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Equilibrium, Auction, Multiple Substitutes and Complements

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Equilibrium, Auction, Multiple Substitutes and Complements[1]

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Abstract: We study a market model where there are n different types of indivisible goods for sale. The goods can be substitutable or complementary. There are multiple units of each good. Each agent may consume several goods and has quasi-linear utilities in money. We introduce a general condition which will be shown to guarantee the existence of a Walrasian equilibrium and generalize several existing conditions such as gross substitutes, strong substitutes, and gross substitutes and complements. We also identify several characterizations of this condition. Furthermore, we propose a price adjustment process which converges globally to a Walrasian equilibrium.

Keywords: Indivisibility, equilibrium, existence, substitutes and complements, auction.

JEL classification: C61, C62, D44, D51.

1 Introduction

Auctions are fundamental market mechanisms for efficiently allocating goods and services. Traditionally, research has focused on auctions for selling a single item; see e.g., Vickrey (1961), Myerson (1981), and Milgrom and Weber (1982). Over the last two decades we have seen a growing use of auctions for selling multiple items. Such auctions have been used to sell spectrum licenses, electrical power, natural resources and pollution permits, etc; see Klemperer (2004) and Milgrom (2004).

The purpose of the current paper is to propose a dynamic auction for selling multiple indivisible goods to finitely many bidders. Goods can be substitutes or complements. There may be several units of each good. To devise an auction mechanism, we first have to identify a suitable condition under which a Walrasian equilibrium exists. Then we establish fundamental characterizations of this condition which will be conducive to our dynamic

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design. The literature on dynamic auction design for selling multiple items starts with a seminar article of Kelso and Crawford (1982). They examine a general labor market where each firm may hire several workers and each worker works for at most one firm. They prove through a salary adjustment process that the market has a nonempty (strict) core and thus possesses a Walrasian equilibrium, as long as every firm views all workers as substitutes. This condition is called *the gross substitutes* (GS) and has become a benchmark condition in auction design, matching and equilibrium literature; see Roth and Sotomayor (1990) and Milgrom (2004).

Gul and Stacchetti (1999) study economies with indivisible goods and introduce two new conditions labeled as *single improvement* (SI) *condition* and *no complementarities* (NC) *condition* both being equivalent to the GS condition. Gul and Stacchetti (2000) propose an ascending auction that leads to the smallest Walrasian equilibrium prices in economies with indivisible goods that satisfy the gross substitutes (GS) condition. For similar economies, Milgrom (2000) develops a different ascending auction that converges to approximate equilibrium prices of any desired degree of accuracy, and has used this auction in the sale of US spectrum licenses. Ausubel (2006) has devised an ingenious strategy-proof dynamic auction under the GS environment.[4] In Gul and Stacchetti's auction design and Ausubel's the SI condition proves to be a very handy and useful property. Sun and Yang (2006) generalize the gross substitute (GS) condition to the gross substitutes and complements (GSC) condition and show that there exists a Walrasian equilibrium in economies with indivisible goods.[5] The GSC states that if all the indivisible goods can be split into two groups and to every agent goods in the same group are substitutable but complementary to those in the other group. Sun and Yang (2009) propose a dynamic auction for economies with indivisible goods under the GSC condition.[6]

The auctions cited above and the one to be introduced in the current paper all use linear and anonymous pricing rules. A related strand of the literature concerns auctions that apply to more general environments but have to use discriminatory and nonlinear pricing rules; see e.g., Ausubel and Milgrom (2002), de Vries, Schummer and Vohra (2007), Mishra and Parkes (2007), Day and Milgrom (2008), Erdil and Klemperer (2010). Another strand of the literature concerns auction design in the environment where bidders have interdependent values and the goods are multiple but homogeneous; see Ausubel (2004), and Perry and Reny (2005).

In this paper extending the model of Sun and Yang (2006), we study a general economy

[4]He also proposes a strategy-proof dynamic auction for economies with perfectly divisible goods in which every bidder's value function is increasing and concave.

[5]Ostrovsky (2008) independently proposes a similar condition for a supply chain network where prices are fixed and instead of the Walrasian equilibrium the notion of stability is used.

[6]See Sun and Yang (2008A) for a detailed account on the auction.

in which a seller wishes to sell two classes S_1 and S_2 of indivisible goods to finitely many bidders. Each class contains several different types of goods. There are multiple units of each type. Units of the same type are perfectly substitutable, and goods of different types in the same class are heterogeneous but still substitutable, whereas goods across two classes are complementary.[7] We call this condition *the generalized gross substitutes and complements* (GGSC) *condition* which will be shown to guarantee the existence of a Walrasian equilibrium in the economy. This condition is specified on the demand correspondence of every bidder. When there is only one class of indivisible goods, i.e., either $S_1 = \emptyset$ or $S_2 = \emptyset$, the GGSC condition becomes identical to *the strong-substitute valuation* of Milgrom and Strulovici (2009) and is similar to the $M^\natural$-GS condition of Murota and Tamura (2003). Further, if each type of good has only one unit, it becomes the gross substitute condition of Kelso and Crawford (1982). Bikhchandani and Mamer (1997) and Ma (1998) provide necessary and sufficient conditions to the existence problem. In contrast to GS, GSC and our current conditions which are imposed upon each individual, their conditions are imposed upon the aggregated behavior of all individuals in the economy. Baldwin and Klemperer (2012) have recently applied tropical geometry to equilibrium analysis.

We identify three fundamental characterizations of the GGSC condition. First, we show that a value function satisfies the GGSC condition if and only if it is a GM-concave function. The GM-concave function is due to Sun and Yang (2008B) and generalizes the $M^\natural$-concave function of Murota and Shioura (1999). Our second characterization gives a totally different property on each bidder's demand correspondence which will play an important role in our auction design. This condition extends the generalized single improvement (GSI) property of Sun and Yang (2009) and is called *the extended generalized single improvement* (EGSI) *condition*. GSI is a generalization of the single improvement property of Gul and Stacchetti (1999). Our proof makes use of several results from Murota and Tamura (2003) and Murota (2003). Our third characterization theorem extending a theorem of Ausubel and Milgrom (2002) establishes that a value function is a GM-concave function if and only if its indirect utility function is a generalized variant of $L^\natural$-convex function by Murota and Shioura (2000, 2004). Furthermore, we extend the auction of Sun and Yang (2008A, 2009) to the current market model and demonstrate that starting from any initial price vector, the auction always converges to a Walrasian equilibrium. This auction works as follows. The auctioneer announces a vector of current prices, every bidder reports his demand set at these prices, and the auctioneer adjusts prices of goods in one class upwards but those of goods in the other class downwards. We call it *a generalized double-track auction*. Furthermore,

[7]It is well recognized in the literature that indivisibility and complementarity pose serious problems for equilibrium existence, stability, and auction design; see e.g., Debreu (1959, p. 36), Scarf (1960), Arrow and Hahn (1971), Samuelson (1974), Kelso and Crawford (1982), Milgrom (2000), Jehiel and Moldovanu (2003), Klemperer (2004), Maskin (2005).

we examine the structure of Walrasian equilibrium price vectors and demonstrate that it exhibits an elegant geometry called $L^\natural$-convexity being a refinement of lattice.

The rest of the paper proceeds as follows. Section 2 introduces the auction model and the GGSC condition. Section 3 presents three characterizations of the GGSC condition. Section 4 establishes several results concerning Walrasian equilibria and their structure. Section 5 deals with the auction process and its convergence. Finally, Section 6 introduces one practical application of the general model. All proofs are given in the appendix.

2 The Model

Consider an auction model where a seller wishes to sell n different kinds of indivisible commodities to a finite group B of bidders. Let $N = \{1, 2, \cdots, n\}$ denote the family of n types, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ the bundle of goods for sale where $\omega_h > 0$ stands for the available units of type $h \in N$. The types can be divided into two classes S_1 and S_2 (i.e., $N = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$). For instance, one can think of S_1 as $|S_1|$ different kinds of tables and of S_2 as $|S_2|$ different kinds of chairs. Items of the same kind are identical and thus perfectly substitutes. Let $\Omega = \{x \in \mathbb{Z}^N_+ \mid x \leq \omega\}$ represent the space of all possible bundles for exchange. Every bidder i has a value or utility function $u^i : \Omega \to \mathbb{R}$ specifying his valuation $u^i(z)$ (in units of money) on each bundle z with $u^i(0) = 0$. We assume that every bidder can pay up to his value and has quasi-linear utilities in money, and that the seller values every bundle at zero.

A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^N$ indicates the price for a unit of each good. Given a price vector $p \in \mathbb{R}^N$, we define every bidder i's demand correspondence $D(u^i, p)$, and the indirect utility function $V^i(p)$, respectively as

$$\begin{aligned} D(u^i, p) &= \arg\max_{z \in \Omega} \{u^i(z) - p \cdot z\}, \\ V^i(u^i, p) &= \max_{z \in \Omega} \{u^i(z) - p \cdot z\}. \end{aligned}$$

When bidder i's value function is represented by u^i, for convenience we may simply use $D^i(p)$ and $V^i(p)$ to denote $D(u^i, p)$ and $V^i(u^i, p)$ respectively by ignoring u^i. It is known that for any value function $u^i : \Omega \to \mathbb{R}$, the indirect utility function V^i is a decreasing and polyhedral convex function; see Rockafellar (1970).

An *allocation* is a distribution $(z^i \mid i \in B)$ of the goods ω among all bidders in B, i.e., $\sum_{i \in B} z^i = \omega$ and $z^i \in \Omega$ for all $i \in B$. Note that $z^i = \mathbf{0}$ is allowed. At the allocation, bidder i receives bundle z^i. An allocation $(z^i \mid i \in B)$ is said to be *efficient* if $\sum_{i \in B} u^i(z^i) \geq \sum_{i \in B} u^i(y^i)$ for every allocation $(y^i \mid i \in B)$. Given an efficient allocation $(z^i \mid i \in B)$, let $R(\omega) = \sum_{i \in B} u^i(z^i)$. We call $R(\omega)$ the *market value* of the goods.

Definition 1: *A Walrasian equilibrium* $(p,(z^i \mid i \in B))$ *consists of a price vector* $p \in \mathbb{R}^N$ *and an allocation* $(z^i \mid i \in B)$ *such that* $z^i \in D^i(p)$ *for every* $i \in B$.

It is well known that every equilibrium allocation is efficient, but an equilibrium may not always exist. In order to ensure the existence of an equilibrium, we need to impose the *generalized gross substitutes and complements* (GGSC) *condition*.[8] We say that a discrete set $C \subseteq \mathbb{Z}^N$ is *discrete convex*[9] if it contains all integral vectors in its convex hull $\mathrm{conv}(C)$, i.e., $C = \mathbb{Z}^N \cap \mathrm{conv}(C)$.

Definition 2: *The value function* u^i *of bidder* i *satisfies the* generalized gross substitutes and complements (GGSC) *condition if*
(i) *for any price vector* $p \in \mathbb{R}^N$, $D^i(p)$ *is a discrete convex set; and*
(ii) *for any price vector* $p \in \mathbb{R}^N$, *any type* $k \in S_j$ *for* $j = 1$ *or* 2, *any* $\delta > 0$, *and any* $y \in D^i(p)$, *there exists* $z \in D^i(p + \delta e(k))$ *such that*

$$z_h \geq y_h \ (\forall h \in S_j \setminus \{k\}), \quad z_h \leq y_h \ (\forall h \in S_j^c),$$
$$\sum_{h \in S_j} z_h - \sum_{h \in S_j^c} z_h \leq \sum_{h \in S_j} y_h - \sum_{h \in S_j^c} y_h.$$

GGSC condition states that bidder i's value function exhibits the concavity property, and the bidder views goods of each type in each set S_j as substitutes, but goods across the two sets S_1 and S_2 as complements, in the sense that if the bidder demands a bundle y at prices p and if now the price of goods of some type $k \in S_j$ is increased, then he would not decrease his demand on goods of each type in S_j except type k, and would not increase his demand on goods of each type in the other group S_j^c, and the difference of demand across two groups at current prices should not exceed that at the previous prices. In the case of $\omega = (1, \cdots, 1)$ which corresponds to a set value function $u^i : 2^N \to \mathbb{R}$, GGSC condition reduces to the gross substitutes and complements (GSC) condition of Sun and Yang (2006, 2009).

Observe that condition (i) will be automatically satisfied when $\omega = (1, \cdots, 1)$. In particular, when either $S_1 = \emptyset$ or $S_2 = \emptyset$, GGSC conditon reduces to the strong substitutes (GS) condition of Milgrom and Strulovici (2009) and is essentially equivalent to the $M^\natural$-GS condition of Murota and Tamura (2003).[10] Furthermore, for the set function u^i, when

[8] The following piece of notation is used throughout the paper. For any integer $k = 1, 2, \ldots, n$, the k-th unit vector is denoted by $e(k) \in \mathbb{Z}^N$. Let $\mathbf{0} = e(0) \in \mathbb{Z}^N$ be the vector of 0's. We also denote by $\mathbf{1} \in \mathbb{Z}^N$ the vector of 1's. For any subset A of N, let A^c denote its complement, i.e., $A^c = N \setminus A$. For any set $D \subseteq \mathbb{R}^N$, $\mathrm{conv}(D)$ denotes its convex hull.

[9] This property is also called *hole-free* in Murota (2003).

[10] In Murota and Tamura (2003), their GS definition does not require $D^i(p)$ to be discrete convex. Instead they need u^i to be concave-extensible which is less natural from a viewpoint of economics. See Murota and Tamura (2003, Theorem 18 (b)).

either $S_1 = \emptyset$ or $S_2 = \emptyset$, GSC condition reduces to the gross substitutes (GS) condition of Kelso and Crawford (1982). The GS condition has been studied extensively in the literature; see e.g., Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), Milgrom (2000), Crawford (2005), and Ausubel (2006).

3 Value Functions and Demand Correspondences

In this section we give three major characterizations of GGSC condition in term of value functions, demand correspondences, and indirect utility functions.

Before presenting two classes of value functions, we first introduce several notations. Let U be the $n \times n$ matrix whose i-th column is given by $e(i), i \in S_1$ and $-e(i), i \in S_2$. That is, if $S_1 = \{1, 2, \ldots, n'\}$ and $S_2 = \{n'+1, n'+2, \ldots, n\}$, then U is the diagonal matrix given as

$$U = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & O & \\ & & 1 & & & \\ & & & -1 & & \\ & O & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

Given $x \in \mathbb{R}^N$, define

$$\mathrm{supp}^+(x) = \{k \in N \mid x_k > 0\}, \qquad \mathrm{supp}^-(x) = \{k \in N \mid x_k < 0\}.$$

We also define

$$\begin{aligned} \mathrm{supp}_g^+(x) &= \mathrm{supp}^+(Ux) = \{k \in S_1 \mid x_k > 0\} \cup \{k \in S_2 \mid x_k < 0\}, \\ \mathrm{supp}_g^-(x) &= \mathrm{supp}^-(Ux) = \{k \in S_1 \mid x_k < 0\} \cup \{k \in S_2 \mid x_k > 0\}. \end{aligned}$$

An *(finite) integer interval* is a set of integral vectors given as

$$\Gamma = \{x \in \mathbb{Z}^N \mid a_i \le x_i \le b_i \ (\forall i = 1, 2, \ldots, n)\},$$

where $a_i, b_i \in \mathbb{Z}$ for each $i = 1, 2, \ldots, n$. Note that $\Gamma = \Omega$ holds if $a_i = 0$ and $b_i = \omega_i$, and $\Gamma = \{0, 1\}^N$ if $a_i = 0$ and $b_i = 1$.

Definition 3: *A value function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ is* GM-concave *if, for every $x, y \in \Gamma$ and $k \in \mathrm{supp}_g^+(x-y)$, there exists some $l \in \mathrm{supp}_g^-(x-y) \cup \{0\}$ such that*

$$u(x) + u(y) \le u(x - U(e(k) - e(l))) + u(y + U(e(k) - e(l))).$$

The class of GM-concave functions is introduced by Sun and Yang (2008B), where they consider functions defined on more general domains, including integer intervals. GM-concave function is referred to as *twisted $M^\natural$-concave function* in Ikebe and Tamura (2012). Sun and Yang (2006) deal with GM-concave functions defined on a special domain of $\{0,1\}^N$. GM-concave functions given by Definition 3 generalize the class of $M^\natural$-concave functions proposed by Murota and Shioura (1999).

DEFINITION 4: *A value function $u:\Gamma\to\mathbb{R}$ defined on a finite integer interval Γ is $M^\natural$-*concave *if, for every $x,y\in\Gamma$ and $k\in\text{supp}^+(x-y)$, there exists some $l\in\text{supp}^-(x-y)\cup\{0\}$ such that*

$$u(x)+u(y)\le u(x-(e(k)-e(l)))+u(y+(e(k)-e(l))).$$

Clearly, if $S_1=N$ and $S_2=\emptyset$, then the concept of GM-concave function coincides with that of $M^\natural$-concave function. The concept of $M^\natural$-concave function is originally defined on more general domains, other than finite integer intervals; see Murota and Shioura (1999) for details.

The following theorem gives the first major characterization of the GGSC condition. Observe that the GGSC condition characterizes a bidder's demand set while the GM-concavity of a function describes a bidder's valuation on every bundle. We note that the GGSC condition can be naturally extended to value functions defined on general finite integer intervals, although the original definition considers value functions only on Ω.

THEOREM 1: *A value function $u:\Gamma\to\mathbb{R}$ defined on a finite integer interval Γ satisfies the GGSC condition if and only if it is GM-concave.*

This result reveals an intimate relationship between two totally distinct objects: demand sets and value functions, generalizing a result of Fujishige and Yang (2003) which shows that any value function $u:\{0,1\}^N\to\mathbb{R}$ satisfies the GS condition if and only if it is $M^\natural$-concave.

The following definition gives another characterization of a bidder's demand behavior.

DEFINITION 5: *A value function $u:\Gamma\to\mathbb{R}$ defined on a finite integer interval Γ satisfies the* extended generalized single improvement (EGSI) *condition if for any $p\in\mathbb{R}^N$ and $x,y\in\Gamma$ with $u(y)-p\cdot y>u(x)-p\cdot x$, there exists $z\in\Gamma$ such that $u(z)-p\cdot z>u(x)-p\cdot x$ and z satisfies one of the following conditions:*
(i) *$z=x-e(k)+e(l)$ for some $k\in\text{supp}^+(x-y)\cup\{0\}$ and $l\in\text{supp}^-(x-y)\cup\{0\}$ so that if $k\neq 0$ and $l\neq 0$, then $k,l\in S_1$ or $k,l\in S_2$;*
(ii) *$z=x-e(k)-e(l)$ for some $k,l\in\text{supp}^+(x-y)$ with $k\in S_1$ and $l\in S_2$;*
(iii) *$z=x+e(k)+e(l)$ for some $k,l\in\text{supp}^-(x-y)$ with $k\in S_1$ and $l\in S_2$.*

The bundle z in the definition is called *an EGSI improvement of* bundle x.

EGSI condition says that every suboptimal bundle x of a bidder at prices p can be strictly improved by either adding a unit of some good to x, or removing one unit of some good from x, or adding one unit of some good in one set S_j and simultaneously removing one unit of another good also in the same set S_j. The bundle x can be also strictly improved by adding simultaneously one unit of some good from each set S_j to it, or removing simultaneously one unit of some good from each set S_j, $j = 1, 2$.

When $\Gamma = \{0,1\}^N$, EGSI condition reduces to the generalized single improvement property of Sun and Yang (2009). Furthermore, if either S_1 or S_2 is empty, EGSI condition coincides with the single improvement (SI) condition of Gul and Stacchetti (1999) which in turn is equivalent to the GS condition. The EGSI condition plays an important role in our adjustment process design as SI condition does in Ausubel (2006), Gul and Stacchetti (2000).

The following theorem shows that GGSC and EGSI conditions are equivalent, although they appear dramatically different from each other.

THEOREM 2: *A value function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ satisfies the GGSC condition if and only if it satisfies the EGSI condition.*

Before presenting the third characterization of the GGSC condition, we need to introduce several additional concepts. Let $x, y \in \mathbb{R}^N$ be any vectors. The standard *meet* and *join* are defined, respectively, by

$$\begin{aligned} x \wedge y &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \cdots, \min\{x_n, y_n\}), \\ x \vee y &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \cdots, \max\{x_n, y_n\}). \end{aligned}$$

We also define the *generalized meet* $s = (s_1, \cdots, s_n) = x \wedge_g y$ and *generalized join* $t = (t_1, \cdots, t_n) = x \vee_g y$ by

$$\begin{aligned} s_k = \min\{x_k, y_k\} \quad &\text{if } \beta_k \in S_1, \quad s_k = \max\{x_k, y_k\} \quad \text{if } \beta_k \in S_2; \\ t_k = \max\{x_k, y_k\} \quad &\text{if } \beta_k \in S_1, \quad t_k = \min\{x_k, y_k\} \quad \text{if } \beta_k \in S_2. \end{aligned}$$

For $x, y \in \mathbb{R}^N$, we introduce a new order by defining $x \leq_g y$ if

$$x_h \leq y_h \text{ for } h \in S_1, \qquad x_h \geq y_h \text{ for } h \in S_2.$$

Given a set $C \subseteq \mathbb{R}^N$, a point $x^* \in C$ is called *a minimal element* if $x^* \leq_g y$ for every $y \in C$. Similarly, a point $y^* \in C$ is called *a maximal element* if $y^* \geq_g x$ for every $x \in C$. In general, there may exist many minimal and maximal elements in C.

DEFINITION 6: *A function $g : \mathbb{R}^N \to \mathbb{R}$*

(i) *is* submodular *if* $g(x \wedge y) + g(x \vee y) \leq g(x) + g(y)$ *for any* $x, y \in \mathbb{R}^N$;

(ii) *is* generalized submodular *if* $g(x \wedge_g y) + g(x \vee_g y) \leq g(x) + g(y)$ *for any* $x, y \in \mathbb{R}^N$;

(iii) *is* polyhedral $L^\natural$-convex *if it is a polyhedral convex function and satisfies*

$$g(x \wedge (y + \alpha\mathbf{1})) + g((x - \alpha\mathbf{1}) \vee y) \leq g(x) + g(y) \qquad (\forall x, y \in \mathbb{R}^N,\ \alpha \in \mathbb{R}_+);$$

(iv) *is* generalized polyhedral $L^\natural$-convex *if it is a polyhedral convex function and satisfies*

$$g(x \wedge_g (y + \alpha\mathbf{1})) + g((x - \alpha\mathbf{1}) \vee_g y) \leq g(x) + g(y) \quad (\forall x, y \in \mathbb{R}^N,\ \alpha \in \mathbb{R}_+).$$

A polyhedron is *integral* if all its vertices are integral. A (generalized) polyhedral $\mathrm{L}^\natural$-convex function $g : \mathbb{R}^N \to \mathbb{R}$ is *integral* if for every $x \in \mathbb{R}^N$, the set $\arg\min\{g(p) - x \cdot p \mid p \in \mathbb{R}^N\}$ is an integral polyhedron (or an empty set).

Clearly, a polyhedral $L^\natural$-convex function is submodular but the reverse need not be true. See Murota (2003) on $L^\natural$-convex functions for details. A generalized polyhedral $L^\natural$-convex function is generalized submodular but the reverse need not be true.

Ausubel and Milgrom (2002, Theorem 10) prove that goods are substitutes for a bidder if and only if his indirect utility function is submodular.

THEOREM 3: *Let $u : \Gamma \to \mathbb{R}$ be a value function defined on a finite integer interval Γ. Then, u is GM-concave if and only if its indirect utility function $V(p) = \max\{u(x) - p \cdot x \mid x \in \Gamma\}$ is a generalized polyhedral $L^\natural$-convex function in $p \in \mathbb{R}^N$. Moreover, u is an integer-valued GM-concave function if and only if V is an integral generalized polyhedral $L^\natural$-convex function.*

In this section we have given three different equivalent conditions of the GGSC condition. In the following we may state anyone of these four as a sufficient condition for the existence of a Walrasian equilibrium. Observe that we do not impose any monotonicity condition, which means the model can accommodate indivisible bads. In this case, equilibrium prices can be negative and the seller pays the bidders for buying bads.

4 Equilibrium Theorems

For the auction model, define the Lyapunov function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}$ by

$$(1) \qquad \mathcal{L}(p) = \sum_{i \in B} V^i(p) + p \cdot \omega,$$

where $V^i(p)$ is the indirect utility of bidder $i \in B$ at prices p and $p \cdot \omega$ is the seller's value of her goods at prices p. This type of function has been used in the literature for economies with divisible goods (see, e.g., Arrow and Hahn (1971) and Varian (1981)) but was only recently explored by Ausubel (2005, 2006), Sun and Yang (2009) in the context of indivisible goods. The Lyapunov function will be used in our auction design. Note that the following two lemmas do not assume that value functions u^i satisfy the GGSC condition. In other words, the lemmas hold for general value functions.

LEMMA 1: *If $(p, (z^i \mid i \in B))$ is an equilibrium, $(z^i \mid i \in B)$ must be efficient, and $(p, (y^i \mid i \in B))$ is an equilibrium for any efficient allocation $(y^i \mid i \in B)$.*

LEMMA 2: *A price vector p^* is a Walrasian equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$ satisfying $\mathcal{L}(p^*) = R(\omega)$.*

We will now demonstrate the existence of a Walrasian equilibrium under the GGSC condition and examine the structure of the set of Walrasian equilibrium price vectors.

DEFINITION 7: *Let $D \subseteq \mathbb{R}^N$ be a nonempty set.*

(i) *D is a lattice if*

$$p \wedge q \in D,\ p \vee q \in D \qquad (\forall p, q \in D).$$

(ii) *D is a generalized lattice if*

$$p \wedge_g q \in D,\ p \vee_g q \in D \qquad (\forall p, q \in D).$$

(iii) *D is $L^\natural$-convex if*

$$(p - \lambda \mathbf{1}) \vee q \in D,\ \ p \wedge (q + \lambda \mathbf{1}) \in D \qquad (\forall p, q \in D,\ \forall \lambda \in \mathbb{R}_+).$$

(iv) *D is generalized $L^\natural$-convex if*

$$(p - \lambda \mathbf{1}) \vee_g q \in D,\ \ p \wedge_g (q + \lambda \mathbf{1}) \in D \qquad (\forall p, q \in D,\ \forall \lambda \in \mathbb{R}_+).$$

Clearly, D is *generalized $L^\natural$-convex* (a generalized lattice) if $D = \{Up \mid p \in D'\}$ for some L$^\natural$-convex set D' (some lattice D'). An (generalized) $L^\natural$-convex set is a (generalized) lattice but the reverse need not be true. Obviously, a nonempty compact L$^\natural$-convex set has unique minimal and maximal vectors with respect to the order $\leq$. Similarly, a nonempty compact generalized L$^\natural$-convex set has unique minimal and maximal vectors with respect to the order $\leq_g$.

LEMMA 3: *Let $D \subseteq \mathbb{R}^N$ be a nonempty closed set.*
(i) *D is $L^\natural$-convex if and only if it is a polyhedron given as*

$$D = \{p \in \mathbb{R}^N \mid a_i \leq p_i \leq b_i \ (i \in N),\ p_i - p_j \leq c_{i,j} \ (i, j \in N,\ i \neq j)\}$$

with numbers a_i, b_i $(i \in N)$ and $c_{i,j}$ $(i, j \in N,\ i \neq j)$ such that $-a_i, b_i \in \mathbb{R} \cup \{+\infty\}$ and $c_{i,j} \in \mathbb{R} \cup \{+\infty\}$. Moreover, the polyhedron is integral if and only if the numbers a_i, b_i, and $c_{i,j}$ are integral (or $+\infty$ or $-\infty$).
(ii) *D is generalized $L^\natural$-convex if and only if it is a polyhedron given as*

$$\begin{aligned} D = \{p \in \mathbb{R}^N \mid a_i \leq p_i \leq b_i \ (i \in N),\ p_i - p_j \leq c_{i,j} \ (i, j \in S_k,\ i \neq j,\ k = 1, 2), \\ -c_{j,i} \leq p_i + p_j \leq c_{i,j} \ (i \in S_1,\ j \in S_2)\} \end{aligned}$$

with numbers a_i, b_i $(i \in N)$ and $c_{i,j}$ $(i, j \in N,\ i \neq j)$ such that $-a_i, b_i \in \mathbb{R} \cup \{+\infty\}$ and $c_{i,j} \in \mathbb{R} \cup \{+\infty\}$. Moreover, the polyhedron is integral if and only if the numbers a_i, b_i, and $c_{i,j}$ are integral (or $+\infty$ or $-\infty$).

The following theorems will be used to prove the convergence of the generalized double-track auction.

THEOREM 4: *Assume that every bidder $i \in B$ has a GM-concave value function u^i. Then the Lyapunov function $\mathcal{L}$ defined by* (1) *is a generalized polyhedral $L^\natural$-convex function. Moreover, if value functions u^i $(i \in B)$ take only integer values, then $\mathcal{L}$ is an integral generalized polyhedral $L^\natural$-convex function.*

The next theorem establishes the existence of a Walrasian equilibrium under the GGSC condition and reveals the structure of the set of Walrasian equilibrium price vectors.

THEOREM 5: *Assume that every bidder $i \in B$ has a GM-concave value function u^i. Let $\Lambda \subseteq \mathbb{R}^N$ be the set of Walrasian equilibrium price vectors.*
(i) *Then Λ is a nonempty compact, generalized $L^\natural$-convex set which is equal to the set of minimizers of the Lyapunov function. In particular, Λ has unique minimal and maximal equilibrium price vectors, denoted by $\underline{p}$ and $\bar{p}$, respectively.*
(ii) *Suppose further that the value functions u^i $(i \in B)$ are integer-valued. Then, Λ is a nonempty integral polyhedron and vectors $\underline{p}$ and $\bar{p}$ are integral.*

Theorem 4 implies that the Lyapunov function is a well-behaved function meaning that a local minimum is also a global minimum, while Theorem 5 asserts that the set of Walrasian equilibrium price vectors possesses an elegant geometry, i.e., the set is a polyhedron described by simple inequalities.

5 The Price Adjustment Process

In this section we present a price adjustment process which can always globally converge to a Walrasian equilibrium in finitely many steps. In the following, we assume that every bidder i's value function u^i takes *integer values*. This assumption is quite standard and natural, since one cannot value a bundle of goods more closely than to the nearest penny; see, e.g., Ausubel (2006) and Sun and Yang (2009). We see from this assumption and Theorem 5 (ii) that there exists an *integral* equilibrium price vector. We can therefore concentrate on integral price vectors in the following discussion.

Define

$$\square_{\mathbb{Z}} = \{\delta \in \mathbb{Z}^N \mid \delta_k \in \{0, +1\}, \forall k \in S_1,\ \delta_l \in \{0, -1\}, \forall l \in S_2\}.$$

For any bidder $i \in B$, any price vector $p \in \mathbb{Z}^N$, and any price variation $\delta \in \square_{\mathbb{Z}}$, choose

$$\tilde{z}^i \in \arg \min_{z^i \in D^i(p)} \delta \cdot z^i. \tag{2}$$

The next lemma asserts that bidder i's optimal bundle $\tilde{z}^i$ in (2) chosen from $D^i(p)$ remains the same for all price vectors on the line segment from p to $p + \delta$.

Lemma 4: *Assume that every bidder $i \in B$ has an integer-valued GM-concave value function u^i. Let $i \in B$, $p \in \mathbb{Z}^N$, and $\delta \in \square_{\mathbb{Z}}$. For every λ with $0 \leq \lambda \leq 1$, the solution $\tilde{z}^i$ of Formula (2) satisfies $\tilde{z}^i \in D^i(p + \lambda\delta)$. Moreover, the Lyapunov function $\mathcal{L}(p + \lambda\delta)$ is linear in λ in the interval $0 \leq \lambda \leq 1$.*

We now describe the basic idea of the price adjustment process. At time $t \in \mathbb{Z}_+$, the auctioneer announces the current price vector $p(t) \in \mathbb{Z}^N$ and then asks every bidder i to report his demand $D^i(p(t))$ at prices $p(t)$. Then the auctioneer uses all bidders' reported demand sets $D^i(p(t))$, $i \in B$, to determine the next price vector $p(t + 1)$. The auctioneer chooses a price adjustment $\delta \in \square_{\mathbb{Z}}$ that can reduce the value of the Lyapunov function $\mathcal{L}$ as much as possible. In other words, the auctioneer needs to find δ that solves the following problem:

$$\max\{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta) \mid \delta \in \square_{\mathbb{Z}}\}. \tag{3}$$

However, because the Lyapunov function $\mathcal{L}$ involves every bidder's private valuation on every bundle of goods, it will be practically impossible for the auctioneer to obtain such information. Instead of tackling the problem (3) directly, the auctioneer can solve the following problem:

$$\max\Big\{\sum_{i \in B}\Big(\min_{z^i \in D^i(p(t))} \delta \cdot z^i\Big) - \delta \cdot \omega \;\Big|\; \delta \in \square_{\mathbb{Z}}\Big\}. \tag{4}$$

Observe that this formula contains only every bidder i's reported demand set $D^i(p(t))$ and nothing else. It states that when the prices are adjusted from $p(t)$ to $p(t+1) = p(t) + \delta(t)$ with $\delta(t)$ being a solution to the problem (4), every bidder i tries to minimize his indirect utility loss $\delta \cdot z^i$ whereas the auctioneer aims to achieve the highest gain. We stress that the problem (4) is actually solved by the auctioneer based upon every bidder's reported demand $D^i(p(t))$.

We show how to derive the formula (4) from the formula (3). By the definition of the Lyapunov function we have

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{i \in B} (V^i(p(t)) - V^i(p(t)+\delta)) - \delta \cdot \omega. \tag{5}$$

Then it follows from Lemma 4 that

$$V^i(p(t)) - V^i(p(t)+\delta) = \min_{z^i \in D^i(p(t))} \delta \cdot z^i.$$

Consequently, the formula (5) becomes

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{i \in B} \Big(\min_{z^i \in D^i(p(t))} \delta \cdot z^i \Big) - \delta \cdot \omega.$$

Hence, the problem (3) can be reduced to the problem (4).

We are now ready to give a formal description of the price adjustment process.

The price adjustment process

Step 1: The auctioneer announces an initial price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to Step 2.

Step 2: The auctioneer asks every bidder i to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer finds a solution $\delta(t)$ to the problem (4). If $\delta(t) = \mathbf{0}$, go to Step 3. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to Step 2.

Step 3: The auctioneer asks every bidder i to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer finds a solution $\delta(t)$ to (4) where $\square_{\mathbb{Z}}$ is replaced by $-\square_{\mathbb{Z}}$. If $\delta(t) = \mathbf{0}$, then the procedure stops. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to Step 3.

Several remarks are in order. First, observe that in Step 2, the auctioneer adjusts the price of each type of good $i \in S_1$ upwards but the price of each type of good $i \in S_2$ download, whereas in Step 3, the auctioneers adjusts prices in the reverse way. This auction extends the double-track auction of Sun and Yang (2008A, 2009) from set utility functions

$u^i : \{0,1\}^N \to \mathbb{Z}$ to more general utility functions $u^i : \Gamma \to \mathbb{Z}$ and is called a *generalized double-track auction*. This type of auction differs crucially from other auctions in that it simultaneously adjusts prices in one family and prices in another family in opposite directions, whereas other existing auctions are either ascending or descending. Second, the process terminates in Step 3 and never goes from Step 3 to Step 2. This is a crucial step quite different from the global auction of Ausubel (2006) which requires repeated implementation of his ascending auction and descending auction one after the other in the GS case. Third, in the current process the auctioneer computes only an optimal solution which need not be the minimal or maximal element as required in the auction of Ausubel (2006). This will improve computational efficiency.

THEOREM 6: *Assume that every bidder $i \in B$ has an integer-valued GM-concave value function u^i. Starting with any integer price vector $p(0) \in \mathbb{Z}^N$, the price adjustment process converges to an equilibrium price vector in a finite number of rounds.*

The following result shows that if the auctioneer starts with a particular price vector $p(0) \in \mathbb{Z}^N$ with very low price for each type of good in S_1 and very high price for each type of good in S_2, say, $p_i(0) = -L$ for every $i \in S_1$ and $p_i(0) = +L$ for every $i \in S_2$ with a sufficiently large positive integer L, the price adjustment process will terminate at the minimal equilibrium price vector $\underline{p}$, provided that in each round t the minimal $\delta(t)$ that solves the problem (4) is chosen. In this case the auction never goes to Step 3. Recall that there exists a unique minimal equilibrium price vector $\underline{p}$ by Theorem 5.

THEOREM 7: *Assume that every bidder $i \in B$ has an integer-valued GM-concave value function u^i. Starting at an integer price vector $p(0) \in \mathbb{Z}^N$ with $p(0) \leq_g \underline{p}$, the price adjustment process converges to the minimal equilibrium price vector $\underline{p}$ in a finite number of rounds if in every round $t \in \mathbb{Z}_+$ the auctioneer chooses the minimal $\delta(t)$ (in the sense $\leq_g$) that solves the problem* (4).

6 A Practical Example

In this section we discuss a practical application of the general model. Consider a manufacturing industry consisting of many firms, workers and machines. Each firm can hire many workers and use many machines to produce goods. Each worker can work at most at one firm and one machine can be used at most in one firm. Let I denote the set of all firms, $W = \{w_1, w_2, \cdots, w_m\}$ the family of all types of workers, and $M = \{m_1, m_2, \cdots, m_n\}$ the family of all types of machines. There are $d_k \in \mathbb{Z}_{++}$ workers of type $w_k \in W$, and $b_l \in \mathbb{Z}_{++}$ units of machine $m_l \in M$. When worker w_k operates machine m_l at firm $i \in I$, they will generate a revenue $R^i(w_k, m_l)$ for firm i. This revenue can be nonnegative or negative. If it is negative, firm i will let w_k and m_l stay idle and thus make no gain and no loss. Without loss of generality, we can therefore assume that $R^i(w_k, m_l)$ is a nonnegative integer. When firm i hires α workers of type w_k and uses β machines of type m_l, the firm can achieve a revenue of $\min\{\alpha, \beta\}R^i(w_k, m_l)$. When firm i has only workers or only machines, it cannot produce anything and thus its revenue is zero. Let $S_1 = W$ and $S_2 = M$. Then a vector $x = (x', x'') \in \mathbb{Z}^{S_1} \times \mathbb{Z}^{S_2}$ denotes the numbers of workers and machines hired or used by a firm. Given such (x', x''), firm $i's$ revenue is uniquely determined by the optimal pair combinations between workers and machines given in (x', x'') and will be denoted by $R^i(x', x'')$. The revenue function R^i is a function defined on the set

$$\Omega = \{(x', x'') \in \mathbb{Z}^{S_1} \times \mathbb{Z}^{S_2} \mid 0 \le x'_k \le d_k,\ k \in S_1,\ 0 \le x''_l \le b_l,\ \in S_2\}.$$

An interesting question arises here: Does there exist a system of competitive wages and prices through which workers and machines can be efficiently allocated to the firms? The following result gives a positive answer.

THEOREM 8: *For the manufacturing industry model, each firm i's revenue function R^i is a GM-concave function. Thus the manufacturing industry possesses at least one Walrasian equilibrium.*

Sun and Yang (2006) consider the special but important case in which $d_k = 1$ for all $k \in W$ and $b_l = 1$ for all $l \in M$, and establish the existence of a Walrasian equilibrium. Shapley (1962) examines the optimization problem of a single firm and shows that the firm's revenue function is submodular with respect to machines only (or workers only) but supermodular with respect to workers and machines.

APPENDIX

This appendix contains the proofs of all results stated in the main body of the paper. The proofs of Theorems 1, 3, and 5 appear most difficult and make extensive use of results from discrete convex analysis for which Murota (2003) and Fujishige (2005) are excellent references.

A Proof of Theorem 1

Recall that U represents the $n \times n$ matrix for which the i-th column is given by $e(i)$ if $i \in S_1$ and by $-e(i)$ if $i \in S_2$ (see Section 3). Note that $U^{-1} = U$ and $U \cdot U$ is equal to the identity matrix, i.e., $U \cdot Ux = x$ for every vector x.

For a function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ, we define a finite integer interval $\tilde{\Gamma}$ and a function $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ by

$$(6) \qquad \tilde{\Gamma} = \{Uy \mid y \in \Gamma\}, \qquad \tilde{u}(x) = u(Ux) \qquad (x \in \tilde{\Gamma}).$$

Then, the following property is easy to see from the definition of GM-concavity.

Proposition 1 *A function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ is GM-concave if and only if the function $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ defined by (6) is M$^\natural$-concave.*

The GGSC condition for function u can be rewritten in terms of the function $\tilde{u}$ given by (6) as follows. For $p \in \mathbb{R}^N$, we define

$$D(\tilde{u}, p) = \arg\max\{\tilde{u}(z) - p \cdot z \mid z \in \tilde{\Gamma}\}.$$

(GGS±)
(i) For every $p \in \mathbb{R}^N$, $D(\tilde{u}, p)$ is a discrete convex set.
(ii) Let $p \in \mathbb{R}^N$, $\delta > 0$, and $x \in D(\tilde{u}, p)$.
(ii-1) For any $k \in S_1$, there exists $y \in D(\tilde{u}, p + \delta e(k))$ such that

$$y_h \geq x_h \quad (\forall h \in N \setminus \{k\}), \qquad \sum_{h \in N} y_h \leq \sum_{h \in N} x_h.$$

(ii-2) For any $k \in S_2$, there exists $y \in D(\tilde{u}, p - \delta e(k))$ such that

$$y_h \leq x_h \quad (\forall h \in N \setminus \{k\}), \qquad \sum_{h \in N} y_h \geq \sum_{h \in N} x_h.$$

Proposition 2 *A function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ satisfies the GGSC condition if and only if the function $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ given by (6) satisfies* (GGS±).

By Propositions 1 and 2, Theorem 1 is equivalent to the following statement:

Theorem 9 *A function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ satisfies the condition* (GGS±) *if and only if it is an M$^\natural$-concave function.*

In the following, we prove Theorem 9 instead of Theorem 1.

A.1 Proof of "If" Part

The "if" part of Theorem 9 is proven first. A *generalized polymatroid* (*g-polymatroid*, for short) is a polyhedron of the form

$$Q = \{x \in \mathbb{R}^N \mid \mu(X) \leq x(X) \leq \rho(X) \ (X \in 2^N)\}$$

given by a pair of submodular/supermodular functions $\rho : 2^N \to \mathbb{R} \cup \{+\infty\}$, $\mu : 2^N \to \mathbb{R} \cup \{-\infty\}$ satisfying the inequality

$$\rho(X) - \mu(Y) \geq \rho(X \setminus Y) - \mu(Y \setminus X) \qquad (\forall X, Y \in 2^N).$$

If ρ and μ are integer-valued, then Q is an integral polyhedron; in such a case, we say that Q is an *integral g-polymatroid.* Note that for any integral polyhedron Q, the set $Q \cap \mathbb{Z}^N$ is a discrete convex set.

PROPOSITION **3 (see, e.g., Murota (2003, Chapter 6))** *Let $u : \Gamma \to \mathbb{R}$ be an $M^\natural$-concave function defined on a finite integer interval Γ. For every $p \in \mathbb{R}^N$, $D(u,p)$ is the set of integral vectors in an integral g-polymatroid; it is a discrete convex set, in particular.*

Proposition 3 immediately implies the condition (i) in (GGS±).

To prove the second condition in (GGS±), we show below a (slightly) stronger statement.

LEMMA 5 *Let $u : \Gamma \to \mathbb{R}$ be an $M^\natural$-concave function defined on a finite integer interval Γ. For $p \in \mathbb{R}^N$, $\delta > 0$, $x \in D(u,p)$, and $k \in N$, the following properties hold:*

(a) *There exists $y \in D(u, p + \delta e(k))$ such that*

$$y_h \geq x_h \quad (\forall h \in N \setminus \{k\}), \qquad \sum_{h \in N} y_h \leq \sum_{h \in N} x_h. \tag{7}$$

(b) *There exists $y \in D(u, p - \delta e(k))$ such that*

$$y_h \leq x_h \quad (\forall h \in N \setminus \{k\}), \qquad \sum_{h \in N} y_h \geq \sum_{h \in N} x_h.$$

Below we prove the claim (a) only since (b) can be done in the same way. Let $q = p + \delta e(k)$, and y be a vector in $D(u,q)$ which minimizes the value $\|y - x\|_1$ among all vectors in $D(u,q)$. We prove that the vector y satisfies the condition (7). The proof below is quite similar to the one for Theorem 8 of Murota and Tamura (2003) and Theorem 6.34 (2) of Murota (2003).

To prove the first inequality in (7), assume, to the contrary, that $y_i < x_i$ for some $i \in N \setminus \{k\}$. By M$^\natural$-concavity of u applied to x, y, and $i \in \mathrm{supp}^+(x-y)$, we have

$$u(x) + u(y) \leq u(x - e(i) + e(j)) + u(y + e(i) - e(j)) \tag{8}$$

for some $j \in \mathrm{supp}^-(x-y) \cup \{0\}$. Since $x \in D(u,p)$ and $y \in D(u,q)$, it holds that

$$u(x) - p \cdot x \geq u(x - e(i) + e(j)) - p \cdot (x - e(i) + e(j)), \tag{9}$$

$$u(y) - q \cdot y \geq u(y + e(i) - e(j)) - q \cdot (y + e(i) - e(j)). \tag{10}$$

From (9) and (10) follows that

$$\begin{aligned} u(x) + u(y) &\geq \{u(x - e(i) + e(j)) - p \cdot (-e(i) + e(j))\} \\ &\qquad + \{u(y + e(i) - e(j)) - q \cdot (e(i) - e(j))\} \\ &= u(x - e(i) + e(j)) + u(y + e(i) - e(j)) - \delta e(k) \cdot (e(i) - e(j)) \\ &= u(x - e(i) + e(j)) + u(y + e(i) - e(j)) + \delta e(k) \cdot e(j) \\ &\geq u(x - e(i) + e(j)) + u(y + e(i) - e(j)), \end{aligned} \tag{11}$$

where the second equality is by the fact that $i \neq k$. From the inequalities (8) and (11) we see that all inequalities in (8), (9), (10), and (11) hold with equality. This implies, in particular, that $y + e(i) - e(j) \in D(u,q)$, a contradiction to the choice of y since

$$\|(y + e(i) - e(j)) - x\|_1 < \|y - x\|_1.$$

Hence, we have $y_h \geq x_h$ for all $h \in N \setminus \{k\}$.

We then prove the second inequality in (7), where we use the following property of M$^\natural$-concave functions:

Proposition 4 (Murota and Shioura (1999) and Murota (2003, Chapter 6)) *Let $u : \Gamma \to \mathbb{R}$ be an M$^\natural$-concave function. For every $x, y \in \Gamma$ with $\sum_{h \in N} x_h < \sum_{h \in N} y_h$, there exists some $j \in \mathrm{supp}^-(x-y)$ such that*

$$u(x) + u(y) \leq u(x + e(j)) + u(y - e(j)).$$

Suppose, to the contrary, that $\sum_{h \in N} y_h > \sum_{h \in N} x_h$. By Proposition 4, there exists some $j \in \mathrm{supp}^-(x-y)$ such that

$$u(x) + u(y) \leq u(x + e(j)) + u(y - e(j)). \tag{12}$$

Since $x \in D(u,p)$ and $y \in D(u,q)$, it holds that

$$u(x) - p \cdot x \geq u(x + e(j)) - p \cdot (x + e(j)), \tag{13}$$

$$u(y) - q \cdot y \geq u(y - e(j)) - q \cdot (y - e(j)). \tag{14}$$

Hence, we have

$$
\begin{aligned}
u(x)+u(y) &\geq \{u(x+e(j))-p\cdot e(j)\}+\{u(y-e(j))+q\cdot e(j)\} \\
&= u(x+e(j))+u(y-e(j))+\delta e(k)\cdot e(j) \\
&\geq u(x+e(j))+u(y-e(j)). \qquad (15)
\end{aligned}
$$

Thus, all inequalities in (12), (13), (14), and (15) hold with equality. This implies, in particular, that $y-e(j) \in D(u,q)$, a contradiction to the choice of y since $\|(y-e(j))-x\|_1 < \|y-x\|_1$. Hence, we have $\sum_{h\in N} y_h \leq \sum_{h\in N} x_h$.

This concludes the proof for "if" part of Theorem 9.

A.2 Proof of "Only If" Part

To prove the "only if" part of Theorem 9, we use a characterization of M$^\natural$-concave functions by demand sets. Recall that conv(C) denotes the convex hull of a set $C \subseteq \mathbb{R}^N$.

PROPOSITION **5 (Theorem 6.30 of Murota (2003))** *A function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ is M$^\natural$-concave if and only if $D(u,p)$ is the set of integral vectors in an integral g-polymatroid for every $p \in \mathbb{R}^N$.*

We now give a proof of the "only if" part of Theorem 9. Let $u : \Gamma \to \mathbb{R}$ be a function defined on a finite integer interval Γ satisfying the condition (GGS$\pm$). We use Proposition 5 to prove the M$^\natural$-concavity of u, i.e., we show that for every $p \in \mathbb{R}^N$, there exists some integral g-polymatroid $Q \subseteq \mathbb{R}^N$ such that $D(u,p) = Q \cap \mathbb{Z}^N$. By the condition (GGS$\pm$), $D(u,p)$ is a discrete convex set. Hence, it suffices to show that conv$(D(u,p))$ is a g-polymatroid; note that conv$(D(u,p))$ is an integral polyhedron. In the proof, the following characterization of g-polymatroids is used.

PROPOSITION **6 (cf. Theorem A.1 of Fujishige and Yang (2003))** *A bounded polyhedron $Q \subseteq \mathbb{R}^N$ is a g-polymatroid if and only if for every edge E of Q and every distinct points $x, y \in E$, $x-y$ is a multiple of a vector taken from the set*

$$\{+e(i) \mid i \in N\} \cup \{-e(j) \mid j \in N\} \cup \{+e(i)-e(j) \mid i,j \in N,\ i \neq j\}. \qquad (16)$$

Let $Q = \text{conv}(D(u,p))$, and E be an edge of Q. Since u is a function defined on a finite set Γ, there exists some $q \in \mathbb{R}^N$ such that $E = \text{conv}(D(u,q))$. Since $D(u,q)$ is a discrete covnex set by the condition (GGS$\pm$), we have $E \cap \mathbb{Z}^N = D(u,q)$.

Let $x, y \in E$ be the two endpoints of the edge E. We have $x, y \in E \cap \mathbb{Z}^N = D(u,q)$ since $E = \text{conv}(D(u,q))$. Since $x \neq y$, we may assume that $x_k > y_k$ for some $k \in N$.

By the choice of x and y, there exists some sufficiently small $\delta > 0$ such that

$$D(u, q + \delta e(k)) = \{y\}, \qquad D(u, q - \delta e(k)) = \{x\}. \tag{17}$$

Since $x, y \in D(u, q)$, the condition (GGS±) (ii) and (17) imply that

$$y_i \geq x_i \ (\forall i \in N \setminus \{k\}), \quad \sum_{i \in N} y_i \leq \sum_{i \in N} x_i; \tag{18}$$

indeed, if $k \in S_1$, then this follows immediately from (ii-1) in (GGS±) and the first equation in (17); if $k \in S_2$, then (18) follows from (ii-2) and the second equation in (17), where the roles of x and y are changed.

If $y_i = x_i$ holds for all $i \in N \setminus \{k\}$, then we are done since $x - y = \alpha e(k)$ for some $\alpha > 0$. Hence, we assume $x_h < y_h$ for some $h \in N \setminus \{k\}$. Then, in a similar way as in the discussion above with the roles of x and y changed, we can show that

$$x_i \geq y_i \quad (\forall i \in N \setminus \{h\}), \qquad \sum_{i \in N} x_i \leq \sum_{i \in N} y_i, \tag{19}$$

which, together with (18), implies

$$x_i = y_i \quad (\forall i \in N \setminus \{h, k\}), \qquad \sum_{i \in N} x_i = \sum_{i \in N} y_i.$$

From this follows that $x - y = \alpha(e(k) - e(h))$ for some $\alpha > 0$.

This concludes the proof for "only if" part of Theorem 9.

B Proof of Theorem 2

Theorem 2 follows from immediately from Theorem 1 and the following theorem.

THEOREM 10: *A function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ satisfies the EGSI condition if and only if it is GM-concave.*

We now prove Theorem 10. For a function $u : \Gamma \to \mathbb{R}$ on a finite integer interval Γ, let $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ be a function defined by (6). We say that a function $\tilde{u}$ satisfies the *single improvement* (SI) condition if it satisfies the following condition:

> for any $p \in \mathbb{R}^N$ and any $x, y \in \tilde{\Gamma}$ with $\tilde{u}(y) - p \cdot y > \tilde{u}(x) - p \cdot x$, there exists $z \in \tilde{\Gamma}$ such that $\tilde{u}(z) - p \cdot z > \tilde{u}(x) - p \cdot x$ and $z = x - e(k) + e(l)$ for some $k \in \mathrm{supp}^+(x - y) \cup \{0\}$ and $l \in \mathrm{supp}^-(x - y) \cup \{0\}$.

The following property can be found in Theorem 7 of Murota and Tamura (2003) and Theorem 11.4 of Murota (2003).

Proposition 7 *A function $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ defined on a finite integer interval $\tilde{\Gamma}$ satisfies the SI condition if and only if it is M$^\natural$-concave.*

It is not difficult to see that function u satisfies the EGSI condition if and only if $\tilde{u}$ satisfies the SI condition. By Proposition 1, function u is GM-concave if and only if $\tilde{u}$ is M$^\natural$-concave. Hence, Theorem 10 follows immediately from Proposition 7. This concludes the proof of Theorem 2.

C Proof of Theorem 3

We give some properties of (generalized) M$^\natural$-concavity and (generalized) polyhedral L$^\natural$-convexity, which are useful in proving Theorem 3.

Proposition 8 **(Theorem 7.10 of Murota (2003))** *Let $f, g : \mathbb{R}^N \to \mathbb{R}$ be (integral) polyhedral L$^\natural$-convex functions, and $x \in \mathbb{R}^N$.*
(i) *The function $f + g$ is (integral) polyhedral L$^\natural$-convex.*
(ii) *The function $g_x(p) = g(p) + x \cdot p$ is (integral) polyhedral L$^\natural$-convex in p.*

Proposition 9 **(see Murota (2003) and Fujishige (2005))** *For a function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ, let $u^\circ : \mathbb{R}^N \to \mathbb{R}$ be a function given by*

$$u^\circ(p) = \min\{p \cdot x - u(x) \mid x \in \Gamma\} \qquad (p \in \mathbb{R}^N). \tag{20}$$

Then, u is an M$^\natural$-concave function if and only if $-u^\circ$ is a polyhedral L$^\natural$-convex function. Moreover, function u is an integer-valued M$^\natural$-concave function if and only if $-u^\circ$ is an integral polyhedral L$^\natural$-convex function.

Propositions above for M$^\natural$-convexity and L$^\natural$-convexity can be rewritten as follows in terms of generalized M$^\natural$-convexity and L$^\natural$-convexity.

Proposition 10 *Let $f, g : \mathbb{R}^N \to \mathbb{R}$ be (integral) generalized polyhedral L$^\natural$-convex functions, and $x \in \mathbb{R}^N$.*
(i) *The function $f + g$ is (integral) generalized polyhedral L$^\natural$-convex.*
(ii) *The function $g_x(p) = g(p) + x \cdot p$ is (integral) generalized polyhedral L$^\natural$-convex in p.*

Proposition 11 *For a function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ, let $u^\circ : \mathbb{R}^N \to \mathbb{R}$ be a function given by (20). Then, u is a GM-concave function if and only if $-u^\circ$ is a generalized polyhedral L$^\natural$-convex function. Moreover, function u is an integer-valued GM-concave function if and only if $-u^\circ$ is an integral generalized polyhedral L$^\natural$-convex function.*

PROOF: Given a function u, we define a function $\tilde{u} : \tilde{\Gamma} \to \mathbb{R}$ by (6). By Proposition 1, function u is GM-concave if and only if $\tilde{u}$ is M$^\natural$-concave.

Similarly to the function u°, we define $\tilde{u}^\circ : \mathbb{R}^N \to \mathbb{R}$ by

$$\tilde{u}^\circ(p) = \min\{p \cdot x - \tilde{u}(x) \mid x \in \tilde{\Gamma}\} \qquad (p \in \mathbb{R}^N).$$

Then, it holds that

$$\begin{aligned} \tilde{u}^\circ(p) &= \min\{p \cdot x - \tilde{u}(x) \mid x \in \tilde{\Gamma}\} \\ &= \min\{p \cdot x - u(Ux) \mid x = Uy,\ y \in \Gamma\} \\ &= \min\{p \cdot Uy - u(U \cdot Uy) \mid y \in \Gamma\} \\ &= \min\{Up \cdot y - u(y) \mid y \in \Gamma\} \ = \ u^\circ(Up). \end{aligned}$$

This equation implies that the function $-u^\circ$ is generalized polyhedral L$^\natural$-convex if and only if $-\tilde{u}^\circ$ is polyhedral L$^\natural$-convex. Hence, the claim follows from Proposition 9. Q.E.D.

We can now prove Theorem 3. Consider a value function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ and its indirect utility function $V : \mathbb{R}^N \to \mathbb{R}$. It holds that

$$\begin{aligned} V(p) &= \max\{u(x) - p \cdot x \mid x \in \Gamma\} \\ &= -\min\{p \cdot x - u(x) \mid x \in \Gamma\} \ = \ -u^\circ(p) \qquad (p \in \mathbb{R}^N), \end{aligned}$$

where u° is given by (20). Hence, Theorem 3 follows immediately from Proposition 11.

D Proofs of Lemmas 1, 2, 3, and Theorem 4

Proof of Lemma 1: Take any allocation $(y^i \mid i \in B)$. Since $z^i \in D^i(p)$, we have

(21) $$u^i(z^i) - p \cdot z^i \geq u^i(y^i) - p \cdot y^i.$$

It follows that

$$\sum_{i \in B} u^i(z^i) - p \cdot (\sum_{i \in B} z^i) \geq \sum_{i \in B} u(y^i) - p \cdot (\sum_{i \in B} y^i),$$

which implies

(22) $$\sum_{i \in B} u^i(z^i) - \sum_{i \in B} u^i(y^i) \geq p \cdot (\sum_{i \in B} z^i) - p \cdot (\sum_{i \in B} y^i) = p \cdot (\omega - \omega) = 0.$$

Therefore, $(z^i \mid i \in B)$ is efficient.

We then assume that the allocation $(y^i \mid i \in B)$ is efficient. Since $(z^i \mid i \in B)$ is also efficient, we have $\sum_{i \in B} u^i(z^i) = \sum_{i \in B} u^i(y^i)$, which, together with (22), implies that the inequality in (21) must hold with equality. Hence, $y^i \in D^i(p)$ holds for every $i \in B$, i.e., $(p, (y^i \mid i \in B))$ is an equilibrium. Q.E.D.

Proof of Lemma 2: Let $(z^i \mid i \in B)$ be an efficient allocation. Then, we have $\sum_{i \in B} u^i(z^i) = R(\omega)$. Also, let $p \in \mathbb{R}^N$. Since $V^i(p) \geq u^i(z^i) - p \cdot z^i$ holds for all $i \in B$, we have

$$\mathcal{L}(p) = \sum_{i \in B} V^i(p) + p \cdot \omega \geq \sum_{i \in B} (u^i(z^i) - p \cdot z^i) + p \cdot \omega = \sum_{i \in B} u^i(z^i) = R(\omega). \tag{23}$$

Suppose that p^* is an equilibrium price vector. Then, Lemma 1 implies that $(p^*, (z^i \mid i \in B))$ is an equilibrium, i.e., $z_i \in D^i(p^*)$ holds for all $i \in B$. Hence, we have $V^i(p^*) = u^i(z^i) - p^* \cdot z^i$ holds for all $i \in B$, which, together with (23), implies that $\mathcal{L}(p^*) = R(\omega)$. Note that it follows from the inequality (23) and $\mathcal{L}(p^*) = R(\omega)$ that p^* is a minimizer of $\mathcal{L}$.

We then assume that p^* satisfies $\mathcal{L}(p^*) = R(\omega)$. Then, the inequality (23) implies that $V^i(p^*) = u^i(z^i) - p^* \cdot z^i$ holds for all $i \in B$, i.e., $z_i \in D^i(p^*)$ for all $i \in B$. This means that $(p^*, (z^i \mid i \in B))$ is an equilibrium. Q.E.D.

Proof of Lemma 3: The statement (i) is already shown in Murota (2003) and Murota and Shioura (2004), while the statement (ii) follows from (i) and the definition of a generalized L$^\natural$-convex set. Q.E.D.

Proof of Theorem 4: Since the Lyapunov function $\mathcal{L}$ is given as $\mathcal{L}(p) = \sum_{i \in B} V^i(p) + p \cdot \omega$, the statement of the theorem follows from Theorem 3 and Proposition 10. Q.E.D.

E Proof of Theorem 5

E.1 Useful Properties

We present several properties which will be used in the proof of Theorem 5. For a convex function $g : \mathbb{R}^N \to \mathbb{R}$ and $p \in \mathbb{R}^N$, the *subdifferential* $\partial g(p)$ of g at p is defined by

$$\partial g(p) = \{x \in \mathbb{R}^N \mid g(q) - g(p) \geq (q - p) \cdot x \ (\forall q \in \mathbb{R}^N)\}.$$

PROPOSITION **12 (Theorem 23.8 of Rockafellar (1970))** *Let* $g_i : \mathbb{R}^N \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ *be convex functions, and define* $g = \sum_{i=1}^m g_i$. *For* $p \in \mathbb{R}^N$, *it holds that*

$$\partial g(p) = \partial g_1(p) + \partial g_2(p) + \cdots + \partial g_m(p) = \{\sum_{i=1}^m x_i \mid x_i \in \partial g_i(p) \ (i = 1, 2, \ldots, m)\}.$$

For a set $C \subseteq \mathbb{R}^N$, we denote $-C = \{-x \mid x \in C\}$. Recall that $\mathrm{conv}(C)$ denotes the convex hull of a set $C \subseteq \mathbb{R}^N$. For a function $u : \Gamma \to \mathbb{R}$ defined on a finite integer interval Γ, its *concave closure* $\bar{u} : \mathrm{conv}(\Gamma) \to \mathbb{R}$ is defined as

$$(24) \qquad \bar{u}(x) = \min\{p \cdot x + \alpha \mid p \in \mathbb{R}^N,\ \alpha \in \mathbb{R},\ p \cdot y + \alpha \geq u(y)\ (\forall y \in \Gamma)\} \quad (x \in \mathrm{conv}(\Gamma)).$$

Note that $\mathrm{conv}(\Gamma)$ is a finite interval and the concave closure $\bar{u}$ is a concave function (see, e.g., Rockafellar (1970)). The right-hand side of (24) can be seen as a linear programming problem, and therefore the minimum always exists for $x \in \mathrm{conv}(\Gamma)$. For $p \in \mathbb{R}^N$, we denote

$$(25) \qquad D_{\mathbf{R}}(\bar{u}, p) = \arg\max\{\bar{u}(z) - p \cdot z \mid z \in \mathrm{conv}(\Gamma)\}.$$

Note that $D_{\mathbf{R}}(\bar{u}, p)$ is a convex set.

Proposition **13 (cf. Rockafellar (1970))** *Let $u : \Gamma \to \mathbb{R}$ be a value function defined on a finite integer interval Γ, and $V : \mathbb{R}^N \to \mathbb{R}$ be its indirect utility function. For $p \in \mathbb{R}^N$, it holds that $\partial V(p) = -D_{\mathbf{R}}(\bar{u}, p)$.*

We call a set $Q \subseteq \mathbb{R}^N$ a *twisted g-polymatroid* if there exists a g-polymatroid $\tilde{Q} \subseteq \mathbb{R}^N$ such that $Q = \{Ux \in \mathbb{R}^N \mid x \in \tilde{Q}\}$. Hence, every twisted g-polymatroid is a polyhedron; we say that a twisted g-polymatroid is *integral* if it is an integral polyhedron. From Proposition 5 and the relationship between GM-concavity and M$^\natural$-concavity, we obtain the following property.

Proposition **14** *Let $u : \Gamma \to \mathbb{R}$ be a GM-concave function defined on a finite integer interval Γ. For every $p \in \mathbb{R}^N$, the set $D_{\mathbf{R}}(\bar{u}, p)$ is an integral twisted g-polymatroid such that $D_{\mathbf{R}}(\bar{u}, p) = \mathrm{conv}(D(u, p))$.*

It is known that the sum of (integral) g-polymatroids is an (integral) g-polymatroid (see Frank and Tardos (1988)). From this fact the following property of twisted g-polymatroids follows.

Proposition **15** *For twisted g-polymatroids $Q_1, Q_2, \ldots, Q_m \subseteq \mathbb{R}^N$, their sum $Q_1 + Q_2 + \cdots + Q_m$ is also a twisted g-polymatroid. Moreover, $Q_1 + Q_2 + \cdots + Q_m$ is integral if each Q_i is integral.*

E.2 Proof

Recall that $\Lambda \subseteq \mathbb{R}^N$ denotes the set of Walrasian equilibrium price vectors.

Lemma 6 *Assume that every bidder $i \in B$ has a GM-concave value function u^i. The set $\arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\}$ is a nonempty compact set and satisfies $\arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\} \subseteq \Lambda$.*

By Lemma 2, it holds that $\Lambda \subseteq \arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\}$, which, combined with Lemma 6, implies that Λ is a nonempty compact set satisfying $\Lambda = \arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\}$. Since the Lyapunov function $\mathcal{L}$ is a generalized polyhedral L$^\natural$-convex function by Theorem 4, the set $\Lambda = \arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\}$ satisfies the following property:

$$(p - \lambda\mathbf{1}) \vee_g q \in \Lambda,\ p \wedge_g (q + \lambda\mathbf{1}) \in \Lambda \qquad (\forall p, q \in W,\ \forall \lambda \in \mathbb{R}_+),$$

i.e., the set Λ is a generalized L$^\natural$-convex set. Since every compact generalized L$^\natural$-convex set contains unique minimal and maximal vectors with respect to the order $\leq_g$ (see Section 4), the claim (i) of Theorem 5 holds. Claim (ii) of Theorem 5 follows from the claim (i) and the latter part of Theorem 4.

We now give a proof of Lemma 6. It is firstly shown that $\arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\}$ is a nonempty compact set. Denote $\Phi^i = \max\{|u^i(x)| \mid x \in \Omega\}$ for $i \in B$ and $\Phi = \max_{i\in B} \Phi^i$.

Lemma 7 *Let $p \in \mathbb{R}^N$.*
(i) *If $p_h > 2\Phi$ holds for some $h \in N$, then the vector $p' \in \mathbb{R}^N$ given by*

$$p'_j = \begin{cases} p_j & (j \in N \setminus \{h\}), \\ 2\Phi & (j = h) \end{cases}$$

satisfies $\mathcal{L}(p') < \mathcal{L}(p)$.
(ii) *If $p_h < -2\Phi$ holds for some $h \in N$, then the vector $p'' \in \mathbb{R}^N$ given by*

$$p''_j = \begin{cases} p_j & (j \in N \setminus \{h\}), \\ -2\Phi & (j = h) \end{cases}$$

satisfies $\mathcal{L}(p'') < \mathcal{L}(p)$.

Proof: We first prove (i). For $i \in B$ and $x \in \Gamma$ with $x_h > 0$, we have

$$\begin{aligned}[u^i(x) - p \cdot x] - [u^i(x - e(h)) - p \cdot (x - e(h))] &\leq u^i(x) - u^i(x - e(h)) - p_h \\ &\leq 2\Phi_i - 2\Phi \leq 0.\end{aligned}$$

This implies that the value $u^i(x) - p \cdot x$ is maximized by a vector x with $x_h = 0$, i.e.,

$$V^i(p) = \max\{u^i(x) - p \cdot x \mid x \in \Gamma,\ x_h = 0\}.$$

Similarly, we have

$$V^i(p') = \max\{u^i(x) - p' \cdot x \mid x \in \Gamma,\ x_h = 0\}.$$

Hence, it holds that

$$\begin{aligned}V^i(p) &= \max\{u^i(x) - p \cdot x \mid x \in \Gamma,\ x_h = 0\} \\ &= \max\{u^i(x) - p' \cdot x \mid x \in \Gamma,\ x_h = 0\} = V^i(p')\end{aligned}$$

for all $i \in B$. It follows from this equation that

$$\mathcal{L}(p) \;=\; \sum_{i \in B} V^i(p) + p \cdot \omega \;=\; \sum_{i \in B} V^i(p') + p \cdot \omega \;>\; \sum_{i \in B} V^i(p') + p' \cdot \omega \;=\; \mathcal{L}(p'),$$

where the strict inequality follows from $\omega_h > 0$ and $p'_h < p_h$.

We then prove (ii). For $i \in B$ and $x \in \Gamma$ with $x_h < \omega_h$, we have

$$\begin{aligned}
[u^i(x) - p \cdot x] - [u^i(x + e(h)) - p \cdot (x + e(h))] &\le u^i(x) - u^i(x + e(h)) + p_h \\
&\le 2\Phi_i - 2\Phi \;\le\; 0.
\end{aligned}$$

This implies that the value $u^i(x) - p \cdot x$ is maximized by a vector x with $x_h = \omega_h$, i.e.,

$$V^i(p) = \max\{u^i(x) - p \cdot x \mid x \in \Gamma,\; x_h = \omega_h\}.$$

Similarly, we have

$$V^i(p'') = \max\{u^i(x) - p'' \cdot x \mid x \in \Gamma,\; x_h = \omega_h\}.$$

Hence, it holds that

$$\begin{aligned}
V^i(p) &= \max\{u^i(x) - p \cdot x \mid x \in \Gamma,\; x_h = \omega_h\} \\
&= \max\{u^i(x) - p'' \cdot x - (p_h + 2\Phi)\omega_h \mid x \in \Gamma,\; x_h = \omega_h\} \\
&= V^i(p'') - (p_h + 2\Phi)\omega_h
\end{aligned}$$

for all $i \in B$. This equation implies that

$$\begin{aligned}
\mathcal{L}(p) - \mathcal{L}(p') &= \left(\sum_{i \in B} V^i(p) + p \cdot \omega\right) - \left(\sum_{i \in B} V^i(p'') + p'' \cdot \omega\right) \\
&= \left(\sum_{i \in B} V^i(p) - \sum_{i \in B} V^i(p'')\right) + (p \cdot \omega - p'' \cdot \omega) \\
&= -|B|(p_h + 2\Phi)\omega_h + (p_h + 2\Phi)\omega_h \;>\; 0,
\end{aligned}$$

where the strict inequality follows from $p_h < -2\Phi < 0$ and $|B| > 1$. Q.E.D.

From Lemma 7 follows that

$$\arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\} = \arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N,\; |p_h| \le 2\Phi \;(\forall h \in N)\},$$

where the set in the right-hand side is nonempty and compact since it is the set of minimizers of a convex function in a bounded region (see, e.g., Rockafellar (1970)). Hence, the first statement of Lemma 6 follows.

We now prove $\arg\min\{\mathcal{L}(p) \mid p \in \mathbb{R}^N\} \subseteq \Lambda$. For this, we show that for every minimizer $p^* \in \mathbb{R}^N$ of $\mathcal{L}$, there exists a set of vectors $z_i \in \Gamma$ $(i \in B)$ such that $(p^*, (z^i \mid i \in B))$ is a Walrasian equilibrium, i.e.,

$$z^i \in D^i(p^*) \;(i \in B), \qquad \sum_{i \in B} z^i = \omega. \tag{26}$$

Since $\mathcal{L}$ is a convex function and p^* is a minimizer of $\mathcal{L}$, the subdifferential of $\mathcal{L}$ at p^* contains the zero vector, i.e., $\mathbf{0} \in \partial\mathcal{L}(p^*)$. By the definition of the Lyapunov function and Propositions 12 and 13, it holds that

$$\partial\mathcal{L}(p^*) = \sum_{i\in B} \partial V^i(p^*) + \{\omega\} = -\sum_{i\in B} D_{\mathbf{R}}(\bar{u}^i, p^*) + \{\omega\}.$$

By Proposition 14, we have $D_{\mathbf{R}}(\bar{u}^i, p) = \mathrm{conv}(D^i(p^*))$ and $D^i(p^*) = D_{\mathbf{R}}(\bar{u}^i, p) \cap \mathbf{Z}^N$ for $i \in B$. Hence, it follows that

$$\begin{aligned} \partial\mathcal{L}(p^*) \cap \mathbf{Z}^N &= \Big(-\sum_{i\in B} D_{\mathbf{R}}(\bar{u}^i, p) + \{\omega\}\Big) \cap \mathbf{Z}^N \\ &= -\sum_{i\in B}(D_{\mathbf{R}}(\bar{u}^i, p) \cap \mathbf{Z}^N) + \{\omega\} \;=\; -\sum_{i\in B} D^i(p^*) + \{\omega\}, \end{aligned}$$

where the first equality is by Proposition 15. This equation and $\mathbf{0} \in \partial\mathcal{L}(p^*)$ imply the existence of a set of vectors $z^i \in \Gamma$ $(i \in B)$ satisfying (26).

F Proofs of Lemma 4, Theorems 6 and 7

The proof of the following lemma is similar to those given in Ausubel (2006) and Sun and Yang (2009).

Proof of Lemma 4: Suppose, to the contrary that, there exists λ such that $0 < \lambda \le 1$ but $\tilde{z}^i \notin D^i(p + \lambda\delta)$. By the EGSI property, for $\tilde{z}^i$ there exists an EGSI improvement bundle z^i with

$$(27) \qquad u^i(z^i) - (p + \lambda\delta) \cdot z^i > u^i(\tilde{z}^i) - (p + \lambda\delta) \cdot \tilde{z}^i.$$

By the choice of $\tilde{z}^i$, we see that

$$u^i(\tilde{z}^i) - (p + \lambda\delta) \cdot \tilde{z}^i \ge u^i(x^i) - (p + \lambda\delta) \cdot \tilde{x}^i \qquad \text{for all } x^i \in D^i(p),$$

and hence $z^i \notin D^i(p)$. It follows from the integrality assumption of value function u^i and $p \in \mathbf{Z}^N$ that

$$(28) \qquad u^i(z^i) - p \cdot z^i \le u^i(\tilde{z}^i) - p \cdot \tilde{z}^i - 1.$$

On the other hand, since $0 \le \lambda\delta_k \le 1$ for any $k \in S_1$ and $-1 \le \lambda\delta_l \le 0$ for any $l \in S_2$, and z^i is an EGSI improvement bundle of $\tilde{z}^i$, we have $|\lambda\delta \cdot (\tilde{z}^i - z^i)| \le 1$, implying that

$$(29) \qquad \lambda\delta \cdot \tilde{z}^i - 1 \le \lambda\delta \cdot z^i.$$

From inequalities (28) and (29) follows that

$$u^i(z^i) - (p + \lambda\delta) \cdot z^i \le u^i(\tilde{z}^i) - (p + \lambda\delta) \cdot \tilde{z}^i,$$

yielding a contradiction to (27).

To prove the second part, observe that the above result $\tilde{z}^i \in D^i(p + \lambda\delta)$ implies

$$\mathcal{L}(p + \lambda\delta) = \mathcal{L}(p) + \lambda(\delta \cdot \omega - \sum_{i \in B} \delta \cdot \tilde{z}^i)$$

for all λ with $0 \leq \lambda \leq 1$. Q.E.D.

We then prove Theorems 6 and 7 concerning the validity of price adjustment prodecure. These theorems can be shown by the results in Kolmogorov and Shioura (2009), where we use the observation that the procedure can be seen as an algorithm for minimizing the Lyapunov function $\mathcal{L}$ which is generalized $L^\natural$-convex by Theorem 4. Instead, we give below more elementary proofs of Theorems 6 and 7. Our proof is based on the generalized submodularity of the Lyapunov function $\mathcal{L}$, which follows from the generalized $L^\natural$-convexity of $\mathcal{L}$ shown in Theorem 4.

Define an n-dimensional cube by

$$\square = \{\delta \in \mathbb{R}^N \mid 0 \leq \delta_k \leq 1, \forall k \in S_1,\ -1 \leq \delta_l \leq 0, \forall l \in S_2\}.$$

Note that $\square_{\mathbb{Z}} = \square \cap \mathbb{Z}^N$. Consider the problem:

$$\max_{\delta \in \square} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\}. \tag{30}$$

The following result shows that the set of solutions to Problem (30) is a generalized $L^\natural$-convex set and both its minimal and maximal elements are integral.

LEMMA 8: *Assume that every bidder $i \in B$ has an integer-valued GM-concave value function. Then the set of solutions to Problem* (30) *is a nonempty generalized $L^\natural$-convex set and both its minimal and maximal elements are integer vectors.*

PROOF: The proof is similar to that of Theorem 5 and is thus omitted. Q.E.D.

This lemma implies, in particular, that Problem (30) has an integral optimal solution, i.e.,

$$\max_{\delta \in \square} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} = \max_{\delta \in \square_{\mathbb{Z}}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\}. \tag{31}$$

Proof of Theorem 6 Theorem 5 shows that the market has a Walrasian equilibrium and an integral vector is a minimizer of the Lyapunov function $\mathcal{L}$ if and only if it is an equilibrium price vector. Since the prices and value functions take only integer values, the Lyapunov function is an integer valued function and it lowers by a positive integer value in each round of the adjustment process. Moreover, the minimum value of the Lyapunov function is finite. This guarantees that the auction terminates in finitely many rounds, i.e., $\delta(t^*) = \mathbf{0}$ in Step 3 for some $t^* \in \mathbb{Z}_+$. Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. Let $\bar{t} \in \mathbb{Z}_+$ be the time when the adjustment process finds $\delta(\bar{t}) = \mathbf{0}$ at Step 2. In the following, we conclude the proof by showing that the vector $p(t^*)$ is a minimizer of the Lyapunov function.

We claim that

$$
\begin{array}{llll}
(32) & \mathcal{L}(p) & \geq & \mathcal{L}(p(\bar{t})) \qquad (\forall p \in \mathbb{Z}^N \text{ with } p \geq_g p(\bar{t})), \\
(33) & \mathcal{L}(p) & \geq & \mathcal{L}(p(t^*)) \qquad (\forall p \in \mathbb{Z}^N \text{ with } p \leq_g p(t^*)).
\end{array}
$$

Since the proofs are quite similar, we prove the inequality (32) only. Suppose, to the contrary, that there exists some $p \geq_g p(\bar{t})$ such that $\mathcal{L}(p) < \mathcal{L}(p(\bar{t}))$. The convexity of function $\mathcal{L}$ implies that there is a strict convex combination p' of p and $p(\bar{t})$ such that $p' \in p(\bar{t}) + \Box$ and $\mathcal{L}(p') < \mathcal{L}(p(\bar{t}))$. By the equation (31) and the choice of $\delta(\bar{t})$, we have

$$\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) \;=\; \min_{\delta \in \Box_{\mathbb{Z}}} \mathcal{L}(p(t) + \delta) \;=\; \min_{\delta \in \Box} \mathcal{L}(p(t) + \delta) \;\leq\; \mathcal{L}(p') \;<\; \mathcal{L}(p(\bar{t})),$$

implying that $\delta(\bar{t}) \neq \mathbf{0}$, which is a contradiction.

By (32), we have $\mathcal{L}(p \vee_g p(\bar{t})) \geq \mathcal{L}(p(\bar{t}))$ for all $p \in \mathbb{Z}^N$ since $p \vee_g p(\bar{t}) \geq_g p(\bar{t})$. We will further show that

$$(34) \qquad \mathcal{L}(p \vee_g p(t)) \geq \mathcal{L}(p(t)) \qquad (\forall t = \bar{t}+1, \bar{t}+2, \cdots, t^*,\ \forall p \in \mathbb{Z}^N).$$

By induction, it is sufficient to prove the case of $t = \bar{t}+1$. Notice that $p(\bar{t}+1) = p(\bar{t}) + \delta(\bar{t})$, where $\delta(\bar{t}) \in -\Box_{\mathbb{Z}}$ is determined in Step 3 of the adjustment process. Assume, to the contrary, that there is some $p \in \mathbb{Z}^N$ such that $\mathcal{L}(p \vee_g p(\bar{t}+1)) < \mathcal{L}(p(\bar{t}+1))$. Then if we start the adjustment process from $p(\bar{t}+1)$, we can by the same previous argument find a nonzero $\widehat{\delta} \in \Box_{\mathbb{Z}}$ in Step 2 such that $\mathcal{L}(p(\bar{t}+1) + \widehat{\delta}) < \mathcal{L}(p(\bar{t}+1))$. Since $\mathcal{L}$ is a generalized submodular function, we have

$$(35) \qquad \mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1) + \widehat{\delta})) + \mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1) + \widehat{\delta})) \leq \mathcal{L}(p(\bar{t})) + \mathcal{L}(p(\bar{t}+1) + \widehat{\delta}).$$

Since $p(\bar{t}) \vee_g (p(\bar{t}+1) + \widehat{\delta}) \geq_g p(\bar{t})$, we have $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1) + \widehat{\delta})) \geq \mathcal{L}(p(\bar{t}))$ by the induction hypothesis, which, together with (35), impies that

$$\mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1) + \widehat{\delta})) \leq \mathcal{L}(p(\bar{t}+1) + \widehat{\delta}) < \mathcal{L}(p(\bar{t}+1)).$$

Observe that

$$p(\bar{t}) \wedge_g (p(\bar{t}+1) + \widehat{\delta}) = p(\bar{t}) \wedge_g (p(\bar{t}) + \delta(\bar{t}) + \widehat{\delta}) = p(\bar{t}) + \delta',$$

where $\delta' = \mathbf{0} \wedge_g (\delta(\bar{t}) + \widehat{\delta}) \in -\Box_{\mathbb{Z}}$. This yields $\mathcal{L}(p(\bar{t}) + \delta') < \mathcal{L}(p(\bar{t}) + \delta(\bar{t}))$ and so $\delta' \neq \delta(\bar{t})$, contradicting the chocie of $\delta(\bar{t})$ since $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) = \min_{\delta \in -\Box_{\mathbb{Z}}} \mathcal{L}(p(\bar{t}) + \delta)$.

We finally show that for all $p \in \mathbb{Z}^N$, $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$ holds. By (33), we have $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ since $p \wedge_g p(t^*) \leq_g p(t^*)$. We also have $\mathcal{L}(p \vee_g p(t^*)) \geq \mathcal{L}(p(t^*))$ by (34). Since $\mathcal{L}$ is a generalized submodular function, we have

$$\mathcal{L}(p) + \mathcal{L}(p(t^*)) \geq \mathcal{L}(p \vee_g p(t^*)) + \mathcal{L}(p \wedge_g p(t^*)) \geq 2\mathcal{L}(p(t^*)),$$

implying that $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$. Hence, we see that $p(t^*)$ is a minimizer of the Lyapunov function $\mathcal{L}$. Q.E.D.

Proof of Theorem 7 The price adjustment process considered in Theorem 7 is a special case of the one in Theorem 6, the termination of the process in a finite number of rounds follows from Theorem 6.

Let $\{p(t) \mid t = 0, 1, \cdots, t^*\}$ be the sequence of price vectors generated by the process. Note that for $t = 0, 1, \cdots, t^*-1$, the vector $p(t+1)$ is an integer vector with $p(t) \leq_g p(t+1)$ and $p(t) \neq p(t+1)$. We also have

$$(36) \qquad \min_{\delta \in \Box'_{\mathbb{Z}}} \mathcal{L}(p(t^*) + \delta) = \mathcal{L}(p(t^*)).$$

In the following, we show that $p(t^*) = \underline{t}$ holds.

We claim that $p(t) \leq_g \underline{p}$ holds for $t = 0, 1, \cdots, t^*$. Proof is done by induction on t. Suppose, to the contrary, that there exists a price vector $p(t)$ such that $p(t) \leq_g \underline{p}$ but $p(t+1) \not\leq_g \underline{p}$. Then, we have $p(t) \wedge_g \underline{p} = p(t)$ and

$$(37) \qquad p(t) \leq_g p(t+1) \wedge_g \underline{p} \leq_g p(t+1) \text{ and } p(t+1) \wedge_g \underline{p} \neq p(t+1)$$

Recall that $\underline{p}$ is a minimizer of $\mathcal{L}$ by Theorem 5. Hence, we have

$$(38) \qquad \mathcal{L}(\underline{p}) \leq \mathcal{L}(p(t+1) \vee_g \underline{p}).$$

Since $\mathcal{L}$ is a generalized submodular function, we have

$$(39) \qquad \mathcal{L}(p(t+1) \vee_g \underline{p}) + \mathcal{L}(p(t+1) \wedge_g \underline{p}) \leq \mathcal{L}(p(t+1)) + \mathcal{L}(\underline{p}).$$

From (38) and (39) follows that $\mathcal{L}(p(t+1) \wedge_g \underline{p}) \leq \mathcal{L}(p(t+1))$. By the construction of $p(t+1)$, this implies that $\mathcal{L}(p(t+1) \wedge_g \underline{p}) = \mathcal{L}(p(t+1))$ and therefore $p(t+1) \leq_g p(t+1) \wedge_g \underline{p}$, contradicting the inequality (37).

By the claim above, we have $p(t^*) \leq_g \underline{p}$, in particular. Suppose, to the contrary, that $p(t^*) \neq \underline{p}$. Then, $p(t^*) \wedge_g \underline{p}$ is less than $\underline{p}$ in at least one component with respect to the order of $\leq_g$. This implies that $p(t^*) \wedge_g \underline{p}$ is not a minimizer of $\mathcal{L}$, i.e.,

$$(40) \qquad \mathcal{L}(\underline{p}) < \mathcal{L}(p(t^*) \wedge_g \underline{p})$$

since $\underline{p}$ is the unique minimal minimizer of $\mathcal{L}$ with respect to the order of $\leq_g$ by Theorem 5. Since $\mathcal{L}$ is a generalized submodular function, we have

$$(41) \qquad \mathcal{L}(p(t^*) \vee_g \underline{p}) + \mathcal{L}(p(t^*) \wedge_g \underline{p}) \leq \mathcal{L}(p(t^*)) + \mathcal{L}(\underline{p}).$$

From (40) and (41) follows that $\mathcal{L}(p(t^*) \vee_g \underline{p}) < \mathcal{L}(p(t^*))$. Hence, there exists a strict convex combination p' of $p(t^*)$ and $p(t^*) \vee_g \underline{p}$ such that $p' - p(t^*) \in \Box$ and $\mathcal{L}(p') < \mathcal{L}(p(t^*))$ due to the convexity of $\mathcal{L}$. On the other hand, (31) and (36) imply that

$$\mathcal{L}(p') \geq \min_{\delta \in \Box} \mathcal{L}(p(t^*) + \delta) = \min_{\delta \in \Box_{\mathbb{Z}}} \mathcal{L}(p(t^*) + \delta) = \mathcal{L}(p(t^*)),$$

a contradiction. Hence, $p(t^*) = \underline{t}$ holds. Q.E.D.

G Proof of Theorem 8

Sun and Yang (2006) establish their Theorem 4.1 by showing that every firm's revenue function satisfies the GSC condition. In the current more general model, we will show that every firm's revenue function is GM-concave and thus satisfies the GGSC condition by Theorem 1. Then it follows from Theorem 5 that the model has a Walrasian equilibrium.

To prove that every firm revenue function R (here we ignore the identity index i) is a GM-concave function, we first construct a directed graph and show that the revenue function can be formulated as the maximum weight circulation problem of a flow network. The graph $G = (V, A)$ is constructed as follows. The set $V = M \cup W \cup \{S\}$ represents all nodes or vertices, where $M = \{m_1, m_2, \cdots, m_n\}$ is the family of all types of machines, $W = \{w_1, w_2, \cdots, w_m\}$ is the family of all types of workers, and S stands for the intersection node. The set $A = \{(S, m_l) \mid m_l \in M\} \cup \{(S, w_k) \mid w_k \in W\} \cup \{(w_k, m_l) \mid w_k \in W, m_l \in M\}$ denotes the set of all arcs in the graph. Here the arc (S, m_l) enters m_l and leaves S. S and m_l are called the tail and the head of the arc, respectively. (S, w_k) and (w_k, m_l) can be explained in a similar manner.

A function ξ that assigns every arc a real value is called a *flow.* A flow $\xi = (\xi(a) \mid a \in A) \in \mathbb{Z}^A$ is called a *circulation* if the flow conservation law holds at every node, i.e.,

$$\sum\{\xi(a) \mid \text{arc } a \text{ enters } v\} = \sum\{\xi(a) \mid \text{arc } a \text{ leaves } v\}, \ \forall v \in V.$$

Suppose that we are given a vector $y = (y', y'') \in \overline{\Omega}$, where

$$\overline{\Omega} = \{(y', y'') \in \mathbb{Z}^{S_1} \times \mathbb{Z}^{S_2} \mid 0 \leq y'_k \leq d_k, \ k \in S_1, \ -b_l \leq y''_l \leq 0, \ l \in S_2\}.$$

For our graph $G = (V, A)$, we have the following capacity constraints and revenues (weights) for the flow in each arc. For each arc $(S, w_k) \in A$, it should satisfy the capacity constraint

$0 \leq \xi(S, w_k) \leq y'_k$ and its revenue $w(S, w_k)$ is equal to 0; for each arc $(S, m_l) \in A$, it should satisfy the capacity constraint $y''_l \leq \xi(S, m_l) \leq 0$ and its revenue $w(S, m_l)$ is 0; for each arc $(w_k, m_l) \in A$, it should satisfy the capacity constraint $0 \leq \xi(w_k, m_l) \leq \infty$ and its revenue $w(w_k, m_l)$ is $R(w_k, m_l)$.

The maximum weight (revenue) circulation problem is to find a circulation $\xi \in \mathbb{Z}^A$ that maximizes the total revenue $\sum_{a \in A} w(a)\xi(a)$ subject to the described capacity constraints. Clearly, this maximum revenue depends on all y'_k and y''_l and can be seen as the function of the variables $(y', y'') \in \mathbb{Z}^{S_1} \times \mathbb{Z}^{S_2}$. We denote this function by $F(y', y'')$. Below we will prove that $F(y', y'')$ is an $M^\natural$-concave function. Since the firm's revenue function is given as $R(x', x'') = F(U(x', x''))$ with $(x', x'') \in \mathbb{Z}^{S_1} \times \mathbb{Z}^{S_2}$, it is GM-concave by Proposition 1 in the Appendix.

We now prove that the function $F(y', y'')$ is an $M^\natural$-concave function. $M^\natural$-concavity of the function $F(y', y'')$ is rewritten as follows:

for every $y = (y', y''), z = (z', z'') \in \overline{\Omega}$ and $k^* \in \mathrm{supp}^+(y - z)$, there exists some $l^* \in \mathrm{supp}^-(y - z) \cup \{0\}$ such that

$$F(y) + F(z) \leq F(y - e(k^*) + e(l^*)) + F(z + e(k^*) - e(l^*)).$$

The proof below is quite similar to the one for Theorem 1.4 of Murota and Shioura (2005). Notice, however, that while they prove $M^\natural$-concavity in upper capacity only or in lower capacity only, here we have to show that $M^\natural$-concavity holds true both for upper capacity and for lower capacity simultaneously.

Let $y = (y', y''), z = (z', z'') \in \overline{\Omega}$ and $k^* \in \mathrm{supp}^+(y - z)$. We here consider the case of $k^* \in S_1$ only since the case of $k^* \in S_2$ can be shown in the same way. Let $\xi \in \mathbb{Z}^A$ and $\zeta \in \mathbb{Z}^A$ be optimal circulations for y and z, respectively, i.e., ξ (resp., ζ) is a circulation satisfying the capacity constraint with respect to y (resp., z) such that $F(y) = \sum_{a \in A} w(a)\xi(a)$ (resp., $F(z) = \sum_{a \in A} w(a)\zeta(a)$).

Suppose that $\xi(S, w_{k^*}) < y_{k^*}$ holds. Then, ξ (resp., ζ) is a circulation satisfying the capacity constraint with respect to $y - e(k^*)$ (resp., $z + e(k^*)$), implying that

$$\sum_{a \in A} w(a)\xi(a) \leq F(y - e(k^*)), \qquad \sum_{a \in A} w(a)\zeta(a) \leq F(z + e(k^*)).$$

Hence, we have

$$F(y) + F(z) = \sum_{a \in A} w(a)\xi(a) + \sum_{a \in A} w(a)\zeta(a) \leq F(y - e(k^*)) + F(z + e(k^*)).$$

We then consider the case with $\xi(S, w_{k^*}) = y_{k^*}$. Then, it holds that

$$\xi(S, w_{k^*}) = y_{k^*} > z_{k^*} \geq \zeta(S, w_{k^*}).$$

Therefore, by a standard network-flow argument (see, e.g., Gale and Politof (1981) and Murota and Shioura (2005)), it can be shown that there exists a sequence of distinct nodes $v_1, v_2, \ldots, v_h$ of graph G satisfying the following conditions:

- $v_1 = S$, $v_2 = w_{k^*}$.
- for each $i = 1, 2, \ldots, h$, either (v_i, v_{i+1}) or (v_{i+1}, v_i) is an arc in A; this means that the arc set

$$C = \{a \in A \mid a = (v_i, v_{i+1}) \text{ or } a = (v_{i+1}, v_i) \text{ for some } i\}$$

is an (undirected) circuit in G.
- for each $i = 1, 2, \ldots, h$, if $(v_i, v_{i+1}) \in A$ then $\xi(v_i, v_{i+1}) > \zeta(v_i, v_{i+1})$ holds; if $(v_{i+1}, v_i) \in A$ then $\xi(v_{i+1}, v_i) < \zeta(v_{i+1}, v_i)$ holds.

Observe that $(v_1, v_2) = (S, w_k)$ and (v_1, v_h) are the only arcs in C incident to the node S, and $v_h \in S_1 \cup S_2$ holds.

Using the circuit C, we define two new flows ξ' and ζ' by

$$\xi'(a) = \begin{cases} \xi(a) - 1 & (\text{if } a = (v_i, v_{i+1}) \text{ for some } i), \\ \xi(a) + 1 & (\text{if } a = (v_{i+1}, v_i) \text{ for some } i), \\ \xi(a) & (\text{otherwise}), \end{cases}$$

$$\zeta'(a) = \begin{cases} \zeta(a) + 1 & (\text{if } a = (v_i, v_{i+1}) \text{ for some } i), \\ \zeta(a) - 1 & (\text{if } a = (v_{i+1}, v_i) \text{ for some } i), \\ \zeta(a) & (\text{otherwise}). \end{cases}$$

Then, the flows ξ' and ζ' are circulations satisfying

$$\sum_{a \in A} w(a)\xi'(a) + \sum_{a \in A} w(a)\zeta'(a) = \sum_{a \in A} w(a)\xi(a) + \sum_{a \in A} w(a)\zeta(a).$$

We then show that there exists some $l^* \in \mathrm{supp}^-(y - z) \cup \{0\}$ such that ξ' (resp., ζ') satisfies the capacity constraint with respect to $y - e(k^*) + e(l^*)$ (resp., $z + e(k^*) - e(l^*)$). We have

$$\begin{aligned}
&0 \le \xi'(S, w_k) = \xi(S, w_k) \le y'_k, \quad 0 \le \zeta'(S, w_k) = \zeta(S, w_k) \le z'_k \quad \text{if } w_k \notin \{w_{k^*}, v_h\}, \\
&0 \le \zeta(S, w_{k^*}) \le \xi'(S, w_{k^*}) = \xi(S, w_{k^*}) - 1 = y'_{k^*} - 1, \\
&0 \le \zeta'(S, w_{k^*}) = \zeta(S, w_{k^*}) + 1 \le z'_{k^*} + 1, \\
&y''_l \le \xi'(S, m_l) = \xi(S, m_l) \le 0, \quad z''_l \le \zeta'(S, m_l) = \zeta(S, m_l) \le 0 \quad \text{if } m_l \ne v_h.
\end{aligned}$$

Suppose that $v_h = w_{l^*}$ for some $l^* \in S_1$. Then, we have

$$\begin{aligned}
&0 \le \xi'(S, w_{l^*}) = \xi(S, w_{l^*}) + 1 \le y'_{l^*} + 1, \\
&0 \le \xi(S, w_{l^*}) \le \zeta'(S, w_{l^*}) = \zeta(S, w_{l^*}) - 1 \le z'_{l^*} - 1,
\end{aligned}$$

implying that ξ' (resp., ζ') satisfies the capacity constraint with respect to $y - e(k^*) + e(l^*)$ (resp., $z + e(k^*) - e(l^*)$).

We then suppose that $v_h = m_{l^*}$ for some $l^* \in S_2$. It holds that

$$\begin{aligned} y''_{l^*} + 1 &\leq \xi(S, m_{l^*}) + 1 = \xi'(S, m_{l^*}) \leq \zeta(S, m_{l^*}) \leq 0, \\ z''_{l^*} - 1 &\leq \zeta(S, m_{l^*}) - 1 = \zeta'(S, m_{l^*}) \leq 0, \end{aligned}$$

implying that ξ' (resp., ζ') satisfies the capacity constraint with respect to $y - e(k^*) + e(l^*)$ (resp., $z + e(k^*) - e(l^*)$).

Hence, we have

$$\begin{aligned} F(y - e(k^*) + e(l^*)) + F(z + e(k^*) - e(l^*)) &\geq \sum_{a \in A} w(a)\xi'(a) + \sum_{a \in A} w(a)\zeta'(a) \\ &= \sum_{a \in A} w(a)\xi(a) + \sum_{a \in A} w(a)\zeta(a) \\ &= F(y) + F(z). \end{aligned}$$

This concludes the proof for M$^\natural$-concavity of F.

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