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# Auctioning risk: The all-pay auction under mean-variance preferences

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# Auctioning risk: The all-pay auction under mean-variance preferences*

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**Abstract**

We develop the idea of using mean-variance preferences for the analysis of the first-price, all-pay auction. On the bidding side, we characterise the optimal strategy in symmetric all-pay auctions under mean-variance preferences for general distributions of valuations and any number of bidders. We find that, in contrast to winner-pay auction formats, only high-type bidders increase their bids relative to the risk-neutral case while low types minimise variance exposure by bidding low. Introducing asymmetric variance aversions across bidders into a Uniform valuations, two-player framework, we show that a more variance-averse type bids always higher than her less variance-averse counterpart. Taking mean-variance bidding behaviour as given, we show that an expected revenue maximising seller may want to optimally limit the number of participants. Although expected revenue for risk-neutral bidders typically dominates revenue under mean-variance bidding, if the seller himself takes account of the variance of revenue, he may find it preferable to attract bidders endowed with mean-variance preferences. (JEL *C7, D7, D81.* Keywords: *Auctions, Contests, Mean-Variance preferences.*)

## 1 Introduction

Mean-variance preferences have long been successfully applied to portfolio choice investment problems where asset managers evaluate alternative portfolios on the basis of the mean and variance of their return. It therefore may be surprising that the mechanism design literature and, specifically, the large literature on auctions has not yet addressed the decision making problem of players endowed with mean-variance preferences over their wealth. The present paper attempts to close this gap by fully characterising bidding and revenue-optimal sales behaviour in one of the standard auction types, the all-pay auction. This auction type may be viewed as an obvious candidate because it exposes a bidder to the inherent risk of either winning the object (potentially at a bargain) or losing one's bid without gaining anything.

We offer a group of motivational examples underpinning the relevance of this approach. All our examples have two random components: i) the 'endogenous' variance centred on the bidding process,

*Thanks to Olivier Bos, Dan Kovenock, and Thomas Giebe for helpful comments and discussions. †University of Zurich, Department of Economics, Blümlisalpstr. 10, CH-8006 Zurich, Switzerland, bettina.klose@econ.uzh.ch. ‡Department of Economics, University of York, Heslington, York YO10 5DD, United Kingdom, paul.schweinzer@york.ac.uk. (19-Nov-2012) i.e., the tension between winning a prize (consisting of a private valuation minus the own bid) and paying the own bid for sure, and ii) the 'exogenous' variance of the prize (or outside option) itself. Examples for both seem to abound in, for instance, a firm's market development decisions while facing competitors involved in similar decision problems or Human Resource decisions on a firm's own outlays to welcome and accommodate job candidates. A further and distinct set of examples lies in the practice of introductory offers ('burning money') with which firms try to ascertain uncertain future market shares through certain upfront losses. An entrepreneur's team composition selection is as applicable to our setup as the research portfolio selection of heads of R&D or similar institutions. Finally, the classical portfolio selection problem seems to be related as, clearly, portfolio choice is usually based on mean-variance considerations and anecdotal evidence is available which underlines all-pay aspects of the fund managing practice.[1]

What is a motivation to consider risk aversion in winner-pay auctions? A bidder in a (first-price) winner-pay auction controls, through her bid, both the probability of winning and the amount she wins. A risk averse bidder is willing to sacrifice some of the payoff (the individual value minus her bid) for a higher probability of winning (the higher bid). Hence, in a first-price, winner-pay auction, risk aversion causes an increase in equilibrium bids relative to the risk neutral case.[2] In all-pay auctions, in addition, increasing one's bid has the direct negative effect of increasing the certain payment *independently* of both other effects. In consequence, an all-pay low type bidder under risk aversion wants to *decrease* her losses while a risk averse, high-type bidder wants to *increase* her probability of winning through more aggressive bidding.

Apart from intellectual curiosity, we field three main arguments in order to justify the attention we place on mean-variance preferences in this paper. First, the typically employed risk-neutral, expected payoff analysis of auctions simply ignores any risk considerations; compared with that, a mean-variance analysis certainly represents progress. Second, if all relevant probability distributions have the same elliptic form, then the mean and variance represent a sufficient statistic to identify the true distribution of returns. Then, the mean-variance approach does not differ from a *full* account of expected utility using a general representation of risk aversion. Third, financial practitioners make the vast majority of their day-to-day portfolio choice decisions on the basis of the mean and variance of portfolios. It would seem likely that this group could benefit from a similar representation of their choices for auctioning activities.

## Literature

To the best of the authors' knowledge there are no existing papers which analyse auctions or incomplete information contests under mean-variance preferences. Most existing work on risk aversion in contests applies to full information Tullock contests. An attempt to model mean-variance preferences in the full information case is Robson (2012) who derives an 'irrelevance result' in the sense

[1] "Five-star funds. Four-star funds. Those seem to be the only mutual funds people want to buy." Investors Business Daily, "Making Money in Mutuals: Don't Focus too Narrowly on Star Ratings," by A. Shell, 22 June 1998, cited in Bagnoli and Watts (2000). Hence, although all funds invest efforts, only the most highly ranked funds obtain large investments.

[2] See Maskin and Riley (1984).

that for two-player Tullock contests bidding behaviour is not affected by the introduction of an aversion to variance. A more general analysis in terms of risk aversion of the same setup is Cornes and Hartley (2010) who focus on existence questions of both symmetric and asymmetric Nash equilibria. (For the case of loss aversion see Cornes and Hartley (2012).) The only existing work on risk aversion for the incomplete information all-pay auction of which we are aware is Fibich, Gavious, and Sela (2006). They show that analytic equilibrium strategies cannot be usually obtained for von Neumann-Morgenstern risk-averse players. Thus, contrasting our fully analytical approach, they turn to perturbation analysis to obtain their mostly numerical results. Esö and White (2004) show that under special conditions on valuations, decreasingly absolute risk averse players prefer the first-price auction to the all-pay auction. Fibich, Gavious, and Sela (2006) extend this ranking to the case of general risk aversion for independent valuations. Their results are limited, however, by the fact that they cannot generally obtain analytic forms of the equilibrium bidding strategies of risk averse players. We can overcome this limitation at the price of focusing attention to the class of mean-variance preferences.

Papers relating to the analysis of risk aversion in general winner-pay auction environments are Maskin and Riley (1984) and Matthews (1987), both discussing risk-averse bidders' behaviour in auctions, Esö and White (2004), analysing precautionary bidding in auctions, and Esö and Futó (1999), who derive the revenue-optimal strategy for a risk-averse seller, and Hu, Matthews, and Zou (2010) who discuss reserve prices. The existing analyses of asymmetric auctions, for instance Maskin and Riley (2000), Fibich, Gavious, and Sela (2004), Kirkegaard (2012), or Kaplan and Zamir (2012), typically employ asymmetric distributions (or supports) while we use our idiosyncratic variance-aversion parameter. To the best of the authors' knowledge, this is the first paper to analyse bidding in a contest when players are asymmetric in their degree of risk-aversion. For accounts of auctions under ambiguity see, for instance, Bose and Daripa (2009), and for a more general approach to mechanism design under ambiguity, see Bodoh-Creed (2012).

In terms of revenue and payoff analysis, Matthews (1987) compares payoffs for risk averse behaviour under bidders exhibiting constant and increasing absolute risk aversion. For CARA, he finds that bidders are indifferent between first- and second-price auctions and that IARA bidders prefer the first-price auction. Smith and Levin (1996) show that this ranking can be reversed under decreasing absolute risk aversion.

Mean-variance preferences can be transformed into the expected utility form under certain assumptions on the location, scale, and concordance parameters of the environment. For the precise relation of von Neumann-Morgenstern preferences to mean-variance preferences, see, for instance, Sinn (1983), Mayer (1987), or, more recently, Eichner and Wagener (2009).

# 2 Model

There is a seller with one indivisible object for sale. The seller's valuation of the item is (normalised to) zero. There are $n \geq 2$ potential buyers with valuations $\theta_i$, $i \in \mathcal{N} = \{1, 2, \ldots, n\}$, respectively. The own valuation is private information of each buyer and all players' valuations, $\theta_i$, $i \in \mathcal{N}$, are assumed to be independent draws from the same increasing and atom-less distribution $F$ over some interval $I \subseteq [0, \infty)$. Let $f(\cdot) = F'(\cdot)$ represent the probability density function and $\underline{\theta}, \bar{\theta} \subseteq I$ its support. The final value of the object may further be influenced by an exogenous shock, $\varepsilon \sim W(0, \hat{\varepsilon}^2)$, which is distributed over some compact interval with mean zero and variance $\hat{\varepsilon}^2$. Similarly, a player's valuation of the state in which she does not win the object may be subject to another exogenous shock $\delta \sim L(0, \hat{\delta}^2)$.

After realising their own (expected) valuations of the object, $\theta_i$, all players simultaneously submit their bids, $b_i$, $i \in \mathcal{N}$. The player with the highest bid receives the object and all players forgo their bids. After the auction has ended, the exogenous shocks realise and player $i$'s payoff is given by

$$\pi_i(b_i, b_{-i}; \theta_i) = \begin{cases} \theta_i + \varepsilon - b_i & \text{if } b_i > b_j \forall j \neq i \\ \frac{1}{m}(\theta_i + \varepsilon) + \frac{m-1}{m}\delta - b_i & \text{if } i \in Q = \{j \in N | b_j = \max_{k \in N} b_k\}, m = |Q|. \\ \delta - b_i & \text{if } \exists j : b_i < b_j \end{cases}$$

In the following we focus on three particular cases.

1. *No exogenous shock*: In this case, both $\varepsilon$ and $\delta$ take the value zero with probability one, i.e., each bidder $i \in \mathcal{N}$, knows with certainty that in case of winning the auction she will obtain a prize of value $\theta_i$ while her valuation of losing is zero.

2. *Winner's uncertainty*: When $\hat{\varepsilon}^2 > 0$, the valuation of the prize is uncertain and $\theta_i$ is merely a signal, the expected value of the object.

3. *Loser's uncertainty*: In case that $\hat{\delta}^2 > 0$ a player faces uncertainty in the event that she does not secure the object for sale.

Notice that the three cases described above do not have any effects on equilibrium bidding behaviour in the standard model of buyers with risk-neutral von Neumann-Morgenstern utility, who simply maximise expected payoffs. In the following we discuss how bidding strategies of buyers with mean-variance preferences are affected in each of the aforementioned scenarios.

When buyers have mean-variance preferences, they maximise an objective function $u_i(\mu_i, \sigma_i^2)$, which is increasing in the expected payoff, $\mu_i$, and decreasing in the variance of their payoff, $\sigma_i^2$. For much of our analysis we use the following simple linear representation of mean-variance preferences[3]

$$u_i(\mu(b, \theta_i), \sigma^2(b, \theta_i)) = \mu(b, \theta_i) - \nu_i \sigma^2(b, \theta_i), \tag{1}$$

where the parameter $\nu_i \in [0, 1]$ accounts for player $i$'s variance-aversion. The case of $\nu_i = 0$ represents the standard case of risk-neutral expected payoff maximisation.

[3] In an empirical paper, Saha (1997) justifies variants of this form as plausible. More recently, Chiu (2010) discusses the applicability of mean-variance preferences of this form to a large set of problems in finance and economics in choice theoretic terms.

# 3 Bidding behaviour

## 3.1 The symmetric case: $n$ identical bidders

Under the first-price, all-pay auction, a type-$\theta_i$ bidder's expected payoff when issuing a bid of $b$ is given by

$$\pi(b,\beta;\theta_i) = \int_0^{\beta^{-1}(b)} \cdots \int_0^{\beta^{-1}(b)} \theta_i f(\theta_1)\cdots f(\theta_{n-1}) d\theta_1 \cdots d\theta_{n-1} - b \tag{2}$$

where $\beta(\theta)$ is the tentative symmetric equilibrium bid issued by a type-$\theta$ player. We conjecture that the function $\beta(\theta)$ is non-decreasing and denote the highest type who submits a bid no higher than $b$ by $\theta = \beta^{-1}(b)$. It is well known that the strategies

$$\beta_{\mathsf{rn}}(\theta) = \theta(F(\theta))^{n-1} - \int_0^{\theta} (F(\vartheta))^{n-1} d\vartheta, \tag{3}$$

maximise (2), hence constituting a symmetric equilibrium if players simply maximise their expected payoffs (i.e., $\nu_i = 0$ for all $i \in N$).

With mean-variance preferences, symmetric players with $\nu = \nu_1 = \cdots = \nu_n$ choose a bidding function which maximises (1), taking into account their payoff variance in addition to their expected payoff. These are given for the first-price, all-pay auction as

$$\mu(b,\theta_i) \ = \theta_i(F(\beta^{-1}(b)))^{n-1} - b; \ \sigma^2(b,\theta_i) = (F(\beta^{-1}(b)))^{n-1}(1-(F(\beta^{-1}(b)))^{n-1})\theta_i^2. \tag{4}$$

Plugging these back into the player's objective gives

$$u_i(b,\theta_i) = \theta_i(F(\beta^{-1}(b)))^{n-1}(1-\nu\theta_i + \nu\theta_i(F(\beta^{-1}(b)))^{n-1}) - b. \tag{5}$$

The first-order condition for maximisation of (5) with respect to $b$ is[4]

$$\theta_i\left(1-\nu\theta_i + 2\nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right)(n-1)(F(\beta^{-1}(b)))^{n-2} f(\beta^{-1}(b))\frac{\partial\beta^{-1}(b)}{\partial b} = 1. \tag{8}$$

In the symmetric equilibrium $b = \beta(\theta_i)$, this yields the first-order differential equation

$$\beta'(\theta_i) = \theta_i\left(1-\nu\theta_i + 2\nu\theta_i(F(\theta_i))^{n-1}\right)(n-1)(F(\theta_i))^{n-2} f(\theta_i). \tag{9}$$

[4] The second-order condition for the case of two players is

$$\frac{2\theta_i\nu - 4\theta_i\nu F(\theta_i) - 1}{\theta_i^2 f(\theta_i)(1-\theta_i\nu + 2\theta_i\nu F(\theta_i))^2} < 0 \tag{6}$$

The condition for the general case of $n > 2$ players is more involved and relegated to the appendix. If $\theta_i\nu \leq 1,$ a sufficient condition for the second-order condition to hold is that

$$F(\theta)^{n-1} > \frac{1}{2} - \frac{1}{4\theta\nu}. \tag{7}$$

If $\nu$ is sufficiently small such that $\theta\nu \leq \frac{1}{2}$ for all $\theta$ in the support of the type distribution $F(\cdot)$, then (7) always holds.

This differential equation together with the boundary condition $\beta(0) = 0$ is solved (through repeated integration by parts) by the bidding function

$$\begin{aligned}\beta_{\mathsf{mv}}(\theta_i) =&\theta_i(F(\theta_i))^{n-1} - \int_0^{\theta_i} (F(\vartheta))^{n-1}\, d\vartheta - \nu\theta_i^2(F(\theta_i))^{n-1} + \\ &\nu\theta_i^2(F(\theta_i))^{2(n-1)} + \nu\int_0^{\theta_i} 2\vartheta(F(\theta))^{n-1}\, d\vartheta - \nu\int_0^{\theta_i} 2\vartheta(F(\theta))^{2(n-1)}\, d\vartheta \\ =&\beta_{\mathsf{rn}} - \nu\left(\theta_i^2\left(F(\theta_i)^{n-1} - (F(\theta_i))^{2(n-1)}\right) - \int_0^{\theta_i} 2\vartheta\left((F(\vartheta))^{n-1} - (F(\vartheta))^{2(n-1)}\right) d\vartheta\right) \\ =&\beta_{\mathsf{rn}} - \nu\int_0^{\theta_i} \vartheta^2(n-1)(F(\vartheta))^{n-2}f(\vartheta)\left(1 - 2(F(\vartheta))^{n-1}\right) d\vartheta.\end{aligned} \tag{10}$$

Notice that, from (9), $\beta_{\mathsf{mv}}$ is an increasing function thus confirming our tentative monotonicity conjecture. Our next result shows that low types submit lower bids under mean-variance preferences, while high types submit higher bids under mean-variance preferences than if they were to maximise expectations.

**Proposition 1** (Single-crossing). *Either $\beta_{mv}(\theta) \leq \beta_{rn}(\theta)$ for all $\theta \in [0,1]$ or there exists a $\hat{\theta}$ in the support of $F$ such that $\beta_{mv}(\theta) \leq \beta_{rn}(\theta)$ for $\theta \leq \hat{\theta}$, $\beta_{mv}(\hat{\theta}) = \beta_{rn}(\hat{\theta})$ and $\beta_{mv}(\theta) > \beta_{rn}(\theta)$ for $\theta > \hat{\theta}$.*

*Proof.* Note that the symmetric equilibrium strategy can be written as

$$\beta_{\mathsf{mv}}(\theta) = \beta_{\mathsf{rn}} - \nu\int_0^{\theta} G(\vartheta)H(\vartheta)d\vartheta,$$

where $G(\vartheta) = \vartheta^2(n-1)(F(\vartheta))^{n-2}f(\vartheta)$ and $H(\vartheta) = 1-2(F(\vartheta))^{n-1}$. $F$ is a cumulative distribution function with density $f$, therefore $G(\vartheta) \geq 0$ for all $\vartheta \in [\underline{\theta}, \bar{\theta}]$. $H(\vartheta)$ is a continuous and decreasing function with $H(\underline{\theta}) = 1$ and $H(\bar{\theta}) = -1$. Hence, $\int_0^{\theta} G(\vartheta)H(\vartheta)d\vartheta > 0$ for sufficiently small $\theta > 0$ and if $\int_0^{\bar{\theta}} G(\vartheta)H(\vartheta)d\vartheta = 0$ for any $\hat{\theta} > 0$, then $\int_0^{\theta} G(\vartheta)H(\vartheta)d\vartheta < 0$ for all $\theta > \hat{\theta}$. □

This result is qualitatively in line with Propositions 1 and 2 in Fibich, Gavious, and Sela (2006). The intuition is that low-valuation bidders expect to lose in a symmetric equilibrium and therefore decrease their bids in order to keep their variance low. High-valuation bidders, by contrast, are likely to win and therefore increase their bids in line with variance compression. Proposition 1 says that there is only a single type of bidder endowed with mean-variance preferences who behaves in exactly the same way as her risk-neutral counter part.

**Corollary 1.** *As the number of participating bidders $n$ expands,*

1. *the convexity of the bidding function $\beta_{mv}(\theta)$ increases, i.e., low types decrease their bids and high types increase their bids relative to the case with a lower number of bidders;*
2. *the type $\hat{\theta}$ who issues the same bid under mean-variance and risk-neutral von Neumann-Morgenstern preferences shifts to the right.*

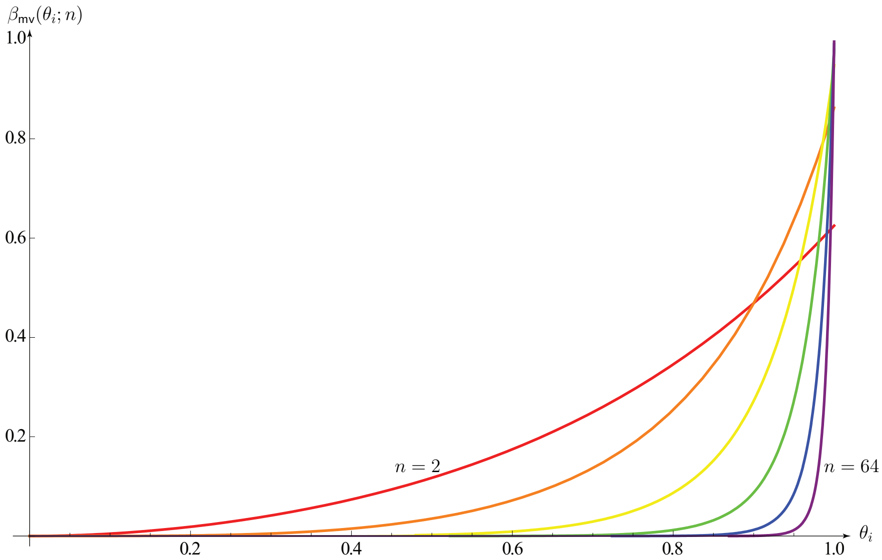


Figure 1: Equilibrium bidding functions for the all-pay auction under mean-variance preferences for uniformly distributed types, $\nu = 3/4$, and $n \in \{2, 4, 8, 16, 32, 64\}$ players, respectively (sorted in the colours of the rainbow from red to violet).

*Proof.* Consider the derivative of (9) with respect to $n$

$$\theta f(\theta)F(\theta)^{n-3}\left[(1-\theta\nu)F(\theta)(1-\kappa)+2\theta\nu F(\theta)^n(1-2\kappa)\right] \tag{11}$$

where $\kappa = -(n-1)\log(F(\theta))$. Notice that $\log(F(\theta)) \leq 0$ and $\log(F(\theta))$ is strictly increasing in $\theta$ with $\log(F(\theta)) \to -\infty$ as $\theta$ approaches the lower bound of the support of its distribution and $\log(F(\theta)) \to 0$ as $\theta$ approaches the upper bound of the support of its distribution. Therefore, for sufficiently small $\theta$, (11) becomes negative. Similarly, for $\theta$ sufficiently large, (11) is positive. □

### 3.1.1 Examples

#### The Uniform distribution

In the following, we exemplarily illustrate our findings for the case of $n$ players when values are drawn from a Uniform distribution over the interval $[0, 1]$. In this case, the expression for the objective of a bidder with mean-variance preferences simplifies to

$$u_i(b, \theta_i) = \theta_i(\beta^{-1}(b))^{n-1}(1 - \nu\theta_i + \nu\theta_i(\beta^{-1}(b))^{n-1}) - b \tag{12}$$

which determines the symmetric equilibrium bidding functions as

$$b^* = \beta(\theta_i) = \frac{n-1}{n}\theta_i^n + \nu\left(\frac{n-1}{n}\theta_i^{2n} - \frac{n-1}{n+1}\theta_i^{n+1}\right) + C \tag{13}$$

for some constant $C$ which is zero because a type-0 will not make a positive bid. Figure 2 compares this equilibrium bidding behaviour with that under standard risk-neutral von Neumann-Morgenstern preferences for two players. As seen before, the bidding behaviour of low-intermediate valuation types is more aggressive under expected payoff maximisation while high types submit higher bids under mean-variance preferences.

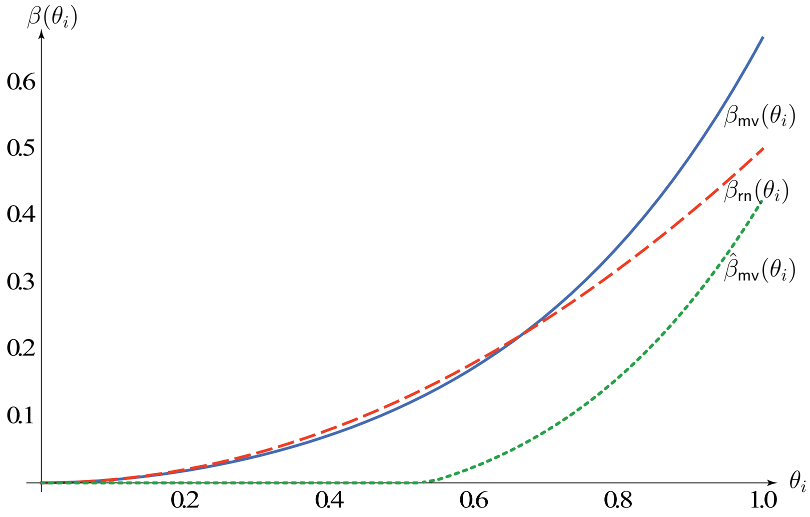


Figure 2: Equilibrium bidding functions for the all-pay auction under risk-neutrality (dashed, $\beta(\theta_i) = \theta_i^2/2$) and mean-variance preferences ($\nu = 1$, solid). The dotted bidding function results under mean-variance preferences if the prize itself is risky $\hat{\varepsilon} - \hat{\delta} = \frac{1}{4}$.

### Other distributions

The table below displays the symmetric, two-player equilibrium bidding functions for the most commonly used distribution functions when players have mean-variance preferences with $\nu = 1$ and values are i.i.d. according to the specified distribution.

| Dist'n | $F(\theta)$ | $f(\theta)$ | $\beta(\theta)$ |
|---|---|---|---|
| Uniform[0,1] | $\theta$ | $1$ | $\frac{\theta_i^2}{2} - \frac{\theta_i^3}{3} + \frac{\theta_i^4}{2}$ |
| Power[0,1] | $\theta^\alpha$ | $\alpha\theta^{\alpha-1}$ | $\frac{\alpha\theta_i^{1+\alpha}\left(2+\alpha-\theta_i-\alpha\theta_i+(2+\alpha)\theta_i^{1+\alpha}\right)}{(1+\alpha)(2+\alpha)}$ |
| Beta(2,2) | $\frac{\int_0^\theta u(1-u)du}{\int_0^1 u(1-u)du}$ | $\frac{\theta(1-\theta)}{\int_0^1 u(1-u)du}$ | $2\theta_i^3 - 3\theta_i^4 + \frac{6\theta_i^5}{5} + 6\theta_i^6 - \frac{60\theta_i^7}{7} + 3\theta_i^8$ |
| Quadratic-U | $4(\theta - {}^1\!/_2)^3 + 4({}^1\!/_2)^3$ | $12(\theta - {}^1\!/_2)^3$ | $\frac{3\theta_i^2}{2} - 5\theta_i^3 + \frac{21\theta_i^4}{2} - 24\theta_i^5 + 40\theta_i^6 - \frac{240\theta_i^7}{7} + 12\theta_i^8$ |
| Exponential | $1 - \exp(-\lambda\theta)$ | $\lambda\exp(-\lambda\theta)$ | $\frac{e^{-2\theta_i\lambda}\left(1+e^{2\theta_i\lambda}(3+2\lambda)+2\theta_i\lambda(1+\theta_i\lambda)-2e^{\theta_i\lambda}(2+\lambda(1+\theta_i(2+\lambda+\theta_i\lambda)))\right)}{2\lambda^2}$ |
| Pareto | $1 - (x+1)^{-2}$ | $2(x+1)^{-3}$ | $2\log(\theta_i + 1) - \frac{\theta_i(6+\theta_i(18+\theta_i(22+7\theta_i)))}{3(1+\theta_i)^4}$ |

## 3.2 Exogenous noise

We want to motivate the analysis of exogenous noise with a separate stylised example. Consider a R&D company engaging in costly research outlays in order to obtain some (patentable) innovation first among a group of competitors. The endogenous variance is grounded, as before, in the uncertain spread between certain outlays and probabilistic winning. The exogenous component may be seen as market uncertainty in case of winning: the firm cannot usually be entirely certain about the market perception and success of its future product.

We now extend the basic model analysed in section 3 with exogenous noise. Consider expected revenue distributed $\mathbb{E}[\mathsf{R}] \sim W[\theta_i, \hat{\varepsilon}^2]$, where the distribution $W$ is elliptical, i.e., completely determined by mean $\theta_i$ and variance $\hat{\varepsilon}^2 \in [0,1]$.[5] Similarly, we allow for the case that a player's valuation, if she does not win the object, is subject to another exogenous shock $\delta \sim L(0, \hat{\delta}^2)$. In the case of R&D competition, $\hat{\delta}^2$ reflects the uncertainty in the company's forecast of the residual demand after the innovation.

A competitor's objective from (1) therefore changes into

$$u_i(\mu(b,\theta_i), \sigma^2(b,\theta_i)) = \mu(b,\theta_i) - \nu\left(\sigma^2(b,\theta_i) + \hat{\varepsilon}^2 F(\beta^{-1}(b))^{n-1} + \hat{\delta}^2(1 - F(\beta^{-1}(b))^{n-1})\right) \quad (14)$$

for $\nu \in [0,1]$. Plugging back the expressions for the mean and variance (4), her objective is

$$\begin{aligned} u_i(b,\theta_i) = &(F(\beta^{-1}(b)))^{n-1}\left[\theta_i\left(1 - \nu\theta_i + \nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right) - \nu\hat{\varepsilon}^2\right] \\ &+ (1 - (F(\beta^{-1}(b))))^{n-1}\hat{\delta}^2 \\ &- b. \end{aligned} \quad (15)$$

The first-order condition for maximisation of this expression with respect to $b$ is

$$\begin{aligned} &\left[\theta_i\left(1 - \nu\theta_i + 2\nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right) - \nu(\hat{\varepsilon}^2 - \hat{\delta}^2)\right] \times \\ &(n-1)(F(\beta^{-1}(b)))^{n-2} f(\beta^{-1}(b))\frac{\partial\beta^{-1}(b)}{\partial b} = 1. \end{aligned} \quad (16)$$

In the symmetric equilibrium $b = \beta(\theta_i)$ for all $i \in \mathcal{N}$, this yields the first-order differential equation

$$\begin{aligned} \beta'(\theta_i) &= \theta_i\left(1 - \nu\theta_i + 2\nu\theta_i(F(\theta_i))^{n-1}\right)(n-1)(F(\theta_i))^{n-2} f(\theta_i) \\ &- (n-1)(F(\theta_i))^{n-2} f(\theta_i)\nu(\hat{\varepsilon}^2 - \hat{\delta}^2). \end{aligned} \quad (17)$$

This differential equation is solved by the bidding function

$$\hat{\beta}_{\mathsf{mv}}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \leq \theta_0 \\ \beta_{\mathsf{mv}}(\theta_i) - (F(\theta_i))^{n-1}\nu(\hat{\varepsilon}^2 - \hat{\delta}^2) & \text{if } \theta_i > \theta_0 \end{cases} \quad (18)$$

where $\beta_{\mathsf{mv}}(\theta_i)$ is defined in (10) and the 'cutoff type' $\theta_0$ is implicitly defined as the solution to

$$\begin{aligned} &\theta_0(F(\theta_0))^{n-1} - \int_0^{\theta_0}(F(\vartheta))^{n-1}d\vartheta \\ &-\nu\int_0^{\theta_0}\vartheta^2(n-1)(F(\vartheta))^{n-2}f(\vartheta)\left(1 - 2(F(\vartheta))^{n-1}\right)d\vartheta - (F(\theta_0))^{n-1}\nu(\hat{\varepsilon}^2 - \hat{\delta}^2) = 0 \end{aligned} \quad (19)$$

for which a closed form solution is generally unavailable. As we restrict bids to be non-negative, the resulting bidding function is still invertible. Similarly to the common practice of normalising the valuation of the outside option to zero, (18) shows that there is a degree of freedom to normalise the variance of one of the two possible outcomes. In the remainder we therefore normalise $\hat{\delta}^2 \equiv 0$ for simplicity.

**Corollary 2.** *Introducing exogenous noise $\hat{\varepsilon}^2 - \hat{\delta}^2 > 0$ on the prize shifts the optimal bidding schedule down, causing low-type bidders to abstain from participating in the auction.*

[5] Elliptical distributions are a generalisation of the normal family containing, among others, the Uniform, Student-t, Logistic, Laplace, symmetric stable, and Normal distributions. A detailed presentation of these distributions is available in Fang, Kotz, and Ng (1987).

### 3.2.1 Example

We round off this section with our usual Uniform, two-bidders example. Consider the equilibrium bidding function

$$b^* = \hat{\beta}(\theta_i) = \frac{\theta_i^2}{2} - \nu\left(\frac{\theta_i^3}{3} - \frac{\theta_i^4}{2} + \theta_i\hat{\varepsilon}^2\right). \tag{20}$$

The bidding behaviour this suggests for the case of a stochastic prize parameterised by $\hat{\varepsilon}^2 = {}^1/_4$, is shown as dotted line in figure 2. Consider now a case in which we auction two objects valued $\theta_1 > \theta_2$ with exogenous prize variance $\hat{\varepsilon}^1 > \hat{\varepsilon}^2$. If (full demand) bidders submit separate bidding functions for each object, then we can get cases where the bid for the high-value/high-risk object is below that of the low-value/low-risk object. An example under Uniform valuations and $\nu = 1$ is $\beta(\theta_1 = 3/4|\hat{\varepsilon}_1^2 = 1/4) = 0.111 < 0.139 = \beta(\theta_2 = 2/3|\hat{\varepsilon}_2^2 = 1/8)$.

## 3.3 Two asymmetric bidders

This section presents the first result we are aware of on all-pay auctions between bidders who are not identical in terms of their risk preferences. Consider the following Uniform, two-players setup featuring asymmetric degrees of variance aversion $\nu_i$ where player $i \in \{1, 2\}$ maximises

$$u_i(\mu(b,\theta_i), \sigma^2(b,\theta_i)) = \mu(b,\theta_i) - \nu_i\sigma^2(b,\theta_i), \;\; \nu_i \in \mathbb{R}_+. \tag{21}$$

We consider the particular case of $\nu_1 = 0$ and $\nu_2 = \nu$, i.e., bidder 1 is risk-neutral while bidder 2 is variance-averse. Therefore, we get

$$\begin{aligned} u_1(\theta_1, b_1) &= \beta_2^{-1}(b_1)\theta_1 - b_1, \\ u_2(\theta_2, b_2) &= \beta_1^{-1}(b_2)\theta_2 - \nu\left(\theta_2^2\beta_1^{-1}(b_2)(1 - \beta_1^{-1}(b_1))\right) - b_2 \end{aligned} \tag{22}$$

with the pair of first-order conditions

$$\begin{aligned} \frac{\partial u_1(\theta_1, b_1)}{\partial b_1} &= \frac{1}{\beta_2'(\beta_2^{-1}(b_1))}\theta_1 - 1 = 0 \\ &\Leftrightarrow \beta_2'(\beta_2^{-1}(b_1)) = \theta_1, \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial u_2(\theta_2, b_2)}{\partial b_2} &= \frac{1}{\beta_1'(\beta_1^{-1}(b_2))}\left(\theta_2 - \nu\theta_2^2(1 - 2\beta_1^{-1}(b_2))\right) - 1 = 0 \\ &\Leftrightarrow \beta_1'(\beta_1^{-1}(b_2)) - 2\nu\theta_2^2\beta_1^{-1}(b_2) = \theta_2 - \nu\theta_2^2. \end{aligned} \tag{24}$$

In equilibrium $b_1 = \beta_1(\theta_1)$ and $b_2 = \beta_2(\theta_2)$. Thus, we substitute $\beta_1^{-1}(b_1) = \theta_1$ into (23) to obtain

$$\beta_2'(\beta_2^{-1}(b)) = \beta_1^{-1}(b). \tag{25}$$

Taking the derivative of $\beta_1^{-1}(b)$ and applying (23) gives

$$\beta_1'(\beta_1^{-1}(b)) = \frac{\beta_2'(\beta_2^{-1}(b))}{\beta_2''(\beta_2^{-1}(b))} \tag{26}$$

where we use $b$ as variable from the joint support of $\beta_1(\cdot)$ and $\beta_2(\cdot)$.[6] Substituting (26) and (25) into (24) yields the following second-order differential equation in $\beta_2$

$$\frac{\beta_2'(\beta_2^{-1}(b))}{\beta''_2(\beta_2^{-1}(b))} - 2\nu\theta^2\beta_2'(\beta_2^{-1}(b)) = \theta - \nu\theta^2. \tag{27}$$

This differential equation can be solved using the boundary condition $\beta_2(0) = 0$ to obtain

$$\begin{aligned} \beta_2(\theta_2) &= \frac{1}{2\sqrt{1+4c}\nu}\Big[\sqrt{1+4c}\left(-1+\theta\nu+\sqrt{1+\theta\nu(-2+\theta\nu+4c\theta\nu)}+\log(2)\right) \\ &+ \log\left(1-\sqrt{1+4c}\right) - \sqrt{1+4c}\log\left(1-\theta\nu+\sqrt{1+\theta\nu(-2+\theta\nu+4c\theta\nu)}\right) \\ &- \log\left(1-\nu\left(\theta+4c\theta+\sqrt{1+4c}\sqrt{\tfrac{1}{\nu^2}+\tfrac{\theta}{\nu}(-2+\theta\nu+4c\theta\nu)}\right)\right)\Big] \end{aligned} \tag{28}$$

for yet undetermined constant of integration $c$. In order to solve for the first player's bidding function, we solve (25) for

$$\beta_1(\theta) = \beta_2\left((\beta_2')^{-1}(\theta)\right) \tag{29}$$

where

$$(\beta_2')^{-1}(\theta) = \frac{\theta}{\nu(\theta-\theta^2+c)}. \tag{30}$$

From an argument similar to the one used in a standard (risk-neutral) all-pay auction follows that the two bidding functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ must share the same support. Intuitively, in equilibrium no player's type can submit a strictly higher bid than the other player's highest type. Setting $\beta_1(1) = \beta_2(1)$ implies that the only possible value for the constant of integration is

$$c = \frac{1}{\nu}. \tag{31}$$

Substituting this constant into (28), we obtain the following pair of bidding functions

$$\begin{aligned} \beta_1(\theta) &= \frac{\log(1-\theta^2\nu+\theta\nu)}{2\nu} - \frac{\theta^2}{\theta\nu(\theta-1)-1} \\ &+ \frac{1}{2\sqrt{\nu(4+\nu)}}\left(\log\left(1-\sqrt{\tfrac{4+\nu}{\nu}}\right) - \log\left(\frac{\sqrt{\frac{4+\nu}{\nu}}+\theta\left(4+\theta\left(\nu+\sqrt{\nu(4+\nu)}\right)\right)-1}{\theta\nu(\theta-1)-1}\right)\right), \\ \beta_2(\theta) &= \frac{1}{2\nu}\Big[\theta\nu-1+\sqrt{1-2\theta\nu+\theta^2\nu(4+\nu)}+\sqrt{\tfrac{\nu}{4+\nu}}\log\left(\sqrt{\tfrac{4+\nu}{\nu}}-1\right) \\ &- \log\left(\tfrac{1}{2}\left(1-\theta\nu+\sqrt{1-2\theta\nu+\theta^2\nu(4+\nu)}\right)\right) \\ &- \sqrt{\tfrac{\nu}{4+\nu}}\log\left(\theta(4+\nu)-1+\sqrt{\tfrac{(4+\nu)(1-2\theta\nu+\theta^2\nu(4+\nu))}{\nu}}\right)\Big], \end{aligned} \tag{32}$$

which are illustrated for the case of $\nu = 1$ in figure 3.

As the figure shows, each positive risk-neutral player type bids less than the corresponding type of her variance-averse opponent. While the $\nu$-variance-averse, asymmetric bidder with bidding function $\beta_2(\cdot)$ always bids more than symmetric risk-neutral bidders $\beta_{\text{rn}}^{\text{sym}}$, the asymmetric risk-neutral bidder with bidding function $\beta_1(\cdot)$ (competing with a variance-averse player) bids up to a cutoff-type $c_2$ below the symmetric risk-neutral bidders and, for types higher than $c_2$, she bids above. Similarly, the

[6] The standard argument applies that in the two-player, all-pay auction the supports of both players' bidding functions coincide.

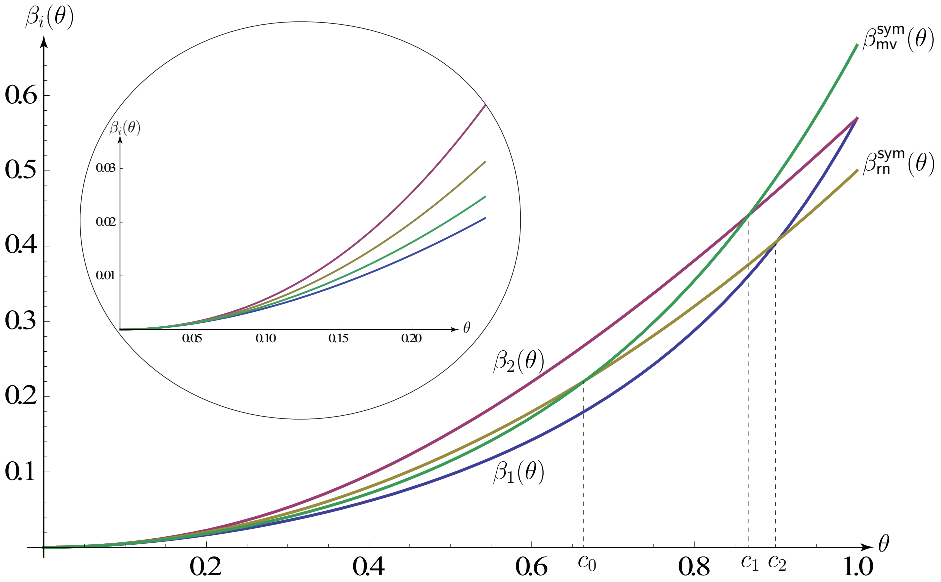


Figure 3: Comparison of asymmetric and symmetric bidding under mean-variance preferences for $\nu = 1$.

asymmetric variance-averse bidder (competing with a risk-neutral bidder) bids up to a cutoff-type $c_1$ above the symmetric $\nu$-variance-averse bidders ($\beta^{sym}_{mv}$) and bids below for types higher than $c_1$. Both properties are qualitatively similar to the single-crossing property with cutoff $\hat{\theta} = c_0$ from proposition 1. The generally high bids of the variance-averse bidder cause low types of the risk-neutral bidder to bid less in comparison to their strategy when faced with risk-neutral opponents. High types of the risk-neutral bidder, on the other hand, increase their bid in reaction to their variance-averse opponent's strategy.

# 4 Revenue valuation

The classical reference for revenue valuation in winner-pay auctions under risk aversion is Holt (1980) who discusses a procurement setup. Revenue equivalence between the standard auction formats breaks down with risk averse bidders. While second-price bidders maintain their dominant strategies of bidding their values, first-price competitors increase their bids with respect to the standard, risk-neutral case. This is due to the fact that raising one's bid in a first-price auction can be seen as partial insurance against loosing. From a risk averse seller's point of view, the first-price auction format is preferable to a second-price auction because it exposes the seller to less revenue risk.[7]

In this section we limit attention to uniformly distributed bidder valuations because our objective lies in the derivation of a series of concrete revenue ranking results. The results, however, are qualitatively similar for the other distributions listed in the table of section 3.1.1. The seller's expected revenue R depends on the bidder's preferences. In the case of risk-neutral bidders with von Neumann-Morgenstern preferences, the seller expects to earn

$$\mathbb{E}[\mathsf{R}_{\mathsf{rn}}] = n \int_0^1 \frac{n-1}{n} \theta^n \, d\theta = \frac{n-1}{n+1}. \tag{33}$$

[7] For references, see Milgrom (2004, p123).

If the bidders exhibit mean-variance preferences, then the seller can expect

$$
\begin{aligned}
\mathbb{E}[\mathsf{R}_{\mathsf{mv}}] &= n \int_0^1 \frac{n-1}{n}\theta^n + \nu \left( \frac{n-1}{n}\theta^{2n} - \frac{n-1}{n+1}\theta^{n+1} \right) d\theta \\
&= \frac{n-1}{n+1} + \nu \left( \frac{n-1}{2n+1} - \frac{n(n-1)}{(n+1)(n+2)} \right).
\end{aligned}
\tag{34}
$$

The revenue limit for $n \to \infty$ is $1 - \nu/2$. This limit, however, is only approached from below for low values of $\nu$. As figure 5 illustrates graphically, for high variance weights $\nu$, there exists a revenue-maximal finite number of bidders $N$. The following corollary states this revenue-optimal exclusion result and figure 4 gives a graphical illustration.

**Corollary 3.** *An expected revenue $\mathbb{E}[R_{mv}]$ maximising seller finds it optimal to limit the number of participants in an all-pay auction if bidders have a sufficiently high variance aversion parameter $\nu$. This optimal number of participants $N$ is decreasing in $\nu$.*

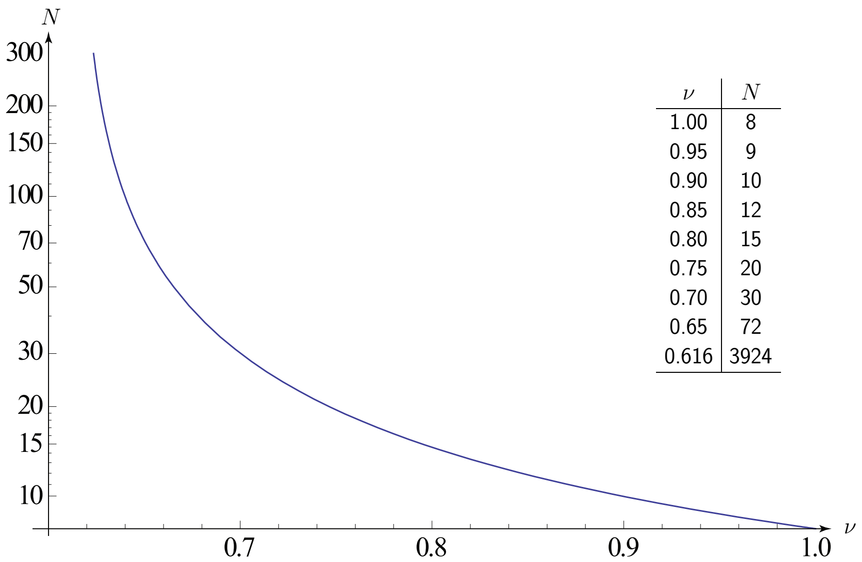


Figure 4: Expected revenue maximising numbers of participants $n$ (on a logarithmic scale) as a function of the variance aversion parameter $\nu$.

Holding the number of players fixed, expected revenue is strictly increasing in $\nu$ for the two-players case and strictly decreasing in $\nu$ for all $n > 2$. This is illustrated in figure 5 and summarised in corollary 4.

**Corollary 4.** *For $n = 2$, the expected revenue is strictly greater if bidders exhibit mean-variance preferences than if they are expected payoff maximisers. For all other $n$, this relationship is reversed.*

If the seller himself should also consider the revenue variance in addition to the revenue's mean, then his preference may be reversed. In the case of risk-neutral bidding, the seller's revenue variance is

$$
\begin{aligned}
\mathbb{V}[\mathsf{R}_{\mathsf{rn}}] &= n \int_0^1 \left( \beta_{\mathsf{rn}}(\theta) - \frac{\mathbb{E}[\mathsf{R}_{\mathsf{rn}}]}{n} \right)^2 d\theta \\
&= \frac{n(n-1)^2}{(2n+1)(n+1)^2}.
\end{aligned}
\tag{35}
$$

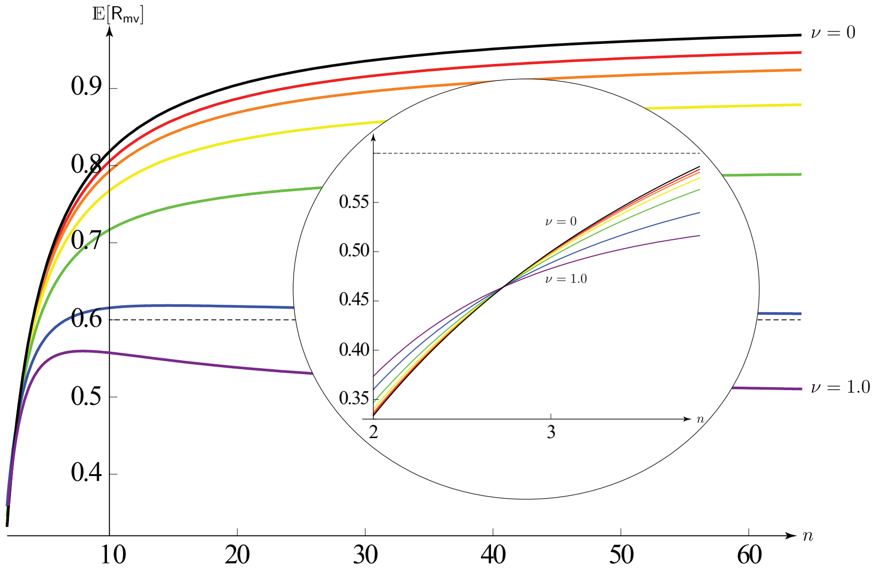


Figure 5: Revenue for $\nu \in \{.05, .1, .2, .4, .8, 1.0\}$ players, respectively (sorted in the colours of the rainbow from red to violet) and $\nu = 0$ in black for $n \in [2, 64]$.

The bidders behaving according to mean-variance preferences cause a revenue variance of

$$\begin{aligned} \mathbb{V}[\mathsf{R_{mv}}] &= n \int_0^1 \left(\beta_{\mathsf{mv}}(\theta) - \frac{\mathbb{E}[\mathsf{R_{mv}}]}{n}\right)^2 d\theta \\ &= \frac{n(n-1)^2}{(1+2n)^2}\left(\frac{1+2n}{(1+n)^2} + \nu\frac{7+n(21-4n(n-3))}{(1+n)^2(2+n)(1+3n)}\right. \\ &\qquad \left. + \nu^2\frac{74+n(151+8n(n-3)(n-1))}{(2+n)^2(3+2n)(2+3n)(1+4n)}\right). \end{aligned} \tag{36}$$

The rate of $\mathbb{V}[\mathsf{R_{mv}}]/\mathbb{V}[\mathsf{R_{rn}}]$ is given by

$$1 + \frac{(7+(7-2n)n)\nu}{(2+n)(1+3n)} + \frac{(1+n)^2(74+n(151+8(n-3)(n-1)n))\nu^2}{(2+n)^2(1+2n)(3+2n)(2+3n)(1+4n)}. \tag{37}$$

For $n = 2, 3, 4$, this ratio is greater than one and increasing in $\nu$. For $n \geq 5$ the variance ratio is decreasing in $\nu$ and below one. This implies the following corollary.

**Corollary 5.**

| | $n = 2$ | $n = 3, 4$ | $n \geq 5$ |
|---|---|---|---|
| $\mathbb{E}[R]$ | $\mathbb{E}[R_{mv}] > \mathbb{E}[R_{rn}]$ | $\mathbb{E}[R_{mv}] < \mathbb{E}[R_{rn}]$ | $\mathbb{E}[R_{mv}] < \mathbb{E}[R_{rn}]$ |
| $\mathbb{V}[R]$ | $\mathbb{V}[R_{mv}] > \mathbb{V}[R_{rn}]$ | $\mathbb{V}[R_{mv}] > \mathbb{V}[R_{rn}]$ | $\mathbb{V}[R_{mv}] < \mathbb{V}[R_{rn}]$ |

Therefore, for $n = 3, 4$, a seller with both types of preferences will prefer bidders maximising expected payoff. In all other cases, a variance-averse seller may prefer bidders with mean-variance preferences, where the exact ranking depends on the degree of the seller's variance aversion.

## 4.1 Optimal reserve prices

We now introduce an exogenous reserve price $p_r > 0$ into our revenue analysis. In the symmetric equilibrium, either one of two symmetric, variance-averse players will participate in the auction if their utility at bidding $p_r$ equals

$$u_i(\mu(b = p_r, \theta_i), \sigma^2(b = p_r, \theta_i)) = \mu(p_r, \theta_i) - \nu\sigma^2(p_r, \theta_i) = 0, \tag{38}$$

where

$$\mu(p_r, \theta_i) = \theta_i\theta_i - p_r, \text{ and } \sigma^2(p_r, \theta_i) = \theta_i(\theta_i(1-\theta_i))^2 + (1-\theta_i)(\theta_i\theta_i)^2. \tag{39}$$

The first participating type in the contest with reserve price $p_r$, $\theta_r$ who solves (38) is implicitly defined by

$$p_r = \theta_r^2 + (\theta_r - 1)\theta_r^3\nu. \tag{40}$$

The solution to this equation, $\theta_r^{-1}(p_r)$, gives this type as a function of the reserve price.[8] Only if a player's type $\theta_i$ is at least as high as $\theta_r^{-1}(p_r)$, she will participate in the auction. Her maximisation problem (15) gives the bidding function $\beta_r(\theta)$ as equivalent of (18) as solution to

$$\beta_r(\theta) = \int_{\theta_r^{-1}(p_r)}^{\theta} \vartheta(1 - \nu\vartheta + 2\nu\vartheta^2)\, d\vartheta + p_r. \tag{41}$$

The seller's expected revenue when setting reserve price $p_r$ is now

$$\mathbb{E}[\mathsf{R}_{\mathsf{mv}}(p_r)] = 2\int_{\theta_r^{-1}(p_r)}^{1} \beta_r(\vartheta)\, d\vartheta; \tag{42}$$

it is shown for various $\nu$ in figure 6. As evident from the figure, the revenue-maximising reserve

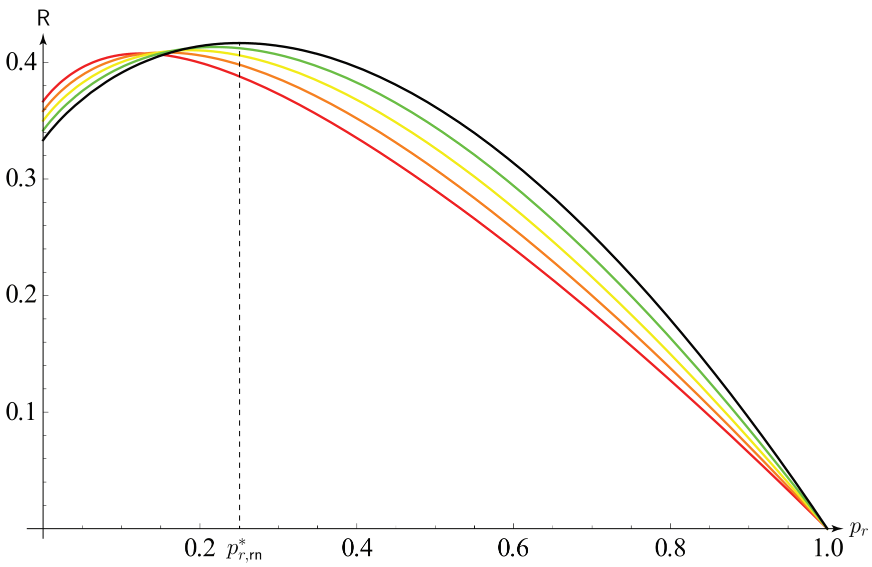


Figure 6: Seller's expected revenue (in the colours of the rainbow from red $\nu = 1.0$ in steps of $.25$ to green $\nu = 0.25$) as a function of the reserve price $p_r$ under variance-averse bidding compared to the case of risk-neutral bidding $\nu = 0$ (black).

price $p_r^*$ is a decreasing function of risk aversion $\nu$.

**Corollary 6.** *The expected revenue in the symmetric, two-player all-pay auction without a reserve price is increasing in the degree of variance aversion. With an optimally set reserve price this ranking is reversed, i.e., the highest expected revenue achievable by setting a reserve price is decreasing in the degree of variance aversion.*

[8] As the explicit form of $\theta_r^{-1}(p_r)$, (48), is rather unappealing it is relegated to the appendix (as are all following expressions which are based on it).

# 5 Concluding remarks

We present first results for the study of all-pay auctions if buyers or sellers are endowed with mean-variance preferences. Our results seem to be relevant since, from the point of view of stochastic dominance, many distributions can be fully characterised using just their mean and variance. We first fully characterise the symmetric equilibrium bidding functions of the all-pay auction with $n$ identical bidders when bidders maximise an additively separable function of their expected payoff and payoff variance. Our first proposition shows that consideration of mean-variance preferences suffices to derive the qualitative properties of the bidding function which Fibich, Gavious, and Sela (2006) obtain in their analysis of a similar environment but considering any von Neumann-Morgenstern utility function which entails risk aversion. In our model of mean-variance preferences, players choose a strategy that maximises the difference between their expected payoff and the payoff variance, which is weighted by a parameter, $\nu$, representing the players' degree of variance aversion. One advantage of this approach is that we obtain closed form solutions for the bidding functions with just a single parameter representing risk aversion. Thus, we can perform comparative statics. Furthermore, this functional form allows us to relax the standard assumption of identical preferences. We exemplary solve for the bidding functions in an all-pay auction with one expected payoff maximiser and one bidder with mean-variance preferences. In contrast to the symmetric equilibrium, we find that the mean-variance bidder of a given type always bids more than her risk neutral opponent of the same type. Although the analysis is only provided for the case of two bidders, the result would look similar if more general sets of $n_1$ risk neutral and $n_2$ mean-variance bidders were competing. Similarly, we conjecture that the qualitative findings from our benchmark case would carry over if the first bidder type was not risk neutral, but just less variance averse than her opponent.

Having obtained the (symmetric) equilibrium bidding function we then turn to the seller's perspective and consider effects of the number of bidders, their degree of variance aversion, and an optimally set reserve price. Corollary 4 shows that the influence of variance aversion on expected revenue depends on the number of players. In particular, we find that considering $n \geq 3$ reverses the ranking found for the two-player case. This finding suggests that under risk-aversion the generalisation from the two-player case to the general case may not always be as intuitive as it is often the case under risk neutrality. Furthermore, we find that the expected revenue is only increasing in the number of players as long as players are not too variance averse. If players exhibit a sufficiently high degree of variance aversion, then a seller would optimally want to limit the number of participants in the contest. One way of doing so could possibly be a multi stage sequential-elimination contest à la **?**—a mean-variance analysis of which is left for future research.

With the exception of the analysis of bidding behaviour of $n$ ex-ante identical players, much of our analysis focuses on the case of valuations that are i.i.d. draws from the Uniform distribution over $[0, 1]$. The resulting simplification of otherwise lengthy expressions and the possibility to analytically obtain solutions has caused us to make this assumption. However, qualitatively similar results can be obtained for other standard distributions.

# Appendix

## A. Second-order condition

The second-order condition is obtained by twice differentiating the objective (2) and supplying (9) for $\frac{\partial \beta^{-1}(b)}{\partial b}$ and $\frac{\partial^2 \beta^{-1}(b)}{\partial b^2} = -\frac{\beta''(\theta)}{\beta'(\theta)^3}$. The resulting expression simplifies to

$$F(\theta_i)^{5-2n} \frac{F(\theta_i)^2(\theta_i\nu - 2\theta_i\nu F(\theta_i) - 1)\,(1 - 2\theta_i\nu + 4\theta_i\nu F(\theta_i)^{n-1}) + \theta_i f(\theta_i) D}{(n-1)^2\theta_i^2 f(\theta_i)\,(F(\theta_i) - \theta_i\nu F(\theta_i) + 2\theta_i\nu F(\theta_i)^n)^3} \tag{43}$$

where

$$\begin{aligned} D = {} & (\theta_i\nu - 1)F(\theta_i)((n-2)(1-\theta_i\nu) + 2(n-3)\theta_i\nu F(\theta_i)) \\ & - 2\theta_i\nu F(\theta_i)^n((2n-3)(1-\theta_i\nu) + 4(n-2)\theta_i\nu F(\theta_i)), \end{aligned} \tag{44}$$

which, under the sufficient condition (7), is negative.

## B. Explicit forms used in revenue derivation

This appendix shows the explicit form of some of the equations kept for presentation reasons from the main text. Define

$$A = \sqrt[3]{27p_r\nu^2 - 2 - 72p_r\nu + \sqrt{4(12p_r\nu - 1)^3 + (2 + 9p_r(8-3\nu)\nu)^2}}, \tag{45}$$

$$B = \sqrt{3 - \frac{8}{\nu} + \frac{4\;2^{1/3}(12p_r\nu - 1)}{\nu A} - \frac{2\;2^{2/3}A}{\nu}}, \tag{46}$$

and

$$C = \sqrt{3}B - 3 - \sqrt{6}\sqrt{3 - \frac{8}{\nu} + \frac{2\;2^{1/3}(1 - 12p_r\nu)}{\nu A} + \frac{2^{2/3}A}{\nu} - \frac{3\sqrt{3}(\nu - 4)}{\nu B}}. \tag{47}$$

Then the inverse of (40) is given explicitly as

$$\theta_r^{-1}(p_r) = \frac{3 - \sqrt{3}B + \sqrt{6}\sqrt{3 - \frac{8}{\nu} + \frac{2\;2^{1/3}(1-12p_r\nu)}{\nu A} + \frac{2^{2/3}A}{\nu} - \frac{3\sqrt{3}(\nu-4)}{\nu B}}}{12}. \tag{48}$$

The explicit version of (41) is

$$\beta_r(\theta_i) = \frac{3p_r + \theta_i^2\,(3 - 2\nu\theta_i + 3\nu\theta_i^2) + \nu\left(\frac{C}{12}\right)^3}{6}. \tag{49}$$

Finally, the explicit version of the revenue of the all-pay auction with two symmetrically variance-averse players with parameter $\nu$ and reserve price $p_r$ (42) is

$$\mathsf{Rev}_{\mathsf{mv}}(p_r) = \frac{10 + 39p_r + \nu + 3p_rC + (4+\nu)\left(\frac{C}{12}\right)^3 - \left(\frac{C}{4}\right)^2}{30}. \tag{50}$$

In principle, the derivative of the last expression with respect to $p_r$ gives an explicit version of the revenue-optimal reserve price $p_r^*$. This derivation is not shown here for reasons of economy of space.

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