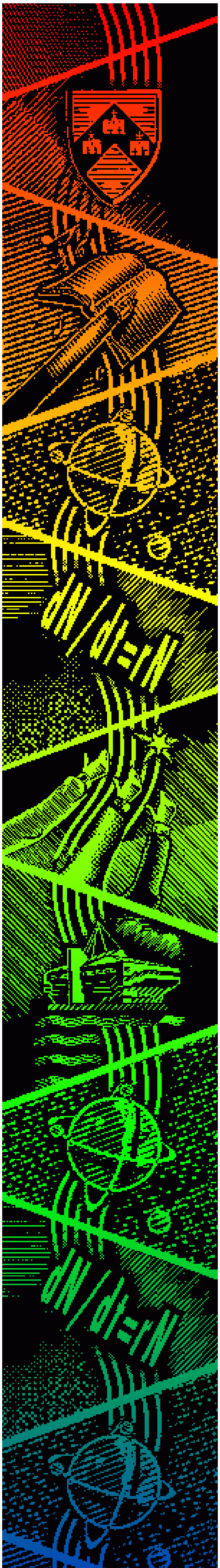

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A Competitive Partnership Formation Process

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Abstract

A group of heterogenous agents may form partnerships in pairs. All single agents as well as all partnerships generate values. If two agents choose to cooperate, they need to specify how to split their joint value among one another. In equilibrium, which may or may not exist, no agents have incentives to break up or form new partnerships. This paper proposes a dynamic competitive adjustment process that always either finds an equilibrium or exclusively proves the nonexistence of any equilibrium in finitely many steps. When an equilibrium exists, partnership and revenue distribution will be automatically and endogenously determined by the process. Moreover, several fundamental properties of the equilibrium solution and the model are derived.

Keywords: Partnership formation, adjustment process, equilibrium, assignment market.
JEL classification: C62, C72, D02.

1. Introduction

Partnership is a fundamental and common pattern observed in many social and economic relations. We consider a group of self-interested agents or firms who individually choose to act alone, or, if it is to their mutual benefit, form partnerships in pairs. Any agent that stays independent generates a value for herself, whereas a cooperating pair must agree upon how to split their jointly generated value. The values together with the corresponding set of agents form the basis of the partnership formation problem. In this competitive environment agents cannot simply choose whom to cooperate with – if they do not offer a sufficiently generous split of the joint value, their potential partner may look elsewhere for agents to cooperate with. Hence, each agent faces a trade-off between trying to maximize her own benefit while still bestowing her partner with a large enough fraction of their joint value. In equilibrium, no agents have incentives to break up or

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form new partnerships. More precisely, no agent gets less than she generates on her own, whereas no two agents get less in total than their joint value.

Two special but prominent cases of the partnership formation problem are the widely studied assignment and marriage matching markets; see Koopmans and Beckmann (1957), Shapley and Shubik (1971), and Becker (1973) among many others. These markets are two-sided in the sense that two disjoint groups are interacting, for example buyers and sellers, firms and workers, or men and women. This separation is erased in the partnership formation problem – agents may well be different (in the values they create), but they are nevertheless all gathered on "the same side" of the market. The assignment markets have been extensively investigated, predominantly with an equilibrium concept that coincides with the core. It is known not only that equilibrium always exists (Koopmans and Beckmann, 1957), but also that the set of equilibria forms a lattice (Shapley and Shubik, 1971; Demange and Gale, 1985). In practice, these markets are likely affected by informational asymmetry: the surplus created by a firm employing a worker is information at best available to that particular firm and worker. For assignment markets, several adjustment processes have been proposed that converge to market equilibrium and in addition do not require agents to disclose all of their private information (which they may be reluctant to). Two of the most important ones are due to Crawford and Knoer (1981), and Demange, Gale, and Sotomayor (1986).[5]

Unlike the assignment markets which always have an equilibrium, there exist partnership formation problems that do not have any equilibrium. For instance, consider the following problem with three agents – called 1, 2, and 3. Assume there is no gain from staying independent, whereas any partnership generates a value of 3 dollars. Formally, a matching is used to keep track of who is cooperating with whom, under the restriction that each agent at most may have one partner. Hence, if agents 1 and 2 form a partnerships, then agent 3 necessarily remains single. Suppose that agents 1 and 2 allocate one dollar to agent 1 and two dollars to agent 2 out of their jointly generated 3 dollars. This situation is not stable as agent 3 can lure agent 1 away from agent 2 by offering him two dollars – this is beneficial for agent 1 (he gets two instead of one) as well as for agent 3 (he keeps one instead of zero). Hence, the partnership between agents 1 and 2 is broken up, and a new one is formed between agents 1 and 3. By symmetry, this situation is no less unstable than the previous. Applying similar arguments to different divisions of the joint values, one can verify that there exists no equilibrium. To ensure the existence of an equilibrium, several necessary and sufficient conditions have been identified. Chiappori, Galichon, and Salanié (2012) and Talman and Yang (2011) use the linear programming approach to examine the existence problem, while Eriksson and Karlander (2001) explore a graph-theoretic method. In particular, Talman and Yang (2011, Theorem 1) introduce a general and natural sufficient condition which is easily satisfied by the assignment markets. Chiappori, Galichon, and Salanié (2012) demonstrate that the existence of an equilibrium is restored if the economy is duplicated by "cloning" each agent.[6]

[5]All papers cited so far have assumed quasi-linear utility. The problems have been analyzed for more general structures on the utility functions, see for instance Demange and Gale (1985), Kaneko and Yamamoto (1986), Quinzii (1984), and Svensson (1983). In turn, there is an extensive literature on matching markets where monetary transfers are not allowed, ranging from the house swapping market of Shapley and Scarf (1974) to the marriage and roommate problems of Gale and Shapley (1962).

[6]Relatedly, Sotomayor (2005) shows that each core outcome coincides with what we call an equilibrium. Klaus

Due to its inherent nature and generality, none of the existing adjustment processes (Crawford and Knoer, 1981; Demange, Gale, and Sotomayor, 1986) can be applied to the partnership formation problem. The main contribution of this paper is to propose a novel adjustment process that can always either find an equilibrium or exclusively disprove the existence of any equilibrium in finitely many steps. This process imitates a kind of bargaining and negotiation in real life business and is built upon two principles: market adjustment and fairness. It can be roughly described as follows. Each agent initially announces her stand-alone value. We assume that each agent does so without any strategic deliberations. At each step of the process, every agent looks for those agents who can bring her the highest payoff. At this point, agents' demand may be biased towards some agents, say because they are more productive and create larger values. To get the market in balance, we alter the payoffs for agents who are overdemanded, where a set of agents S is defined to be overdemanded if there are fewer agents in S than there are agents only demanding agents in S. By increasing the payoffs for these agents, we make them less attractive compared to the other agents. By repeatedly increasing payoffs for minimal overdemanded sets, we eventually get rid of all overdemand. We prove that this process always terminates in a finite number of steps. At the last step, if it is possible to find a matching among the agents such that everyone is matched to someone they demand, then there exists an equilibrium. On the other hand, if such a matching does not exist, neither does an equilibrium.

The lattice structure of the equilibrium price vectors plays an important role in the analysis of Demange, Gale, and Sotomayor (1986) for the assignment market. Their auction always ends up with the minimum equilibrium price vector. However, the set of equilibrium payoff vectors in the partnership formation problem need not be a lattice. We instead find that the set of payoff vectors that induce no overdemanded sets is a lower semilattice. We show that our adjustment process finds the unique minimum element of this set. Moreover, we show how this particular payoff vector can be used to prove or disprove the existence of an equilibrium and is therefore a stepping stone on the path to equilibrium if one exists. Our adjustment process can be seen as an innovative and significant generalization of Crawford and Knoer (1981) and Demange, Gale, and Sotomayor (1986) from the assignment markets to the partnership formation models. Similar to their processes, an important feature of our process is that it does not require agents to disclose their entire information on values.

The paper is organized as follows. In Section 2, we present the model. In Section 3, the adjustment process is introduced accompanied by the four main theorems. In Section 4, we discuss the relation between our findings and the assignment markets. We conclude in Section 5. Most of the proofs are deferred to the Appendix.

2. The Model

Consider a competitive environment where a finite group of self-interested agents (or firms) wish to make joint ventures. Let $N = \{1, \ldots, n\}$ denote the set of agents. Agent $i \in N$ can work alone and generate a *stand-alone value* of $v(\{i\})$, or form a partnership with some other agent j and generate a *joint value* of $v(\{i, j\})$. It is natural to assume that all values $v(\{i\})$ and $v(\{i, j\})$ are

and Nichifor (2010) study properties of equilibrium using a consistency axiom.

integers, e.g., that the revenues are specified to the nearest hundred dollars or nearest dollar. The pair (N, v) is called a *partnership formation problem*.

A widely accepted solution to any competitive model is the notion of (competitive) equilibrium. For the partnership formation problem, an equilibrium consists of a matching and a payoff distribution. A *matching* μ on the set N of agents is a one-to-one mapping from N to N satisfying $\mu(i) = j$ if and only if $\mu(j) = i$. $\mathcal{M}$ denotes the set of all matchings on N. We call agent $i \in N$ the partner of agent $j \in N$ if $\mu(i) = j$. With some abuse of notation, we denote $\mu(S) = \{\mu(i) \mid i \in S\}$ for $S \subseteq N$. Note that $\#\mu(S) = \#S$. A payoff vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ specifies a payoff p_i for each agent $i \in N$.

Definition 1. A pair $(\mu, p^*) \in \mathcal{M} \times \mathbb{R}^n$ is an *equilibrium* for the partnership formation problem (N, v) if for all $i \in N$ it holds that

(i) $p_i^* \geq v(\{i\})$ (*individual rationality*),
(ii) $p_i^* + p_k^* \geq v(\{i, k\})$ for all $k \neq i$ (*stability*),
(iii) $p_i^* = v(\{i\})$ if $\mu(i) = i$, and $p_i^* + p_j^* = v(\{i, j\})$ if $j = \mu(i)$ and $j \neq i$ (*feasibility*).

The *expected payoff* for agent $i \in N$ of being matched to agent $k \in N$, $k \neq i$, at payoff vector $p \in \mathbb{R}^n$, is given by

$$P(i, k, p) = p_i + \frac{v(\{i, k\}) - p_i - p_k}{2} = \frac{v(\{i, k\}) + p_i - p_k}{2}, \tag{1}$$

i.e., if the partnership is formed, both agents expect to receive their payoff and then split the remaining surplus equally among them. The expected payoff of an agent $i \in N$ being single is given by $P(i, i, p) = v(\{i\})$.

At a given payoff vector, each agent demands the agents from whose partnership she gets the highest expected payoff. Hence, the demand correspondence of agent $i \in N$, at payoff vector $p \in \mathbb{R}^n$, is given by

$$D_i(p) = \{j \in N \mid P(i, j, p) \geq P(i, k, p) \text{ for all } k \in N\}.$$

An agent $j \in N$ is *demanded* by agent $i \in N$ at payoff vector $p \in \mathbb{R}^n$ if $j \in D_i(p)$, and a matching $\mu \in \mathcal{M}$ is demanded by $S \subseteq N$ at p if $\mu(i) \in D_i(p)$ for all $i \in S$.

Those agents who demand only agents in $S \subseteq N$ at payoff vector $p \in \mathbb{R}^n$ form the set $O(S, p) = \{i \in N \mid D_i(p) \subseteq S\}$. If $\#O(S, p) > \#S$, the set S is *overdemanded* at payoff vector p. Note that neither the empty set nor N can be overdemanded at any payoff vector. The set of individually rational payoff vectors at which there are no overdemanded sets is given by

$$\mathcal{H} = \{p \in \mathbb{R}^n \mid p_i \geq v(\{i\}) \text{ for all } i \in N \text{ and } \#O(S, p) \leq \#S \text{ for all } S \subseteq N\}.$$

An overdemanded set $S \subseteq N$ at payoff vector $p \in \mathbb{R}^n$ is *minimal* if no proper subset $T \subset S$ is overdemanded at p. Similarly, $U(S, p) = \{i \in N \mid D_i(p) \cap S \neq \emptyset\}$ represents the set of agents that demand some agent in S at p. If $\#U(S, p) < \#S$, the set S is *underdemanded* at p.

The following auxiliary results concerning overdemanded or underdemanded sets are interesting on their own right and will be used to prove our major results in Section 3.

Lemma 1. If $S \subset N$ is minimal overdemanded at p, then $\#[U(T,p) \cap O(S,p)] \geq \#T$ for all $T \subseteq S$.[7]

Proof. Note first that the statement is trivially true if $T = \emptyset$. For this reason it is henceforth assumed that $T \neq \emptyset$. Next, note that $U(T,p) \cap O(S,p) = O(S,p) \setminus O(S \setminus T, p)$. Because $O(S \setminus T, p) \subset O(S,p)$, it follows that $\#[U(T,p) \cap O(S,p)] = \#O(S,p) - \#O(S \setminus T, p)$. Note that $\#O(S,p) > \#S$ and $\#O(S \setminus T, p) \leq \#(S \setminus T)$, as S is minimal overdemanded at p and $T \neq \emptyset$. These observations together with the above condition and that $T \subseteq S$ give

$$\begin{aligned} \#[U(T,p) \cap O(S,p)] &= \#O(S,p) - \#O(S \setminus T, p) \\ &> \#S - \#O(S \setminus T, p) \\ &\geq \#S - \#(S \setminus T) \\ &= \#S - (\#S - \#T) \\ &= \#T, \end{aligned}$$

yielding the desired conclusion. □

Lemma 2. A set $S \subseteq N$ is overdemanded at p if and only if its complement $N \setminus S$ is underdemanded at p.

Proof. Take any payoff vector p and any $S \subseteq N$. By definition it holds that $U(N \setminus S, p) = N \setminus O(S,p)$. Note that the statement of this lemma is equivalent to

$$\#O(S,p) > \#S \Leftrightarrow \#U(N \setminus S, p) < \#(N \setminus S). \tag{2}$$

Because $O(S,p) \subseteq N$, we obtain $\#U(N \setminus S, p) = \#N - \#O(S,p)$. Since $\#N = \#S + \#(N \setminus S)$, this implies $\#O(S,p) - \#S = \#(N \setminus S) - \#U(N \setminus S, p)$. Then condition (2) holds. □

3. Main Results

This section starts by demonstrating that the set $\mathcal{H}$ is nonempty and that it contains a unique minimum payoff vector. The key in proving these properties is a dynamic procedure called Process 1. This process can be seen as a bidding procedure where each agent starts by announcing her stand-alone value. A fictitious "auctioneer" then asks the agents to report their demand sets given the revealed information. Based on the reported demand sets, the auctioneer checks if there is any overdemanded set. If there is no overdemanded set, the process stops. Otherwise, the auctioneer identifies a minimal overdemanded set, increases the payoff of every agent in the minimal overdemanded set by one unit of money (recall that all values are assumed to be integers) and keeps the payoff of any other agent unchanged. Then each agent reports her demand set at the updated payoff vector, and so on. Formally, the procedure can be described as follows.

Process 1. Initialize $t := 0$ and let $p^0 = (v(\{1\}), \ldots, v(\{n\}))$.

[7] Andersson, Andersson, and Talman (2012, Theorem 2) demonstrated that this property holds for any minimal overdemanded set also at the assignment market. Lemma 1 can be seen as a generalization of their result as the assignment market is a special case of the partnership formation problem (see Section 4).

Iteration	p^t	$D_1(p^t)$	$D_2(p^t)$	$D_3(p^t)$	$D_4(p^t)$	$D_5(p^t)$	MOD set
t = 0	$(2,1,0,0,0)$	{4}	{3}	{4}	{3}	{3}	{3}
1	$(2,1,1,0,0)$	{4}	{3,5}	{4}	{3}	{3}	{3}
2	$(2,1,2,0,0)$	{4}	{5}	{4}	{1,3}	{3}	{4}
3	$(2,1,2,1,0)$	{4}	{5}	{4,5}	{1,3}	{3}	{4,5}
4	$(2,1,2,2,1)$	{4}	{3,5}	{4,5}	{1,3}	{3}	{1,3,4,5}
5	$(3,1,3,3,2)$	{4}	{2,3,5}	{4,5}	{1,3}	{3}	—

Table 1: Summary of the iterations in Process 1 for Example 1.

1. Collect the demand correspondences $D_i(p^t)$ for all $i \in N$.
2. If there is no overdemanded set at p^t, terminate the process. Otherwise, pick a minimal overdemanded set S^t, compute the updated payoff vector p^{t+1} whose components are given by:
$$p_i^{t+1} = \begin{cases} p_i^t + 1 & \text{if } i \in S^t \\ p_i^t & \text{otherwise,} \end{cases}$$
set $t := t + 1$ and go to Step 1.

Because Process 1 is central for our analysis, it is instructive to illustrate it via an example.

Example 1. Suppose that $N = \{1, 2, 3, 4, 5\}$ and construct the value function v such that $v(\{i, j\})$ is the $(i, j)^{\text{th}}$ entry of the matrix
$$\begin{pmatrix} 2 & 3 & 6 & 8 & 4 \\ 3 & 1 & 4 & 0 & 3 \\ 6 & 4 & 0 & 8 & 7 \\ 8 & 0 & 8 & 0 & 3 \\ 4 & 3 & 7 & 3 & 0 \end{pmatrix}.$$
The process starts at the payoff vector $p^0 = (2, 1, 0, 0, 0)$, being the vector of diagonal elements representing the stand-alone values. At this step, agents 1 and 3 both only demand agent 4, whereas agents 2, 4, and 5 only demand agent 3. There are in total eleven different overdemanded sets, though only two are minimal: {3} and {4}. It is with no loss that we choose the one over the other; the one that is not chosen remains a minimal overdemanded set in the upcoming iteration. Hence, say {3} is chosen as the minimal overdemanded set for which payoffs are increased. The process then reaches payoff $p^1 = (2, 1, 1, 0, 0)$. See Table 1 for the demand sets and the selected minimal overdemanded sets (MOD set) in each step. Notice that {4} remains an overdemanded set at p^1. The process eventually terminates at payoff vector $p^5 = (3, 1, 3, 3, 2)$, where no set of agents is overdemanded.

Theorem 1. The set $\mathcal{H}$ is nonempty for any partnership formation problem.

Proof. Take any partnership formation problem (N, v). We show that Process 1 terminates in a finite number of iterations and that the final payoff vector belongs to the set $\mathcal{H}$.

Let S^t denote the selected minimal overdemanded set at Step t of the process, and let the set $R^t \subseteq N$ contain all agents whose payoffs are greater than their stand-alone value at Step t:

$$R^t \equiv S^0 \cup \cdots \cup S^{t-1} = \{i \in N \mid p_i^t > v(\{i\})\} \text{ for } t \geq 0.$$

At $t = 0$, $p_i^0 = v(\{i\})$ for all $i \in N$, by construction, so indeed $R^0 = \emptyset$. By induction on t, we first show that no subset of R^t is underdemanded at any $t \geq 0$, i.e., that $\#U(T, p^t) \geq \#T$ for all $T \subseteq R^t$ and all $t \geq 0$. Note that this condition holds trivially for $R^0 = \emptyset$.

Next, we make the induction assumption that $\#U(T, p^t) \geq \#T$ for all $T \subseteq R^t$ and some $t \geq 0$. Take now an arbitrary $T \subseteq R^{t+1} = R^t \cup S^t$. We need to show that $\#U(T, p^{t+1}) \geq \#T$. Partition T into $T_1 = T \setminus S^t \subseteq R^t$ and $T_2 = T \cap S^t \subseteq S^t$.

As prices are weakly increasing throughout the process, we have $p_j^{t+1} \geq p_j^t$ for all $j \in N$. In addition, as $T_1 \cap S^t = \emptyset$, we have $p_i^{t+1} = p_i^t$ for all $i \in T_1$. Hence, if $i \in T_1$ and $i \in D_j(p^t)$, then $i \in D_j(p^{t+1})$. Consequently, $U(T_1, p^t) \subseteq U(T_1, p^{t+1})$. Therefore

$$\begin{aligned} U(T, p^{t+1}) &= U(T_1, p^{t+1}) \cup U(T_2, p^{t+1}) \\ &\supseteq U(T_1, p^t) \cup U(T_2, p^{t+1}) \\ &\supseteq U(T_1, p^t) \cup (U(T_2, p^{t+1}) \cap O(S^t, p^t)). \end{aligned}$$

This observation together with elementary laws of set theory gives

$$\begin{aligned} \#U(T, p^{t+1}) &\geq \#U(T_1, p^t) \cup (U(T_2, p^{t+1}) \cap O(S^t, p^t)) \\ &= \#U(T_1, p^t) + \#U(T_2, p^{t+1}) \cap O(S^t, p^t) \\ &- \#U(T_1, p^t) \cap (U(T_2, p^{t+1}) \cap O(S^t, p^t)). \end{aligned}$$

Note that $U(T_1, p^t) \cap O(S^t, p^t) = \emptyset$, because if an agent only demands agents in S^t she cannot demand any agent in T_1 as $T_1 \cap S^t = \emptyset$. This means that the last term in the above equality equals zero. Hence,

$$\#U(T, p^{t+1}) \geq \#U(T_1, p^t) + \#U(T_2, p^{t+1}) \cap O(S^t, p^t). \tag{3}$$

Next, as $T_1 \subseteq R^t$, we have by the induction assumption that

$$\#U(T_1, p^t) \geq \#T_1. \tag{4}$$

As $T_2 \subseteq S^t$ and S^t is minimal overdemanded at p^t, it follows from Lemma 1 that

$$\#U(T_2, p^t) \cap O(S^t, p^t) \geq \#T_2. \tag{5}$$

Moreover, if $i \in O(S^t, p^t)$, then $D_i(p^t) \subseteq D_i(p^{t+1})$ by monotonicity. Hence, if $i \in U(T_2, p^t) \cap O(S^t, p^t)$, then $i \in U(T_2, p^{t+1}) \cap O(S^t, p^t)$. This together with condition (5) gives

$$\#U(T_2, p^{t+1}) \cap O(S^t, p^t) \geq \#U(T_2, p^t) \cap O(S^t, p^t) \geq \#T_2. \tag{6}$$

Conditions (3), (4), and (6) and $T_1 \cup T_2 = T$, $T_1 \cap T_2 = \emptyset$ yield the desired conclusion, i.e., $\#U(T, p^{t+1}) \geq \#T$. Consequently, no subset of R^t is underdemanded at any $t \geq 0$ by induction.

We now show that there exists an agent $i \in N$ such that $p_i^t = v(\{i\})$ for all t. Suppose that t is the first iteration such that $p_i^{t+1} > v(\{i\})$ for all $i \in N$. Then $N \setminus S^t \subseteq R^t$. By the above conclusion, $N \setminus S^t$ is not underdemanded at Step t. By Lemma 2, S^t is then not overdemanded at Step t which contradicts that S^t is minimal overdemanded at Step t. Hence, there exists an agent $i \in N$ such that $p_i^t = v(\{i\})$ for all t.

Finally, we prove that Process 1 terminates in a finite number of iterations. Define the finite integer M by

$$M = 1 + \max_j \{v(\{j\}), \max_{k \neq j} \{v(\{j,k\}) - v(\{j\}), M_{j,k}\}\},$$

where, for $k \neq j$,

$$M_{j,k} = \max_{i \neq j,k} \{v(\{i,k\}) - v(\{i,j\}) + v(\{j\})\}.$$

Suppose at step t it holds that $p_k^t = M$ for for some $k \in N$. Let $i \in N$ be such that $p_i^t = v(\{i\})$. By definition of M it holds that $i \neq k$. Moreover, $P(j,k,p^t) < P(j,i,p^t)$ for any $j \neq k$, and therefore $k \notin D_j(p^t)$ for $j \neq k$. This implies that k cannot be an element of any minimal overdemanded set at p^t and therefore either the process terminates at p^t or $p_k^{t+1} = p_k^t$. This proves that the process cannot generate payoff vectors p^t with $p_i^t > M$ and since payoffs increase monotonically the process must terminate in a finite number of iterations with some payoff vector in $\mathcal{H}$. □

As established next, the set $\mathcal{H}$ is a lower semilattice, i.e., it always contains a unique minimum payoff vector. This minimum payoff vector can be identified using Process 1, it is integer valued and the payoff for at least one agent equals her stand-alone value.

Theorem 2. There exists a payoff vector $p^{\min} \in \mathcal{H}$ such that $p^{\min} \leq p$ for any $p \in \mathcal{H}$. Moreover

(i) $p^{\min}$ is identified in Process 1,

(ii) $p^{\min}$ is integer valued,

(iii) $p_i^{\min} = v(\{i\})$ for some $i \in N$.

As already described in Section 1, an equilibrium may not always exist. However, several necessary and sufficient conditions for the equilibrium existence have been introduced in the literature as mentioned in Section 1. Here we provide a quite different characterization that makes use of the minimum payoff vector.

Theorem 3. There exists an equilibrium (μ, p^*) if and only if μ is demanded by N at $p^{\min}$.

The main innovation of this paper is a dynamic adjustment procedure, called the Partnership Formation Process, that identifies an equilibrium whenever one exists, and otherwise proves the non-existence of any equilibrium. This procedure is formally described next.

The Partnership Formation Process.

1. Find $p^{\min}$ using Process 1.
2. If no matching μ is demanded by N at $p^{\min}$, there exists no equilibrium and terminate the process. Otherwise take any such matching, ask agents $i = \mu(j)$ and $j = \mu(i)$ to report $v(\{i, j\})$. Define p^* by
$$p_i^* = \begin{cases} P(i, \mu(i), p^{\min}) & \text{if } \mu(i) \neq i \\ p_i^{\min} & \text{otherwise,} \end{cases}$$
and terminate the process.

We remark that Step 2 of the Partnership Formation Process may be written differently. As is apparent from Theorem 3, the Partnership Formation Problem identifies an equilibrium matching μ (or prove that no such matching exists) in Step 2. Thus, to find an equilibrium, only the payoff vector needs to be determined. This task can also be achieved by splitting the process into k different processes where k represents the number of partnerships at μ. In each process, the payoff of two partners i and $j = \mu(i)$ is then increased by $1/2$ until $p_i + p_j = v(\{i, j\})$. Because this is equivalent to asking agents i and j to report $v(\{i, j\})$ whenever $i = \mu(j)$ and $j = \mu(i)$ at Step 2, we adopt the above more straightforward specification of the Partnership Formation Process to avoid introducing unnecessary notation.

Theorem 4. The Partnership Formation Process either finds an equilibrium or proves the nonexistence of an equilibrium in finitely many iterations.

Example 2. Recall from Example 1 that $p^5 = (3, 1, 3, 3, 2)$. By Theorem 2(i), $p^5 = p^{\min}$. This payoff vector is identified in Step 1 of the Partnership Formation Process using Process 1. The Partnership Formation Process then continues to Step 2 where it is possible to find a matching μ such that $\mu(1) = 4$, $\mu(2) = 2$, and $\mu(3) = 5$ that is demanded by N. Applying the formulas specified in Step 2 of the Partnership Formation Process yields

$$\begin{aligned}
p_1^* &= p_1^{\min} + \frac{v(\{1,4\}) - p_1^{\min} - p_4^{\min}}{2} = 3 + \frac{8-3-3}{2} = 4, \\
p_2^* &= p_2^{\min} = 1, \\
p_3^* &= p_3^{\min} + \frac{v(\{3,5\}) - p_3^{\min} - p_5^{\min}}{2} = 3 + \frac{7-3-2}{2} = 4, \\
p_4^* &= p_4^{\min} + \frac{v(\{1,4\}) - p_1^{\min} - p_4^{\min}}{2} = 3 + \frac{8-3-3}{2} = 4, \\
p_5^* &= p_5^{\min} + \frac{v(\{3,5\}) - p_3^{\min} - p_5^{\min}}{2} = 2 + \frac{7-3-2}{2} = 3.
\end{aligned}$$

By Theorem 4, (μ, p^*) is an equilibrium.

4. Relation to the Assignment Markets

This section relates the findings from the current paper to some results previously established for the classical assignment markets (e.g. Koopmans and Beckmann, 1957; Shapley and Shubik, 1971). We start by describing how the latter type of market is related to the partnership formation problem and, in particular, what role Process 1 plays in this relationship.

In the assignment market, the role of agents is exogenously given, and all agents in N are exogenously split into two disjoint groups, N_1 and N_2 (with $N_1 \cup N_2 = N$) where agents in the same group cannot be partners. For example, the agents on one side of the market may be buyers or workers whereas the agents on the other side may be sellers or firms. As observed by Talman and Yang (2011), an equilibrium in the assignment market models resembles the one described in Definition 1 but it is often described in terms of demand. However, Definition 1 can be easily interpreted in terms of demand. Thus, it is not difficult to see that the assignment market model is a special case of the partnership formation problem. Moreover, the assignment market model automatically satisfies the oddness condition in Talman and Yang (2011, Theorem 1) and thus always has an equilibrium (see Talman and Yang, ibid., Theorem 3). In fact, an equilibrium exists in the assignment market at a payoff vector p if and only if there are neither overdemanded sets nor underdemanded sets at p as established by Mishra and Talman (2010, Theorem 1). However, this result does not carry over to the Partnership Formation Problem as illustrated in the following example and as a result shows another fundamental and inherent difference between the two models.

Example 3. Let $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 3$. For payoff vector $p = (r, r, r)$ where r is an arbitrary real number, the demand sets are given by $D_1(p) = \{2,3\}$, $D_2(p) = \{1,3\}$, $D_3(p) = \{1,2\}$. There are neither overdemanded nor underdemanded sets of agents at payoff p. However, there is no equilibrium either.

Given that an equilibrium always exists at the assignment market, a number of constructive existence proofs has been introduced. For example Crawford and Knoer (1981), Demange, Gale, and Sotomayor (1986), Andersson, Andersson, and Talman (2012), and Andersson and Erlanson (2012) have proposed convergent dynamic processes. In particular, the auction mechanism of Demange, Gale, and Sotomayor (1986) can be seen as a special case of our Process 1 by specifying the corresponding parameters as follows:

(a) the payoff vector at $t = 0$ represents the reservation payoffs for the agents in N_2,

(b) the demand sets are collected only for the agents in N_1 at any step,

(c) the demand set for an arbitrary agent $i \in N_1$ must be a subset of $N_2 \cup \{i\}$.

Demange, Gale, and Sotomayor (1986) prove that their mechanism always converges to the unique minimum equilibrium price vector $q^{\min}$. Because of restrictions (a)–(c), these findings can be regarded as a corollary to the more general results presented in Theorem 1 and cases (i) and (ii) of Theorem 2. Furthermore, in the case when $\#N_1 = \#N_2$, it is known that $q_i^{\min} = v(\{i\})$ for some $i \in N_2$ (see e.g. Sun and Yang, 2003, Theorem 2.5). This result can be regarded as a special case of Theorem 2(iii).

5. Concluding Remarks

This paper has provided several fundamental properties of the partnership formation problem. Based on these properties, a dynamic competitive adjustment process is proposed, which offers a natural process for agents to form partnerships. It is shown that this process always either converges to an equilibrium or refutes the existence of any equilibrium. When an equilibrium is found, partnerships and payoff distribution will be endogenously determined.

We assume that all agents are well informed as often is the case in the literature on the assignment markets. It will be significantly important but also difficult to study the case of imperfect information in which agents do not have precise knowledge about their joint revenues. We believe that our study provides a first necessary and important step to examine the possible effects of imperfect information. A second question closely relates to Kelso and Crawford (1982). They examine a job assignment model in which each firm can hire many workers and every worker can have at most one job. They propose a salary adjustment process that always converges to an equilibrium, provided that every firm views all workers as substitutes (see also Hart and Kurz, 1983; Qin, 1996). A challenging open question is whether it is possible to develop a similar process that always either finds an equilibrium or disproves the existence of equilibrium in a general setting where coalitions allow any number of agents in a way as we move from the assignment market to the partnership formation model.

Appendix: Proofs

This Appendix gives all the remaining proofs and contains three additional technical lemmas which will be used for proving Theorems 2 and 3.

Lemma 3. Let $p, q \in \mathbb{R}^n$ and let

$$C_m \quad = \quad C_{m-1} \cup \{j \in N \mid p_j - q_j \geq p_k - q_k \text{ for all } k \in N \setminus C_{m-1}\},$$

for $m = 1, \ldots, h$, where $C_0 = \emptyset$ and $h \in \{1, \ldots, n\}$ is such that $C_h = N$. Then

(i) $U(A \cap C_1, p) \cap O(A, p) \subseteq O(A \cap C_1, q)$ for all $A \subseteq N$,
(ii) $U(C_m, p) = O(C_m, q)$ for $m = 1, \ldots, h$ if $p, q \in \mathcal{H}$.

Proof. Note first that if $P(i, j, p) \geq P(i, k, p)$ and $p_j - q_j \geq p_k - q_k$ hold with at least one strict inequality, then $P(i, j, q) > P(i, k, q)$. This follows as $p_j - q_j \geq p_k - q_k$ implies

$$(p_j - p_i + q_i - q_j)/2 \geq (p_k - p_i + q_i - q_k)/2.$$

Adding this inequality to $P(i, j, p) \geq P(i, k, p)$ gives $P(i, j, q) > P(i, k, q)$ if one of the above two inequalities is strict.

Part (i). Let $i \in U(A \cap C_1, p) \cap O(A, p)$. Then $D_i(p) \subseteq A$ and there exists $j \in D_i(p) \cap A \cap C_1$. Take any $k \notin A \cap C_1$. Suppose first that $k \notin A$. Then $D_i(p) \subseteq A$ implies $k \notin D_i(p)$. Consequently, $P(i, j, p) > P(i, k, p)$. From $j \in C_1$ it follows that $p_j - q_j \geq p_k - q_k$. Hence, $P(i, j, q) > P(i, k, q)$ by the above conclusion, which means that $k \notin D_i(q)$. Suppose next that $k \in A \setminus C_1$. Then

$j \in D_i(p)$ implies $P(i, j, p) \geq P(i, k, p)$, and $p_j - q_j > p_k - q_k$ as $j \in C_1$ and $k \notin C_1$. Hence, again $P(i, j, q) > P(i, k, q)$ and $k \notin D_i(q)$. Therefore, $k \notin D_i(q)$ whenever $k \notin A \cap C_1$, which implies $i \in O(A \cap C_1, q)$.

Part (ii). Let $p, q \in \mathcal{H}$ and take $i \in U(C_m, p)$. Then there exists $j \in D_i(p) \cap C_m$. Take any $k \notin C_m$. Then $j \in D_i(p)$ implies $P(i, j, p) \geq P(i, k, p)$, and $p_j - q_j > p_k - q_k$ as $j \in C_m$ and $k \notin C_m$. Hence, $P(i, j, q) > P(i, k, q)$, and so $k \notin D_i(q)$. Therefore, $i \in O(C_m, q)$. Consequently, $U(C_m, p) \subseteq O(C_m, q)$. As $p, q \in \mathcal{H}$, C_m is neither underdemanded at p nor overdemanded at q. Thus $\#O(C_m, q) \leq \#C_m \leq \#U(C_m, p)$. Since $U(C_m, p) \subseteq O(C_m, q)$, this implies $U(C_m, p) = O(C_m, q)$. □

Theorem 2. There exists a payoff vector $p^{\min} \in \mathcal{H}$ such that $p^{\min} \leq p$ for any $p \in \mathcal{H}$. Moreover

(i) $p^{\min}$ is identified in Process 1,

(ii) $p^{\min}$ is integer valued,

(iii) $p_i^{\min} = v(\{i\})$ for some $i \in N$.

Proof. Before proving parts (i)–(iii) of the theorem, we first demonstrate that there exists a payoff vector $p^{\min} \in \mathcal{H}$ such that $p^{\min} \leq p$ for any $p \in \mathcal{H}$. To prove this, we show that for any two vectors $p, q \in \mathcal{H}$ the vector r defined by $r_i = \min\{p_i, q_i\}$ for all $i \in N$ is an element of $\mathcal{H}$.

Suppose that $r \notin \mathcal{H}$. Then there exists a minimal overdemanded set A at r. Hence, $\#O(A, r) > \#A$. Let now

$$C_1^p = \{i \in N \mid r_i - p_i \geq r_j - p_j \text{ for all } j \in N\},$$
$$C_1^q = \{i \in N \mid r_i - q_i \geq r_j - q_j \text{ for all } j \in N\}.$$

Note the similarity to the construction of C_1 in Lemma 3. Since, by the construction of r, $C_1^p \cup C_1^q = N$, it holds that among the agents that demand only in A at r, some demand only in C_1^p, some only in C_1^q, and the others in both, i.e.

$$O(A, r) = [U(A \cap C_1^p, r) \cap O(A, r)] \cup [U(A \cap C_1^q, r) \cap O(A, r)].$$

It follows that

$$\#O(A, r) = \#[U(A \cap C_1^p, r) \cap O(A, r)] + \#[U(A \cap C_1^q, r) \cap O(A, r)] \tag{7}$$
$$- \#[U(A \cap C_1^p, r) \cap U(A \cap C_1^q, r) \cap O(A, r)]. \tag{8}$$

Note that if $i \in U(A \cap C_1^p \cap C_1^q, r) \cap O(A, r)$, then $i \in U(A \cap C_1^p, r) \cap U(A \cap C_1^q, r) \cap O(A, r)$. Therefore

$$\#[U(A \cap C_1^p, r) \cap U(A \cap C_1^q, r) \cap O(A, r)] \geq \#[U(A \cap C_1^p \cap C_1^q, r) \cap O(A, r)].$$

Moreover, by Lemma 1 and as A is a minimal overdemanded set at r

$$\#[U(A \cap C_1^p \cap C_1^q, r) \cap O(A, r)] \geq \#(A \cap C_1^p \cap C_1^q).$$

In conclusion

$$\#O(A,r) \leq \#[U(A \cap C_1^p, r) \cap O(A,r)] + \#[U(A \cap C_1^q, r) \cap O(A,r)] - \#(A \cap C_1^p \cap C_1^q).$$

By Lemma 3(i), we get $U(A \cap C_1^p, r) \cap O(A,r) \subseteq O(A \cap C_1^p, p)$. As $p \in \mathcal{H}$, the set $A \cap C_1^p$ is not overdemanded at p. Hence $\#O(A \cap C_1^p, p) \leq \#(A \cap C_1^p)$. In conclusion

$$\#[U(A \cap C_1^p, r) \cap O(A,r)] \leq \#O(A \cap C_1^p, p) \leq \#(A \cap C_1^p).$$

By similar arguments, we obtain

$$\#[U(A \cap C_1^q, r) \cap O(A,r)] \leq \#O(A \cap C_1^q, q) \leq \#(A \cap C_1^q).$$

Note next that

$$\#(A \cap C_1^p) + \#(A \cap C_1^q) = \#[A \cap (C_1^p \cup C_1^q)] + \#(A \cap C_1^p \cap C_1^q).$$

As $C_1^p \cup C_1^q = N$ and $A \subseteq N$, this can be simplified to

$$\#(A \cap C_1^p) + \#(A \cap C_1^q) = \#A + \#(A \cap C_1^p \cap C_1^q).$$

Hence

$$\#O(A,r) \leq \#A + \#(A \cap C_1^p \cap C_1^q) - \#(A \cap C_1^p \cap C_1^q) = \#A.$$

This contradicts A being overdemanded at r. Therefore, there exists no overdemanded set at r. It follows that $r \in \mathcal{H}$.

Part (i): The payoff vector $p^{\min}$ is the unique minimum payoff vector in $\mathcal{H}$ by the above conclusions. Because Process 1 terminates in a finite number it iterations (see the proof of Theorem 1), it is henceforth assumed that it converges at iteration τ. Moreover, Process 1 cannot terminate at iteration τ if $p^\tau \leq p^{\min}$ and $p_i^\tau < p_i^{\min}$ for some $i \in N$ because this would contradict that $p^{\min}$ is minimum in $\mathcal{H}$ and the process only terminates when there are no overdemanded sets. Thus, it needs to be established that $p^t \leq p^{\min}$ for $t = 0, \dots, \tau$, as this then implies that $p^\tau = p^{\min}$.

Suppose now, in order to obtain a contradiction, that there is an iteration $t \geq 0$ such that $p^t \leq p^{\min}$ but $p_i^{t+1} > p_i^{\min}$ for some $i \in N$. Note that $p^0 \leq p^{\min}$. Let A be the minimal overdemanded set selected in Step 2 of iteration t and define

$$C_1 = \{j \in N \mid p_j^t - p_j^{\min} \geq p_k^t - p_k^{\min} \text{ for all } k \in N\}.$$

As $p_i^{t+1} > p_i^{\min}$ can only occur if $p_i^t = p_i^{\min}$ and $i \in A \cap C_1$, this implies $A \cap C_1 \neq \emptyset$. By Lemma 3(i), $U(A \cap C_1, p^t) \cap O(A, p^t) \subseteq O(A \cap C_1, p^{\min})$. Now, as $p^{\min} \in \mathcal{H}$, $A \cap C_1$ is not overdemanded at $p^{\min}$. That is, $\#O(A \cap C_1, p^{\min}) \leq \#(A \cap C_1)$. It follows that

$$\#[U(A \cap C_1, p^t) \cap O(A, p^t)] \leq \#O(A \cap C_1, p^{\min}) \leq \#(A \cap C_1). \tag{9}$$

If $A \subseteq C_1$, this reduces to $\#O(A, p^t) \leq \#A$, contradicting that A is overdemanded at p^t. If $A \nsubseteq C_1$, then $A \setminus C_1$ is a nonempty proper subset of A and we will show that $A \setminus C_1$ is overdemanded at p^t. Since $A \setminus C_1 = A \setminus (A \cap C_1)$ and $A \cap C_1 \subset A$, it holds that

$$\#O(A \setminus C_1, p^t) = \#O(A, p^t) - \#[U(A \cap C_1, p^t) \cap O(A, p^t)].$$

As A is overdemanded, $\#O(A, p^t) > \#A$. Thus

$$\#O(A \setminus C_1, p^t) > \#A - \#[U(A \cap C_1, p^t) \cap O(A, p^t)]. \tag{10}$$

From (9) and (10) and the fact that $A \setminus (A \cap C_1) = A \setminus C_1$, it follows that

$$\#O(A \setminus C_1, p^t) > \#A - \#(A \cap C_1) = \#(A \setminus C_1),$$

implying that $A \setminus C_1$ is overdemanded at p^t, which contradicts that A is a minimal overdemanded set at p^t.

Part (ii): To prove this part of the theorem, it suffices to demonstrate that if $p \in \mathcal{H}$ and $q_i = \lfloor p_i \rfloor$ for $i \in N$, where $\lfloor p_i \rfloor$ is the largest integer less than or equal to p_i, then $q \in \mathcal{H}$. Note first that for all $l \in N$ it holds that $p_l = \lfloor p_l \rfloor + y_l$ for some $0 \leq y_l < 1$. Hence, for $i \in N$ and $k \neq i$

$$\begin{aligned} P(i, k, p) = (v(\{i, k\}) + p_i - p_k)/2 &= (v(\{i, k\}) + \lfloor p_i \rfloor - \lfloor p_k \rfloor + y_i - y_k)/2 \\ &= (v(\{i, k\}) + q_i - q_k + y_i - y_k)/2 \\ &= P(i, k, q) + (y_i - y_k)/2 \\ &= P(i, k, q) + x^i_k, \end{aligned}$$

for some $-1/2 < x^i_k < 1/2$. Note next that $P(i, k, p) = v(\{i\}) = P(i, k, q)$ if $i = k$. Thus, $P(i, k, p) = P(i, k, q) + x^i_k$ for all $i, k \in N$ and some $-1/2 < x^i_k < 1/2$. It is next proved that $D_i(p) \subseteq D_i(q)$ for all $i \in N$. Let $j \in D_i(p)$ and $k \in N$. Then $P(i, j, p) \geq P(i, k, p)$ or equivalently

$$P(i, j, q) + x^i_j \geq P(i, k, q) + x^i_k. \tag{11}$$

Suppose that $P(i, j, q) < P(i, k, q)$. Because $P(i, j, q)$ and $P(i, k, q)$ are integers, this means that $P(i, k, q) - P(i, j, q) \geq 1$, and so $P(i, j, q) + x^i_j < P(i, k, q) + x^i_k$ as $-1/2 < x^i_l < 1/2$ for $l = j, k$. But this contradicts (11). Hence, $j \in D_i(p)$ implies $P(i, j, q) \geq P(i, k, q)$ for all $k \in N$, i.e., $j \in D_i(q)$. Consequently, $D_i(p) \subseteq D_i(q)$ for all $i \in N$.

Consider now an arbitrary $A \subseteq N$ and let $i \in O(A, q)$. Then $D_i(p) \subseteq D_i(q) \subseteq A$ by construction and the above conclusion. Hence, $i \in O(A, p)$, and consequently $O(A, q) \subseteq O(A, p)$. Therefore, $\#O(A, q) \leq \#O(A, p) \leq \#A$ as $p \in \mathcal{H}$. Because A is arbitrary and is not overdemanded at q, it follows that $q \in \mathcal{H}$.

Part (iii): It has been shown in the proof of Theorem 1. □

Lemma 4. If (μ, p) is an equilibrium of the partnership formation problem (N, v), then μ is demanded by N at p and $p \in \mathcal{H}$.

Proof. Suppose that (μ, p) is an equilibrium. We first demonstrate that μ is demanded by N at p, i.e., $\mu(i) \in D_i(p)$ for all $i \in N$.

Let first $i \in N$ be such that $\mu(i) = i$. Feasibility and stability means that $p_i = v(\{i\})$ and $p_i \geq v(\{i, k\}) - p_k$ for all $k \neq i$. By combining these two conditions and using the fact that $P(i, i, p) = v(\{i\})$, we obtain $P(i, k, p) \geq P(i, l, p)$ for all $k \neq i$. Thus $i \in D_i(p)$ if $\mu(i) = i$.

Let now $i \in N$ be such that $\mu(i) \neq i$ and let $j = \mu(i)$. Feasibility, individual rationality and stability imply

$$2p_i = v(\{i, j\}) + p_i - p_j = 2P(i, j, p), \tag{12}$$

$$2p_i \geq 2v(\{i\}) = 2P(i, i, p), \tag{13}$$

$$2p_i \geq v(\{i, k\}) + p_i - p_k = 2P(i, k, p) \text{ for all } k \neq i, \tag{14}$$

respectively. Conditions (12) and (13) give $P(i, j, p) \geq P(i, i, p)$, and conditions (12) and (14) yield $P(i, j, p) \geq P(i, k, p)$ for all $k \neq i$. Consequently, $P(i, j, p) \geq P(i, k, p)$ for all $k \in N$, and thus $j \in D_i(p)$. Therefore, $\mu(i) \in D_i(p)$ for all $i \in N$.

We will show that there are no overdemanded sets at p. Pick an arbitrary nonempty $S \subset N$. For any $i \in \mu(S)$ it holds that $\mu(i) \in S$ and, by the above conclusion that μ is demanded by N at p, $\mu(i) \in D_i(p)$. Consequently, $i \in U(S, p)$. Hence, $\mu(S) \subseteq U(S, p)$. Therefore, $\#S = \#\mu(S) \leq \#U(S, p)$. This implies that S is not underdemanded at p. From Lemma 2 it then follows that $N \setminus S$ is not overdemanded at p. Because S is arbitrary, there is no overdemanded set at p. Hence, $p \in \mathcal{H}$. □

Lemma 5. Let $p, q \in \mathcal{H}$. If μ is demanded by N at p, then μ is demanded by N at q.

Proof. Construct $C_0, C_1, \ldots, C_h$ as in Lemma 3. Take $i \in N$ and let $m \in \{1, \ldots, h\}$ be such that $i \in \mu(C_m)$ and $i \notin C_{m-1}$. As μ is demanded by N at p, $\mu(i) \in D_i(p)$. Hence, $i \in U(C_m, p)$. Therefore, $\mu(C_m) \subseteq U(C_m, p)$. By Lemma 3(ii), $U(C_m, p) = O(C_m, q)$. As $q \in \mathcal{H}$, C_m is not overdemanded at q, i.e., $\#O(C_m, q) \leq \#C_m$. In conclusion

$$\#C_m = \#\mu(C_m) \leq \#U(C_m, p) = \#O(C_m, q) \leq \#C_m.$$

Hence $\mu(C_m) = O(C_m, q)$, and, in particular, $i \notin O(C_{m-1}, q)$ as $i \notin \mu(C_{m-1})$. Then there must exist an agent $k \in D_i(q)$ such that $k \notin C_{m-1}$. Now, as $j = \mu(i) \in D_i(p)$

$$P(i, j, p) = (v(\{i, j\}) + p_i - p_j)/2 \geq (v(\{i, k\}) + p_i - p_k)/2 = P(i, k, p). \tag{15}$$

Moreover, $p_j - q_j \geq p_k - q_k$ because $j \in C_m$ and $k \notin C_{m-1}$. But the latter inequality implies

$$(p_j - p_i + q_i - q_j)/2 \geq (p_k - p_i + q_i - q_k)/2. \tag{16}$$

Adding inequalities (15) and (16) yields $P(i, j, q) \geq P(i, k, q)$. So $j \in D_i(q)$. Repeating for all $i \in N$, we find that μ is demanded by N at q. □

Theorem 3. There exists an equilibrium (μ, p^*) if and only if μ is demanded by N at $p^{\min}$.

Proof. Suppose that there exists an equilibrium (μ, p^*). By Lemma 4, μ is demanded by N at p^* and $p^* \in \mathcal{H}$. Also, there exists a minimum payoff vector $p^{\min}$ by Theorem 2. But then μ is demanded by N at $p^{\min}$ by Lemma 5.

To complete the proof, we need to demonstrate that there exists an equilibrium if μ is demanded by N at $p^{\min}$. For this purpose, let $i = \mu(j)$ and $j = \mu(i)$, and define p^* as

$$p_i^* = (v(\{i, j\}) + p_i^{\min} - p_j^{\min})/2 \text{ if } i \neq j, \tag{17}$$

$$p_i^* = p_i^{\min} = v(\{i\}) \text{ if } i = j. \tag{18}$$

We will demonstrate that (μ, p^*) is an equilibrium, i.e., that all three requirements from Definition 1 are satisfied at (μ, p^*).

Individual rationality. If agent i is single, individual rationality follows directly from condition (18). Suppose instead that agent i is matched with some distinct agent j at μ. Because μ is demanded by N at $p^{\min}$, it is clear that $P(i, j, p^{\min}) \geq P(i, i, p^{\min})$. Using the above specification of p_i^* and the identity $P(i, i, p^{\min}) = v(\{i\})$, it follows that $p_i^* = P(i, j, p^{\min}) \geq v(\{i\})$. Hence, individual rationality is satisfied also in this case.

Stability. Note first that for any two distinct agents i and k it holds that $P(i, k, p^{\min}) + P(k, i, p^{\min}) = v(\{i, k\})$ by construction of the expected payoffs. Suppose now that $\mu(i) = j$ and $\mu(k) = l$. As μ is demanded by N at $p^{\min}$, it follows that $P(i, j, p^{\min}) \geq P(i, k, p^{\min})$ and $P(k, l, p^{\min}) \geq P(k, i, p^{\min})$. Adding these two inequalities and using the above identity yields

$$P(i, j, p^{\min}) + P(k, l, p^{\min}) \geq P(i, k, p^{\min}) + P(k, i, p^{\min}) = v(\{i, k\}).$$

Now the conclusion follows directly from the above inequality and the observation that $p_i^* = P(i, j, p^{\min})$ and $p_k^* = P(i, k, p^{\min})$.

Feasibility. This follows directly from conditions (17) and (18).

In summary, because (μ, p^*) satisfies all requirements of Definition 1, it must be an equilibrium. □

Theorem 4. The Partnership Formation Process either finds an equilibrium or proves the nonexistence of an equilibrium in finitely many iterations.

Proof. From Theorem 1 we know that Process 1 converges after finitely many iterations. Then because Step 2 of the Partnership Formation Process requires only one additional iteration, the Partnership Formation Process converges after a finite number of iterations. If an equilibrium exists, there is a matching μ that is demanded by N at $p^{\min}$ by Theorem 3. Because Step 2 of the Partnership Formation Process identifies a payoff vector identical to p^* specified in the proof of Theorem 3 it is clear that the Partnership Formation Process finds an equilibrium whenever it exists. In the case when there exists no matching μ that is demanded by N in Step 2, the Partnership Formation Process will prove the non-existence of an equilibrium by Theorem 3. □

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