THE UNIVERSITY *of York*

Discussion Papers in Economics

No. 13/06

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Zaifu Yang and Dongmo Zhang

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

On a Fundamental Property of Talman-Yang's Auction under Price Control[1]

Zaifu Yang[2] and Dongmo Zhang[3]

Abstract: Recently Talman and Yang (2008) examined an assignment market under price control. In the market a number of heterogeneous items are to be sold to several bidders. Each bidder has a valuation on each item. The seller has a reservation price for every item. Meanwhile every item has a ceiling price imposed by a central planner. Due to price controls, there usually do not exist market-clearing prices. To deal with this allocation problem, Talman and Yang proposed a dynamic auction with rationing that always yields a constrained equilibrium. In this paper we establish that this dynamic auction can actually find a core allocation in finite steps, resulting in a Pareto efficient outcome. A core allocation consists of an assignment of items and a supporting price vector for the assignment.

Keywords: Ascending auction, multi-item auction, housing market, price control, Pareto efficiency, core.

JEL classification: D44.

1 Introduction

We examine a general assignment market under price control. In this market a seller wishes to sell a number of items, say, houses, to many potential buyers. Each buyer has a valuation on every item. The seller has a reservation price for every item. Meanwhile, a central planner imposes a ceiling price on every item. The basic question is how to

[1]Part of this research was done while the first author was visiting the University of Western Sydney, Australia. He wishes to thank the Institute and ARC (DPDP0988750) for the hospitality and financial support.

[2]Department of Economics and Related Studies, University of York, York, UK, `zaifu.yang@york.ac.uk`

[3]Intelligent Systems Laboratory, University of Western Sydney, Australia, `d.zhang@uws.edu.au`

allocate these items as efficiently as possible through a price system satisfying both the reservation and ceiling price constraints. Typical examples include public housing and on-campus housing. Price control has been also used on other occasions. For example, to prevent breakdown of stock markets, often ceilings and floors are imposed upon the price of each stock; price controls are used to reduce inflation or deflation; and minimum wages are employed to protect certain groups of the society. See e.g., Drèze (1975), Cox (1980), and Arnott (1995).

The assignment market under price control we study here reduces to the famous assignment market investigated by Koopmans and Beckmann (1957) and Shapley and Shubik (1971) where there is no price or rent control, i.e., the ceiling price of each item is sufficiently large. It is well-known that the assignment market has at least one Walrasian equilibrium (Koopmans and Beckmann, 1957) and the set of Walrasian equilibrium price vectors forms a lattice (Shapley and Shubik, 1971). Due to price controls, a Walrasian equilibrium usually fails to exist. Traditionally, to deal with this problem, a weaker solution called constrained equilibrium has been developed. In this solution a rationing scheme must be used to curb consumers' demand or seller's supply on certain goods. The interested reader could refer to Drèze (1975), Cox (1980), van der Laan (1980), Kurz (1982), Azariadis and Stiglitz (1983), Dehez and Drèze (1984), Weddepohl (1987), Herings, Talman and Yang (1996), Talman and Yang (2008) among many others. Rationing will help prices to facilitate the allocation of goods among agents. Unfortunately, it is known that a constrained equilibrium need not be Pareto efficient; see e.g., Böhm and Müller (1977), Herings and Konovalov (2009).

While the literature on price or rent control has focused almost entirely on economic models with divisible goods, Talman and Yang (2008) recently have studied the assignment market under price control where items for sale are inherently indivisible, such as houses or apartments. Modifying the processes of Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986), Talman and Yang (2008) propose an ascending auction with rationing that always generates a constrained equilibrium.[4] The constrained equilibrium consists of

[4]Andersson and Svensson (2012) examined a similar problem and proposed a different constrained equilibrium which depends on an exogenously given priority-ordering on buyers.

an assignment of items, a price vector and a rationing scheme. In this paper instead of using the notion of constrained equilibrium, we adopt the concept of core allocation to the current model. The obvious advantage of this solution over the constrained equilibrium is that firstly every core allocation is Pareto efficient and stable against any coalition deviation, secondly the core allocation is conceptually simpler, more intutitive and more straightforward than the constrained equilibrium, and thirdly the core allocation has been widely used in general exchange economies and game theoretic models; see e.g., Scarf (1967). A core allocation consists of only an assignment of items and a supporting price vector for the assignment. It does not use any rationing scheme. We demonstrate that the auction proposed by Talman and Yang (2008) always finds a core allocation in finitely many steps, thus resulting in a Pareto efficient outcome.

The rest of the paper proceeds as follows. Section 2 sets up the model. Section 3 presents the main results.

2 The Model

A seller wishes to sell n heterogenous items, denoted by 1, 2, $\cdots$, n, to m potential bidders indexed by 1, 2, $\cdots$, m. We use the set $N = \{0, 1, 2, \cdots, n\}$ to represent all items including the dummy good 0 and the set $M = \{1, 2, \cdots, m\}$ to stand for all bidders. The dummy item 0 has no value but does no harm and can be assigned to any number of bidders. Every bidder $i \in M$ has a valuation $V^i(j) \in \mathbb{Z}_+$ in monetary units on each item $j \in N$ with $V^i(0) = 0$. The seller has a reservation price $\underline{p}_j \in \mathbb{Z}_+$ for each item $j \in N$ below which the item will not be sold. In addition, a central planner, say, the government imposes rent controls. That is, each item $j \in N$ has a ceiling price $\bar{p}_j \in \mathbb{Z}_+$ above which the item is not permitted to sell. By convention $\underline{p}_0 = \bar{p}_0 = 0$. When there are no rent controls, i.e., $\bar{p}_j = \infty$ for every $j = 1, 2, \cdots, n$, the model reduces to the famous assignment market model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971).

A *feasible assignment* π assigns every bidder $i \in M$ an item $\pi(i)$ such that no item in $N \setminus \{0\}$ is assigned to more than one bidder. Note that a feasible allocation may assign the dummy good to several bidders and a real item $j \neq 0$ may not be assigned to any bidder at all. An item $j \geq 1$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let $U(\pi)$ denote the set of all unassigned items at π.

A price vector $p \in \mathbb{R}^N_+$ indicates a price for every good. The price of each good $j \in N \setminus \{0\}$ is not completely flexible and is restricted to an interval $[\underline{p}_j, \bar{p}_j]$, where $\underline{p}_j$ and $\bar{p}_j$ are integers and $0 \leq \underline{p}_j < \bar{p}_j$. The set

$$P = \{p \in \mathbb{R}^N \mid p_0 = 0,\ \underline{p}_j \leq p_j \leq \bar{p}_j,\ j = 1, \cdots, n\ \}$$

denotes the set of admissible prices. The price of the dummy good is always fixed at zero.

A *feasible allocation* (π, p) consists of a feasible assignment π and an admissible price vector p such that $p_j = \underline{p}_j$ for every unassigned item $j \in U(\pi)$.

A feasible assignment π^* is *socially optimal* if

$$\sum_{i \in M} V^i(\pi^*(i)) + \sum_{j \in U(\pi^*)} \underline{p}_j \geq \sum_{i \in M} V^i(\pi(i)) + \sum_{j \in U(\pi)} \underline{p}_j$$

for every feasible assignment π.

A feasible allocation (π, p) is *Pareto efficient* if there does not exist another feasible allocation (ρ, q) such that $V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}$ for every bidder $i \in M$, and $\sum_{h \in N} q_h > \sum_{h \in N} p_h$ for the seller.

Without rent controls (or price ceilings), the demand set of bidder $i \in M$ at a price vector $p \in \mathbb{R}^N_+$ is given by

$$D^i(p) = \{j \mid V^i(j) - p_j \geq V^i(k) - p_k \text{ for every } k \in N\}.$$

A *Walrasian equilibrium* consists of a feasible assignment π and a price vector $p \in \mathbb{R}^N_+$ such that $\pi(i) \in D^i(p)$ for all $i \in M$ and $p_j = \underline{p}_j$ for any unassigned good j at π. It is well-known from Koopmans and Beckmann (1957), Shapley and Shubik (1971) that a Walarasian equilibrium exists in the economy when there are no rent controls, i.e., $\bar{p}_j = +\infty$ for all j. Furthermore, Crawford and Knoer (1981) and Demange, Gale and Sotomayor (1986) propose two elegant auctions which can always find a Walrasian equilibrium.

In the case of rent control, a Walrasian equilibrium may not exist since the equilibrium price vector may not be admissible. This can be easily seen from the following example. Suppose that a seller wishes to sell one item 1 to two bidders. The reservation price is zero and the ceiling price is $\bar{p}_1 = 3$. Bidder 1's valuation is 5 and bidder 2's valuation is 7. It is clear that at any admissible price p_1, $0 \leq p_1 \leq 3$, that the seller can charge, the item is always in excess demand. Therefore it is impossible to have a Walrasian equilibrium.

To deal with rent controls or price rigidities which fail the existence of Walrasian equilibrium, a weaker solution called constrained equilibrium has been explored. In this solution rationing is used to curb either demand or supply but never both. There is a rich literature on this subject; see e.g., Drèze (1975), Dehez and Drèze (1984), Cox (1980), van der Laan (1980), Kurz (1982), Azariadis and Stiglitz (1983), Weddepohl (1987), Arnott (1995), Herings, Talman and Yang (1996), and Talman and Yang (2008) among many others. Unfortunately, it is known that a constrained equilibrium need not be Pareto efficient; see e.g., Böhm and Müller (1977), Herings and Konovalov (2009).

In this paper we adopt the notion of core to the current model. This concept is more general than that of Walrasian equilibrium and is a widely used solution for general exchange economies and non-transferable utility games, see Scarf (1967). It does not use rationing and is more intuitive and more straightforward than the constrained equilibrium. It is also more familiar to economists.

Definition 2.1 *A feasible allocation* (π, p) *is* a core allocation *if there do not exist a coalition* S *of bidders and another feasible allocation* (ρ, q) *such that* $\rho(i) = 0$ *for every* $i \in M \setminus S$*, and* $V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}$ *for every* $i \in S$*, and* $\sum_{h\in N} q_h > \sum_{h\in N} p_h$.

Clearly, a core allocation must be Pareto efficient.

3 The Talman-Yang Dynamic Auction

Talman and Yang's auction works roughly as follows: The auctioneer starts the auction at the reservation prices of the items for sale. Then the bidders respond with their demand sets. The auctioneer accordingly eliminates over-demanded items by increasing their prices or by a lottery to determine who should be assigned the item when its price has reached its ceiling price. The auction stops when there are no over-demanded items at which a core allocation will be shown to exist.

A set of real items $S \subseteq N \setminus \{0\}$ is *over-demanded* at a price vector $p \in \mathbb{R}^N$, if the number of bidders who demand only items in S is strictly greater than the number of items in S, i.e., $|\{i \in M \mid D^i(p) \subseteq S\}| > |S|$. An over-demanded set S is said to be *minimal* if no strict subset of S is an over-demanded set. Now we are ready to describe the Talman-Yang dynamic auction under price rigidities. Note that in the auction process, since the set of bidders and the set of items are shrinking, the demand set of each bidder and the over-demanded sets need to be adapted accordingly.

The Talman-Yang Auction

Step 1: The auctioneer announces the set of items $N = \{0, 1, \cdots, n\}$ for sale, the reservation price vector $\underline{p}$ and the ceiling price vector $\bar{p}$. The bidders, denoted by $M = \{1, \cdots, m\}$, come to bid. Let $t := 0$ and $p^t := \underline{p}$. Go to Step 2.

Step 2: The auctioneer asks every remaining bidder i to report his demand set $D^i(p^t)$ on the remaining items and checks whether there is any over-demanded set of items at p^t. If there is no over-demanded set of items, the auction stops. Otherwise, there is at least one over-demanded set. The auctioneer first chooses an over-demanded set S of items and next checks whether the price of any item in the set S has reached its ceiling price. Let $\bar{S} := \{j \in S \mid p^t_j = \bar{p}_j\}$. If $\bar{S}$ is empty, the auctioneer increases the price of each item in S by one unit and keeps the prices of all other items unchanged. Let $t := t + 1$ and return to Step 2. If $\bar{S}$ is not empty, go to Step 3.

Step 3: The auctioneer picks an item at random from $\bar{S}$ and asks all bidders who demand the item to draw lots for the right to buy the item. Then the (unique) winning bidder gets the item by paying its current price and exits from the auction. Delete this bidder from M and delete his won item from N. If $M = \emptyset$ or $N = \emptyset$, the auction stops. Otherwise, let $t := t + 1$ and return to Step 2.

Note that in Step 2 the auctioneer just needs to choose an over-demanded set. This rule is more general and more flexible than the original one of Talman and Yang (2008) which requires a minimal over-demanded set.

Before proving the convergence of the auction, we illustrate by example how the auction actually operates.

Example 1: Suppose that there are five bidders (1, 2, 3, 4, 5) and four items (0, 1, 2, 3, 4) in a market. The reservation and ceiling price vectors are $\underline{p} = (0, 5, 4, 1, 5)$, and $\bar{p} = (0, 6, 6, 4, 7)$. Bidders' values are given in Table 1.

Table 1: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	3	10	7
Bidder 2	0	7	6	20	3
Bidder 3	0	5	5	40	7
Bidder 4	0	9	4	40	2
Bidder 5	0	6	2	40	10

The auction starts at the price vector $p^0 = (0, 5, 4, 1, 5)$. Then bidders report their demand sets: $D^1(p^0) = \{3\}$, $D^2(p^0) = \{3\}$, $D^3(p^0) = \{3\}$, $D^4(p^0) = \{3\}$ and $D^5(p^0) = \{3\}$. The set $S = \{3\}$ is a minimal over-demanded set and the auctioneer adjusts p^0 to $p^1 = (0, 5, 4, 2, 5)$. The demand sets and price vectors and other relevant data generated by the auction are illustrated in Table 2. In Step 3, the price of item 3 has reached its upper bound 4. The auctioneer assigns randomly item 3, say, to bidder 1. So bidder 1 gets item 3 by paying 4 dollars and leaves the auction. Then we have $M = \{2, 3, 4, 5\}$ and $N = \{0, 1, 2, 4\}$. The auctioneer adjusts p^3 to $p^4 = (0, 5, 4, 5)$. In Step 7, there is no over-demanded set of items at $p^7 = \{6, 4, 7\}$ and the auctioneer can assign item 2 to bidder 2, item 1 to bidder 4, and item 4 to bidder 5. In the end, bidder 1 gets item 3 and pays 4; bidder 2 gets item 2 and pays 5; bidder 3 gets nothing and pays nothing; bidder 4 gets item 1 and pays 6; bidder 5 gets item 4 and pays 7. Let $p^* = (0, 6, 5, 4, 7)$ and $\pi^* = (3, 2, 0, 1, 4)$. The auction ends up with the allocation (π^*, p^*).

Let us give a simple argument showing that (π^*, p^*) is a core allocation. At (p^*, π^*), the seller's revenue is 22, bidder 1's profit is 6, bidder 2's is 1, bidder 3's is 0, bidder 4's is 3, and bidder 5's is 3. Suppose to the contrary that (π^*, p^*) is not a core allocation. There must exist a blocking coalition consisting of the seller and some bidders who can make themselves better off. In order for the seller to be better off, her revenue must be greater than 22. Notice that all prices p_j^* except p_2^* have reached the ceiling prices. So the seller has to sell her goods at a new price vector $q = (q_0, q_1, q_2, q_3, q_4)$ that must satisfy $q_0 = 0$, $q_1 = p_1^* = 6$, $p_2^* = 5 < q_2 \leq 6$, $q_3 = p_3^* = 4$, and $q_4 = p_4^* = 7$. It means that all items must be sold and the possible blocking coalition must have at least 4 bidders. Observe that item 2 can be sold only to bidder 2. But then bidder 2's profit will be less than what he gets from (π^*, p^*). This shows that (π^*, p^*) cannot be blocked by any coalition and thus must be a core allocation.

Table 2: The data generated by the auction for the Example.

Step	Prices	N	M	S	$D^1(p)$	$D^2(p)$	$D^3(p)$	$D^4(p)$	$D^5(p)$
0	$(0,5,4,1,5)$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
1	$(0,5,4,2,5)$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
2	$(0,5,4,3,5)$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
3	$(0,5,4,4,5)$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\emptyset$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$(0,5,4,5)$	$\{0,1,2,4\}$	$\{2,3,4,5\}$	$\{4\}$		$\{1,2\}$	$\{4\}$	$\{1\}$	$\{4\}$
5	$(0,5,4,6)$	$\{0,1,2,4\}$	$\{2,3,4,5\}$	$\{2,4\}$		$\{1,2\}$	$\{1,4\}$	$\{1\}$	$\{4\}$
6	$(0,6,4,7)$	$\{0,1,2,4\}$	$\{2,3,4,5\}$	$\{2\}$		$\{2\}$	$\{2\}$	$\{1\}$	$\{4\}$
7	$(0,6,5,7)$	$\{0,1,2,4\}$	$\{2,3,4,5\}$	$\emptyset$		$\{1,2\}$	$\{0,2,4\}$	$\{1\}$	$\{4\}$

The following finite convergence theorem establishes a fundamental property of the auction. That is, the allocation found by the Talman-Yang auction is always in the core.

Theorem 3.1 *The Talman-Yang auction always finds a core allocation in a finite number of steps.*

Proof: We first show that the auction terminates in finitely many steps. This follows immediately from the fact that the auction is ascending and the valuation of every bidder on each item is finite. Let (π, p) be the allocation generated by the auction. It remains to prove that (π, p) is a core allocation.

Suppose to the contrary that (π, p) were not a core allocation. Then there would exist a group S of bidders and an allocation (ρ, q) blocking (π, p). So for the seller, we have

$$\begin{aligned}\sum_{j\in N} q(j) &= \sum_{i\in S} q_{\rho(i)} + \sum_{j\in U(\rho)} \underline{p}_j \\ &> \sum_{i\in M} p_{\pi(i)} + \sum_{j\in U(\pi)} \underline{p}_j \\ &= \sum_{j\in N} p_j\end{aligned}$$

It is clear that there exists some $j^* \in N$ such that

$$q_{j^*} > p_{j^*} \tag{3.1}$$

This means that some bidder $i^* \in S$ must be assigned item j^* at ρ, i.e., $\rho(i^*) = j^*$. On the other hand, for every bidder $i \in S$, we have

$$V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)} \tag{3.2}$$

For bidder i^*, it follows from (3.1) and (3.2) that

$$\begin{aligned}V^{i^*}(j^*) - p_{j^*} &> V^{i^*}(j^*) - q_{j^*} \\ &> V^{i^*}(\pi(i^*)) - p_{\pi(i^*)}\end{aligned}$$

But then this inequality would imply that at prices p, bidder i^* should have rejected item $\pi(i^*)$ in favor of item j^*, yielding a contradiction. Observe that this argument is valid, because item j^* cannot be any item that has reached its upper price level $\bar{p}_{j^*}$ and has been sold before the auction stops, for otherwise, then we would have $q_{j^*} = p_{j^*} = \bar{p}_{j^*}$ which then contradicts inequality (3.1).

This demonstrates that (π, p) is indeed a core allocation. □

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