THE UNIVERSITY *of York*

Discussion Papers in Economics

No. 12/22

An Efficient Double-Track Auction for Substitutes and Complements

Ning Sun and Zaifu Yang

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

An Efficient Double-Track Auction for Substitutes and Complements[1]

Ning Sun[2] and Zaifu Yang[3]

Abstract: We propose a dynamic auction mechanism for efficiently allocating multiple heterogeneous indivisible items. These goods can be split into two distinct sets so that items in each of the two sets are substitutes but are complementary to items in the other. The seller has a reserve value for each bundle of goods. In each round of the auction, the auctioneer announces the current prices for all items, bidders respond by reporting their demands at these prices, and then the auctioneer adjusts simultaneously the prices of items in one set upwards but those of items in the other downwards. We prove that despite the fact that bidders are not assumed to be price-takers and thus can strategically exercise their market power, this dynamic auction always yields an efficient outcome and induces the bidders to bid truthfully and at the same time protects them from fully exposing their private values.

Keywords: Dynamic auction, gross substitutes and complements, incentives, efficiency, indivisibility.

JEL classification: D44

1 Introduction

Our purpose is to provide a dynamic auction that can efficiently allocate multiple heterogeneous indivisible goods to many bidders and at the same time induces bidders to behave truthfully. An important feature of the auction is that it can handle significant complementarity among the goods. Traditionally research has focused on examining auctions for

[1]The second author gratefully acknowledges financial support by the University of York, by the Ministry of Education, Science and Technology of Japan, and by CentER for Economic Research, Tilburg University. We wish to thank participants at several seminars and workshops for their helpful feedback. In particular, we are extremely grateful to Larry Samuelson for inspiring us to design this incentive-compatible auction and providing also constructive suggestions and comments. The usual disclaimer applies.

[2]N. Sun, School of Economics, Shanghai University of Finance and Economics, Shanghai, China; nsun@mail.shufe.edu.cn

[3]Z. Yang, Department of Economics and Related Studies, University of York, Heslington, York, YO10 5DD, UK; zaifu.yang@york.ac.uk.

selling a single item. However, over the last twenty years auctions for selling multiple items have become popular and widespread use, see e.g., Klemperer (2004) and Milgrom (2004) on auctioning spectrum rights. The past study has deepened our understanding of how the design of auction affects its outcome and also how the environment influences the auction design.

In a recent seminar paper, Ausubel (2006) develops an ingenious dynamic auction mechanism for selling heterogeneous goods. His auction yields an efficient outcome, induces the bidders to bid sincerely, and at the same time protects bidders' private values from being fully exposed. Therefore this auction not only maintains the important strategy-proof property of the well-known Vickrey-Clarke-Groves (VCG) mechanism but also overcomes the informational inefficiency problem facing the VCG mechanism. More specifically, the VCG mechanism requires all bidders to report their entire values over all possible bundles, whereas Ausubel's only needs bidders to report their demands at several price vectors along a finite path towards equilibrium. As pointed out by Rothkopf, Teisberg, and Kahn (1990), Rothkopf (2007), Ausubel (2004, 2006), Perry and Reny (2005), and Milgrom (2007) among others, the VCG mechanism requires significant amounts of information from bidders and has thus made it rarely used in practice, because in reality businessmen are generally reluctant to reveal their private value or cost and in fact always tend to use such information very prudently. Perry and Reny (2005) also argue that the VCG mechanism uses information wastefully and can negatively cause bidders to submit less accurate information. Generating an efficient outcome, Ausubel (2006)'s auction not only provides a remedy for the VCG's defects but also preserves its strategic property. Ausubel (2006) examines two auction models: In his first model, the goods are assumed to be perfectly divisible and bidders have strictly concave value functions, whereas in his second model, all goods are indivisible and are viewed as substitutes by every bidder.[4] His analysis mainly concentrates on the first model.

The purpose of this paper is to show that we can extend and generalize Ausubel's auction from the setting with substitutable indivisible goods to a more general and more practical setting that permits complementarities among goods. More precisely, we examine an auction market where a seller wishes to sell two disjoint sets S_1 and S_2 of heterogeneous items to many bidders and has a value for every bundle of goods. The seller trades her products in order to maximize revenues. Generally, items in the same set S_i are substitutes but are complementary to items in the other set S_j. This relation is introduced by Sun and Yang (2006) and called *gross substitutes and complements (GSC)*.[5] This fundamental

[4]Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000) have also developed dynamic auctions for similar models with indivisible and substitutable goods. The major difference between their auctions and Ausubel's is that the latter possesses the strategy-proof property.

[5]This condition subsumes and extends the well-known gross substitutes (GS) of Kelso and Crawford

pattern captures many familiar and important situations. For instance, in the view of manufacturing firms, workers and machines are typically complements, whereas workers are substitutes and so are machines.[6] In our earlier analysis (Sun and Yang (2009)), we propose a price adjustment process and show that this process always yields a Walrasian equilibrium if all bidders are assumed to be price-takers. Thus the important strategic and incentive issues have not been addressed. In contrast, in the current model, we assume that every bidder has a private value on each bundle of the goods and may have an incentive to economize on his private information. So in this setup, bidders are not assumed to behave as price-takers and thus may strategically exercise their market power. Now the central issue is how to design a dynamic auction that can induce bidders to bid truthfully and at the same time provides an efficient outcome in a complex environment where items for sale are indivisible and can create strong synergies when being used together.[7]

Built upon and improving the adjustment process of Sun and Yang (2009), we will develop a strategy-proof dynamic auction design for the environment. Roughly speaking, the auction works as follows. Starting from an arbitrary price vector, the auctioneer calls out the current price vector, bidders submit their demands at these prices, and then the auctioneer adjusts the prices of over-demanded items in one set S_1 (or S_2) upwards but those of under-demanded items in the other set S_2 (or S_1) downwards. We call this *a double-track auction* because it simultaneously updates prices in two opposite directions (ascending and descending). We show that this allocation mechanism always induces bidders to bid sincerely and finds an efficient outcome in finitely many rounds. In particular, this auction exhibits a significant strategic property that sincere bidding by every bidder is an ex post strongly perfect equilibrium of the dynamic game of incomplete information induced by the auction. More specifically, this means that after the auction has run up to any time t^*, no matter what has happened up to t^* and no matter whether it is now on or off an equilibrium path, sincere bidding is an optimal strategy for every bidder i, as long as from t^* on, every his opponent j bids sincerely according to a certain fixed GSC utility function $\tilde{u}^j$ which need not be his true GSC utility function u^j. This auction is also detail-free, robust against any regret and independent of any probability distribution. Another attractive feature of this auction is that it is simple, transparent and privacy-preserving. This auction does

(1982). GS is a benchmark condition for the existence of a competitive equilibrium in a market where goods for sale are indivisible but are substitutes to the consumers.

[6]Ostrovsky (2008) independently presents a similar condition for a supply chain model where prices of goods are fixed and a non-Walrasian equilibrium solution is used.

[7]Complementarities or synergies among items are known as a difficult issue in auction design and well-documented in Milgrom (2000, 2004), Jehiel and Moldovanu (2003), Klemperer (2004), and Maskin (2005) among others. As pointed out by Kelso and Crawford (1982), complementarity can even cause problems with existence of competitive equilibrium in the presence of indivisibilities. Nonetheless, GS or GSC guarantees existence of competitive equilibrium in economies with indivisibilities.

not only subsume and generalize Ausubel's from the setting with substitutes to the setting with both substitutes and complements, but also improves Ausubel's itself.[8] Aside from the theoretical interest and general applicability of this dynamic auction, our analysis complements Ausubel's which focuses on the model of divisible goods. Furthermore, our analysis is quite elementary and intuitive.

The remainder of this paper goes as follows. Section 2 presents the auction model. Section 3 describes the price adjustment process. Section 4 provides the main results.

2 The Auction Model

A seller (denoted by 0) wishes to auction a set $N = \{\beta_1, \beta_2, \cdots, \beta_n\}$ of n indivisible items to a finite group I of bidders. The items may be heterogeneous and can be divided into two sets S_1 and S_2 (i.e., $N = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$). For example, one can think of S_1 as tables and of S_2 as chairs. Items in the same set can be also heterogeneous. Let $I_0 = I \cup \{0\}$ denote the set of all agents (bidders and seller) in the market. Every agent $i \in I_0$ has a value function $u^i : 2^N \to \mathbb{R}$ specifying his/her valuation $u^i(B)$ (in units of money) on each bundle B with $u^i(\emptyset) = 0$, where 2^N denotes the family of all bundles of items.[9] It is standard to assume that u^i is weakly increasing, and that every bidder can pay up to his value, and every agent has quasi-linear utilities in money. The seller is a revenue-maximizer while the bidders are profit-maximizers. Here we allow the seller to have a utility function u^0 and so we can accommodate more practical situations than the usual situation of assuming u^0 to be always zero.

A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ indicates a price p_h for each item $\beta_h \in N$. Agent i's demand correspondence $D^i(p)$, the net utility function $v^i(A, p)$, and the indirect utility function $V^i(p)$, are defined respectively by

$$\begin{array}{rcl} D^i(p) & = & \arg\max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}, \\ v^i(A,p) & = & u^i(A) - \sum_{\beta_h \in A} p_h, \ \text{ and} \\ V^i(p) & = & \max_{A \subseteq N}\{u^i(A) - \sum_{\beta_h \in A} p_h\}. \end{array} \tag{2.1}$$

Because the seller is a revenue-maximizer, the family of her retaining bundles at prices p are given by

$$S(p) = \arg\max_{A \subseteq N}\{u^0(A) + \sum_{\beta_h \in N \setminus A} p_h\}.$$

[8] In each step of his auction, the auctioneer needs to compute the smallest or largest solution of an optimization problem which typically has multiple solutions. We will show that this cumbersome computation is not needed. This improvement is very useful for practical auction design. See Section 3 in detail.

[9] The seller's value function u^0 actually denotes her reservation price function. This function is not assumed to be separable and additive.

We first have the following basic observation which will be used later. The proof of the next result, and Lemma 2.3 and Theorem 3.1 will be relegated to the Appendix.

Lemma 2.1 *For the seller, it holds that* $S(p) = D^0(p)$.

An allocation of items in N is a *partition* $\pi = (\pi(i), i \in I_0)$ of items among all agents in I_0, i.e., $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{i \in I_0} \pi(i) = N$. Note that $\pi(i) = \emptyset$ is allowed. At allocation π, agent i receives bundle $\pi(i)$. An allocation π is *efficient* if $\sum_{i \in I_0} u^i(\pi(i)) \geq \sum_{i \in I_0} u^i(\rho(i))$ for every allocation ρ. Given an efficient allocation π, let $R(N) = \sum_{i \in I_0} u^i(\pi(i))$. We call $R(N)$ *the market value* of the items which is the same for all efficient allocations.

Let $\mathcal{M}$ denote the market with the set I_0 of agents and the set N of items, and for each bidder $i \in I$, let $\mathcal{M}_{-i}$ denote the market $\mathcal{M}$ *without bidder* i. Let $I_{-i} = I_0 \setminus \{i\}$ for every bidder $i \in I$, and for convenience also let $\mathcal{M}_{-0} = \mathcal{M}$ and $I_{-0} = I_0$.

Next, we introduce two solutions for this auction model: the Walraisian equilibrium and the Vickrey-Clarke-Groves (VCG) outcome.

Definition 2.2 *A Walrasian equilibrium* (p, π) *consists of a price vector* $p \in \mathbb{R}^n_+$ *and an allocation* π *such that* $\pi(i) \in D^i(p)$ *for every bidder* $i \in I$ *and* $\pi(0) \in S(p)$ *for the seller.*

In equilibrium (p, π), the seller retains the bundle $\pi(0)$ of goods and collects the payment $\sum_{j \in I} \sum_{\beta_h \in \pi(j)} p_h$ from her sold goods and thus her equilibrium revenue is $u^0(\pi(0)) + \sum_{j \in I} \sum_{\beta_h \in \pi(j)} p_h$. Notice that in Gul and Stacchetti (1999, 2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2006, 2009) it is assumed the seller values every bundle of goods at zero and consequently in equilibrium all goods will be sold to bidders. In the current model, because the seller has reservation value for every bundle, we need to slightly modify the notion of equilibrium. The following lemma shows that the modification is appropriate.

Lemma 2.3 *Let* (p, π) *be a Walrasian equilibrium. Then* π *is an efficient allocation.*

The following defines the Vickrey-Clarke-Groves mechanism. The definition is slightly more general than its standard one because here we permit the seller to have her own utility function. The standard one usually assumes that the seller values everything at zero.

Definition 2.4 *The VCG outcome is the outcome of the following procedure: every agent* $i \in I_0$ *reports his/her value function* u^i. *Then the auctioneer computes an efficient allocation* π *with respect to all reported* u^i *and assigns bundle* $\pi(i)$ *to bidder* $i \in I$ *and charges him a payment of* $q_i^* = u^i(\pi(i)) - R(N) + R_{-i}(N)$, *where* $R(N)$ *and* $R_{-i}(N)$ *are the market values of the markets* $\mathcal{M}$ *and* $\mathcal{M}_{-i}$ *based on* u^i $(i \in I_0)$*, respectively. Bidder* i*'s VCG payoff equals* $R(N) - R_{-i}(N)$, $i \in I$.

To ensure the existence of a Walrasian equilibrium, it will be necessary for us to impose some conditions. The most important one is known as *gross substitutes and complements condition*, which is introduced and used in Sun and Yang (2006, 2009), and defined as follows.[10]

Definition 2.5 *The value function u^i of agent i satisfies the* gross substitutes and complements (GSC) *condition if for any price vector $p \in \mathbb{R}^n$, any item $\beta_k \in S_j$ for $j = 1$ or 2, any $\delta \geq 0$, and any $A \in D^i(p)$, there exists $B \in D^i(p+\delta e(k))$ such that $[A \cap S_j] \setminus \{\beta_k\} \subseteq B$ and $[A^c \cap S_j^c] \subseteq B^c$.*

GSC says that agent i views items in each set S_j as substitutes, but items across the two sets S_1 and S_2 as complements. In particular, when either $S_1 = \emptyset$ or $S_2 = \emptyset$, GSC reduces to the gross substitutes (GS) condition of Kelso and Crawford (1982). GS requires that all the items be substitutes, and thus excludes any complementarity among items. The GS case has been studied extensively in the literature; see e.g., Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), Milgrom (2000, 2004), and Ausubel (2006).

The following three assumptions will be maintained throughout:

(A1) *Integer Private Values for Bidders*: Every bidder i's value function $u^i : 2^N \to \mathbb{Z}_+$ takes integer values and is his private information.

(A2) *Integer Public Values for Seller*: The seller's value function $u^0 : 2^N \to \mathbb{Z}_+$ takes integer values and is public information.

(A3) *Gross Substitutes and Complements*: The value function u^i of every agent $i \in I_0$ satisfies the GSC condition with respect to the two sets S_1 and S_2.

In the literature, the value of the seller over each bundle is usually assumed to be zero and this information is made public. Here A2 is more general and can accommodate more realistic situations where the seller's reservation value over her goods for sale need not be zero and may vary from one bundle to another.

3 The Price Adjustment Process

In a dynamic auction, at each time $t \in \mathbb{Z}_+$ and with respect to a price vector $p(t) \in \mathbb{R}^n$, each bidder i selects a bid $C^i(t)$, a subset of 2^N. We say that bidder i *bids sincerely*

[10] The following piece of notation will be used. For any positive integer $k \leq n$, $e(k)$ denotes the kth unit vector in $\mathbb{R}^n$. Let $\mathbb{Z}^n$ stand for the integer lattice in $\mathbb{R}^n$ and 0 the n-vector of 0's. For any subset A of N, let $e(A) = \sum_{\beta_k \in A} e(k)$. When $A = \{\beta_k\}$, we also write $e(A)$ as $e(k)$. For any subset A of N, let A^c denote its complement, i.e., $A^c = N \setminus A$. For any finite set A, $\sharp(A)$ denotes the number of elements in A.

relative to value function u^i if his bid always equals his true demand correspondence, i.e., $C^i(t) = D^i(p(t)) = \arg\max_{A\subseteq N}\{u^i(A) - \sum_{\beta_h\in A} p_h(t)\}$.

In this section we assume that bidders are price-takers and thus bid sincerely. We will present a modified version of the double-track adjustment process introduced by Sun and Yang (2009). This process always yields an equilibrium and provides a key ingredient for the auction design in Section 4 where bidders are not assumed to behave as price-takers and thus may act strategically. Throughout the paper, in the price adjustment process and in the auction mechanism, at the beginning the seller reports her reserve price function u^0 to the auctioneer who then uses u^0 to calculate the seller's demand correspondence $D^0(p(t))$ at prices $p(t)$ in every round t. Thus, the auctioneer acts as a proxy bidder for the seller. Recall that since by Lemma 2.1, $D^0(p(t)) = S(p(t))$, the seller can act as a bidder. In the sequel, the seller may be also called a bidder. Nevertheless, remember that this proxy bidder always acts sincerely.

While the existing auctions typically adjust all prices simultaneously in one direction (either ascending or descending), the current process adjusts simultaneously prices of items in S_1 and S_2 respectively in opposite directions (ascending for one set S_1 (S_2) but descending for the other S_2 (S_1)). Therefore, we can define an n-dimensional cube for price adjustment as

$$\square = \{\delta \in \mathbb{R}^n \mid 0 \leq \delta_k \leq 1, \forall \beta_k \in S_1,\ -1 \leq \delta_l \leq 0, \forall \beta_l \in S_2\}.$$

Let $\Delta = \square \cap \mathbb{Z}^n$ be the discrete set and $\square^* = -\square$, $\Delta^* = -\Delta$. Through $\square$ (Δ), we lower prices of items in S_2 but raise prices of items in S_1, while through $\square^*$ (Δ^*), we lower prices of items in S_1 but raise prices of items in S_2.

To describe how prices will be adjusted, we define the Lyapunov function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ for the auction model as

$$\mathcal{L}(p) = \sum_{\beta_h\in N} p_h + \sum_{i\in I_0} V^i(p) \tag{3.2}$$

where V^i is the indirect utility function of agent $i \in I_0$. This type of function has been explored in Ausubel (2005, 2006), and Sun and Yang (2009). Here the Lyapunov function includes also the seller's indirect utility function V^0 and is more general than those previously used in the literature.

Now we discuss in detail how prices should be adjusted based on bidders' reported demands in each round of the process. Given a current price vector $p(t) \in \mathbb{Z}^n$, the auctioneer first asks every bidder i to report his demand $D^i(p(t))$. Then she uses every bidder's reported demand $D^i(p(t))$ to determine the next price vector $p(t+1)$. The underlying rationale for the auctioneer is to choose a direction $\delta \in \square$ so as to reduce the value of the

Lyapunov function $\mathcal{L}$ as much as possible. To achieve this, she needs to solve the following problem

$$\max_{\delta\in\square}\{\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)\} \tag{3.3}$$

Note that the above formula involves every bidder's valuation of every bundle of goods, so it involves private information. Apparently, it is impossible for the auctioneer to know such information unless the bidders tell her. Fortunately, she can fully infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t)+\delta)$ just from the reported demands $D^i(p(t))$ and the price variation δ. To see this, we know from the definition of the Lyapunov function that for any given $p(t)\in\mathbb{Z}^n$ and $\delta\in\square$, the difference is given by

$$\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)=\sum_{i\in I_0}(V^i(p(t))-V^i(p(t)+\delta))-\sum_{\beta_h\in N}\delta_h \tag{3.4}$$

As shown in Sun and Yang (2009), when prices move from $p(t)$ to $p(t)+\delta$, the change in indirect utility for every bidder i is unique and given by

$$V^i(p(t))-V^i(p(t)+\delta)=\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h=\sum_{\beta_h\in\tilde{S}^i}\delta_h \tag{3.5}$$

where $\tilde{S}^i$ is a solution given by

$$\tilde{S}^i\in\arg\min_{S\in D^i(p(t))}\{\sum_{\beta_h\in S}\delta_h\}, \tag{3.6}$$

for bidder i with respect to price vector $p(t)\in\mathbb{Z}^n$ and price variation $\delta\in\Delta$.

Consequently, the equation (3.4) becomes the following simple formula whose right side involves only price variation δ and optimal choices at $p(t)$:

$$\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)=\sum_{i\in I_0}\Big(\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h\Big)-\sum_{\beta_h\in N}\delta_h=\sum_{i\in I_0}\sum_{\beta_h\in\tilde{S}^i}\delta_h-\sum_{\beta_h\in N}\delta_h \tag{3.7}$$

In Sun and Yang (2009), it is shown that solving the continuous optimization problem (3.3) is equivalent to solving the following discrete optimization problem in the right hand:

$$\max_{\delta\in\square}\{\mathcal{L}(p(t))-\mathcal{L}(p(t)+\delta)\}=\max_{\delta\in\Delta}\Big\{\sum_{i\in I_0}\Big(\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h\Big)-\sum_{\beta_h\in N}\delta_h\Big\} \tag{3.8}$$

The max-min relation in the formulas (3.8) admits a meaningful economic interpretation: when the auctioneer adjusts the prices from $p(t)$ to $p(t+1)=p(t)+\delta(t)$, she acts in an elaborate manner so that the seller can make a maximal gain whereas every bidder can achieve a minimal loss in indirect utility. Observe that the auctioneer is responsible for executing the computation of (3.8) based on bidders' reported demands $D^i(p(t))$. It is fairly easy to calculate the value $(\min_{S\in D^i(p(t))}\sum_{\beta_h\in S}\delta_h)$ for each given $\delta\in\Delta$ or Δ^* and bidder i. We can now summarize the steps of the adjustment process as follows:

The improved double-track (IDT) adjustment process

Step 1: The seller reports her reserve price function u^0 to the auctioneer, who announces the initial price vector $p(0) \in \mathbb{Z}^n_+$. Let $t := 0$ and go to Step 2.

Step 2: The auctioneer asks every bidder $i \in I_0$ (this also includes the proxy bidder 0) to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes a solution $\delta(t)$ to the problem (3.8). If $\delta(t) = 0$, go to Step 3. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to Step 2.

Step 3: The auctioneer asks every bidder $i \in I_0$ to report his demand $D^i(p(t))$ at $p(t)$. Then based on reported demands $D^i(p(t))$, the auctioneer computes a solution $\delta(t)$ to the problem (3.8) where Δ is replaced by Δ^*. If $\delta(t) = 0$, then the auction stops. Otherwise, set the next price vector $p(t+1) := p(t) + \delta(t)$ and $t := t+1$. Return to Step 3.

Observe that in both Step 2 and Step 3 the auctioneer needs only an arbitrary solution to the problem (3.8) with respect to Δ or Δ^*. This improves considerably the original process of Sun and Yang (2009) which requires to take the smallest or largest solution to the same problem if there are several solutions. (In fact, the set of solutions to the problem (3.8) is a nonempty lattice and typically has multiple solutions.) From a practical point of view, this improvement is extremely useful and important for practical auction design and makes the implementation very easy and fast. Consequently, it also improves the auction of Ausubel (2006). Recall that in his auction model with indivisible goods, all goods are assumed to be substitutes, i.e., $S_1 = \emptyset$ or $S_2 = \emptyset$ in the current model. In each step of his auction, the auctioneer must compute the smallest or largest solution of an optimization problem which typically has multiple solutions. The above process shows that this cumbersome computation is no longer needed.

The following theorem shows the global convergence of the IDT adjustment process.

Theorem 3.1 *For the market model under Assumptions* (A1), (A2) *and* (A3), *starting with any integer price vector, the IDT adjustment process converges to an equilibrium price vector in a finite number of rounds.*

4 The Strategy-Proof Dynamic Auction Mechanism

We now address the strategic issue such as When confronting an auction, is honesty the best policy for every bidder? More specifically, does sincere bidding constitute a Nash equilibrium (or its variants) of the auction game? If it is the case, the auction is said to be *strategy-proof.* The (sealed-bid) Vickrey-Clarke-Groves (VCG) auction is strategy-proof. The dynamic auction of Ausubel (2006) not only possesses this important strategy-proof property but also offers advantages of informational efficiency, transparency and privacy preservation. One may wonder whether it is possible to design an auction that can deal with the current more general environment but still possess all properties of Ausubel's. We will provide a positive answer to this question. To do so we divide this section into two subsections. Section 1 introduces the mechanism and section 2 analyzes strategic properties of the mechanism.

4.1 The Auction Mechanism Design

We now introduce the following strategy-proof dynamic auction mechanism in which every bidder may act strategically and thus may not behave as a price-taker. The mechanism runs the IDT adjustment process for all markets $\mathcal{M}_{-m}$ ($m \in I_0$) simultaneously in parallel and in coordination. The IDT adjustment process works for every market $\mathcal{M}_{-m}$ exactly as described in Section 3 but needs the following modifications: Consider any market $\mathcal{M}_{-m}$. At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n_+$, every bidder $i \in I_{-m}$ reports a bid $C^i_{-m}(t) \subseteq 2^N$ (which need not be his demand set $D^i(p^{-m}(t))$[11]) and the problem (3.8) becomes the next one for Δ or Δ^* respectively,

$$\max_{\delta \in \Delta(\text{ or } \Delta^*)} \Big\{ \sum_{i \in I_{-m}} \Big(\min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h \Big) - \sum_{\beta_h \in N} \delta_h \Big\} \tag{4.9}$$

If the auctioneer finds a solution $\sigma^{-m}(t)$ of (4.9) for Δ (Δ^*), she obtains the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$ whenever $\delta^{-m}(t) \neq 0$. We say the IDT adjustment process *finds an allocation* π^{-m} in $\mathcal{M}_{-m}$ if $\delta^{-m}(t) = 0$ for Δ^* (i.e., in Step 3 of the auction) and $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$. The IDT adjustment process needs to go back to Step 2 from Step 3 if $\delta^{-m}(t) = 0$ for Δ^* but it finds no allocation π^{-m} in $\mathcal{M}_{-m}$ such that $\pi^{-m}(i) \in C^i_{-m}(t)$ for all $i \in I_{-m}$—this modification is meant to tolerate minor mistakes or manipulations committed by bidders. The IDT adjustment process *detects serious manipulation* if it finds $p^{-m}_h(t+1) < 0$ for some $\beta_h \in N$, or if it never finds an allocation in $\mathcal{M}_{-m}$ in which case the auction is said to *stop at time* ∞. Now we have

The strategy-proof double-track (SPDT) auction

Step 1: Run the IDT adjustment process simultaneously in parallel for every market $\mathcal{M}_{-m}$ ($m \in I_0$) by starting with a common initial price vector $p^{-m}(0) = p(0) \in \mathbb{Z}^n_+$.

[11] However, the proxy bidder 0 (the seller) always bids honestly by reporting her demand set $C^0_{-m}(t) = D^0(p^{-m}(t))$.

At $t \in \mathbb{Z}_+$ and $p^{-m}(t) \in \mathbb{Z}^n$, every bidder $i \in I_{-m} \setminus \{0\} = I \setminus \{m\}$ reports a bid $C^i_{-m}(t) \subseteq 2^N$, the proxy bidder 0 bids truthfully by reporting $C^0_{-m}(t) = D^0(p^{-m}(t))$, and the auctioneer finds the next price vector $p^{-m}(t+1) = p^{-m}(t) + \delta^{-m}(t)$. If the IDT adjustment process detects serious manipulations in any market, go to Step 3. Otherwise, the IDT adjustment process continues until it finds an allocation π^{-m} in every market $\mathcal{M}_{-m}$ ($m \in I_0$) at $p^{-m}(T^{-m}) \in \mathbb{Z}^n_+$, and $T^{-m} \in \mathbb{Z}_+$. Go to Step 2.

Step 2: In this case all markets are clear. For every $m \in I_0$, every agent $i \in I_{-m}$ and every $t = 0, 1, \cdots, T^{-m} - 1$, let $\Delta_i^{-m}(t)$ denote the "indirect utility change" of agent i in I_{-m} when prices move from $p^{-m}(t)$ to $p^{-m}(t+1)$, where

$$\Delta_i^{-m}(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h^{-m}(t) \tag{4.10}$$

Every bidder $i \in I$ is assigned the bundle $\pi^{-0}(i)$ of the allocation π^{-0} found in the market $\mathcal{M}_{-0} = \mathcal{M}$ and required to pay q_i and then the auction stops, where

$$q_i = \sum_{j \in I_{-i}} \Big(\sum_{t=0}^{T^{-0}-1} \Delta_j^{-0}(t) - \sum_{t=0}^{T^{-i}-1} \Delta_j^{-i}(t) \Big) + \sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \tag{4.11}$$

Step 3: In this case every bidder $i \in I$ receives no item but is assigned a payoff of $-\infty$. The auction stops.

The payment q_i of bidder $i \in I$ has an intuitive interpretation: q_i is equal to the accumulation of "indirect utility changes" of his opponents $l \in I_{-i}$ (also including the proxy bidder 0) along the path from $p^{-i}(T^{-i})$ to $p(0)$ (in the market $\mathcal{M}_{-i}$) and the path from $p(0)$ to $p^{-0}(T^{-0})$ (in the market $\mathcal{M}$) by subtracting $\sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}$–the equilibrium payments by bidder i's opponents in the market $\mathcal{M}$, and adding $\sum_{\beta_h \in N} p_h^{-i}(T^{-i})$–the equilibrium payments by bidder i's opponents in the market $\mathcal{M}_{-i}$.

It is simple but important to observe that the SPDT auction tolerates minor mistakes or manipulations committed by bidders and allows them to correct so that for any time $t^* \in \mathbb{Z}_+$, no matter what has happended before t^*, as long as from t^* on every bidder i bids according to his GSC value function u^i, the auction will find a Walrasian equilibrium in every market in finitely many rounds and thus terminates in Step 2, because the IDT adjustment process converges to a Walrasian equilibrium from any integer price vector.

4.2 Incentive and Strategic Issues

To study the incentive and strategic properties of the SPDT auction mechanism, we will formulate this auction as an extensive-form dynamic game of incomplete information in which bidders are players. Prior to the start of the (auction) game, nature reveals to every player $i \in I$ only his own value function $u^i \in \mathcal{U}$ of private information and a joint probability distribution $F(\cdot)$ from which the profile $\{u^i\}_{i\in I}$ is drawn, where $\mathcal{U}$ denotes the family of all value functions $u : 2^N \to \mathbb{Z}_+$ satisfying Assumptions (A1) and (A2). Let H_i^t be the part of the information (or history) of play that player i has observed just before he submits his choice sets at time $t \in \mathbb{Z}_+$. A natural and sensible specification is that H_i^t comprises the complete set of all observable price vectors and all players' choice sets, i.e.,

$$H_i^t = \{p^{-m}(t), p^{-m}(s), C^j_{-m}(s) \mid m \in I_0, j \in I, 0 \le s < t, m \ne j\}$$

Note that $H_i^t = H_j^t$ for all $i, j \in I$, namely, all bidders share a common history just like in an English auction. Let T^* be the time when the SPDT auction stops at Steps 2 or 3. If the auction has found an allocation in any $\mathcal{M}_{-m}$, for consistency and convenience, we define $C^i_{-m}(t) = C^i_{-m}(T^{-m})$ and $p^{-m}(t) = p^{-m}(T^{-m})$ for any $i \in I_{-m}$ and any $t \in \mathbb{Z}_+$ between T^{-m} and T^*. After any history H_i^t and at any time $t \in \mathbb{Z}_+$, each player i updates his posterior beliefs $\mu_i(\cdot \mid t, H_i^t, u^i)$ over opponents' value functions; see also Ausubel (2006). We stress that even after the auction is finished, player i may not know his opponents' value functions precisely.

A (dynamic) *strategy* σ_i of player $i(i \in I)$ is a set-valued function $\{(t, m, H_i^t, u^i) \mid t \in \mathbb{Z}_+, m \in I_{-i}, u^i \in \mathcal{U}\} \to 2^N$, which tells him to bid $\sigma_i(t, m, H_i^t, u^i) \subseteq 2^N$ for every market $\mathcal{M}_{-m}(m \in I_{-i})$ at each time $t \in \mathbb{Z}_+$ when he observes H_i^t. Let Σ_i denote player $i's$ strategy space of all such strategies σ_i. We say that σ_i is *a regular bidding strategy* for player i if irrespective of his true utility function u^i, he always reports his choice set $C^i_{-m}(t)$ according to some utility function $\tilde{u}^i \in \mathcal{U}$ for any $m \in I_{-i}$, $t \in \mathbb{Z}_+$, $p^{-m}(t) \in \mathbb{Z}^n$, and H_i^t, i.e.,

$$\sigma_i(t, m, H_i^t, u^i) = C^i_{-m}(t) = \arg\max_{A\subseteq N}\{\tilde{u}^i(A) - \sum_{\beta_h\in A} p_h^{-m}(t)\}$$

Note that $\tilde{u}^i$ may or may not be his true utility function u^i. We denote such a regular bidding strategy by $\sigma_i^{\tilde{u}^i}$. Thus, every GSC utility function $\tilde{u}(\tilde{u} \in \mathcal{U})$ determines a regular bidding strategy for each player. For simplicity, we also use $\mathcal{U}$ to denote the family of all such strategies. Clearly, $\mathcal{U} \subseteq \Sigma_i$. A regular bidding strategy σ_i is *sincere bidding* (strategy) for player i if he always reports his demand set $D^i(p^{-m}(t))$ as defined by (2.1)with respect to his true utility function u^i, i.e., $\sigma_i(t, m, H_i^t, u^i) = C^i_{-m}(t) = D^i(p^{-m}(t)) = \arg\max_{A\subseteq N}\{u^i(A) - \sum_{\beta_h\in A} p_h^{-m}(t)\}$ for all $t \in \mathbb{Z}_+$, $m \in I_{-i}$ and $p^{-m}(t) \in \mathbb{Z}^n$. The strategy space Σ_i of player i contains regular bidding strategies, sincere bidding strategies and also various other strategies.

Given the auction rules, the outcome of this auction game depends entirely upon the realization of utility functions and the strategies the bidders take. When every bidder $i \in I$ takes a strategy σ_i and the SPDT auction terminates in Step 2, then bidder $i \in I$ receives bundle $\pi^{-0}(i)$ and pays q_i given by (4.11). When every bidder $i \in I$ takes a strategy σ_i and the SPDT auction stops in Step 3, every bidder gets nothing but a payoff of $-\infty$. In summary, every player $i's$ payoff function $W_i(\cdot, \cdot)$ is given by

$$W_i\Big(\{\sigma_j\}_{j\in I}, \{u^j\}_{j\in I}\Big) = \begin{cases} u^i(\pi^{-0}(i)) - q_i & \text{if the auction stops in Step 2,} \\ -\infty & \text{if the auction stops in Step 3.} \end{cases}$$

We now introduce notions of equilibrium to the current dynamic auction games of incomplete information. Following Ausubel (2004, 2006), the $\sharp(I)$-tuple $\{\sigma_i\}_{i\in I}$ is an ex post perfect equilibrium[12] if for any time $t \in \mathbb{Z}_+$, any history profile $\{H_i^t\}_{i\in I}$, and any realization $\{u^i\}_{i\in I}$ of profile of utility functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i^t, u^i)$ of every player $i \in I$ (i.e., $\sigma_i(s, m, H_i^s \mid t, H_i^t, u^i) \subseteq 2^N$ for all $s \geq t$, $m \in I_{-i}$ and H_i^s) constitutes his best response against the continuation strategies $\{\sigma_j(\cdot \mid t, H_j^t, u^j)\}_{j\in I_{-i}}$ of player i's opponents of the game even if the realization $\{u^i\}_{i\in I}$ becomes common knowledge.

We shall define and use the following stronger equilibrium solution. A strategy σ_i of player i constitutes *an ex post strongly perfect strategy* for him if for any time $t \in \mathbb{Z}_+$, any history profile $\{H_j^t\}_{j\in I}$, and any realization $\{u^j\}_{j\in I}$ of profile of utility functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i^t, u^i)$ of player i is his best response against *all continuation regular bidding strategies* $\{\sigma_j^{\tilde{u}^j}(\cdot \mid t, H_j^t, u^j)\}_{j\in I_{-i}}$ of player i's opponents, even if the realization $\{u^i\}_{i\in I}$ becomes common knowledge. The $\sharp(I)$-tuple $\{\sigma_i\}_{i\in I}$ of regular bidding strategies comprises *an ex post strongly perfect (Nash) equilibrium* if for every player $i \in I$, his regular bidding strategy σ_i is an ex post strongly perfect strategy. Clearly, every ex post strongly perfect equilibrium is an ex post perfect equilibrium but the reverse may not be true. Stronger than Bayesian equilibrium or perfect Bayesian equilibrium, ex post (strongly) perfect equilibria have a number of additional desirable properties, i.e., they are not only robust against any regret but also independent of any probability distribution. Furthermore, in the complete information case, ex post perfect equilibrium simply coincides with to the familiar notion of subgame perfect equilibrium.

In the current auction game, although the auctioneer knows that every bidder $i \in I$ possesses a GSC utility function u^i, she has no precise knowledge of u^i. This implies that as long as a bidder reports his demand according to some fixed GSC utility function $\tilde{u}^i$ not necessarily being his true utility function, it is extremely hard if not impossible to prove whether he bids truthfully or not. According to Hurwicz (1973, p.23) on mechanism design, "it is conceivable that the participants would cheat without openly violating the rules." This is why we focus on "all regular bidding strategies" instead of "all dynamic strategies" of all opponents of every bidder $i \in I$ in the definition of the proposed solution.

[12]In (static or sealed-bid) auction games of incomplete information, the ex post equilibrium was used by Crémer and McLean (1985) and Krishna (2002).

Regular bidding strategies are safe, whereas irregular ones are unsafe in the sense that they have a high probability of being detected for open violation of the auction rules.

Now we are prepared to establish our major theorem.

Theorem 4.1 *Suppose that the market $\mathcal{M}$ satisfies Assumptions* (A1), (A2) and (A3).
(i) *When every bidder bids sincerely, the SPDT auction converges to a Walrasian equilibrium and yields a Vickrey-Clarke-Groves outcome for the market $\mathcal{M}$ in a finite number of rounds.*
(ii) *Sincere bidding is an ex post strongly perfect equilibrium in the SPDT auction.*

Proof: We first prove (i). By the argument in Section 3, we see that when every bidder i bids sincerely according to his true GSC function u^i , the auction terminates at Step 2 and finds a Walrasian equilibrium $(p^{-m}(T^{-m}), \pi^{-m})$ in every market $\mathcal{M}_{-m}$, $m \in I_0$. By the rules, every bidder i receives bundle $\pi^{-0}(i)$ and pays q_i of (4.11). It follows from (3.5) that

$$\Delta_i^{-m}(t) = \min_{S \in C^i_{-m}(t)} \sum_{\beta_h \in S} \delta_h^{-m}(t) = V^i(p^{-m}(t)) - V^i(p^{-m}(t+1))$$

for all $i \in I$ and $m \in I_0$ $(i \neq m)$, where $C^i_{-m}(t) = D^i(p^{-m}(t))$. Using these equations, we will show that q_i coincides with the VCG payment $q_i^* = u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N)$, where $R(N) = \sum_{j \in I} u^j(\pi^{-0}(j))$ and $R_{-i}(N) = \sum_{j \in I_{-i}} u^j(\pi^{-i}(j))$. Observe that payment q_i of (4.11) satisfies

$$\begin{aligned}
q_i &= \textstyle\sum_{j \in I_{-i}} \Big(\sum_{t=0}^{T^{-0}-1} (V^j(p^{-0}(t)) - V^j(p^{-0}(t+1))) \\
&\qquad - \textstyle\sum_{t=0}^{T^{-i}-1} (V^j(p^{-i}(t)) - V^j(p^{-i}(t+1))) \Big) \\
&\quad + \textstyle\sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \\
&= \textstyle\sum_{j \in I_{-i}} \Big((V^j(p^{-0}(0)) - V^j(p^{-0}(T^{-0}))) - (V^j(p^{-i}(0)) - V^j(p^{-i}(T^{-0}))) \Big) \\
&\qquad + \textstyle\sum_{\beta_h \in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \\
&= \Big(\textstyle\sum_{j \in I_{-i}} V^j(p^{-i}(T^{-0})) + \sum_{\beta_h \in N} p_h^{-i}(T^{-i}) \Big) \\
&\qquad - \Big(\textstyle\sum_{j \in I_{-i}} V^j(p^{-0}(T^{-0})) + \sum_{\beta_h \in N \setminus \pi^{-0}(i)} p_h^{-0}(T^{-0}) \Big) \\
&= \textstyle\sum_{j \in I_{-i}} u^j(\pi^{-i}(j)) - \sum_{j \in I_{-i}} u^j(\pi^{-0}(j)) \\
&= u^i(\pi^{-0}(i)) - R(N) + R_{-i}(N) \\
&= q_i^*.
\end{aligned}$$

Bidder $i's$ payoff $u^i(\pi^{-0}(i)) - q_i$ equals his VCG payoff $R(N) - R_{-i}(N)$.

Now we prove (ii). It suffices to show that sincere bidding is every player $i's$ ex-post partially dominant strategy. Consider any time $t^* \in \mathbb{Z}_+$, any history profile $\{H_j^{t^*}\}_{j \in I}$ (which may be on or off the equilibrium path), and any realization $\{u^j\}_{j \in I}$ of profile of utility functions in $\mathcal{U}^I$ of private information.[13] Suppose that from this time t^* on every

[13] In this case, the outcome of the game depends on the histories $H_j^{t^*}$ and the strategies that all bidders will take in the continuation game starting from t^*. Bidders cannot change histories but can influence the path of the future from t^* on.

opponent $j(j \in I_{-i})$ will report his bids according to a regular bidding strategy. That is, every player $j(j \in I_{-i})$ according to some $\tilde{u}^j \in \mathcal{U}$ reports his $C^j_{-m}(t)$ at every round $t(t \geq t^*)$, namely,

$$\sigma_j^{\tilde{u}^j}(t, m, H_j^t, u^j) = C^j_{-m}(t) = \arg\max_{A \subseteq N}\{\tilde{u}^j(A) - \sum_{\beta_h \in A} p_h^{-m}(t)\}$$

for every $m \in I_{-j}$. Of course, it is possible that $\tilde{u}^j \neq u^j$. Clearly, in this continuation game from time t^*, when all opponents of player i choose regular bidding strategies, because of the payoff of $-\infty$, bidder i strictly prefers a strategy which results in the auction terminating at Step 2, to any other strategies which result in the auction stopping at Step 3. Therefore, it sufficient to compare the sincere bidding strategy with any other strategies which also result in the auction finishing at Step 2. Suppose that $\sigma_i'(\cdot \mid t^*, H_i^{t^*}, u^i)$ (σ_i' in short) is such a continuation strategy of player i resulting in an allocation ρ for $\mathcal{M}$, and that bidder i's (continuation) sincere bidding strategy results in an allocation π for $\mathcal{M}$. Without any loss of generality, we assume that by the time t^*, the auction for the markets $\mathcal{M}$ and $\mathcal{M}_{-i}$ has not yet finished, i.e., $t^* < T^{-0}$ and $t^* < T^{-i}$. When player i chooses the strategy σ_i', his payment q_i' given by (4.11) is

$$\begin{aligned}
q_i' &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}\Delta_j^{-0}(t) + \sum_{t=t^*}^{T^{-0}-1}[\tilde{V}^j(p^{-0}(t)) - \tilde{V}^j(p^{-0}(t+1))] \\
&\qquad - \textstyle\sum_{t=0}^{t^*-1}\Delta_j^{-i}(t) - \sum_{t=t^*}^{T^{-i}-1}[\tilde{V}^j(p^{-i}(t)) - \tilde{V}^j(p^{-i}(t+1))]\Big) \\
&\quad + \textstyle\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) - \sum_{\beta_h\in N\setminus\rho(i)} p_h^{-0}(T^{-0}) \\
&= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t) - \Delta_j^{-i}(t)] + \tilde{V}^j(p^{-0}(t^*)) + \tilde{V}^j(p^{-i}(T^{-i})) - \tilde{V}^j(p^{-i}(t^*))\Big) \\
&\qquad + \textstyle\sum_{\beta_h\in N} p_h^{-i}(T^{-i}) \\
&\quad -\Big(\textstyle\sum_{j\in I_{-i}}\tilde{V}^j(p^{-0}(T^{-0})) + \sum_{\beta_h\in N\setminus\rho(i)} p_h^{-0}(T^{-0})\Big) \\
&= \mathit{constant} - \textstyle\sum_{j\in I_{-i}}\tilde{u}^j(\rho(j)),
\end{aligned}$$

where $\tilde{V}^j$ is bidder j's indirect utility function based on $\tilde{u}^j$ and *constant* is given by

$$\begin{aligned}
\mathit{constant} &= \textstyle\sum_{j\in I_{-i}}\Big(\sum_{t=0}^{t^*-1}[\Delta_j^{-0}(t) - \Delta_j^{-i}(t)]\Big) \\
&\quad + \textstyle\sum_{j\in I_{-i}}\Big(\tilde{V}^j(p^{-0}(t^*)) + \tilde{V}^j(p^{-i}(T^{-i})) - \tilde{V}^j(p^{-i}(t^*))\Big) + \sum_{\beta_h\in N} p_h^{-i}(T^{-i})
\end{aligned}$$

Observe that *constant* is totally determined by the history profile $\{H_j^{t^*}\}_{j\in I}$ and the market $\mathcal{M}_{-i}$ without bidder i, and does not depend on player i's strategy σ_i', (and that $\Delta_j^{-0}(t)$ and $\Delta_j^{-i}(t)$ for $t < t^*$ cannot be expressed by $\tilde{V}^j$, because player j may not have bid according to $\tilde{u}^j$ before t^*). Analogously we can show that when bidder i uses the (continuation) sincere bidding strategy, his payment $\tilde{q}_i$ will be $\tilde{q}_i = \mathit{constant} - \sum_{j\in I_{-i}}\tilde{u}^j(\pi(j))$, where *constant* is the same as the previous one. Furthermore, we know from the argument in Section 3 that (in the continuation game) when bidder i bids sincerely according to his utility function u^i and every his opponent $j(j \in I_{-i})$ bids according to a regular bidding strategy $\sigma_j^{\tilde{u}^j}$ (i.e., according to a GSC utility function $\tilde{u}^j \in \mathcal{U}$), the resulted allocation π must be efficient for $\mathcal{M}$ w.r.t. u^i and $\tilde{u}^j, j \in I_{-i}$. This implies that

$$u^i(\pi(i)) + \sum_{j\in I_{-i}}\tilde{u}^j(\pi(j)) \geq u^i(\rho(i)) + \sum_{j\in I_{-i}}\tilde{u}^j(\rho(j)).$$

Consequently, for bidder $i's$ payoff $\tilde{W}_i$ with the sincere bidding strategy and his payoff W_i' with the strategy σ_i', we have

$$\begin{aligned}\tilde{W}_i &= u^i(\pi(i)) - \tilde{q}_i = u^i(\pi(i)) - (constant - \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\pi(j)))\\ &= u^i(\pi(i)) + \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\pi(j)) - constant\\ &\geq u^i(\rho(i)) + \textstyle\sum_{j\in I_{-i}} \tilde{u}^j(\rho(j)) - constant = u^i(\rho(i)) - q_i'\\ &= W_i'.\end{aligned}$$

This shows that every player's sincere bidding strategy is his ex post strongly perfect strategy, so sincere bidding is an ex post strongly perfect equilibrium. □

The current dynamic procedure yields the same outcome as that of the VCG auction, but offers several advantages over the VCG auction: First, it utilizes information from every buyer efficiently and judiciously in that it only requires him to report his demand sets on a number of price vectors, whereas the VCG auction is sealed-bid and requires every buyer to report his entire values. In reality, businessmen generally do not like to reveal their values even if truth-telling may be theoretically a dominant strategy; see e.g., Milgrom (2007), Rothkopf (2007), Rothkopf, Teisberg and Kahn (1990). In fact, Hurwicz (1973) has stressed the importance of informational efficiency in mechanism design. Second, the current procedure gives a simple and transparent way of computing efficient allocations, equilibrium prices and VCG payments using observable information, whereas the VCG auction tells only a way of computing VCG payments assuming that all buyers' values and efficient allocations are already given.

While both the current dynamic procedure and Ausubel's (2006) compute a Walrasian equilibrium in every market $\mathcal{M}_{-m}$ ($m \in I_0$) somehow like the VCG auction that needs to compute every market $\mathcal{M}_{-m}$ value $R_{-m}(N)$, the current procedure and analysis differ from Ausubel's in several aspects: First, the current procedure applies to the environment with both complements and substitutes, while Ausubel's applies to the environment with substitutes. Second, the current procedure attains the ex post strongly perfect equilibrium which is stronger than Ausubel's ex post perfect equilibrium. Third, his procedure and payment rule are not symmetric, whereas the current procedure and payment rule are symmetric and simpler.[14] Fourth, Ausubel's analysis on the VCG outcome focuses on economies with divisible goods and relies on calculus and Theorem 1 of Krishna and Maenner (2001) but

[14]More precisely, the current auction starts with the same initial price vector $p(0)$ for all markets $\mathcal{M}$ and $\mathcal{M}_{-i}$, $i \in I$, whereas Ausubel's (Ausubel (2006, pp.615-616)) starts with the same initial price vector $p(0)$ only for the markets $\mathcal{M}_{-i}$, $i \in I$, but for the market $\mathcal{M}$ his auction starts with the equilibrium price vector p^{-k^*} of any chosen market $\mathcal{M}_{-k^*}$. In Ausubel's auction, the VCG payment of bidder k^* is given by Equation (7) (Ausubel (2006, p.611)) using the price vectors along the path from p^{-k^*} to p^*. The VCG payment of bidder i ($i \in I_{-k^*}$) is also given by Equation (7) but using the price vectors along the path from p^{-i} to p^0; the path from p^0 to p^{-k^*}; and the path from p^{-k^*} to p^*.

he mentioned that his analysis can be analogously done for his model with indivisible goods under the GS condition, whereas the current analysis is quite different from his and in fact very elementary and simple.

Appendix

Proof of Lemma 2.1 Because, at any given prices p,

$$\begin{aligned} \max_{A\subseteq N}\{u^0(A)+\sum_{\beta_h\in N\setminus A}p_h\} &= \max_{A\subseteq N}\{u^0(A)-\sum_{\beta_h\in A}p_h+\sum_{\beta_h\in A}p_h+\sum_{\beta_h\in N\setminus A}p_h\} \\ &= \max_{A\subseteq N}\{u^0(A)-\sum_{\beta_h\in A}p_h\}+\sum_{\beta_h\in N}p_h, \end{aligned}$$

clearly we have $S(p)=D^0(p)$. $\square$

Proof of Lemma 2.3 Take any Walrasian equilibrium (p,π) and any allocation ρ. By definition, we have for any bidder $i\in I$

$$u^i(\pi(i))-\sum_{\beta_h\in\pi(i)}p_h\geq u^i(\rho(i))-\sum_{\beta_h\in\rho(i)}p_h$$

and for the seller

$$u^0(\pi(0))+\sum_{\beta_h\in N\setminus\pi(0)}p_h\geq u^0(\rho(0))+\sum_{\beta_h\in N\setminus\rho(0)}p_h$$

Summing up the two inequalities yields

$$\sum_{i\in I_0}u^i(\pi(i))\geq\sum_{i\in I_0}u^i(\rho(i)).$$

This shows that π is efficient. $\square$

For the proof of the following result, we need to introduce several notations. Let $p,q\in\mathbb{R}^n$ be any vectors. With respect to the two given sets S_1 and S_2, we define their generalized *meet* $s=(s_1,\cdots,s_n)=p\wedge_g q$ *and join* $t=(t_1,\cdots,t_n)=p\vee_g q$ by

$$\begin{aligned} s_k &= \min\{p_k,q_k\},\quad \beta_k\in S_1,\quad s_k=\max\{p_k,q_k\},\quad \beta_k\in S_2; \\ t_k &= \max\{p_k,q_k\},\quad \beta_k\in S_1,\quad t_k=\min\{p_k,q_k\},\quad \beta_k\in S_2. \end{aligned}$$

Notice that the two operations are different from the standard meet and join operations. For $p,q\in\mathbb{R}^n$, we introduce a new order by defining $p\leq_g q$ if and only if $p_h\leq q_h$ for all $\beta_h\in S_1$ and $p_h\geq q_h$ for all $\beta_h\in S_2$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is *a generalized submodular function* if $f(p\wedge_g q)+f(p\vee_g q)\leq f(p)+f(q)$ for all $p,q\in\mathbb{R}^n$.

Proof of Theorem 3.1 By Theorem 3.1 of Sun and Yang (2006) the market has a Walrasian equilibrium and by Lemma 1 of Sun and Yang (2009) the Lyapunov function $\mathcal{L}(\cdot)$ attains its mimimum value at any equilibrium price vector and is bounded from below.

Since the prices and value functions take only integer values, the Lyapunov function is an integer valued function and it lowers by a positive integer value in each round of the IDT adjustment process. This guarantees that the auction terminates in finitely many rounds, i.e., $\delta(t^*) = 0$ in Step 3 for some $t^* \in \mathbb{Z}_+$.

Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. Let $\bar{t} \in \mathbb{Z}_+$ be the time when the IDT adjustment process finds $\delta(\bar{t}) = 0$ at Step 2. We claim that $\mathcal{L}(p) \geq \mathcal{L}(p(\bar{t}))$ for all $p \geq_g p(\bar{t})$. Suppose to the contrary that there exists some $p \geq_g p(\bar{t})$ such that $\mathcal{L}(p) < \mathcal{L}(p(\bar{t}))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3 (i) of Sun and Yang (2009), there is a strict convex combination p' of p and $p(\bar{t})$ such that $p' \in p(\bar{t}) + \square$ and $\mathcal{L}(p') < \mathcal{L}(p(\bar{t}))$. From equation (3.8) we know that $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) < \mathcal{L}(p(\bar{t}))$, and so $\delta(\bar{t}) \neq 0$ in Step 2 of the IDT adjustment process, yielding a contradiction. Therefore, we have $\mathcal{L}(p \vee_g p(\bar{t})) \geq \mathcal{L}(p(\bar{t}))$ for all $p \in \mathbb{R}^n$, because $p \vee_g p(\bar{t}) \geq_g p(\bar{t})$ for all $p \in \mathbb{R}^n$. We will further show that $\mathcal{L}(p \vee_g p(t)) \geq \mathcal{L}(p(t))$ for all $t = \bar{t}+1, \bar{t}+2, \cdots, t^*$ and $p \in \mathbb{R}^n$. By induction, it sufficies to prove the case of $t = \bar{t}+1$. Notice that $p(\bar{t}+1) = p(\bar{t}) + \delta(\bar{t})$, where $\delta(\bar{t}) \in \Delta^*$ is determined in Step 3 of the IDT adjustment process. Assume by way of contradiction that there is some $p \in \mathbb{R}^n$ such that $\mathcal{L}(p \vee_g p(\bar{t}+1)) < \mathcal{L}(p(\bar{t}+1))$. Then if we start the IDT adjustment process from $p(\bar{t}+1)$, we can by the same previous argument find a $\delta(\neq 0) \in \Delta$ in Step 2 such that $\mathcal{L}(p(\bar{t}+1) + \delta) < \mathcal{L}(p(\bar{t}+1))$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function by Theorem 3 (i) of Sun and Yang (2009), we have $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1)+\delta)) + \mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta)) \leq \mathcal{L}(p(\bar{t}) + \mathcal{L}(p(\bar{t}+1)+\delta)$. Recall that $\mathcal{L}(p(\bar{t}) \vee_g (p(\bar{t}+1)+\delta)) \geq \mathcal{L}(p(\bar{t}))$. It follows that $\mathcal{L}(p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta)) \leq \mathcal{L}(p(\bar{t}+1)+\delta) < \mathcal{L}(p(\bar{t}+1))$. Observe that $\delta' = 0 \wedge_g (\delta(\bar{t}) + \delta) \in \Delta^*$ and $p(\bar{t}) \wedge_g (p(\bar{t}+1)+\delta) = p(\bar{t}) + \delta'$. This yields $\mathcal{L}(p(\bar{t}) + \delta') < \mathcal{L}(p(\bar{t}) + \delta(\bar{t}))$ and so $\delta' \neq \delta(\bar{t})$, contradicting the definition of $\delta(\bar{t}) \in \Delta^*$ by which $\mathcal{L}(p(\bar{t}) + \delta(\bar{t})) = \min_{\delta \in \Delta^*} \mathcal{L}(p(\bar{t}) + \delta)$.

Next we prove that $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. To see this, we first show that $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Suppose to the contrary that there exists some $p \leq_g p(t^*)$ such that $\mathcal{L}(p) < \mathcal{L}(p(t^*))$. By the convexity of $\mathcal{L}(\cdot)$ via Theorem 3 (i) of Sun and Yang (2009), there is a strict convex combination p' of p and $p(t^*)$ such that $p' \in \{p(t^*)\} - \square$ and $\mathcal{L}(p') < \mathcal{L}(p(t^*))$. Because of the symmetry between Step 2 and Step 3, Lemma 3 (where $\square$ is replaced by $\square^* = -\square$) and Step 3 of the GDDT procedure imply that $\mathcal{L}(p(t^*) + \delta(t^*)) = \min_{\delta \in \square^*} \mathcal{L}(p(t^*) + \delta) = \min_{\delta \in \Delta^*} \mathcal{L}(p(t^*) + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p(t^*))$ and so $\delta(t^*) \neq 0$, contradicting the fact that the GDDT procedure stops in Step 3 with $\delta(t^*) = 0$. So we have $\mathcal{L}(p) \geq \mathcal{L}(p(t^*))$ for all $p \leq_g p(t^*)$. Because $p \wedge_g p(t^*) \leq_g p(t^*)$ for all $p \in \mathbb{R}^n$, it follows that $\mathcal{L}(p \wedge_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$.

We also proved above that $\mathcal{L}(p \vee_g p(t^*)) \geq \mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. Since $\mathcal{L}(\cdot)$ is a generalized submodular function by Theorem 3 (i) of Sun and Yang (2009), we have $\mathcal{L}(p) + \mathcal{L}(p(t^*)) \geq \mathcal{L}(p \vee_g p(t^*)) + \mathcal{L}(p \wedge_g p(t^*)) \geq 2\mathcal{L}(p(t^*))$ for all $p \in \mathbb{R}^n$. This shows that

$\mathcal{L}(p(t^*)) \leq \mathcal{L}(p)$ holds for all $p \in \mathbb{R}^n$ and by Lemma 1 of Sun and Yang (2009), $p(t^*)$ is an equilibrium price vector. $\square$

REFERENCES

Ausubel, L. (2004): "An efficient ascending-bid auction for multiple objects," *American Economic Review*, 94 (5), 1452-1475.

——— (2005): "Walrasian tatonnement for discrete goods," preprint.

——— (2006): "An efficient dynamic auction for heterogeneous commodities," *American Economic Review*, 96 (3), 602-629.

Cremér, J., and R.P. Mclean (1985): "Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent," *Econometrica*, 53, 345-362.

Gul, F., and E. Stacchetti (1999): "Walrasian equilibrium with gross substitutes," *Journal of Economic Theory*, 87, 95-124.

——— (2000): "The English auction with differentiated commodities," *Journal of Economic Theory*, 92, 66–95.

Hurwicz, L.(1973): "The design of mechanisms for resource allocation," *American Economic Review*, 63 (2), 1-39.

Jehiel, P., and B. Moldovanu (2003): "An economic perspective on auctions," *Economic Policy*, April, pp. 269-308.

Kelso, A., and V.P. Crawford (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica*, 50, 1483-1504.

Klemperer, P. (2004): *Auctions: Theory and Practice*, Princeton University Press, Princeton.

Krishna, V. (2002): *Auction Theory*, Academic Press, New York.

Krishna, V., and E. Maenner (2001): "Convex potentials with an application to mechanism design," *Econometrica*, 69 (4), 1113-1119.

Maskin, E. (2005): "Recent contributions to mechanism design: a highly selective review," preprint.

Milgrom, P. (2000): "Putting auction theory to work: the simultaneous ascending auction," *Journal of Political Economy*, 108, 245-272.

——— (2004): *Putting auction theory to work*, Cambridge University Press, New York.

——— (2007): "Package auctions and exchanges," *Econometrica*, 75, 935-965.

Ostrovsky, M. (2008): "Stability in supply chain network," *American Economic Review*, 98, 897-923.

Perry, M., and P. Reny (2005): "An efficient multi-unit ascending auction," *Review of Economic Studies*, 72, 567-592.

Rothkopf, M.H. (2007): "Thirteen reasons why the Vickrey-Clarke-Groves process is not practical," *Operations Research*, 55, 191-197.

Rothkopf, M.H., T.J. Teisberg, and E.P. Kahn (1990): "Why are Vickrey auctions rare?" *Journal of Political Economy*, 98, 94-109.

Sun, N., and Yang, Z. (2006): "Equilibria and indivisibilities: gross substitutes and complements," *Econometrica*, 74, 1385-1402.

——- (2009): "A double-track adjustment process for discrete markets with substitutes and complements," *Econometrica*, 77, 933-952.