpha_2 < 0$. So, the general solution to the equation $\mathscr{L}_{(X,Y)}\varphi = r\varphi$ is

$$\varphi(x,y) = B_1 x^{\alpha_1} y^{1-\alpha_1} + B_2 x^{\alpha_2} y^{1-\alpha_2},$$

where B_1 and B_2 are constants.

In order to satisfy the boundary conditions $\varphi(x, 0) = \varphi(0, y) = 0$, it needs to hold that $B_2 = 0$ and $\alpha_1 < 1$, respectively. The latter condition is fulfilled if $r < \mu_1$, which implies that the growth rate of the construction process should exceed the discount rate. This makes intuitive sense, for if $r > \mu_1$, then the expected revenues are discounted faster than the rate of progress on the construction, which implies that the construction costs fully drown out the revenues.

Note that I(x) = c/r, so that we find

$$\Delta I(x) = \left[1 - \left(\frac{x}{x^*}\right)^{\gamma_1}\right] \frac{c}{r}$$

It also follows that the present value of the profits of the project is

$$D(y) = \mathsf{E}_y \left[\int_0^\infty e^{-rt} (Y_t - oc) dt \right] = \frac{y}{r - \mu_2} - \frac{oc}{r}$$

The boundary condition $\varphi(x^*, y) = D(y)$ then gives

$$B_1 = \left[\frac{y}{r - \mu_2} - \frac{oc}{r}\right] (x^*)^{-\alpha_1} y^{\alpha_1 - 1},$$

so that

$$\varphi(x,y) = \left[\frac{y}{r-\mu_2} - \frac{oc}{r}\right] \left(\frac{x}{x^*}\right)^{\alpha_1}$$

Note that $\varphi_x' > 0, \, \varphi_y' > 0$, and $\varphi_{yy}'' \le 0$.

Since all the conditions of Proposition 2 are met, the optimal value of the project can be obtained by finding a stationary point y^* of

$$\frac{1}{\psi(\hat{y})} \left[\varphi(\hat{x}, \hat{y}) - \left(1 - \left(\frac{\hat{x}}{x^*} \right)^{\gamma_1} \right) \frac{c}{r} \right]$$

Standard computations yield that

$$y^* = \frac{\beta_1}{\beta_1 - 1} (r - \mu_2) \left[\frac{oc}{r} + \frac{c}{r} \left(1 - \left(\frac{\hat{x}}{x^*}\right)^{\gamma_1} \right) \left(\frac{\hat{x}}{x^*}\right)^{-\alpha_1} \right].$$

The threshold BCR beyond which investment is optimal then can be computed as

$$BCR^* = 1 + \frac{1}{\beta_1 - 1} \left[1 + \frac{oc}{c} \frac{(\hat{x}/x^*)^{\alpha_1}}{1 - (\hat{x}/x^*)^{\gamma_1}} \right]$$

The specifics of the underlying stochastic construction process determines the expected construction time, provided it exists. For the model described in this section the following lemma, the proof of which can be found in Appendix C, describes this operator. The lemma uses the *digamma function*, ψ , defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$.

Lemma 1. Suppose that $(X_t)_{t>0}$ follows the diffusion (5) with Lévy measure (6). If

$$\mu_1 + \lambda \left(\frac{a}{a+b} + \psi(b) - \psi(a+b)\right) > \frac{1}{2}\sigma_1^2,\tag{7}$$

then $E_x[|\tau(x^*)|] < \infty$, for any $x < x^*$, and

$$\boldsymbol{E}_{x}[\tau(x^{*})] = \frac{\log(x^{*}/)}{\mu_{1} + \lambda \left(\frac{a}{a+b} + \psi(b) - \psi(a+b) - \frac{1}{2}\sigma_{1}^{2}\right)}.$$

6 An Application: High-Speed Rail in the UK

As an application of the model presented in Section 5, this section will look at a particular case study: investment in Phase 1 of HS2, a high-speed rail link between London and Birmingham in the UK. A recently published "strategic case" provides the figures used below, which are taken at face value and used merely

for illustrative purposes.⁶ This report estimates the (present value in 2011 prices) benefits of this rail link to be £28bn (this includes £4.3bn in wider economic benefits), whereas the costs are estimated to be £15.65bn. Operating costs are estimated to have a present value of £8.2bn. The report includes capital spending such as replacement of rolling stock, etc., which will be ignored here. The report then provides a BCR of 1.7, which renders this a "medium value" project in government parlance.

The estimate of the BCR is obtained using traditional methods as described in Section 2. As a result, the parameters for the model described here have to be calibrated and "guesstimated" based on the information provided. The estimate of the (present value of the) construction costs are given with the upper bound of a 95% prediction interval of £21.4bn, with an estimated time to completion of 8 years. The prediction interval is consistent with a volatility of $\sigma_1 = .994$.⁷ The current state of construction is taken to be $\hat{x} = 1$. Birmingham is 150 miles from London, so the average construction speed over 8 years is 18.75 miles of track p.a. This is commensurate with an average growth rate of $\mu_1 = 2.36$. Since it is estimated that this distance will be covered in 8 years, the inferred value for x^* (i.e. the value that gives $\mathsf{E}_{\hat{x}}[\tau(x^*)] = 8$) is 4.48. The cost flow is then inferred to be $c = \pounds 2.24$ bn p.a.

The discount rate used in the report is 3.5%, which is transformed to the continuous rate r = .0344. The present value of the railway is estimated to be £28bn. No clear growth rate of revenues is mentioned in the report, so it will be assumed here that $\mu_2 = .022$, which is the assumed growth rate of passenger numbers. A present value of £8.2bn leads for operating costs leads to a constant operating cost flow of oc = .28. This implies that the estimated value of Y_8 is

$$e^{8r}\left(\frac{Y_7}{r-\mu_2}-\frac{oc}{r}\right)\iff Y_8=.3742,$$

which implies that $Y_0 = e^{-8\mu_2} = .3138$. The volatility of revenues accruing from HS2 is taken to be $\sigma_2 = .2.^8$

Since we assume that the construction process can be modeled as a stochastic process $(X_t)_{t\geq 0}$ which follows a GBM with Beta distributed negative jumps, the jump component of the process $(X_t)_{t\geq 0}$ still needs to be determined. We take the expected jump rate to be 3/7 (i.e. a = 1.5 and b = 2). As a baseline case for the frequency of an unexpected delay we assume that they occur, on average, once a year, i.e. $\lambda = 1$. A sample path for both $(X_t)_{t>0}$ and $(Y_t)_{t>0}$ is given in Figure 1.

⁶All figures quoted in this paper are taken from "The Strategic Case for HS2", published on October 29, 2013 by the Department for Transport (DfT) and High Speed Two (HS2) Ltd. The report can be downloaded from https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/254360/strategic-case.pdf.

⁷This assumes that the report's authors used a normal distribution. No distributional assumptions are given in the report.

⁸This figure is not given in the report, but corresponds to a values used regularly in the literature, see Dixit and Pindyck (1994).



Figure 1: Sample paths for $(X_t)_{t>0}$ and $(Y_t)_{t>0}$, based on base-case parameters

Using these data we find the following estimate of the current BCR, BCR_0 , and the threshold BCR, BCR^* , as well as the expected time to completion, expected construction costs, and expected cost overrun.

λ	BCR_0	BCR^*	$E[\tau(x^*)]$	$\Delta I(\hat{x})$	Cost overrun (%)
0	.99	6.62	8	15.58	42
1	.87	6.39	9.24	17.55	12.11

Even if the construction process does not suffer from unexpected shocks, but just from day-to-day risk, (i.e. if $\lambda = 0$) three conclusions can be drawn. First, the expected costs are actually somewhat lower than DfT estimates. This is because risk can lead to construction slowdowns, but also to positive shocks. So, it is possible that construction finishes before 8 years have passed, in which case expected costs are discounted less. Secondly, the estimated BCR of 1.7 is wide off the mark and comes in at .99. This happens because the revenues are discounted much more than the report allows for. This, in turn, is due to the fact there is uncertainty over the benefits while the railway line is active, but also *while construction is taking place*. Under current DfT practice a BCR of .99 would put the project just on the cusp of being "medium value for money". However, and this is the third conclusion, the BCR threshold beyond which a project can be called value for money is not unity, but, in this case, 6.62. In fact, one can compute the probability with which this

threshold is reached in, say, the next 10 years: 1.37%.⁹

As is to be expected, the picture is worse if there are unexpected negative shocks to the construction process. If, on average, there is one such event per year, the current BCR estimate drops further to .87, while the BCR threshold decreases to 6.39. The latter decrease might seem surprising, but is due to the fact that we are working with a compensated Poisson process and the compensator is positive, because the shocks are negative. The decrease in the current estimate is bigger, however, so that, on balance, the effect of unexpected shocks is negative. This can be seen from the expected construction time which goes up to 9.24 years, whereas the present value of expected costs goes up to \pounds 17.55bn, i.e. an expected cost overrun of 12.11%. The probability that the BCR threshold is reached within the next 10 years also goes down, to .90%.

Note, by the way, that even if one does not take construction uncertainty into account and accepts $BCR_0 = 1.7$, then threshold used in Section 2 still implies that the project is value for money only if the threshold $BCR^* = 6.81$ is reached.

The effect of the frequency of unexpected negative shocks on several quantities of interest can be found in Figure 2. The panel labeled "Stochastic discount factor" gives the ratio of the expected discount factors with which revenues and costs are multiplied, respectively. So, for $\lambda = 0$, the benefits are discounted by a factor that is 4.3 times higher than the construction costs. This dichotomy is due to the fact that costs precede benefits and, as can be seen the effect is substantial. Also note that the expected cost overrun (relative to the DfT estimate of 15.65bn) is steeply increasing in λ . In fact, the average rate of 28% found by Flyvbjerg et al. (2002) corresponds to $\lambda \approx 2$, whereas the 45% rate reported for railways corresponds to $\lambda \approx 2.85$.

Comparative statics for σ_2 are given in Figure 3. The threshold BCR is increasing in σ_2 . This happens because an increase in the volatility of the revenues increases the option value of the project. As is well known (see Sarkar, 2000) this does not necessarily imply that the probability of investment is also increasing. The probability of the threshold being reached within the next 10 years is reported in the right panel of Figure 3. This probability is non-monotonic in σ_2 . It also shows that the likelihood that the case for HS2 will be economically sensible in the next ten years is fairly low.

Figure 4 plots comparative statics for the *convenience yield*, $\delta = r - \mu_2$. This rate can be thought of as the "dividend" rate that one forgoes while not investing in the project. Note that the probability that investment will be optimal in the next 10 years in fairly sensitive to this parameter. This is due to the sensitivity of results to the discount rate, as can be seen in Figure 5. This sensitivity is well-documented and shows the

⁹See Thijssen (2010) for an explicit formula for this probability.



Figure 2: Comparative statics for frequency of unexpected shocks, λ .



Figure 3: Comparative statics for revenue volatility, σ_2 .

importance of a careful study into its effects.

Finally, Figure 6 gives the comparative statics for the present value of the wider economic benefits. The threshold BCR is insensitive to this value and equals $BCR^* = 6.62$. The current BCR estimate and



Figure 4: Comparative statics for convenience yield, δ .



Figure 5: Comparative statics for discount rate, r.

probability that the threshold will be reached within 10 years are fairly sensitive to this value.

7 Conclusion

This paper presents a model of investment under uncertainty where the time of construction is influenced by a stochastic process and revenues only start accruing when the construction process is finalized. As a result the expected discount factor applied to revenues is higher than the expected discount factor applied to costs. This, in turn, increases the threshold BCR beyond which investment is optimal. A case study using



Figure 6: Comparative statics for wider economic benefits.

data from a report on the development of high-speed rail in the UK points to a few effects. Firstly, both the threshold BCR and the current estimate of the BCR are increasing in the volatility of the construction process. This is a standard result in the real options literature and is due to the fact that an increase in the volatility increases the option value of the project. In fact, the probability of the BCR reaching the threshold within 10 years can be increasing in volatility.

The presence of unpredictable negative shocks to the construction process, however, reduces the current estimate of the BCR and the threshold BCR. The reduction in the current estimate is higher than in the threshold, which implies a lower probability of investment being optimal within a certain period. It has been shown numerically that this reduction in BCR can be dramatic.

Several caveats can be added to the application of the model to HS2. First, the parameter values used may not be appropriate given that they have been "backed out" from an analysis that is not suited for the approach presented here. Secondly, it is not obvious that exponential Lévy processes are best suited to model the uncertainty in construction and benefits. Perhaps arithmetic processes would be better suited. However, the main points that standard practice overestimates the BCR and underestimates the threshold BCR are undisputed. It is important to realise that these effects are not due to risk aversion or the application of the precautionary principle. The decision-maker in this model has been modeled as being risk-neutral. The results are entirely due to the dynamic uncertainty in costs and benefits. This shows that the likelihood of unpredictable construction delays is ignored at some peril.

Appendix

A Proof of Proposition 1

First note that F can be written as

$$F(x,y) = -\mathsf{E}_x \left[\int_0^\infty e^{-rt} c(X_t) dt \right] + \mathsf{E}_{(x,y)} \left[\int_{\tau_x(x^*)}^\infty e^{-rt} \left(f(Y_t) + c(Y_t) \right) dt \right]$$

= $\mathsf{E}_{(x,y)} \left[e^{-r\tau(x^*)} D(Y_{\tau(x^*)}) \right] - \left(1 - \mathsf{E}_x \left[e^{-r\tau(x^*)} \right] \right) I(x).$

Since $\mathsf{E}_x[\tau(x^*)] < \infty$ by assumption, an application of Dynkin's formula gives

$$\mathsf{E}_x\left[e^{-r\tau(x^*)}\zeta(X_{\tau(x^*)})\right] = \zeta(x) + \mathsf{E}_x\left[\int_0^{\tau(x^*)} e^{-rt}\left(\mathscr{L}_X\zeta(X_t) - r\zeta(X_t)\right)dt\right] = \zeta(x).$$

So, since $(X_t)_{t\geq 0}$ is spectrally negative,

$$\mathsf{E}_x\left[e^{-r\tau(x^*)}\right] = \frac{\zeta(x)}{\zeta(x^*)}.$$

Therefore,

$$\left(1 - \mathsf{E}_x\left[e^{-r\tau(x^*)}\right]\right)I(x) = \left(1 - \frac{\zeta(x)}{\zeta(x^*)}\right)I(x).$$

Another application of Dynkin's formula gives that

$$\begin{aligned} \mathsf{E}_{(x,y)} \left[e^{-r\tau(x^*)} \varphi(X_{\tau(x^*)}, Y_{\tau(x^*)}) \right] = &\varphi(x, y) \\ &+ \mathsf{E}_{(x,y)} \left[\int_0^{\tau(x^*)} e^{-rt} \left(\mathscr{L}_Z \varphi(X_t, Y_t) - r\varphi(X_t, Y_t) \right) dt \right] \\ = &\varphi(x, y). \end{aligned}$$

Since $\varphi(X_{\tau(x^*)}, Y_{\tau(x^*)}) = D(Y_{\tau(x^*)})$, $\mathsf{P}_{x,y}$ -a.s., it holds that

$$\mathsf{E}_{(x,y)}\left[e^{-r\tau(x^*)}D(Y_{\tau(x^*)})\right] = \varphi(x,y).$$

This establishes F.

B Proof of Proposition 2

The proof is established in several steps.

1. Recall from Proposition 1 that $F(x,y) = \varphi(x,y) - \Delta I(x)$. Since $\varphi(\hat{x},a_2) = 0 < \Delta I(\hat{x})$ and $\zeta' > 0$,

assumption (iii) implies that there is a unique $\bar{y} \in \mathcal{Y}$ such that $F(\hat{x}, \bar{y}) = 0$.

2. On $[y^*, b_2)$ it holds that $F^*(\cdot) = F(\hat{x}, \cdot)$. Since $\psi(a_2) = 0 > F(\hat{x}, a_2) = -\Delta I(\hat{x}), \varphi'_y > 0, \psi' > 0$, and

$$\lim_{y \uparrow y^*} F^*(y) = F(\hat{x}, y),$$

it holds that $F^*(\cdot) > F(\hat{x}, \cdot)$ on $[y^*, b_2)$. So, $F^*(\cdot) \ge F(\hat{x}, \cdot)$ on \mathcal{Y} .

Denote

$$C = \{ y \in \mathcal{Y} \mid F^*(y) > F(\hat{x}, y) \}.$$

This set is also called the *continuation region* where waiting is optimal.

3. We show that C is a connected set, such that $(a_2, \bar{y}] \subset C$. Suppose that (1) has a solution. From Peskir

and Shiryaev (2006, Theorem 2.4) we know that $F^*(\cdot)$ is the least superharmonic majorant of $F(\hat{x}, \cdot)$ on \mathcal{Y} and that the first exit time of C,

$$\tau_C = \inf \{ t \ge 0 \mid Y_t \notin C \},\$$

is the optimal stopping time.

We first show that $(a_2, \bar{y}) \subset C$. Let $y \leq \bar{y}$ and let

$$\tau = \inf \{ t \ge 0 \mid F(\hat{x}, Y_t) \ge 0 \}.$$

Note that it is possible that $\mathsf{P}_y(\tau = \infty) > 0$. It holds that

$$\mathsf{E}_{y}\left[e^{-r\tau}F(\hat{x},Y_{\tau})\right] \ge 0 > F(\hat{x},\bar{y}).$$

So, it cannot be optimal to stop at y and, hence, $(a_2, \overline{y}] \in C$.

We now show that C is a connected set. Suppose not. Then there exist points

$$y_1 > \overline{y}$$
, and $y_2 > y_1$,

such that

$$y_1 \in \mathcal{Y} \setminus C$$
, and $y_2 \in C$.

Let $\tau = \inf \{ t \ge 0 \mid Y_t \ge y_2, Y_t \in \mathcal{Y} \setminus C \}$. Since $F^*(\cdot)$ is a superharmonic majorant of $F(\hat{x}, \cdot)$ it holds that

$$F(\hat{x}, y_1) = F^*(y_1) \ge \mathsf{E}_{y_1}\left[F^*(Y_\tau)\right] = \mathsf{E}_{y_1}\left[F(\hat{x}, Y_\tau)\right] > \mathsf{E}_{y_1}\left[F(\hat{x}, y_2)\right] = F(\hat{x}, y_2).$$

But this contradicts the fact that $F(\hat{x}, \cdot)$ is an increasing function.

4. Since the continuation set is a connected set, problem (1) can now be reduced to a maximization problem

over thresholds:

$$\begin{split} F^*(y) &= \sup_{\tau} \mathsf{E}_y \left[e^{-r\tau} F(\hat{x}, Y_{\tau}) \right] \\ &= \sup_{\hat{y} \in \mathcal{Y}} \mathsf{E}_y \left[e^{-r\tau(\hat{y})} F(\hat{x}, \hat{y}) \right] \\ &= \sup_{\hat{y} \in \mathcal{Y}} \mathsf{E}_y \left[e^{-r\tau(\hat{y})} \right] F(\hat{x}, \hat{y}) \end{split}$$

where the last equality follows from the spectral negativity of $(Y_t)_{t\geq 0}$. 5. From Dynkin's formula and spectral negativity of $(Y_t)_{t\geq 0}$ it follows that

$$\mathsf{E}_{y}\left[e^{-r\tau(\hat{y})}\right] = \frac{\psi(y)}{\psi(\hat{y})}.$$

Therefore, problem (1) can be rewritten as

$$F^{*}(y) = \psi(y) \sup_{\hat{y} \in \mathcal{Y}} \frac{1}{\psi(\hat{y})} F(\hat{x}, \hat{y}).$$
(B.1)

6. If (B.1) has a solution it must be a stationary point y^* of (3), i.e. it should solve f(y) = 0, where

$$f(y) = \varphi'_y(\hat{x}, y^*)\psi(y^*) - \psi'(y^*) \left[\varphi(\hat{x}, y^*) - \Delta I(\hat{x})\right] = 0.$$
(B.2)

Since $y^* > \bar{y}$ it holds that $\varphi(\hat{x}, y^*) > \Delta I(\hat{x})$. Note that $f(\bar{y}) > 0$ and

$$f'(y) = \psi(y)\varphi_{yy}''(\hat{x}, y) - \psi''(y)F(\hat{x}, y) < 0,$$

on $[\bar{y}, b_2)$. So, if f(y) = 0 has a solution it is unique and is a maximum location of (3) and, hence, solves (1). 7. If f(y) = 0 has no solution than the maximum for (B.1) is not attained on \mathcal{Y} and, thus, $C = \mathcal{Y}$.

C Proof of Lemma 1

Applying the characteristic operator of $(X_t)_{t\geq 0}$ to the function $f(x) = \log(x)$ gives

$$\mathscr{L}_X f(x) = -\frac{1}{2}\sigma_1^2 + \mu_1 + \int_{\mathbb{R}} \left[\log(x - ux) - \log(x) + u \right] m_1(du)$$

= $-\frac{1}{2}\sigma_1^2 + \mu_1 + \lambda \frac{a}{a+b} + \lambda \mathsf{E}[\log(1 - U)].$

Let B denote the Beta function. A straightforward computation yields that

$$\mathsf{E}[\log(1-U)] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 \frac{\partial}{\partial b} u^{a-1} (1-u)^{b-1} du$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial}{\partial b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{1}{B(a,b)} \frac{\partial}{\partial b} B(a,b) = \frac{\partial}{\partial b} \log[B(a,b)]$$

$$= \frac{\partial}{\partial b} \log[\Gamma(b)] - \frac{\partial}{\partial b} \log[\Gamma(a,b)] \equiv \psi(b) - \psi(a+b).$$

Therefore,

$$\mathscr{L}_X f(x) = -\frac{1}{2}\sigma_1^2 + \mu_1 + \lambda \frac{a}{a+b} + \lambda [\psi(b) - \psi(a+b)].$$

Under (7) it holds that $\mathscr{L}_X f \leq 0$, so that Dynkin's formula gives that

$$\log(x^*) = \log(x) + \mathsf{E}_x \left[\int_0^{\tau(x^*)} \mathscr{L}_X \log(X_t) dt \right]$$
$$= \log(x) + \left[\mu_1 - \frac{1}{2}\sigma_1^2 + \lambda \left(\frac{a}{a+b} + \psi(b) - \psi(a+b) \right) \right] \mathsf{E}_x[\tau(x^*)],$$

from which the result follows.

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