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Beyond the Horizon: Attainability of Pareto Optimality when the Indefinite Future Matters

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Abstract

In this paper we study the attainability of Pareto optimal allocations in infinite-dimensional exchange economies where agents have utility functions that value consumption at infinity. Such a model can be used to model economic settings where the indefinite future matters. The commodity space that we use is the space of all convergent sequences. We derive a necessary and sufficient condition for the attainability of the Pareto optimal allocations, which states that, for each pair of consumers, the ratio of the weights they place on utility in finite time periods should converge to the ratio of their utility weights at infinity. This, in turn, implies that efficiency can only be attained if consumers' valuations of time are very similar. We extend the model to include consumers with Rawlsian preferences and find that this does not change the attainability of Pareto optimal allocations.

Keywords: Infinite horizon exchange economy, Pareto optimality, non-discounting preferences
JEL classification: D51

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1 Introduction

The standard way to describe economic situations with an essentially open-ended time-line is by using an infinite-dimensional commodity-price duality. This is then usually followed by specifying objective functions for the agents, who are assumed to discount future utility. In applied work on, for example, environmental policy this can lead to debates about the appropriate discount rate of a social planner. Indeed, when the Stern report on climate change was published in the UK in 2006 many commentators[1] argued that discounting the utility of future generations at all is undesirable.[2]

Discounting future utility is a mathematical necessity to build coherent models in the vein of the seminal contribution by Bewley (1972). For the canonical model of an exchange economy with one perishable consumption good, the work of Brown and Lewis (1981) and Araujo (1985) shows that existence of equilibrium in such an economy can only be guaranteed when all consumers discount the future. In fact, Araujo (1985) shows that in an economy with a discounting and a non-discounting consumer, Pareto optimal allocations are not attainable. This leads to the paradoxical situation that in order to build a model in which the long-run matters, the agents should not care about the long-run.

In this paper we wish to build a model of an infinite-dimensional exchange economy where agents are allowed to value the indefinite future. In particular, we are interested in conditions under which the Pareto optimal allocations are attainable. This ensures the existence of a quasi-equilibrium, as has been shown by, for example, Mas–Colell and Zame (1991). In light of the Brown and Lewis (1981) and Araujo (1985) results this means that we cannot use the basic Bewley (1972) infinite horizon model. In order to allow agents to value the indefinite future, we need a mathematical structure that allows us to measure consumption at infinity. The simplest such structure is the space of all convergent sequences. Although this space is smaller than the conventionally used space of bounded sequences, from the point of view of the utility function it makes little difference: a conventional Bewley-type model allows, for example, for bounded fluctuations into the indefinite future, but a discounting utility function assigns little value to these time periods. In fact, one could argue that the model is in some sense richer than the Bewley model: Bewley (1972) approximates bounded consumption streams by sequences that are eventually zero; we, on the other hand, approximate by sequences which are eventually constant.[3] This amounts to a finite dimensional approximation to an infinite sequence, which can achieve any degree of precision. Such approximations do not, in general, exist for bounded sequences. This

[1] See, for example, the Wikipedia entry on the Stern report, `http://en.wikipedia.org/wiki/Stern_Review`.

[2] The Stern report itself used a small but positive discount rate.

[3] This constant could be thought of as a long-run average.

is important for applied (computational) work. Another advantage of using convergent sequences is that the mathematical apparatus that we use is familiar from finite-dimensional analysis: the implicit function theorem and the theorem of Lagrange. The only difference between their use in finite-dimensional analysis and their use here is in some additional technical conditions that need to be checked before they can be applied.

Our main result is to provide a necessary and sufficient condition for the attainability of the Pareto frontier for a fairly large class of utility functions. This class consists of those utility functions where consumers put a non-zero weight on consumption at infinity, but where the utility weights placed on consumption at individual points in time vanishes far into the future. Such preferences can be interpreted as ones where a consumer values long-run average consumption, but does not care about deviations from this average far into the future.

This necessary and sufficient condition, which we call *time value consistency*, requires that for each pair of consumers their ratio of utility weights on consumption far into the future is consistent with the ratio of the weights they put on consumption at infinity. So, there must be a strong agreement among all consumers about the value of time. This is very different from the analysis of finite-dimensional economies. There consumers can be truly atomistic and an economy is just a collection of individuals who live their separate lives, but are guided by the invisible hand to social optimum. In our model of an infinite-dimensional economy with consumption at infinity, there must be some form of agreement about the value of time for the market to work efficiently. In other words, Adam Smith's butcher, brewer, and baker do not have to be altruists, but they should agree to some extent about the value of time. The main result can be extended to economies where some consumers have Rawlsian preferences. It turns out that their presence does not affect the attainability of Pareto optimal allocations.

This paper fits in a renewed interest in the fundamentals of infinite-dimensional economies. The papers most closely related to our work are Araujo et al. (2011) and Chichilnisky (2012a,b). Araujo et al. (2011) allow for "wariness" in consumers' preferences. This means that consumers can be ambiguity-averse. Such consumers can be willing to act as creditors at infinity, which can only occur if there is an asset bubble of some kind. This implies that equilibrium may fail to exist, unless wary consumers are not subjected to a transversality condition.

Chichilnisky (2012a,b) extends the price space to the space of all boundedly additive sequences. That way she can allow for preferences that value infinity *and* ensure equilibrium existence. A disadvantage of this approach, however, is that non-summable price sequences are very difficult to interpret economically. For example, there is no algorithm that allows a social planner or a Walrasian auctioneer to construct such prices. In fact, the existence of such price functionals depends crucially on the axiom of choice, i.e. such prices can only exist by making an uncountable number of arbitrary choices. In our approach, the use of convergent sequences to represent commodity bundles avoids this problem, because the dual space of prices consists of summable sequences.

The paper is organized as follows. Section 2 provides an introduction to the main mathematical differences between the Bewley (1972) world and our approach. Section 3 describes the main ingredients of infinite-dimensional models of exchange economies, followed by a description of the Bewley (1972) set-up and the role of myopic preferences in that model. Readers who are interested mainly in the economic content of the paper can skip these sections. Section 4 describes an exchange economy with the space of convergent sequences as the commodity space. The notion of time value consistency is introduced and discussed here as well. We prove that time value consistency is a necessary and sufficient condition for attainability of the Pareto frontier when consumers have long-run concerns. Some examples and remarks, which give some insight in the economic interpretation of the main theorem can be found in Section 5. In Section 6 we discuss the case where several consumers have Rawlsian preferences and Section 7 provides some concluding remarks.

2 An Introduction to the Main Mathematical Structures

The simplest case of a Bewley (1972) economy is an exchange economy over an infinite time horizon with one consumption good. At the heart of the Bewley (1972) approach lies a specific commodity-price duality. A consumption bundle in this set-up is any bounded sequence and price functionals are represented by summable sequences. Unlike in finite-dimensional economies, one has to be careful in choosing an appropriate topology on the commodity space. This topology should be *consistent* with the commodity-price duality, which means that the price functionals are exactly the continuous functionals. In general, there are many topologies consistent with any given duality, which are, unlike in the finite-dimensional case, genuinely different.[4]

Bewley (1972) chooses the strongest topology (in the sense of making as many functions continuous as possible) that is consistent with the $\langle \ell^\infty, \ell^1 \rangle$ duality. This topology is called the *Mackey topology*. However, in this topology many potentially interesting utility functions are not continuous. Brown and Lewis (1981) show that the topology generated by myopic preferences (i.e. discounted utility) coincides with the Mackey topology, whereas Araujo (1985) shows that attainability of Pareto optimal allocations can not be guaranteed in topologies stronger than the Mackey topology. This result is a clear consequence of the choice of commodity-price duality. If, namely, the price functional is a summable sequence, then a necessary condition is that it converges to zero. An equilibrium price converges to

[4]In finite-dimensional Euclidean space, the topologies generated by $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, etc. are all the same. In infinitely many dimensions, this is not the case.

zero only if the good is not desirable to any consumer. Under standard monotonicity assumptions on preferences this can happen only if consumers discount the future sufficiently.

Our approach is to keep ℓ^1 as the price space (augmented with a price at infinity), but to restrict the commodity space to the set of all convergent sequences, c. The advantage of this is that the norm topology on c is consistent with the $\langle c, \ell^1 \rangle$ duality, which gives us many more continuous preferences, including non-myopic or even Rawlsian ones. The cost of this is that we lose a lot of the mathematical machinery that makes the Bewley (1972) approach work. Our analysis, therefore, differs substantially from the standard literature. In particular, in the Bewley world, closedness of the utility possibility set (i.e. attainability of the Pareto frontier) follows from continuity of utility functions and compactness of the set of attainable allocations. This latter property, in turn, follows from the Alaoglu theorem, which can be applied because ℓ^∞ is the (topological) dual of ℓ^1 (even though ℓ^1 is not the dual of ℓ^∞). So, in the Bewley (1972) set-up continuity in the Mackey topology is not always easily verified, but closedness almost comes for free; whereas in our set-up it is just the other way around. This is a well-known phenomenon in infinite-dimensional analysis: there is a trade-off between continuity and compactness that does not exist in the finite-dimensional case.

This means that we have to prove closedness of the utility possibility set directly, rather than using the Alaoglu theorem. We do this by appealing to infinite-dimensional versions of the implicit function theorem and the theorem of Lagrange. This has several advantages. First, it allows us to understand the social planner problem in great detail. Second, the methods that we use are very similar to methods one would use in finite-dimensional settings. The main difference is that the infinite-dimensional versions of these theorems require a bit more care than in the finite-dimensional case. First, derivatives are in the sense of Fréchet and, second, in order to show that a linear mapping is a bijection (the full-rank condition on the Jacobian in the finite-dimensional case) one needs to show both injectivity and surjectivity, whereas in the finite-dimensional case only one of these properties suffices.[5]

Finally, a word about the different view on the "long-run" that separates Bewley (1972) from our approach. In the traditional literature many proofs are based on a truncation argument: a certain property is proved for sequences truncated at time T and zero afterwards (these are called *terminating sequences*), after which one uses the fact that the space of terminating sequences is weak*-dense in ℓ^∞. In our approach we use the fact that sequences which are eventually constant are $\|\cdot\|_\infty$-dense in c. Note that the closure of the set of eventually constant sequences is exactly c, so that approximations of this type cannot be used in any larger space. In fact, ℓ^∞ is inseparable, so no such finite dimensional approximation scheme is possible in the whole of ℓ^∞.

[5]An injective linear map on $\mathbb{R}^n$ is automatically surjective and vice versa.

3 Infinite-Dimensional Economies, Pareto Optimality, and Equilibrium

Let X be a locally convex topological vector space representing a commodity space. A *decentralized market system* for X consists of a space P, interpreted as the *price system* and a mapping $(x,p) \mapsto \langle x,p\rangle$, for all $x \in X$ and $p \in P$, interpreted as the *value* of consumption bundle x at prices p, such that (i) $\langle \cdot,\cdot\rangle$ is a bilinear form and (ii) $\langle \cdot,\cdot\rangle$ puts X and P in a duality $\langle X,P\rangle$.[6]

We define an economy consisting of N consumers as a collection

$$\mathscr{E} = \left(\langle X,P\rangle, \tau, (X^i, \succeq^i, \omega^i)_{i=1}^N\right). \tag{1}$$

where $\langle X,P\rangle$ is a decentralized market system, τ is a topology that is consistent with the duality $\langle X,P\rangle$, and $X^i \subseteq X_+$, $\succeq^i$ and $\omega^i \in X$ are the consumption set, preference relation on X and initial endowments of consumer i, $i = 1,\ldots,N$, respectively. Here X_+ denotes the positive cone of the commodity space X, i.e. $x \leq y$ implies $y - x \in X_+$ for all $x,y \in X$.

Throughout this paper we will assume that the preference relation of each consumer satisfies the standard axioms of (i) *complete pre-order* ($\succeq^i$ is complete and transitive), (ii) *continuity* ($\succeq^i$ is continuous in τ), (iii) *strong monotonicity* (for all $x \in X^i$ and all $\alpha > 0$, there exists $x_0 \in X_+ \setminus \{0\}$, such that $x + \alpha x_0 \succ^i x$), and (iv) *convexity* (for every $x \in X^i$, the set $\{y \in X^i \mid y \succeq^i x\}$ is convex). Under these assumptions $\succeq^i$ can be represented by a continuous and monotonic utility function $u^i : X^i \to \mathbb{R}$. In the remainder we will work directly from such a utility function.

Let

$$Z = \left\{x = (x^1,\ldots,x^N) \in X^1 \times \cdots \times X^N \;\middle|\; \sum_{i=1}^N x^i \leq \omega\right\},$$

where $\omega = \sum_{i=1}^N \omega^i$ is the set of *attainable allocations*. It is assumed here that consumers can freely dispose of goods. The *utility possibility set* of the economy $\mathscr{E}$ is then given by

$$\begin{aligned} U &= \{u \in \mathbb{R}^N \mid u \leq u(x) = (u^1(x^1),\ldots,u^N(x^N)), \text{ for some } x \in Z\} \\ &= u(Z) - \mathbb{R}^N_+. \end{aligned}$$

The utility vector $u \in U$ is a *weak (Pareto) optimum* if there is no $\hat{u} \in U$ such that $\hat{u}^i \geq u^i$ for all i with strict inequality for at least one i. The set of Pareto optimal allocations is essentially the positive boundary of the utility possibility set, $\partial U \cap \mathbb{R}^N_+$.

[6] A *dual system* (see Schaefer, 1999) is a tuple $(X, P, \langle\cdot,\cdot\rangle)$, where (X,P) is a pair of vector spaces and $\langle\cdot,\cdot\rangle$ is a bilinear form on $X \times P$, which satisfies the properties: (i) if $\langle x_0, p\rangle = 0$, for all $p \in P$, then $x_0 = 0$, and (ii) if $\langle x, p_0\rangle = 0$, for all $x \in X$, then $p_0 = 0$.

Assuming that ω is in the interior of X_+, Mas–Colell and Zame (1991) show that closedness of U is sufficient for the existence of a quasi-equilibrium. Closedness of the utility possibility set also means that every Pareto optimum is attainable. Given that the rhetoric of market-based economics is firmly based on the Pareto optimality of Walrasian equilibria, the first test of an economic model based on decentralized markets is to investigate whether the Pareto frontier is attainable. So, closedness of U is of interest in its own right and it is the focus of the remainder of the paper.

In the approach of Bewley (1972), closedness of U follows immediately from the Alaoglu Theorem, via compactness of Z in the Mackey topology on $X = \ell^\infty$, $P = \ell^1$ and $\langle x, p \rangle = \sum_{t=1}^\infty x_t p_t$. However, Brown and Lewis (1981) show that the only preferences that are continuous in this topology are those that are strongly myopic: roughly, if the tail of a consumption sequence does not change the preference order, no matter what that tail is[7].

In addition, Araujo (1985) showed in a seminal paper that existence of equilibrium can only be ensured in topologies that are no stronger than the Mackey topology. Combining these two results implies that for an equilibrium to exist in an infinite-dimensional economy with commodity space ℓ^∞ and price space ℓ^1, all consumers must have strongly myopic preferences. The canonical example of such preferences is a utility function which is time-separable and exhibits discounting:

$$u(x) = \sum_{t=1}^{\infty} \delta^{t-1} v(x_t),$$

for some continuous, monotonic, and quasi-concave function $v : \mathbb{R}_+ \to \mathbb{R}$ and a discount rate $\delta \in (0, 1)$.

To see what problems can arise with non-myopic preferences, we consider an example from Araujo (1985). There are two agents with consumption sets $X^1 = X^2 = \ell^\infty_+$, initial endowments $\omega^1 = \omega^2 = (1, 1, \dots)$, and preferences

$$u^1(x^1) = \sum_{t \in \mathbb{N}} \left(\frac{1}{2}\right)^t x^1_t, \quad \text{and} \quad u^2(x^2) = \liminf(x^2),$$

respectively. So,

$$\begin{aligned} Z &= \{(x^1, x^2) \in \ell^\infty_+ \times \ell^\infty_+ \mid 0 \le x^1_t + x^2_t \le 2, \text{ all } t \in \mathbb{N}\}, \quad \text{and} \\ U &= \{(u^1, u^2) \in \mathbb{R}^2 \mid u^1 < 2, u^2 \le 2 \text{ or } u^1 \le 2, u^2 \le 0\}. \end{aligned}$$

As Araujo (1985) observes, U is not closed and u^2 is not Mackey continuous. Exactly the same example can be set up in the space of convergent sequences with the norm topology: after all, $\omega^1 = \omega^2$ is a convergent sequence. In this set-up, U is still not closed, despite the

[7] So, even if a bundle x has extremely high levels of consumption far into the future as compared to a bundle y, then a myopic consumer still prefers y to x if early consumption in y is high enough relative to x.

utility function $u^2(x^2) = \lim_{t\to\infty} x_t^2$ being norm continuous. This is due to the fact that we can give consumer 2 his highest possible utility by making consumer 1 only marginally worse off. This in turn is because consumer 2 cares only about the indefinite future and not about what happens in the short-run. We will show in Section 4 that in order for the utility possibility set to be closed, consumers must have closely aligned valuations of the indefinite future.

4 An Infinite-Dimensional Exchange Economy with Convergent Endowments

As we have seen, putting ℓ^1 in duality with ℓ^∞ implies shrinking the set of continuous functionals. Increasing the set of continuous functions can be achieved by strengthening the topology. Since the topology must be consistent with the chosen duality, that means extending the price space or restricting the commodity space. We choose to base our price space on ℓ^1 because the topological dual of ℓ^∞, ba, is a very complicated space and, because of Axiom of Choice issues, is very difficult to interpret economically. Rather than expanding the price space, we choose to shrink the commodity space to the set of convergent sequences c.

4.1 The Commodity-Price Duality, Preferences, and Endowments

The commodity space $X = c$ is a Banach space with norm

$$\|x\|_\infty = \sup_{t\in\mathbb{N}} |x_t|.$$

The price space that we associate with this commodity space is $P = \ell^1 \times \mathbb{R}$, and the bilinear form that puts X and P in duality is

$$\langle x, p\rangle = \sum_{t=1}^{\infty} p_t x_t + p_\infty x_\infty,$$

where $p_\infty \in \mathbb{R}$ and $x_\infty = \lim_{t\to\infty} x_t$.[8]

Our interest is in the study of preferences that reflect concerns about the indefinite future. In order to stay close to the Bewley world and the recent literature, in particular Araujo et al. (2011), we consider utility functions of the form

$$u^i(x^i) = \sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i \lim_{t\to\infty} v^i(x_t^i), \quad x^i \in X^i = c_+ \tag{2}$$

[8]Note that p_∞ is not a limit. It is the price of limit consumption.

defined on the positive cone of the space c. Here, for each i, $(\delta_t^i)_{t\in\mathbb{N}}$ is a strictly positive, summable sequence, $\zeta^i > 0$ is the weight that the consumer places on consumption at infinity (see Section 4.4 for some observations about the cases $\delta_t^i = 0$ for some t and $\zeta^i = 0$) and v^i is defined on an open set containing $[0,\infty)$, i.e. on $(-\varepsilon,\infty)$ for some $\varepsilon > 0$, and is twice continuously differentiable. We also assume that $v^i(0) = 0$ and that for $x \in [0,\infty)$, $(v^i)'(x) > 0$, and $(v^i)''(x) < 0$.

Preferences of the form (2) value both individual time periods and the indefinite future. One way to think about such preferences is to reinterpret limit consumption x_∞ as long-run *average* consumption. The parameter ζ^i measures the weight consumer i places on average consumption relative to deviations from the average at each individual point in time. These deviations are discounted over time.

The total endowment at time t is denoted by ω_t; we assume that this converges to a strictly positive limit ω_∞ as $t \to \infty$ (the case $\omega_\infty = 0$ is different in character, and considerably simpler; see Section 4.4). Also note that discounting in the utility function of the form (2) is not necessarily geometric. In fact, it may well be that the sequence $(\delta_t^i)_{t\in\mathbb{N}}$ is increasing for several t. The only requirement is that $\delta_t^i \to 0$ fast enough for the sum in (2) to be finite, i.e. that $\sum_{t=1}^\infty \delta_t^i < \infty$ for all i.

4.2 Time Value Consistency, Pareto Optimality and the Main Theorem

We begin with some terminology and notation about the utility possibility set U and some ways in which it can be decomposed. As defined, U contains non-positive vectors which, because of our normalization $v^i(0) = 0$, do not represent feasible allocations. Since we are more concerned with allocatable, i.e. non-negative, elements of U and of its boundary, we make the following definition.

Definition 1. The *positive part* of the utility possibility set is defined by $U^+ = U \cap \mathbb{R}_+^N$ or, more constructively,

$$U^+ = \{\, (u^i(x^i))_{i=1}^N \mid x_t^i \geq 0, x_t^i + \cdots + x_t^N \leq \omega_t \; (1 \leq i \leq N, t \in \mathbb{N}) \,\}.$$

Similarly, the *positive boundary* of U is defined by $\partial^+ U = (\partial U) \cap \mathbb{R}_+^N$.

Recall that the *pointwise* or *Minkowski* sum of sets of vectors in $\mathbb{R}^N$ is defined by $A + B = \{\, a + b \mid a \in A, b \in B \,\}$. Because of the time-separable nature of our utility functions, there are various ways of decomposing U^+ into Minkowski sums. The most important is, for some $T \in \mathbb{N}$,

$$
\begin{aligned}
U_{T-} &= \left\{ \left(\sum_{t=1}^{T} \delta_t^i v^i(x_t^i) \right)_{i=1}^{N} \middle| \, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, 1 \leq t \leq T) \right\} \\
U_{T+} &= \left\{ \left(\sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) \right)_{i=1}^{N} \middle| \, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t > T) \right\} \\
U^+ &= U_{T-} + U_{T+}.
\end{aligned}
\tag{3}
$$

Here we decompose U^+ into the utilities attained up to time period T — an essentially finite-dimensional object — plus the utilities attained from time $T+1$ onwards, including utility attained at ∞. Another occasionally useful decomposition is

$$
\begin{aligned}
U_{\mathrm{F}} &= \left\{ \left(\sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) \right)_{i=1}^{N} \middle| \, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t \in \mathbb{N}) \right\} \\
U_\infty &= \left\{ \left(\zeta^i v^i(x_\infty^i) \right)_{i=1}^{N} \middle| \, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t \in \mathbb{N}) \right\} \\
U^+ &= U_{\mathrm{F}} + U_\infty
\end{aligned}
\tag{4}
$$

which we can think of a decomposition of U^+ into the utilities attained over all finite time times, plus the utilities attained at ∞. Here U_∞ is essentially finite-dimensional.

At this point, it is helpful to give a concrete description of the closure of U^+ as an infinite Minkowski sum, and to mention an important strict convexity property.

Lemma 1. *For $t \in \mathbb{N}$, let*

$$U_t = \{(\delta_t^i v^i(x_t^i))_{i=1}^N \mid x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t\},$$

and let

$$\check{U} = \left\{ \left(\zeta^i v^i(\check{x}^i) \right)_{i=1}^{N} \middle| \, \check{x}^i \geq 0, \check{x}^1 + \cdots + \check{x}^N \leq \omega_\infty \right\}.$$

Then the closure of the positive part of the utility possibility set is given by

$$\bar{U}^+ = \left\{ \left(\sum_{t=1}^{\infty} y_t \right) + \check{y} \middle| \, y_t \in U_t \quad (t \in \mathbb{N}), \check{y} \in \check{U} \right\}.$$

If $y \in \partial^+ U$, then any supporting hyperplane for $\bar{U}$ through y has no other points of intersection with $\bar{U}$, i.e. y is an exposed point *of $\bar{U}$.*

The proof of this lemma is in Appendix A. In the finite-dimensional setting, the hyperplane property follows directly from strict convexity of the utility functions but, in the

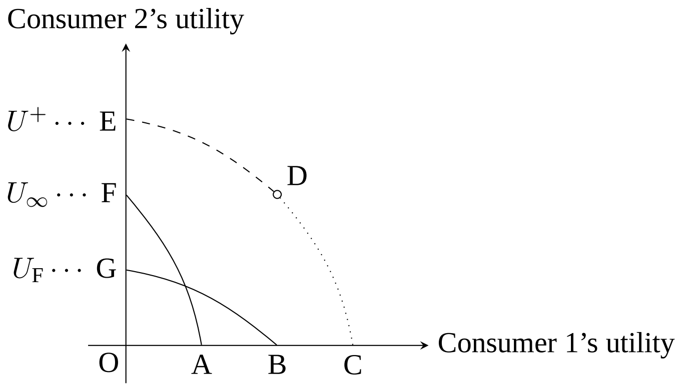

Figure 1: U^+, U_∞ and U_F in a two-consumer case.

infinite-dimensional setting, it is a little more delicate: even though $(v^i)''$ is bounded away from zero, $\delta_t^i (v^i)''$ becomes arbitrarily small for large enough t, making it more difficult to deduce strict convexity results about the closure.

To illustrate why U^+ might not be closed, consider the following example.

Example 1. Here we consider two consumers who value time in different ways. Consumer 1 values finite time periods more highly than the indefinite future; Consumer 2 has exactly the opposite view. For a concrete example, suppose ω_t is constant and that the two utility functions are

$$u^1(x^1) = \frac{2}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^1) + \frac{1}{3}v(x_\infty^1)$$
$$u^2(x^2) = \frac{1}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^2) + \frac{2}{3}v(x_\infty^2)$$

Then the possible utilities can be represented on a diagram as in Figure 1, where we are using the decomposition described in (4). The convex region OBGO represents U_F, the possible utilities summed over all finite time periods, while OAFO represents U_∞, the possible utilities at infinity. According to Lemma 1, the closure of the positive part of the utility possibility set is the sum of these two regions, shown as OCDEO. The positive boundary, $\partial^+ U$, is in two parts: CD is parallel to AF, while DE is parallel to BG. We can see from this diagram, without any calculation, that the utility possibility set is not closed. The simplest observation is that point D is not included: this point can be represented as the sum of an element of U_F (OBGO) and an element of U_∞ (OAFO) in only one way, namely as the sum of B and F. This represents an allocation where all endowments are given to Consumer 1 in all finite time periods (B), and all endowments are given to Consumer 2 at infinity (F); because allocations are convergent sequences, this is not possible and the utility possibility set is not closed. The crucial point is that we cannot treat infinity as just another time period: consumption at infinity is determined by consumption in the far, but finite, future.

More generally, any point on the open arc CD can be represented in only one way as the sum of an element of U_{F} (OBGO) and an element of U_∞ (OAFO): namely, the point B plus a point on the open arc AF. But point B represents the allocation of all endowments in all time periods to Consumer 1; in such a case, Consumer 2 has no utility in any time period, and hence no utility in the limit at infinity. The consumers' utilities at infinity thus lie on OA, not on AF. Points on the open arc CD thus do not represent allocatable utilities.

In fact, in an example of this type, we should expect the whole of the open arc from C through D to E to be missing from the utility possibility set, but this cannot so easily be seen from the diagram. We return to this point in Example 2 in Section 5.

In contrast, if both consumers placed the same relative weights on finite and infinite times, say

$$u^1(x^1) = \frac{1}{2}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x^1_t) + \frac{1}{2}v(x^1_\infty)$$
$$u^2(x^2) = \frac{1}{2}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x^2_t) + \frac{1}{2}v(x^2_\infty)$$

then the sets of utilities at finite time and at infinite time would be (because ω_t is constant) exactly the same closed, convex set; the sum of this set with itself would be the same set, scaled by a factor of 2, and therefore closed. $\triangleleft$

In the first part of this example, the problem is an inconsistency between the values placed by the consumers on the far, but finite, future, and the indefinite future. This leads us towards an important concept related to attainability of Pareto optimal allocations: *time value consistency*.

Definition 2. Let u^i and u^j be utility functions of the form (2). The preferences represented by u^i and u^j are *time value consistent* if

$$\frac{\delta^i_t}{\delta^j_t} \to \frac{\zeta^i}{\zeta^j}, \quad \text{as} \quad t \to \infty. \tag{5}$$

This condition holds if a pair of consumers value consumption in the far future consistently with consumption in the indefinite future. This condition is very strong: requiring the ratio of these sequences to be convergent means that the two consumers' time value weighting sequences have very similar decay rates.[9]

Note that in the case of geometric discounting, $\delta^i_t = (\delta^i)^t$, time value consistency is equivalent to $\delta^i = \delta^j$ and $\zeta^i = \zeta^j$, for all i and j. In addition, by rescaling the v^i, the same

[9]Note that since both $(\delta^i_t)_{t\in\mathbb{N}}$ and $(\delta^j_t)_{t\in\mathbb{N}}$ are summable sequences, they converge to zero.

consistent functions u^i can be represented with weights such that $\delta_t^i/\delta_t^j \to 1$ as $t \to \infty$ and $\zeta^i = \zeta^j$. Finally, if we have N consumers, then they are all time value consistent if and only if they are all consistent with some chosen one: for example, if $\delta_t^1/\delta_t^j \to \zeta^1/\zeta^j$ as $t \to \infty$ for all j, then $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ as $t \to \infty$ for all i and j.

It turns out that time value consistency is a necessary and sufficient condition for the attainability of Pareto efficient allocations. This is our main theorem:

Theorem. *Suppose there are N consumers with utility functions of the form (2) with $\zeta^i > 0$, $\delta_t^i > 0$, $t \in \mathbb{N}$, with $\sum_{t=1}^{\infty} \delta_t^i < \infty$, for each $i = 1, \dots, N$. Assume that the sequence of total endowments lies in the interior of the cone c_+, so $\omega_t > 0$ for all t and $\omega_\infty > 0$. Then the utility possibility set is closed if and only if for each i and j, the utility functions u^i and u^j are time value consistent.*

4.3 Proof of the main Theorem

Before we prove the Theorem, we introduce some notation and technical results needed in the proof. Let ι represent the constant sequence $(1)_{t\in\mathbb{N}}$. The constant sequence $(\xi)_{t\in\mathbb{N}}$ can thus be denoted $\xi\iota$. If $(x_t)_{t\in\mathbb{N}}$ is a convergent sequence, its limit will be denoted by x_∞. The norm on $\mathbb{R}^N$ with which we work is the ∞-norm

$$\|y\|_\infty = \max_{1\le i\le N} |y^i|.$$

Throughout the proof, we are working in a Banach space which we denote c^N. An element x of this space is defined by the real numbers x_t^i, where $1 \le i \le N$ and $t \in \mathbb{N}$, representing an allocation of x_t^i to consumer i at time t. We require x_t^i to converge to a limit as $t \to \infty$, and denote this limit by x_∞^i. The norm on this space is given by $\|x\|_\infty = \sup_{1\le i\le N, t\in\mathbb{N}} |x_t^i|$. There are natural projections of this space onto c and $\mathbb{R}^N$: for any given i, x^i will denote the convergent real sequence, $(x_t^i)_{t\in\mathbb{N}}$; for any given t, x_t will denote the vector in $\mathbb{R}^N$, $(x_t^i)_{i=1}^N$.

An element of the dual space $(c^N)^*$ can be represented as a sequence $(\mu_t^i)_{t\in\mathbb{N},1\le i\le N}$ and a vector $(\nu^i)_{i=1}^N$, where for each i, $\sum_{t=1}^{\infty} \mu_t^i$ converges, i.e. $(\mu_t^i)_{t\in\mathbb{N}} \in \ell^1$. The bilinear form expressing the duality is

$$\langle x, (\mu, \nu)\rangle = \sum_{i=1}^{N} \left[\sum_{t=1}^{\infty} \mu_t^i x_t^i + \nu^i \lim_{t\to\infty} x_t^i \right]. \tag{6}$$

In proving that the utility possibility set generated by utility functions of the form (2) is closed, our basic technical tool is the following result, which is proved in Appendix A.

Lemma 2. *The utility possibility set is closed if and only if for any allocation $x \in c^N$, with $u^j(x^j) = y^j > 0$ ($1 \le j \le N$) and for any i ($1 \le i \le N$), we can find an allocation* *which maximizes u^i subject to the constraints $u^j(x^j) = y^j$ ($1 \leq j \leq N, j \neq i$), $x^j \geq 0$ ($1 \leq j \leq N$) and $x^1_t + \cdots + x^N_t = \omega_t$ ($t \in \mathbb{N}$).*

The following result, also proved in Appendix A, essentially states that, in showing that the utility possibility set is closed, we can discard the first T time periods and work only with the tail of the economy.

Lemma 3. *Consider the positive part U^+ of the utility possibility set and, for $T \in \mathbb{N}$, the sets U_{T-} and U_{T+} described in* (3)*, so $U^+ = U_{T-} + U_{T+}$. Then U^+ is closed if and only if U_{T+} is closed; equivalently, U is closed.*

The proof of the Theorem proceeds in four stages: we show that (i) time value consistency is necessary, then (ii) that it is sufficient in some special cases, then (iii) that it is sufficient in a neighbourhood of these special cases, and finally (iv) that it is sufficient in general.

(i) Proof of necessity. Suppose the utility possibility set is closed and choose $T \in \mathbb{N}$ so that for all i,

$$\sum_{t=T+1}^{\infty} \delta^i_t v^i(\omega_t) < \zeta^i v^i(\omega_\infty/N). \tag{7}$$

By Lemma 3, the set

$$U_{T+} = \left\{ \left(\sum_{t=T+1}^{\infty} \delta^i_t v^i(x^i_t) + \zeta^i v^i(x^i_\infty) \right)_{i=1}^{N} \;\middle|\; x^i_t \geq 0, \sum_{i=1}^{N} x^i_t \leq \omega_t \right\}$$

is closed. For $1 \leq i \leq N$, let

$$y^i = \sum_{t=T+1}^{\infty} \delta^i_t v^i(\omega_t/N) + \zeta^i v^i(\omega_\infty/N).$$

Clearly, $y \in U_{T+}$. Now consider the maximization problem

$$\begin{aligned} &\max_x \sum_{t=T+1}^{\infty} \delta^1_t v^1(x^1_t) + \zeta^1 v^1(x^1_\infty) \\ &\text{s.t. } x^i_t \geq 0, \quad \sum_{i=1}^{N} x^i_t = \omega_t \quad (t > T), \\ &\sum_{t=T+1}^{\infty} \delta^i_t v^i(x^i_t) + \zeta^i v^i(x^i_\infty) = y^i \quad (2 \leq i \leq N), \end{aligned} \tag{8}$$

which, by Lemma 2, has a solution. Inequality (7) shows that any allocation with $x^i_\infty = 0$ for some $i \geq 2$ cannot meet these constraints, so we must have $x^i_\infty > 0$ for all $i \geq 2$. Moreover,

$$\sum_{t=T+1}^{\infty} \delta^1_t v^1(x^1_t) + \zeta^1 v^1(x^1_\infty) \geq y^1,$$

so (7) shows that $x^1_\infty > 0$. It follows that there exists T' such that $x^i_t > 0$ for all i with $1 \le i \le N$ and all $t > T'$. Now fix x_t for $t \le T'$ and consider the same maximization problem (8) as a function only of $\{x_t \mid t > T'\}$. Of course, we have the same solution; but, as $x^i_t > 0$ for $t > T'$ and $x^i_\infty > 0$, the solution as a function of $\{x_t \mid t > T'\}$ is in the interior of the cone $\{(z_t)_{t>T'} \mid z_t \ge 0\}$. The Lagrangian $L : c^N \times \mathbb{R}^{N-1} \times c^* \to \mathbb{R}$ associated with this maximization problem is given by (see Appendix C)

$$\begin{aligned} &L(x, \lambda^2, \dots, \lambda^N, \mu, \nu) = \\ &\quad u^1(x^1) - \sum_{i=2}^N \lambda^i (u^i(x^i) - y^i) - \sum_{t=1}^\infty \mu_t \left(\sum_{i=1}^N x^i_t - \omega_t \right) - \nu \left(\sum_{i=1}^N x^i_\infty - \omega_\infty \right). \end{aligned} \tag{9}$$

Note that the constraints can be written as $G(x) = (y^2, y^3, \dots, y^N, \omega)$, where $G : c^N \to \mathbb{R}^{N-1} \times c$ is defined by

$$G(x) = (u^2(x^2), u^3(x^3), \dots, u^N(x^N), x^1 + x^2 + \dots + x^N).$$

Differentiating equation (9) with respect to $(x_t)_{t>T'}$ (see Lemma 5 in Appendix B) gives the following first-order conditions, which must be satisfied at an interior maximum (see Appendix C; surjectivity of the derivative is easy to check):

$$\begin{aligned} \sum_{t=T'+1}^\infty \delta^1_t (v^1)'(x^1_t) h^1_t + \zeta^1 (v^1)'(x^1_\infty) h^1_\infty - \sum_{i=2}^N \left[\lambda^i \sum_{t=T'+1}^\infty \delta^1_t (v^i)'(x^i_t) h^i_t + \zeta^i (v^i)'(x^i_\infty) h^i_\infty \right] - \\ \sum_{t=T'+1}^\infty \mu_t \sum_{i=1}^N h^i_t - \nu \sum_{i=1}^N h^i_\infty = 0 \qquad (h \in c^N_{T'}). \end{aligned}$$

where $c^N_{T'}$ is the space of all sequences $(h^i_t)_{i=1,\dots,N,t>T'}$ which converge for all i as $t \to \infty$. Writing these in the same form as (6) we have

$$\sum_{i=1}^N \left[\sum_{t=T'+1}^\infty \left(\left\{ {}^{\;1}_{-\lambda^i} \right\} \delta^i_t (v^i)'(x^i_t) - \mu_t \right) h^i_t + \left(\left\{ {}^{\;1}_{-\lambda^i} \right\} \zeta^i (v^i)'(x^i_\infty) - \nu \right) h^i_\infty \right] = 0 \quad (h \in c^N_{T'}),$$

where $\left\{ {}^{\;1}_{-\lambda^i} \right\}$ is 1 if $i = 1$ or $-\lambda^i$ if $i > 1$. For this to be zero for all $h \in C^N_{T'}$, all the coefficients of the h^i_t must be zero. This gives the equations

$$\delta^1_t (v^1)'(x^1_t) - \mu_t = 0 \qquad (t > T') \tag{10}$$
$$-\lambda^i \delta^i_t (v^i)'(x^i_t) - \mu_t = 0 \qquad (2 \le i \le N) \tag{11}$$
$$\zeta^1 (v^1)'(x^1_\infty) - \nu = 0 \qquad (t > T') \tag{12}$$
$$-\lambda^i \zeta^i (v^i)'(x^i_\infty) - \nu = 0 \qquad (2 \le i \le N). \tag{13}$$

We can now eliminate μ_t from the first two equations and ν from the second two (note that this step is reversible; $\mu_t = \delta^1_t (v^1)'(x^1_t)$ defines a summable series because δ^1_t is summable and $(v^1)'(x^1_t)$ converges to a non-zero limit):

$$\delta^1_t(v^1)'(x^1_t) = -\lambda^i(v^i)'(x^i_t)\delta^i_t \qquad (t > T') \tag{14}$$

$$\zeta^1(v^1)'(x^1_\infty) = -\lambda^i\zeta^i(v^i)'(x^i_\infty) \qquad (2 \le i \le N). \tag{15}$$

Because $\zeta^i, \delta^i_t, (v^i)' > 0$, it follows from these equations that $\lambda^i < 0$ for all i. We can rearrange to give

$$\frac{\delta^1_t}{\delta^i_t} = -\lambda^i\frac{(v^i)'(x^i_t)}{(v^1)'(x^1_t)} \qquad (t > T')$$

$$\frac{\zeta^1}{\zeta^i} = -\lambda^i\frac{(v^i)'(x^i_\infty)}{(v^1)'(x^1_\infty)} \qquad (2 \le i \le N).$$

Letting $t \to \infty$ in the first equation and comparing with the second, we have $\delta^1_t/\delta^i_t \to \zeta^1/\zeta^i$ as $t \to \infty$. Taking two different values of i and dividing, we must have for all i, j:

$$\frac{\delta^i_t}{\delta^j_t} \to \frac{\zeta^i}{\zeta^j} \qquad (t \to \infty),$$

as claimed (this could also be established by maximizing u^j subject to the other u^i being fixed).

We also note at this point that any solution of the Lagrange equations (14) and (15) with $T' = 0$, i.e. for all $t \in \mathbb{N}$, leads to a global maximum of u^1, subject to the given constraints. To see this, suppose x, λ^i, μ and ν are a solution. Suppressing the dependency on λ, μ, ν, which are now fixed, if $x + h$ satisfies the constraints, then

$$u^1(x+h) = L(x+h) = L(x)+L'(x)h+\frac{1}{2}L''(x+\theta h)(h,h) = u^1(x)+\frac{1}{2}L''(x+\theta h)(h,h),$$

for some $\theta \in (0,1)$, because x satisfies the constraints and $L'(x) = 0$. It is therefore enough to show that $L''(x + \theta h)(h,h) \le 0$. This follows easily from Lemma 5 in Appendix B:

$$L''(x+\theta h)(h,h) = \sum_{t=1}^{\infty}\delta^1_t(v^1)''(x^1_t+\theta h^1_t)(h^1_t)^2 + \zeta^1(v^1)''(x^1_\infty+\theta h^1_\infty)(h^1_\infty)^2 - \sum_{i=2}^{N}\lambda^i\sum_{t=1}^{\infty}\delta^i_t(v^i)''(x^i_t+\theta h^i_t)(h^i_t)^2 + \zeta^i(v^i)''(x^i_\infty+\theta h^i_\infty)(h^i_\infty)^2, \tag{16}$$

which is negative because $(v^i)'' < 0$, $\lambda^i < 0$, $\delta^i_t > 0$ and $\zeta^i > 0$. ■

(ii) Proof of sufficiency: constant total allocations and equal weighting. We now consider the special case where ω is constant, say $\omega = \omega_0\iota$, and for each i and j, δ^i_t/δ^j_t is constant in t. By rescaling the v^i, we can assume that all the δ^i_t are equal, say $\delta^i_t = \delta_t$; in accordance with time value consistency, all the ζ^i must also be equal, say $\zeta^i = \zeta$. We shall show that the Lagrange equations derived in stage (i) have a unique solution in this case; it will then follow from Lemma 2 that the utility possibility set is closed. In fact, apart from the original constraint equations, we need only solve equation (14): equation (15) follows from that and the hypotheses that $\delta_t^i = \delta_t^j$ and $\zeta^i = \zeta^j$; equations (10)–(13) then follow from these, as remarked in stage (i). After cancelling δ_t, equation (14) reads

$$(v^1)'(x_t^1) = -\lambda^i (v^i)'(x_t^i). \tag{17}$$

Notice that this is independent of t: each $x_t \in \mathbb{R}^N$ satisfies the same system of equations. The same is true of the constraint $x_t^1 + \cdots + x_t^N = \omega_0$. We know from the previous stage that in any solution to these equations we have $\lambda^i < 0$ for all i, so $-\lambda^i(v^i)'(x_t^i)$ is a strictly decreasing function of x_t^i; similarly, $(v^1)'(x_t^1)$ is a strictly decreasing function of x_t^1. It follows from Lemma 6 in Appendix B that, for any fixed $(\lambda^i)_{i=1}^N$, these equations have at most one solution; because each x_t satisfies them, any solution to equation (17) must be constant in t.

We may therefore consider a reduced problem involving only constant sequences: maximize $u^1(\xi^1\iota)$ ($\xi^1 \in \mathbb{R}_+$) subject to $u^i(\xi^i\iota) = y^i$ ($\xi^i \in \mathbb{R}_+$) and $\xi^1 + \cdots + \xi^N = \omega_0$. If we let $\Delta = \sum_{t=1}^\infty \delta_t + \zeta$, then we wish to maximize $\Delta v^1(\xi^1)$ subject to $\Delta v^i(\xi^i) = y^i$. This is essentially trivial: because v^i is strictly increasing, the equation $\Delta v^i(\xi^i) = y^i$ uniquely determines ξ^i for $2 \le i \le N$; ξ^1 is then uniquely determined by $\xi^1 + \cdots + \xi^N = \omega_0$ ($0 \le \xi^1 \le \omega_0$ because the y^i can be allocated). Finally, we let $\lambda^i = -(v^1)'(\xi^1)/(v^i)'(\xi^i)$.

The constant sequences $\xi^i\iota$ now satisfy the constraints $u^i(\xi^i\iota) = y^i$ ($2 \le i \le N$), $\xi^i\iota \ge 0$, $\xi^1\iota + \cdots + \xi^N\iota = \omega_0\iota$ and the Lagrange equation (17); that is, we have a critical point of the Lagrangian which is allocatable and satisfies all constraints. As observed at the end of stage (i), this is a global maximum of u^1.

We chose to maximize u^1 for notational convenience; we could equally have maximized any other u^i. It now follows from Lemma 2 that the utility possibility set is closed. ■

(iii) Proof of sufficiency: near-constant total allocations and near-equal weighting. Suppose the consumers are time value consistent, so $\delta_t^i/\delta_t^1 \to \zeta^i/\zeta^1$ as $t \to \infty$. As before, by rescaling v^i, we can assume that $\delta_t^i/\delta_t^1 \to 1$ and that $\zeta^i = \zeta^1$.

We shall now perturb the solution from the previous result, to show that if ω is close to a constant sequence and each δ^i is close to a constant multiple of δ^1 then the utility possibility set is closed. More precisely, we shall show that, given ω_0 and δ^1, there exists $r > 0$ such that if for all t, $|\omega_t - \omega_0| < r$ and for all t and i, $|\delta_t^i/\delta_t^1 - 1| < r$, then the utility possibility set is closed.

For notational convenience, we shall write $\delta_t^1 = \delta_t$, $\delta_t^i = (1 + \varepsilon_t^i)\delta_t$ for $2 \le i \le N$ (so $\varepsilon_t^i \to 0$ as $t \to \infty$) and $\zeta^i = \zeta$ for all i. Equation (14) now has the form

$$(v^1)'(x_t^1) = -\lambda^i(1 + \varepsilon_t^i)(v^i)'(x_t^i) \qquad (t \in \mathbb{N}),$$

and equation (15) follows on letting $t \to \infty$. Given ω and ε^i ($2 \le i \le N$), we need to solve this for x and λ in combination with the original constraint equations

$$\sum_{t=1}^{\infty} \delta_t(1+\varepsilon_t^i)v^i(x_t^i) + \zeta v^i(x_\infty^i) = y^i \qquad (2 \le i \le N),$$

and

$$\sum_{i=1}^{N} x_t^i = \omega_t \qquad (t \in \mathbb{N}).$$

We know from the above that we can do this if ω is constant, say $\omega = \omega_0\iota$, and $\varepsilon^i = 0$ for all i; the solution is of the form $x^i = \xi^i\iota$, and each λ^i some negative real number. We now start from these solutions and use the Implicit Function Theorem in a Banach space context to show that for any sequence $(\omega_t)_{t\in\mathbb{N}}$ which is sufficiently close to being constant, and any sequences $(\varepsilon_t^i)_{t\in\mathbb{N}}$ which are sufficiently small, we can solve the Lagrange equations.

The Banach spaces are set up as follows:

$$G : c \times c_0^{N-1} \times \mathbb{R}^{N-1} \times c^N \to \mathbb{R}^{N-1} \times c \times c^{N-1},$$

where c_0^{N-1} is the space of all sequences in $\mathbb{R}^{N-1}$ converging to zero, and for each $(\omega, \varepsilon^2, \ldots, \varepsilon^N, \lambda^2, \ldots, \lambda^N, x) \in c \times c_0^{N-1} \times \mathbb{R}^{N-1} \times c^N$,

$$G(\omega, \varepsilon^2, \ldots, \varepsilon^N, \lambda^2, \ldots, \lambda^N, x) = \Big(\underbrace{(u^2(x^2), \ldots, u^N(x^N))}_{\in\mathbb{R}^{N-1}}, \underbrace{x^1 + \cdots + x^N - \omega}_{\in c}, \underbrace{((v^1)'(x_t^1) + \lambda^i(v^i)'(x_t^i))_{2\le i\le N, t\in\mathbb{N}}}_{\in c^{N-1}}\Big). \tag{18}$$

It follows from Lemma 4 in Appendix B that G is continuously differentiable. We wish to solve (given ω and ε, find λ and x) the equation

$$G(\omega, \varepsilon, \lambda, x) = (y^2, \ldots, y^N, (0)_{t\in\mathbb{N}}, (0)_{2\le i\le N, t\in\mathbb{N}}).$$

We know that we have a solution when $\omega = \omega_0\iota$ is a constant sequence, $\varepsilon^i = 0$ for all i and $y^2, \ldots, y^N$ are allocatable. According to the Implicit Function Theorem (Deimling, 1985, Theorem 15.2), there will be a ball in $c \times c_0^{N-1}$ centred around $(\omega_0\iota, 0)$ in which the problem has a unique solution, provided the partial derivative of G with respect to (λ, x) at the established solution defines an invertible mapping from $\mathbb{R}^{N-1} \times c^N$ to $\mathbb{R}^{N-1} \times c \times c^{N-1}$. The radius of this ball gives us the required $r > 0$. We calculate the derivative as follows:

$$\begin{aligned} G_{\lambda,x}(\omega, \varepsilon, \lambda, x)(\mu, h) = \bigg(\Big(\sum_{t=1}^{\infty} \delta_t(1+\varepsilon_t^i)(v^i)'(x_t^i)h_t + \zeta(v^i)'(x_\infty^i)h_\infty^i \Big)_{i=2}^{N}, \\ h^1 + h^2 + \cdots + h^N, \big((v^1)''(x_t^1)h_t^1 + (v^i)'(x_t^i)\mu^i + \lambda^i(v^i)''(x_t^i)h_t^i\big)_{2\le i\le N, t\in\mathbb{N}} \bigg). \end{aligned} \tag{19}$$

The essential structure of this operator from $\mathbb{R}^{N-1} \times c \times c^{N-1}$ to $\mathbb{R}^{N-1} \times c^N$ is

$$T(\mu^2, \ldots, \mu^N, h) = \left(\phi^2(h^2), \ldots, \phi^N(h^N), h^1 + \cdots + h^N, (M^1 h^1 + M^i h^i + \mu^i a)_{i=2}^N\right),$$

where $\phi^i \in c^*$ is a strictly positive functional, M^1 is an operator of multiplication by a negative sequence, bounded and bounded away from zero, M^i for $2 \leq i \leq N$ is (because $\lambda^i < 0$), an operator of multiplication by a positive sequence, bounded and bounded away from zero, and a is a fixed, positive element of c. We can explicitly calculate the inverse of T by solving the equations:

$$\phi^i(h^i) = k^i \qquad (k^i \in \mathbb{R}, 2 \leq i \leq N) \tag{20}$$

$$h^1 + \cdots + h^N = s \qquad (s \in c) \tag{21}$$

$$M^1 h^1 + M^i h^i + \mu^i a = b^i \qquad (b^i \in c, 2 \leq i \leq N). \tag{22}$$

We first find h^1. Because the multiplier sequences are bounded away from zero, the multiplication operators M^i are all invertible. We can therefore multiply (22) by $(M^i)^{-1}$, $2 \leq i \leq N$, and sum to give

$$((M^2)^{-1} + \cdots + (M^N)^{-1}) M^1 h^1 + (h^2 + \cdots + h^N) = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a).$$

Using (21), this becomes

$$((M^2)^{-1} + \cdots + (M^N)^{-1}) M^1 h^1 + (s - h^1) = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a),$$

or, with I representing the identity operator,

$$[((M^2)^{-1} + \cdots + (M^N)^{-1}) M^1 - I] h^1 = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a) - s.$$

Now, M^1 represents multiplication by a negative sequence and the other M^i multiplication by positive sequences, all bounded away from zero; it follows that $[((M^2)^{-1} + \cdots + (M^N)^{-1}) M^1 - I]$ represents multiplication by a negative sequence, bounded away from zero, and hence invertible. This gives us an explicit formula for h^1. Next, we find μ^i by applying $(M^i)^{-1}$ followed by ϕ^i to (22) and substituting from (20):

$$\phi^i((M^i)^{-1} M^1 h^1) + k^i + \mu^i \phi^i((M^i)^{-1} a) = \phi^i((M^i)^{-1} b^i)$$

This gives us an explicit formula for μ^i, provided $\phi^i((M^i)^{-1} a) \neq 0$, which holds because a is a strictly positive sequence, M^i is a strictly positive multiplier, and ϕ^i is a strictly positive functional. Finally, we can find all the remaining h^i by applying $(M^i)^{-1}$ to (22) and rearranging.

This shows that, if ω is sufficiently close to being constant and ε^i is sufficiently small then the Lagrange equations have a unique solution. As observed at the end of stage (i), this gives us a global maximum. We also need to check that the allocations in the solution are positive: this is true for sufficiently small $\omega - \omega_0 \iota$ and ε, because the unperturbed solution $\xi^1 \iota$ lies in the interior of the positive cone and the perturbed solution depends continuously on ω and ε.

We chose to maximize u^1 for notational convenience; we could equally have maximized any other u^i. It now follows from Lemma 2 that the utility possibility set is closed. ■

(iv) Proof of sufficiency: general case. From Lemma 1, we know that any point y on the positive boundary $\partial^+ U = \partial U \cap \mathbb{R}^N_+$ is an exposed point of $\bar{U}$: that is, any supporting hyperplane for $\bar{U}$ which passes through y does not intersect $\bar{U}$ at any other point. Consider arbitrary $\omega \in c_+ \setminus \partial c_+$ (i.e., such that $\omega_t > 0$ and $\omega_t \to \omega_\infty > 0$) and arbitrary utility functions of the form (2), satisfying the time value consistency condition (5), i.e. $\delta^i_t / \delta^1_t \to \zeta^i / \zeta^1$. As in the earlier stages, rescale the v^i so that $\delta^i_t / \delta^1_t \to 1$ and $\zeta^i = \zeta$ for all i. Choose $T \in \mathbb{N}$ such that for $2 \le i \le N$ and $t > T$ we have $|\delta^i_t / \delta^1_t - 1| < r$ and $|\omega_t - \omega_\infty| < r$, where r is the radius obtained in stage (iii). Consider a perturbed economy with total endowments and utility functions

$$\tilde{\omega}_t = \begin{cases} \omega_\infty & (t \le T) \\ \omega_t & (t > T), \end{cases}$$

$$\tilde{u}^i(x^i) = \sum_{t=1}^{T} \delta^1_t v^i(x^i_t) + \sum_{t=T+1}^{\infty} \delta^i_t v^i(x^i_t) + \zeta v^i(x^i_\infty).$$

In this economy, by the results of the previous stage, the utility possibility set $\tilde{U}$ is closed.

To establish the corresponding result for the unperturbed economy, we consider three different sets of partial utility allocations:

$$U_1 = \left\{ \Big(\sum_{t=1}^{T} \delta^1_t v^i(x^i_t) \Big)_{i=1}^{N} \;\Big|\; x^i_t \ge 0, \sum_{i=1}^{N} x^i_t \le \omega_\infty \ (1 \le t \le T, 1 \le i \le N) \right\}$$

$$U_2 = \left\{ \Big(\sum_{t=T+1}^{\infty} \delta^i_t v^i(x^i_t) + \zeta v^i(x^i_\infty) \Big)_{i=1}^{N} \;\Big|\; x^i_t \ge 0, \sum_{i=1}^{N} x^i_t \le \omega_t \ (t > T, 1 \le i \le N) \right\}$$

$$U_3 = \left\{ \Big(\sum_{t=1}^{T} \delta^i_t v^i(x^i_t) \Big)_{i=1}^{N} \;\Big|\; x^i_t \ge 0, \sum_{i=1}^{N} x^i_t \le \omega_t \ (1 \le t \le T, 1 \le i \le N) \right\}.$$

Note that $\tilde{U}^+ = U_1 + U_2$, and $U^+ = U_2 + U_3$. By Lemma 3,

$$U_1 + U_2 \text{ is closed} \iff U_2 \text{ is closed} \iff U_3 + U_2 \text{ is closed}.$$

The property of closedness of the utility possibility set is thus equivalent in the two economies; since it is closed in the perturbed economy, it is closed in the unperturbed economy. ■

4.4 Some Extensions and Variations

In this section we discuss some extensions and variations on the theorem that were hinted at earlier in the paper, to show that similar results hold in some more general contexts. Some of these extensions are needed in Section 6, where Rawlsian utility functions are introduced.

Endowments Equalling or Tending to Zero

The case where $\omega_t \to 0$ as $t \to \infty$ is somewhat different in character. Here, because $0 \leq x_t^i \leq \omega_t$, the set of possible allocations forms a closed, bounded and equiconvergent family of sequences, and is, hence, compact in the norm topology on c^N (see Dunford and Schwartz (1957), IV.13.9). Any norm continuous utility functions therefore lead to a closed utility possibility set. Utility functions of the form (2) reduce to a myopic form: we necessarily have $x_t^i \to 0$, so the values of ζ^i are irrelevant.

We can also consider the case where $\omega_t = 0$ for some values of t. Such time periods make no contribution to any utility function, so they can be removed to give an economy with the same utility possibility set and total endowments $\tilde{\omega}_t > 0$ for all t. Assuming this economy has infinitely many time periods, the time value consistency condition works much as above: if $\omega_\infty = 0$, no further condition is needed for the utility possibility set to be closed, and if $\omega_\infty > 0$ then we require $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ as $t \to \infty$ through those t for which $\omega_t \neq 0$. In the extreme case where $\omega_t > 0$ for only finitely many t, the economy is essentially finite-dimensional and closedness follows from the Heine-Borel Theorem.

Purely Myopic Preferences

Suppose some of the ζ^i are zero and some non-zero; for definiteness, say $\zeta^1 > 0$ and $\zeta^i = 0$ for some i. Then (15) cannot be satisfied, so the utility possibility set is not closed. If, however, we have $\zeta^i = 0$ for all i then (15) is trivially satisfied. The consistency condition $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ is needed precisely to ensure that (15) holds; in the event that $\zeta^i = 0$ for all i it can therefore be abandoned, with the rest of the proof of the main theorem showing that the utility possibility set is closed. This is reminiscent of Bewley (1972), where we have an equilibrium provided all consumers are myopic.

5 Some examples and further remarks

In this section we discuss some further examples to illustrate the concept of time value consistency and its implications for the attainability of Pareto optimal allocations. The following example revisits Example 1 in somewhat more detail.

Example 2. We begin by revisiting Example 1, in which total endowments are constant at ω_0 and the two consumers' utility functions are of the form

$$u^1(x^1) = \frac{2}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^1) + \frac{1}{3}v(x_\infty^1)$$
$$u^2(x^2) = \frac{1}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^2) + \frac{2}{3}v(x_\infty^2).$$

We saw earlier that, in this case, U is not closed. We now give a more detailed analysis, which exactly describes U. Let

$$U_0 = \left\{ \frac{2}{3}v(\xi^1), \frac{1}{3}v(\xi^2) \;\middle|\; \xi^1, \xi^2 \geq 0,\ \xi^1 + \xi^2 \leq \omega_0 \right\}.$$

The set of possible utilities at any finite time t is given by $2^{-t}U_0$, and the set U_{F} of possible utilities at all finite times, OBGO in Figure 1 is (see (4)):

$$U_{\mathrm{F}} = \sum_{t=1}^{\infty} 2^{-t}U_0 = U_0.$$

Suppose y_0 lies in the positive boundary of U_0. Then there is a supporting hyperplane of U_0 passing through y_0; that is, a linear functional ϕ whose maximum value over U_0 is attained at y_0 and, because of the strict concavity of v, at no other point of U_0. We can write $y_0 = \sum_{t=1}^{\infty} 2^{-t}y_0$, and this is the only way of decomposing y_0 as the sum over $t \in \mathbb{N}$ of elements of $2^{-t}U_0$: any other decomposition $y_0 = \sum_{t=1}^{\infty} 2^{-t}y_t$ would lead to the contradiction

$$\phi(y_0) = \sum_{t=1}^{\infty} 2^{-t}\phi(y_t) < \sum_{t=1}^{\infty} 2^{-t}\phi(y_0) = \phi(y_0).$$

Moreover, the strict monotonicity of v shows that there is only one allocation $(\xi, \omega_0 - \xi)$ such that $((2/3)v(\xi), (1/3)v(\omega_0 - \xi)) = y_0$. This shows that the only allocations leading to utilities on the arc BG in Figure 1 are constant. Now, any point on the open arc DE must be the sum of the point F with a point on the open arc BG; this corresponds to constant allocations in which Consumer 2 receives all the endowments at ∞. Consumer 2 thus receives all endowments in all time periods, leading to point E. Points on the open arc DE therefore cannot be allocated.

We can now see that U^+ consists of the figure OCEO, including the closed lines OC and EO but excluding the open arc CE. $\triangleleft$

The next example illustrates how the time value weights in (2) determine which parts of $\partial^+ U$ can or cannot be allocated.

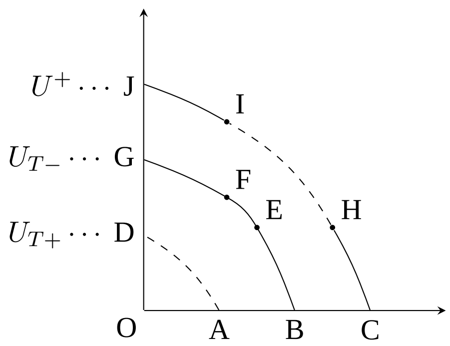

Figure 2: Utility possibility set for two inconsistent consumers

Example 3. In Example 2, all strictly positive points on the boundary of the utility possibility set are excluded. To illustrate the fact that this is not always the case, consider a slight modification of the example, in which we have

$$u^1(x^1) = v(x^1_1) + \frac{2}{3}\sum_{t=2}^{\infty}\frac{1}{2^t}v(x^1_t) + \frac{1}{3}v(x^1_\infty)$$
$$u^2(x^2) = v(x^2_1) + \frac{1}{3}\sum_{t=2}^{\infty}\frac{1}{2^t}v(x^2_t) + \frac{2}{3}v(x^2_\infty)$$

and total endowments

$$\omega_t = \begin{cases}\omega_1 & \text{if } t = 1\\ \omega_0 & \text{if } t > 1.\end{cases}$$

We can decompose the utility possibility set into two parts, $U = U_{T-} + U_{T+}$, with $T = 1$ (so U_{T-} represents utilities at time $t = 1$ and U_{T+} represents utilities at times $t \geq 2$, including $t = \infty$). Apart from a scale factor, the set U_{T+} is exactly as described in Example 2. The gradients of the boundary curve of U_{T-} as it crosses the vertical and horizontal axes are respectively $-v'(\omega_1)/v'(0)$ and $-v'(0)/v'(\omega_1)$; the corresponding gradients for U_{T+} are $-v'(\omega_0)/(2v'(0))$ and $-2v'(0)/v'(\omega_0)$. Under the additional hypothesis that $v'(x) \to 0$ as $x \to \infty$, we can choose ω_1 to be large enough that we have $v'(\omega_1) < v'(\omega_0)/2$. The boundary of U_{T-} therefore meets the vertical axis at a shallower angle than the boundary of U_{T+}, and the horizontal axis at a steeper angle. The possible utilities are illustrated in Figure 2. Here, the positive boundaries of U_{T-}, U_{T+} and U are the arcs BG, AD and CJ, respectively. Points E and F are those at which the positive boundary of U_{T-} is parallel to the positive boundary of U_{T+} at A and D, respectively. The arcs CH and IJ are parallel to BE and FG.

Now, CH is the sum of the point A, which is included in U_{T+}, with the arc BE, which is included in U_{T-}. All of these points are therefore included in U. Similarly, IJ is the sum of FG with D, and is included in U^+. However, the points on the open arc HI can only be represented as sums of points from the open arcs AD and EF; since EF is excluded, these points are excluded.

In summary, the positive boundary of the utility possibility set is the arc CJ; the closed arcs CH and IJ can be allocated, but the open arc HI cannot.

Note that this example is not "less serious" than Example 1, where the entire boundary is not allocatable. Even though it is true that certain Pareto efficient utility allocations can be attained through redistributing initial endowments, there is no guarantee that the other boundary (utility) allocations can be allocated consistently to any degree of precision. In addition, there is no guarantee that equilibria or "near-equilibria" exist. $\triangleleft$

The next example illustrates what happens in an economy with more than two consumers, where some consumers are time value consistent and some are not.

Example 4. With the help of the main theorem, we can extend Examples 1 and 2 to any number of consumers, with different utility functions, provided we retain constant total endowments and identical sequences $(\delta_t^i)_{t\in\mathbb{N}}$. Specifically, suppose we have N consumers with utility functions

$$u^i(x^i) = \sum_{t=1}^{\infty} \delta_t v^i(x_t^i) + \zeta^i v^i(x_\infty^i)$$

which have the same weights at each finite time, but possibly different weights at ∞, and that total endowments are constant with $\omega_t = \omega_0$. As in (4), we can decompose the utility possibility set into the sum of the utilities obtained at finite times, and utilities obtained at ∞: $U^+ = U_{\mathrm{F}} + U_\infty$. Any strictly positive point in the boundary of U^+ decomposes uniquely into the sum of two strictly positive points in the boundaries of U_{F} and U_∞ (in general, an extreme point of a Minkowski sum is the sum of uniquely-determined extreme points of the summands). In the same way as Example 2, we can introduce the set

$$U_0 = \left\{\, (v^i(\xi^i))_{i=1}^N \;\middle|\; \xi^i \geq 0 \; (1 \leq i \leq N),\; \xi^1 + \cdots + \xi^N \leq \omega_0 \,\right\}$$

so the set of possible utilities at any finite time t is given by $\delta_t U_0$, and the set of possible utilities summed over all finite times is

$$U_{\mathrm{F}} = \left(\sum_{t=1}^{\infty} \delta_t \right) U_0.$$

For exactly the same reasons as in Example 2, any strictly positive point y_0 in the boundary of U_F can be written as

$$\sum_{t=1}^{\infty} \delta_t y_0$$

and in no other way as a sum of elements of $\delta_t U_0$. There is a unique (Lemma 6 in Appendix B) allocation $(\xi^1, \ldots, \xi^N)$ such that $v^i(\xi^i) = y^i$, $\xi^i \geq 0$ and $\xi^1 + \cdots + \xi^N = \omega_0$, so any allocation leading to a strictly positive boundary point of U, and hence of U_1, must be constant.

If all these constant sequences are positive, then we have an interior maximum in the proof of the main Theorem, so the consistency condition holds (the ζ^i are all equal) and the utility possibility set is closed. In other words, the attainability of one strictly positive point implies the attainability of the whole positive boundary. Contrapositive, if the ζ^i are not all equal then any positive boundary point must be associated with an allocation in which at least one consumer has zero allocation in every time period. This can be thought of as a separate, smaller economy, excluding the consumers with no allocations; the above result can then be applied iteratively, to give a kind of simplicial decomposition of the positive boundary, in which some simplices are included and some excluded. In the most extreme case, the points representing the allocation of all endowments in all time periods to the same consumer are always attainable.

For example, suppose we have three consumers, where Consumers 1 and 2 are compatible with each other but Consumer 3 is not. Then the utility possibility set will look something like Figure 3. Because the three consumers are not compatible with each other, the open face on the positive boundary cannot be allocated. One dimension down, Consumers 1 and 2 are compatible, so the arc joining their axes can be allocated. Consumer 3, on the other hand, is not compatible with either of the other two consumers, so the two boundary arcs from Consumer 3's axis cannot be allocated. Finally, each consumer can be allocated all endowments at all time, so the three boundary points on the axes can be allocated. $\triangleleft$

Remark. Our final observation concerns the scale of the omitted part of the boundary, in the case that consumers have inconsistent utility functions. The message of stage (i) of the proof of the main theorem (section 4.3) is that if an allocation x such that $x^i_\infty > 0$ for all i gives rise to a strictly positive element of ∂U, then the time value consistency condition (5) is satisfied. Contrapositive, if the consistency condition is not satisfied then any strictly positive element of ∂U which is attained, is attained in such a way that at least one consumer's allocation tends to zero at ∞.

Suppose we have N consumers, who do not satisfy the consistency condition. We can decompose the economy into two parts: a finite-dimensional part representing time periods $1, \ldots, T$ and an infinite-dimensional part representing time periods $T+1, \ldots, \infty$, inclusive. This corresponds to the decomposition of the utility possibility set $U^+ = U_{T-} + U_{T+}$ described in (3).

In the infinite-dimensional part of the decomposition, we have truncated utility functions

$$u^i_{T+}(x^i) = \sum_{t=T+1}^{\infty} \delta^i_t v^i(x^i_t) + \zeta^i v^i(x^i_\infty).$$

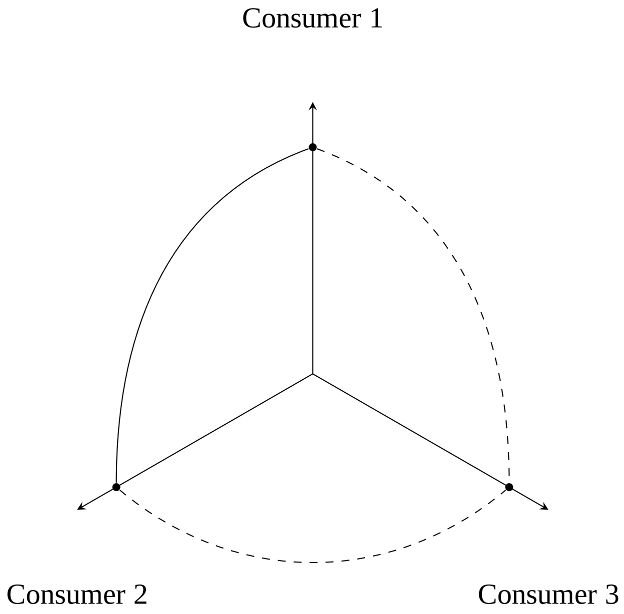

Figure 3: Utility possibility set, when consumers 1 and 2 are time value consistent, but consumer 3 is not.

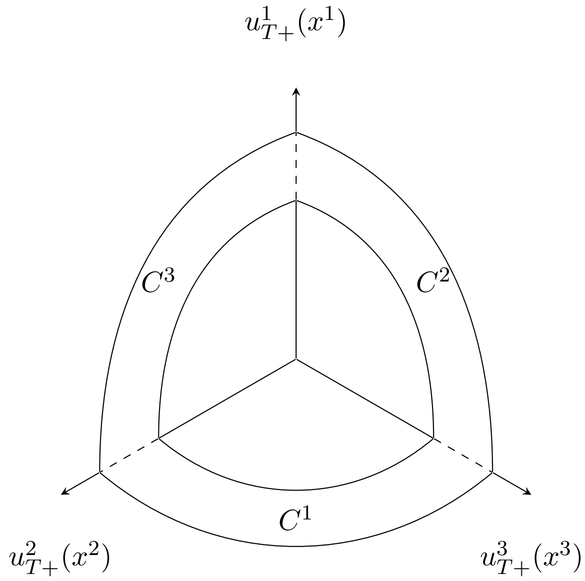

Figure 4: Utility possibility set beyond some large T, when time value consistency not satisfied.

As T increases, for all i, $\sum_{t=T+1}^{\infty} \delta_t^i v^i(\omega_t) \to 0$. In the infinite-dimensional part of the decomposition, for large T, the utility which can be assigned to any consumer in finite time periods is thus much smaller than can be assigned at ∞. Any strictly positive boundary point is attained by an assignment x in which $x^i_\infty = 0$ for at least one i, so $u^i_{T+}(x^i)$ must be small for at least one i. The attainable boundary points are therefore confined to a strip around the edge of the positive boundary of the utility possibility set; see Figure 4. In this diagram, the strip C^i represents small utility to consumer i, consistent with zero allocation at ∞. The boundary region enclosed by the strips cannot be allocated.

As $T \to \infty$, the closure of the utility possibility set in the infinite-dimensional part of the decomposition converges to the set of utilities which can be attained at ∞. Relative to this, the size of the potentially allocatable strip tends to zero so, as $T \to \infty$, in some sense the boundary utilities available at ∞ cannot (because of the lack of consistency) be allocated. Interpreting this in the original economy requires some care: essentially, it means that there is a “hole” on the boundary of the utility possibility set, corresponding to the utilities at ∞. We can see this in Example 1: the positive boundary of the set of utilities at ∞ is the arc AF, which corresponds to the unallocatable arc CD in the utility possibility set. As that example shows, this need not describe all of the unallocatable points on the boundary: the arc DE is missing for different reasons.

6 Rawlsian Preferences

In this section we present a corollary to the main theorem, in which we consider a mixture of time-separable utility functions of the form (2) and Rawlsian utility functions, in which utility depends only on the infimum of allocation. Precisely, we consider utility functions u^i where

$$u^i(x^i) = \begin{cases} \sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) & (1 \le i \le N) \\ v^i(\inf_{t \in \mathbb{N}} x_t^i) = \inf_{t \in \mathbb{N}} v^i(x_t^i) & (N+1 \le i \le M) \end{cases}$$

and δ_t^i, ζ^i and v^i satisfy the conditions stated after (2). As in the main theorem, we assume that the total allocations $(\omega_t)_{t \in \mathbb{N}}$ satisfy $\omega_t > 0$ and $\omega_\infty > 0$. Let $\omega_{\min} = \inf_{t \in \mathbb{N}} \omega_t$, so $\omega_{\min} > 0$.

Corollary. *For the economy described above, the utility possibility set is closed if and only if the time-separable utility functions satisfy the time value consistency condition* (5).

Given the main theorem, the proof that this condition is necessary is straightforward. We prove sufficiency in the same way as in the main theorem, showing that if we fix the utilities of all but one consumer, then we can maximize the utility of the remaining consumer; closedness then follows from Lemma 2. Because there are two forms of utility function, there are two maximization arguments: one for a time-separable utility function, one for a Rawlsian utility function.

Proof of necessity. Suppose the utility possibility set U is closed. Then its positive part U^+ is also closed and hence so is

$$U' = \{(y^1, \dots, Y^N) : y \in U^+ \text{ and } y^{N+1} = \cdots = y^M = 0\}.$$

But this is the positive part of the utility possibility set of the reduced economy consisting only of consumers $1, \dots, N$. This economy has only time-separable utility functions which, by the main theorem, must satisfy the time value consistency condition. ■

Proof of sufficiency: maximizing a time-separable utility function. Suppose the time-separable consumers satisfy the time value consistency condition. We wish to maximize one of the time-separable utilities, say $u^1(x^1)$, subject to the attainable constraints $u^i(x^i) = y^i$ for $2 \le i \le M$.

For $N+1 \le i \le M$ (the Rawlsian consumers), let $\xi^i = (v^i)^{-1}(y^i)$. Any allocation meeting the constraints $u^i(x^i) = y^i$ for $N+1 \le i \le M$ must satisfy $x_t^i \ge \xi^i$ for $N+1 \le i \le M$ and all $t \in \mathbb{N}$; it follows that $\xi^{N+1} + \cdots + \xi^M \le \omega_{\min}$, otherwise no feasible allocation would meet these constraints.

Now consider the reduced economy consisting of N consumers with utility functions $u^1, \dots, u^N$ and total endowments $\tilde{\omega}_t = \omega_t - \xi_{N+1} - \cdots - \xi_M$; note that $\tilde{\omega}_t \ge 0$ because $\xi^{N+1} + \cdots + \xi^M \le \omega_{\min} \le \omega_t$.

There is an allocation x in the original economy which satisfies the constraints $u^i(x^i) = y^i$ ($2 \leq i \leq M$); in this allocation, we have, for $N+1 \leq i \leq M$ and $t \in \mathbb{N}$, $x^i_t \geq \xi^i$ and hence $x^{N+1}_t + \cdots + x^M_t \leq \omega_t - \xi_{N+1} - \cdots - \xi_M = \tilde{\omega}_t$; it follows that $(x^1, \ldots, x^N)$ is a feasible allocation in the reduced economy, satisfying $u^i(x^i) = y^i$ for $2 \leq i \leq N$. Because the consumers in the reduced economy satisfy the time value consistency condition, it is possible to maximize $u^1(x^1)$ subject to the constraints $u^i(x^i) = y^i$ for $2 \leq i \leq N$.

If we now let $x^i_t = \xi^i$ for $N+1 \leq i \leq M$ and all $t \in \mathbb{N}$, then $(x^1, \ldots, x^M)$ is a feasible allocation in the original economy and satisfies the constraints $u^i(x^i) = y^i$ for $2 \leq i \leq M$. We shall now show that this allocation maximizes $u^1(x^1)$ subject to the given constraints.

Suppose w is any other feasible allocation such that $u^i(w^i) = y^i$ for $2 \leq i \leq M$. Then, for $N+1 \leq i \leq M$, $u^i(w^i) = y^i$ so $\inf_{t \in \mathbb{N}} w^i_t = \xi^i$. Define a new allocation z by

$$z^i_t = \begin{cases} w^1_t + (w^{N+1}_t - \xi^{N+1}) + \cdots + (w^M_t - \xi^M) & \text{if } i = 1 \\ w^i_t & \text{if } 2 \leq i \leq N \\ \xi^i & \text{if } N+1 \leq i \leq M. \end{cases}$$

This preserves the total allocation in each time period, so it is a feasible allocation, and it satisfies $u^i(z^i) = y^i$ for $2 \leq i \leq M$; we also have $z^1_t \geq w^1_t$ for all t, so $u^1(z^1) \geq u^1(w^1)$. Now, $z^1_t + \cdots + z^N_t = w^1_t + \cdots + w^M_t - \xi_{N+1} - \cdots - \xi^M \leq \omega_t - \xi_{N+1} - \cdots - \xi^M = \tilde{\omega}_t$, so $(z^1, \ldots, z^N)$ is a feasible allocation in the reduced economy, and for $2 \leq i \leq N$ we have $u^i(z^i) = u^i(w^i) = y^i$. By construction of x, $u^1(z^1) \leq u^1(x^1)$. We already know that $u^1(w^1) \leq u^1(z^1)$, so we have $u^1(w^1) \leq u^1(x^1)$, showing that x does indeed maximize $u^1(x^1)$ subject to the given constraints. ■

Proof of sufficiency: maximizing a Rawlsian utility function. Assuming again that time-separable consumers satisfy the time value consistency condition, we wish to maximize a Rawlsian utility, say $u^M(x^M)$, subject to the attainable constraints $u^i(x^i) = y^i$ for $1 \leq i \leq M-1$. Let S be the set of all non-negative real numbers ξ such that there is a feasible allocation x with $u^i(x^i) = y^i$ for $1 \leq i \leq N$ and $x^1_t + \cdots + x^N_t \leq \omega_t - \xi$ for all t.

For $N+1 \leq i \leq M-1$, let $\xi^i = (v^i)^{-1}(y^i)$. Because the constraints are attainable, there is a feasible allocation x such that $u^i(x^i) = y^i$ for $N+1 \leq i \leq M-1$. In this allocation, $\inf_{t \in \mathbb{N}} x^i_t = \xi^i$ for $N+1 \leq i \leq M-1$, so $x^1_t + \cdots + x^N_t + \xi^{N+1} + \ldots \xi^{M-1} \leq \omega_t$ for all t; this shows that $\xi_{N+1} + \cdots + \xi_{M-1} \in S$. Also, $\omega_{\min}$ is an upper bound for S, because if $\xi > \omega_{\min}$ then $\omega_t - \xi < 0$ for some t. We can therefore let $\Xi = \sup(S)$; necessarily, $\xi^{N+1} + \cdots + \xi^{M-1} \leq \Xi \leq \omega_{\min}$, and $[0, \Xi) \subseteq S$.

We shall now show that $\Xi \in S$. If $\Xi = 0$ then this is trivial, so assume not.

Let $\tilde{U}$ be the positive part of the utility possibility set in the reduced economy comprising N consumers with utility functions $u^1, \ldots, u^N$ and total endowments $\tilde{\omega}_t = \omega_t - \Xi$. This involves only weighted consumers satisfying the time value consistency condition, so $\tilde{U}$ is closed.

Suppose $\varepsilon > 0$. It follows from the differentiability hypotheses on the v^i that each u^i is uniformly continuous on any bounded subset of c_+^N. Hence, there exists $\delta > 0$ such that if x and z are feasible allocations (in the original economy) such that $\|x - z\|_\infty \leq \delta$ (i.e. $|x_t^i - z_t^i| \leq \delta$ for all i and t) then, for all i, $|u^i(x^i) - u^i(z^i)| < \varepsilon$. Without loss of generality, we may assume that $\delta < \Xi$, so $\Xi - \delta \in S$. By definition of S, there exists an allocation w such that $u^i(w^i) = y^i$ ($1 \leq i \leq N$) and $w_t^1 + \cdots + w_t^N \leq \omega_t - \Xi + \delta$. Let

$$z_t^i = \begin{cases} \frac{\omega_t - \Xi}{\omega_t - \Xi + \delta} w_t^i & (1 \leq i \leq N) \\ w_t^i & (N+1 \leq i \leq M) \end{cases}$$

Clearly, this is a feasible allocation and $z_t^1 + \cdots + z_t^N \leq \omega_t - \Xi$ for all t. Moreover, if $N + 1 \leq i \leq M$ then $|w_t^i - z_t^i| = 0$ and if $1 \leq i \leq N$ then

$$|w_t^i - z_t^i| = \left(1 - \frac{\omega_t - \Xi}{\omega_t - \Xi + \delta}\right) w_t^i = \frac{\delta}{\omega_t - \Xi + \delta} w_t^i \leq \delta$$

because $w_t^i \leq w_t^1 + \cdots + w_t^N \leq \omega_t - \Xi + \delta$. It follows that, for all i, $|u^i(w) - u^i(z)| < \varepsilon$, i.e. $|y^i - u^i(z)| < \varepsilon$.

Now, $(z^1, \ldots, z^N)$ is a feasible allocation in the reduced economy, so $(u^1(z^1), \ldots, u^N(z^N)) \in \tilde{U}$. We have therefore shown that for any $\varepsilon > 0$ there is an element of $\tilde{U}$ closer than ε to $(y^1, \ldots, y^N)$. Because $\tilde{U}$ is closed, we have $(y^1, \ldots, y^N) \in \tilde{U}$, so there is an allocation $(x^1, \ldots, x^N) \geq 0$ such that $u^i(x^i) = y^i$ ($1 \leq i \leq N$) and $x_t^1 + \cdots + x_t^N \leq \omega_t - \Xi$ for all t; that is, $\Xi \in S$.

Now let $\xi^M = \Xi - \xi^{N+1} - \cdots - \xi^{M-1}$ and for $t \in \mathbb{N}$ and $N + 1 \leq i \leq M$ let $x_t^i = \xi^i$, so $(x^1, \ldots, x^M)$ is a feasible allocation in the original economy and satisfies the constraints $u^i(x^i) = y^i$ ($1 \leq i \leq M - 1$). We claim that this allocation maximizes $u^M(x^M)$ subject to the constraints on $u^i(x^i)$. To see this, notice that any larger value of $u^M(x^M)$ would require a value of x^M with a larger infimum, so we would have $h > 0$ such that $x_t^M \geq \xi^M + h$ for all t. For $N+1 \leq i \leq M-1$, to meet the constraints $u^i(x^i) = y^i$ we must have $x_t^i \geq \xi^i$ for all t; we therefore have $x_t^{N+1} + \cdots + x_t^M \geq \xi^{N+1} + \cdots + \xi^M + h = \Xi + h$; correspondingly, $x_t^1 + \cdots + x_t^N \leq \omega_t - (\Xi + h)$ for all t. But $\Xi + h > \Xi = \sup(S)$, so $\Xi + h \notin S$ and by definition of S no such $x^1, \ldots, x^N$ can satisfy $u^i(x^i) = y^i$. This shows that no allocation with larger $u^M(x^M)$ can meet all the constraints, so we have indeed maximized $u^M(x^M)$. ■

Note that the reduced economies used in the maximization arguments could specialize into various non-generic forms: specifically, we could have $\tilde{\omega}_\infty = 0$, or $\tilde{\omega}_t = 0$ for some t. As discussed in Section 4.4, this does not cause any difficulties.

If all consumers are Rawlsian, we can easily adapt the Rawlsian maximization argument to show that the utility possibility set is closed: replace the construction of Ξ with the definition $\Xi = \omega_{\min}$, define $\xi^1, \ldots, \xi^N$ in exactly the same way and allocate $x_t^i = \xi^i$ for all i and t. Any larger value of $u^1(x^1)$ would lead to $x_t^1 + \cdots + x_t^N > \omega_t$ for some t.

7 Concluding Remarks

In this paper we have built a model of an infinite-dimensional exchange economy where consumers care about the indefinite future. We restrict attention to consumption bundles that are convergent. These can be interpreted as bundles which consist of a long-run average component and, for each individual period of time, a deviation from that average. The novelty of this paper is that this long-term average consumption, or "consumption at infinity", needs to be priced. Since limit consumption depends on the tail of the consumption sequence, this "price at infinity" is related to the prices at finite time periods. We find that closedness of the utility possibility set (a sufficient condition for the existence of quasi-equilibrium) can be guaranteed if and only if the preferences of all consumers are time value consistent. This implies that consumers' (utility) valuation of the indefinite future should be closely aligned, which, in turn, means that a completely atomistic view of decentralized market economies can not be combined with claims that such market interactions necessarily lead to efficient allocations.

From a mathematical point of view, the paper shows that infinite-dimensional economic models can be analyzed using the infinite-dimensional versions of techniques that are well-known to economists schooled in finite-dimensional analysis; in particular the implicit function theorem and the theorem of Lagrange. The advantage of using this toolbox as opposed to the more abstract and indirect route that is usually taken (via Alaoglu's theorem) is that the model presented here opens up the possibility of developing a computational variant that can be used in applied economic analysis.

In addition, the model presented here may open up an avenue for alternative general equilibrium approaches, not requiring myopic preferences, of branches of economics that are naturally formulated in the language of the indefinite future. We think, in particular, about possible applications in environmental economics, the theory of economic growth, and financial economics.

Appendix

A Proofs of Lemmas

Lemma 1. *For $t \in \mathbb{N}$, let*

$$U_t = \{(\delta_t^i v^i(x_t^i))_{i=1}^N \mid x_t^i \geq 0, \sum_{i=1}^N x_t^i \leq \omega_t\},$$

and let

$$\breve{U} = \left\{ \left(\zeta^i v^i(\breve{x}^i)\right)_{i=1}^N \,\middle|\, \breve{x}^i \geq 0, \breve{x}^1 + \cdots + \breve{x}^N \leq \omega_\infty \right\}.$$

Then the closure of the positive part of the utility possibility set is given by

$$\bar{U}^{+} = \left\{ \left(\sum_{t=1}^{\infty} y_t \right) + \check{y} \;\middle|\; y_t \in U_t \quad (t \in \mathbb{N}), \check{y} \in \check{U} \right\}.$$

If $y \in \partial^{+} U$, then any supporting hyperplane for $\bar{U}$ through y has no other points of intersection with $\bar{U}$, i.e. y is an exposed point *of $\bar{U}$.*

Proof of Lemma 1. Because $(v^i)'' < 0$ and $\{x_t \mid x_t^i \geq 0, x_t^1 + \cdots + x_t^N \leq \omega_t\}$ is compact, each U_t has the property that a supporting hyperplane intersecting U_t at a strictly positive point intersects U_t at no other point.

Let $\sigma_t = \sup\{\|y\| \mid y \in U_t\}$; because the δ_t are summable and the ω_t are bounded, the σ_t are summable. Let

$$U' = \left\{ \left(\sum_{t=1}^{\infty} y_t \right) + \check{y} \;\middle|\; y_t \in U_t \quad (t \in \mathbb{N}),\ \check{y} \in \check{U} \right\}$$

(these series all converge because the σ_t are summable). Any element of U^{+} certainly lies in U'; however, elements of U' do not obviously lie in U^{+}: roughly speaking, because the associated sequence $(y_t)_{t\in\mathbb{N}}$ might not converge to the associated $\check{y}$. We shall show that, in fact, U' is the closure of U^{+}. We begin by showing that U' is compact. Consider a sequence

$$\left(\left(\sum_{t=1}^{\infty} y_{t,n} \right) + \check{y}_n \right)_{n\in\mathbb{N}}$$

in U'. By a Cantor diagonal argument, we can extract subsequences such that as $k \to \infty$, $y_{t,n_k} \to y_t$ for all $t \in \mathbb{N}$ and $\check{y}_{n_k} \to \check{y}$. Since $y_{t,n_k} \in U_t$ and U_t is compact, $y_t \in U_t$; similarly, $\check{y} \in \check{U}$. We also have for all n and t, that $\|y_{t,n_k}\| \leq \sigma_t$ and $\sum_{t=1}^{\infty} \sigma_t < \infty$. It now follows from the Dominated Convergence Theorem (often known as Tannery's Theorem in the case of infinite sums, rather than more general integrals), that

$$\left(\sum_{t=1}^{\infty} y_{t,n_k} \right) + \check{y}_{n_k} \to \left(\sum_{t=1}^{\infty} y_t \right) + \check{y} \in U'$$

as $k \to \infty$, showing that U' is compact.

We now show that U^{+} is dense in U'. Fix some $z \in U'$, say

$$z = \left(\sum_{t=1}^{\infty} z_t \right) + \check{z}.$$

Given $\varepsilon > 0$, choose T such that $\sum_{t=T+1}^{\infty} \sigma_t < \varepsilon/2$. Now define $y_t = z_t$ for $1 \leq t \leq T$ and $\check{y} = \check{z}$. Choose any allocation at ∞ giving rise to utility $\check{y}$ and any allocations at times t with

$t > T$ which do not exceed the total endowments and converge to the chosen allocation at ∞; these will give rise to utilities y_t for $t > T$ such that $y_\infty = \check{y}$, so we have

$$y = \left(\sum_{t=1}^{\infty} y_t\right) + y_\infty \in U^+.$$

Now,

$$\|y - z\| \leq \sum_{t=T+1}^{\infty} \|y_t - z_t\| \leq \sum_{t=T+1}^{\infty} 2\sigma_t < \varepsilon.$$

This shows that U^+ is dense in U'; since U' is closed, U' is the closure of U^+.

Suppose z lies in the positive boundary of U'. Then, because U' is compact and convex, there is a supporting functional ϕ such that $\phi(y) \leq \phi(z)$ for all $y \in \bar{U}$. The problem of maximizing $\phi(y)$ over U' has a unique solution, namely $z = (\sum_{t=1}^{\infty} z_t) + \check{z}$ where z_t ($t \in \mathbb{N}$) is the unique point of U_t at which ϕ attains its maximum over U_t, and $\check{z}$ is the corresponding point for $\check{U}$. There is thus no other point $y \in U'$ for which $\phi(y) = \phi(z)$, so z is an exposed point of $U' = \bar{U}$. ■

Lemma 2. *The utility possibility set is closed if and only if for any allocation $x \in c^N$, with $u^j(x^j) = y^j > 0$ ($1 \leq j \leq N$) and for any i ($1 \leq i \leq N$), we can find an allocation which maximizes u^i subject to the constraints $u^j(x^j) = y^j$ ($1 \leq j \leq N, j \neq i$), $x^j \geq 0$ ($1 \leq j \leq N$) and $x_t^1 + \cdots + x_t^N = \omega_t$ ($t \in \mathbb{N}$).*

The proof of Lemma 2 depends on the following result about convex sets. The crucial property (*) means, essentially, that any line parallel to a coordinate axis intersects the set K in a closed line segment.

Lemma. *Suppose $K \subseteq \mathbb{R}_+^N$ is a non-empty and comprehensive set (i.e., if $y \in K$ and $0 \leq z \leq y$, then $z \in K$). Then K is closed if and only if:*

(*) *for each $y \in K$ and $1 \leq i \leq N$, $K \cap \left\{ z \in \mathbb{R}^N \;\middle|\; z^j = y^j \ (j \neq i) \right\}$ is closed.*

Proof. One direction is trivial: if K is closed then its intersection with any closed set, in particular any line, is closed. Suppose, then, that (*) holds and that y lies in the closure of K; we need to show that $y \in K$. This is trivial if $y = 0$, so assume $y \neq 0$.

Suppose $z \in \mathbb{R}_+^N$ is such that

$$z^i \begin{cases} < y^i & \text{if } y^i > 0 \\ = 0 & \text{if } y^i = 0. \end{cases}$$

Let $r = \min_{y^i \neq 0} y^i - z^i$, so for each i we have either $z^i = y^i = 0$ or $z^i \leq y^i - r$. Now, since y lies in the closure of K, we can choose $w \in K$ such that $\|y - w\|_\infty < r$, i.e. $|y^i - w^i| < r$ for all i. We can assume that $w^i = 0$ whenever $y^i = 0$ (this will make w smaller, so still in K, and will if anything make $\|y-w\|_\infty$ smaller). The inequality $|y^i - w^i| < r$ can be rewritten as $y^i - r < w^i < y^i + r$, which leads to $z^i < w^i$ for all i such that $y^i \neq 0$ and $z^i = w^i = 0$ for all i such that $y^i = 0$. We now have $0 \leq z \leq w \in K$, so $z \in K$.

For simplicity, we initially consider the case where $y^i > 0$ for all i, so if $0 < h^i < y^i$ for all i then $y - h = (y^1 - h^1, y^2 - h^2, \dots, y^N - h^N) \in K$. Since this lies in K for all $h^1 \in (0, y^1)$, it follows from (*) with $i = 1$ that $(y^1, y^2 - h^2, y^3 - h^3, \dots, y^N - h^N) \in K$. Since this lies in K for all $h^2 \in (0, y^2)$, it follows from (*) with $i = 2$ that $(y^1, y^2, y^3 - h^3, \dots, y^N - h^N) \in K$. Continuing in this way, we see that $y \in K$. For a point y with $y^i = 0$ for some i, choose h such that $h^i = 0$ if $y^i = 0$ and $0 < h^i < y^i$ if $y^i > 0$, and argue in the same way for each i such that $y^i \neq 0$. ■

Proof of Lemma 2. The utility possibility set is, by definition, $U^+ - \mathbb{R}^N_+$, where

$$U^+ = \{(u^1(x^1), \dots, u^N(x^N)) : x^i_t \geq 0, \sum_{i=1}^{N} x^i_t \leq \omega_t\}.$$

It is clear from this that closedness of the utility possibility set and of U^+ are equivalent.

The set U^+ has, by continuity and monotonicity of the utility functions and the normalization $v^i(0) = 0$, the property that if $y \in U^+$ and $0 \leq z \leq y$ then $z \in U^+$. For any point $y \in U^+$, the line through y parallel to the ith coordinate axis intersects U^+ in a line segment. One end of this has the ith coordinate equal to zero; this lies in U^+ because it is less than or equal to y. The intersection is therefore closed if and only if the other end point lies in U^+. This corresponds to maximizing $u^i(x^i)$ subject to the constraints $u^j(x^j) = y^j$ ($1 \leq j \leq N, j \neq i$), $x^j_t \geq 0$ ($1 \leq j \leq N$, $t \in \mathbb{N}$) and $x^1_t + \cdots + x^N_t \leq \omega_t$ ($t \in \mathbb{N}$). By strict monotonicity, a maximum cannot occur if $x^1_t + \cdots + x^N_t < \omega_t$ for some t, so we can replace the constraint $x^1_t + \cdots + x^N_t \leq \omega_t$ with $x^1_t + \cdots + x^N_t = \omega_t$, as claimed. Closedness now follows from the preceding lemma.

We need only consider $y^j > 0$, because $y^j = 0$ is exactly equivalent to an allocation of 0 to consumer j in all time periods; we can therefore consider the lower-dimensional problem involving only those consumers for which $y^j > 0$ and then, where $y^j = 0$, assign $x^j_t = 0$ for all t. The resulting set of utility values is closed if and only if the set of utility values in the lower-dimensional problem is closed. ■

Lemma 3. *Consider the positive part U^+ of the utility possibility set and, for $T \in \mathbb{N}$, the sets U_{T-} and U_{T+} described in* (3)*, so $U^+ = U_{T-} + U_{T+}$. Then U^+ is closed if and only if U_{T+} is closed; equivalently, U is closed.*

Proof of Lemma 3. Note first that U_{T-} is compact, because it is the image of a compact set under a continuous mapping. If U_{T+} is closed, then (Heine-Borel) it is compact, and the sum of two compact sets is easily seen to be compact, and hence closed.

Now suppose that U^+ is closed, and hence compact, and for a contradiction that U_{T+} is not closed. Then there is a point y_0 in the positive boundary of U_{T+} which does not lie in U_{T+} itself. By Lemma 1, y_0 is an exposed point of the closure $\bar{U}_{T+}$ of U_{T+}, so there is functional ϕ such that $\phi(y) \leq \phi(y_0)$ for all $y \in \bar{U}_{T+}$ and $\phi(y) < \phi(y_0)$ for all $y \in \bar{U}_{T+}$ with $y \neq y_0$; in particular, $\phi(y) < \phi(y_0)$ for all $y \in U_{T+}$.

Choose $z_0 \in U_{T-}$ such that $\phi(z_0) = \max_{z \in U_{T-}} \phi(z)$ (such z_0 exists because U_{T-} is compact), and note that for any $y \in U^+$ we have $y = \hat{y} + \check{y}$ for some $\hat{y} \in U_{T-}$, $\check{y} \in U_{T+}$, so

$$\phi(y) = \phi(\hat{y}) + \phi(\check{y}) < \phi(z_0) + \phi(y_0) \tag{23}$$

Because $y_0 \in \bar{U}_{T+}$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in U_{T+} converging to y_0. The sequence $(z_0 + y_n)_{n\in\mathbb{N}}$ lies in $U^+ = U_{T-} + U_{T+}$ and converges to $z_0 + y_0$ so, because U^+ is closed, we have $z_0 + y_0 \in U^+$. But $\phi(z_0 + y_0) = \phi(z_0) + \phi(y_0)$, contradicting (23). ■

B Some Supporting Technical Material

For a map between Banach spaces, there are various non-equivalent ideas of differentiability. We need only the idea of differentiability in the sense of Fréchet: briefly, if A is an open subset of a Banach space X, then $F : A \to Y$ is differentiable at $x \in A$ if there is a continuous linear mapping from X to Y, denoted $F'(x)$, with the property

$$F(x + h) = F(x) + F'(x)h + r_x(h)$$

where $\|r_x(h)\|/\|h\| \to 0$ as $h \to 0$. We say that F is continuously differentiable on A if it is differentiable at each point of A and the mapping $x \mapsto F'(x)$ is continuous. See, for example, (Deimling (1985), Section 7.7) for a much fuller description. For completeness, we now find the Fréchet derivatives of the functions used most frequently in our calculations.

Lemma 4. *Let c^N be the space of all convergent sequences in $\mathbb{R}^N$. Suppose A is an open subset of $\mathbb{R}^N$ and $f : A \to \mathbb{R}^M$ is k times differentiable, and therefore has a Taylor expansion*

$$f(\xi + \eta) = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(\xi)(\eta, \ldots, \beta) + r_\xi(\eta).$$

(Here $f^{(j)}$ is a j-linear map from $(\mathbb{R}^N)^j$ to $\mathbb{R}^M$ and $r_x(h) = o(|h|^{k-1})$ as $h \to 0$). Define a subset $\mathcal{A}$ of c^N by $\mathcal{A} = \{x \in c^N \mid x_t \in A \text{ for all } t\}$, and a mapping $F : \mathcal{A} \to c^M$ by $[F(x)]_t = f(x_t)$. Then F is k times differentiable and has a Taylor expansion

$$[F(x + h)]_t = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(x_t)(h_t, \ldots, h_t) + r_{x_t}(h_t). \tag{24}$$

If $f^{(k)}$ is bounded in some neighbourhood of $\lim_{t\to\infty} x_t$, then the error term is $O(\|h\|^k)$ as $h \to 0$. If $f^{(k)}$ is bounded on A, then the error is $O(\|h\|^k)$ uniformly for $x \in \mathcal{A}$.

Proof. Although it is clear that (24) is a valid identity, we need to check that its elements correspond to bounded multilinear forms between c^N and c^M and that the error term has the correct decay, in the norm on c^N. For simplicity, we work with the case $M = 1$; the M-dimensional case is then a direct sum of M 1-dimensional cases. Fix $x \in \mathcal{A}$. For $1 \le j \le k-1$, define a mapping from $(c^N)^j$ to the space of all real sequences by

$$[(F^{(j)}(x))(z)]_t = (f^{(j)}(x_t))(z_t).$$

It is clear that this is j-linear. Because $f^{(j)}$ is continuous and $x \in c^N$, $f^{(j)}(x_t)$ converges as $t \to \infty$; we also know that z_t converges as $t \to \infty$. It is now a straightforward consequence of multilinearity that $(f^{(j)}(x_t))(z_t)$ converges as $t \to \infty$, so $F^{(j)}$ maps $(c^N)^j$ to c. We also see that $\|F^{(j)}\| \le \sup_{t\in\mathbb{N}} \|f^{(j)}(x_t)\|$; this is finite because $f^{(j)}(x_t)$ converges as $t \to \infty$. This shows that the multilinear forms in (24) map continuously between the correct spaces.

It remains to show that the error term has the correct order. We have for some $\theta \in (0,1)$,

$$r_{x_t}(h_t) = \frac{1}{k!} f^{(k)}(x_t)(\theta h_t, \ldots, \theta h_t),$$

from which we have

$$|r_{x_t}(h_t)| \le \frac{1}{k!} \|f^{(k)}(x_t)\| \|h\|^k. \tag{25}$$

If $f^{(k)}$ is bounded in a neighbourhood of $\lim_{t\to\infty} x_t$, then clearly there is an upper bound for all the $\|f^{(k)}(x_t)\|$ terms, showing that the error is $O(\|h\|^k)$ as $h \to 0$. If there is a bound for $\|f^{(k)}\|$ on A, then this gives a uniform $O(\|h\|^k)$ estimate for the whole of $\mathcal{A}$. ■

Note that the Taylor approximation to the original function f has a remainder which is $O(|h|^k)$ at each point of A, provided $f^{(k)}$ exists on A. For the infinite-dimensional remainder to be $O(\|h\|^k)$, we also require local boundedness of $f^{(k)}$. This is because the remainder involves $f^{(k)}(x_t)$ at every point x_t of a convergent sequence, not just at a single point x. This boundedness hypothesis is not redundant: even in one dimension, everywhere differentiable functions can have locally unbounded derivatives, e.g. $x^2 \sin(1/x^2)$.

Lemma 5. *Suppose A is an open subset of $\mathbb{R}^N$, $v : A \to \mathbb{R}$ is twice differentiable with bounded second derivative, $(\delta_t)_{t\in\mathbb{N}}$ is a positive, summable sequence and $\zeta \in \mathbb{R}$. Define $\mathcal{A} \subseteq c$ as in Lemma 4 and a mapping from $u : \mathcal{A} \to \mathbb{R}$ by*

$$u(x) = \sum_{t=1}^{\infty} \delta_t v(x_t) + \zeta \lim_{t\to\infty} v(x_t).$$

Then u is continuously differentiable on $\mathcal{A}$ and

$$(F'(x))h = \sum_{t=1}^{\infty} \delta_t v'(x_t) h_t + \zeta \lim_{t\to\infty} v'(x_t) \lim_{t\to\infty} h_t.$$

Proof. Define V in the same way as F in Lemma 4: $[V(x)]_t = v(x_t)$, and let

$$\phi(x) = \sum_{t=1}^{\infty} \delta_t x_t + \zeta \lim_{t\to\infty} x_t,$$

so $\phi \in c^*$ and $u = \phi \circ V$. Because ϕ is linear, $\phi'(x) = \phi$ for all x. By the chain rule and Lemma 4, u is differentiable on $\mathcal{A}$ and $u'(x) = \phi \circ V'(x)$, i.e.

$$(F'(x))h = \sum_{t=1}^{\infty} \delta_t v'(x_t) h_t + \zeta \lim_{t\to\infty} (v'(x_t)h_t).$$

This is the result claimed, except that $\lim_{t\to\infty}(v'(x_t)h_t)$ has been rewritten as $\lim_{t\to\infty} v'(x_t) \lim_{t\to\infty}(h_t)$, to fit with the usual way of describing elements of c^*. ■

Lemma 6. *Suppose $f^1, \ldots, f^N$ are strictly decreasing functions on an interval $[0, \omega] \subseteq \mathbb{R}$. Then the equations*

$$f^1(x^1) = f^2(x^2) = \cdots = f^N(x^N); \quad x^1 + x^2 + \cdots + x^N = \omega,$$

($x^i \in [0, \omega]$) have at most one solution.

Proof. Let $R^i = f([0, \omega])$. Because f^i is strictly decreasing, there is a well-defined, strictly decreasing inverse mapping $(f^i)^{-1} : R^i \to [0, \omega]$. Let $R = \cap_{i=1}^N R^i$, so each $(f^i)^{-1}$ is defined on R. Suppose we have two solutions to the stated equations, one with $f^i(x^i) = a$ for all i and one with $f^i(y^i) = b$ for all i. Then $a, b \in R$ and we have $(f^1)^{-1}(a) + \cdots + (f^N)^{-1}(a) = (f^1)^{-1}(b) + \cdots + (f^N)^{-1}(b) = \omega$. But $(f^1)^{-1} + \cdots + (f^N)^{-1}$ is strictly decreasing, so we must have $a = b$. This gives $f^i(y^i) = f^i(x^i)$ for all i; because f is strictly decreasing, $x^i = y^i$ for all i. ■

C Lagrange multipliers in Banach spaces

The well-known method of Lagrange multipliers generalizes without great difficulty from the finite-dimensional to the infinite-dimensional world. We give here a brief description of the main result; for details see, for example, (Deimling (1985), Theorem 26.1).

Suppose A is an open subset of a real Banach space X and that we wish to maximize or minimize $F : A \to \mathbb{R}$, subject to the constraint $G(x) = y_0$, where y_0 is an element of another real Banach space Y and $G : A \to Y$.

As in the finite-dimensional case, we assume that F and G are continuously differentiable; this is in the sense of Frećhet, so $F'(x) \in B(X, \mathbb{R}) = X^*$ and $G'(x) \in B(X, Y)$ (here $B(X, Y)$ denotes the space of continuous linear mappings from X to Y). The infinite-dimensional Lagrange theorem now states that if $x_0 \in A$ is a constrained maximum or minimum and $G'(x_0)$ is surjective (the analogue of the full-rank condition in the finite-dimensional setting), then there is a Lagrange multiplier $\lambda \in Y^*$ such that

$$F'(x_0) + (G'(x_0))^* \lambda = 0. \tag{26}$$

Here $(G'(x_0))^* : Y^* \to X^*$ is the Banach space adjoint operator of the derivative $G'(x_0)$. The multiplier equation can be rewritten by expanding out the definition of the adjoint, leading to the alternative and often more directly useful form

$$F'(x_0)h + \lambda(G'(x_0)h) = 0 \qquad (h \in X). \tag{27}$$

The main difference from the finite-dimensional setting is that the multiplier is now a vector in Y^*. This effectively allows us to work with infinitely many scalar constraints, something which is meaningless in the finite-dimensional world but perfectly sensible in infinitely many dimensions. If Y is finite-dimensional then so is Y^*, and we can think of λ as a finite vector of multipliers, just as in the finite-dimensional case.

Although the theorem has been stated with a single constraint, it easily accommodates more. For example, if we have N constraint functions, say $G_n : A \to Y_n$, then we let $Y = Y_1 \times \cdots \times Y_N$, so $Y^* = Y_1^* \times \cdots \times Y_N^*$ and combine the functions into a single mapping $G : A \to Y$ given by $G(x) = (G_1(x), \ldots, G_N(x))$. The derivative of this map is given by $G'(x_0)h = (G_1'(x_0)h, \ldots, G_N'(x_0)h)$, and a Lagrange multiplier is an element of Y^*, i.e. a vector $(\lambda_1, \ldots, \lambda_N)$ where $\lambda_n \in Y_n^*$. The Lagrange equation because (in the formulation of (27))

$$F'(x_0)h + \sum_{n=1}^{N} \lambda_n (G_n'(x_0)h) = 0 \qquad (h \in X).$$

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