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**Abstract**

This article extends the theory of spatial competition by allowing firms to endogenously select their operating costs within a Hotelling (1929) framework. A three-stage duopoly model is examined in which the firms compete in cost reduction, locations and finally prices. Furthermore, it is assumed that firms are identical except with respect to their cost reducing technologies and one firm has a Stackelberg leadership advantage in the cost-reduction stage.

The model implies two results that are unique within the literature. First, if a firm possesses both an efficieny and investment timing advantage, it *always* becomes the dominant firm in the product market in all relevant respects. Second, if an *ex ante* inefficient firm has an investment timing advantage it can only become the *ex post* market leader if and only if the *a priori* efficiency gap is not too large. Consequently, these results suggest that a firm's ability to innovate - in terms of both efficiency and timing - play a large part in determining the composition of the final product market.

**Key words** Location model; Asymmetric firms; Stackelberg game; Endogenous cost selection

**JEL Classification** L13; R32

# 1 Introduction

This paper examines the effects of endogenous cost selection on a firm's product and pricing decisions where the firms are heterogeneous with respect to their *ex ante* investment efficiencies and investment timing. In doing so, it is possible to examine the importance of both investment timing and efficiencies in determining the composition of the final product market or, more interestingly, whether it is possible for an *ex ante* inefficient firm to become an *ex post* market leader. Thus, we extend the linear spatial competition literature in two ways: first, firms are a priori heterogeneous in respect to their effectiveness at reducing costs and the timing of their investments and; second, the *ex post* cost differential is endogenous. The results offer two interesting additions to the existing literature: ($i$) if a firm possesses both an investment timing and efficiency advantage it always becomes the *ex post* efficient firm and; ($ii$) if a firm possesses only an investment timing advantage, it can become the *ex post* efficient firm if and only if the *a priori* efficiency gap is not too large.

Hotelling (1929) developed a simple model for examining firms' product choices in a spatial setting, finding that firms would have a preference to produce homogenous goods. The result suggested that firms would be driven to agglomerate as, by moving towards their rival in the location space, firms could steal a rival firm's market share. d'Aspremont *et al.* (1979) undermined this result arguing that no pure strategy solution existed in Hotelling's original model specification finding, instead, that equilibrium only exists if transport costs are quadratic.[1] The use of quadratic transport costs, proposed by DGT, ensures a stable equilibrium for all price-location pairs but, in contrast to Hotelling's (1929) conclusion, suggested that firms would find it optimal to produce maximally differentiated products. Furthermore, were firms able to locate outside of the city's boundaries, it would be optimal for them do do so (Tabuchi and Thisse, 1995; Lambertini, 1997).

One criticism to be levelled at this body of work is that the firms are assumed to face symmetric production costs. The specific consideration of exogenous cost differentials has been examined within a linear spatial framework by Schulz and Stahl (1985), Ziss (1993) and Matsumura and Matsushima (2009). Ziss (1993) introduced heterogeneous production costs into a DGT framework, finding that maximum product differentiation is the only pure strategy Nash equilibrium solution, and that this (pure strategy) equilibrium outcome exists if and only if the cost differential is not too large. Matsumura and Matsushima (2009) extend this analysis to consider cases in which this small cost differential assumption is violated. They find, where cost differentials are large, the firms face contrasting incentives regarding location with the stronger (weaker) firm having an incentive to minimise (maximise) product differentiation. Thus, a mixed strategy Nash equilibrium is ensured even where the heterogeneity between firms undermines a pure stragey solution.

It is for this reason that the Hotelling (1929) model "is [assumed] inconvenient for investigating endogenous production costs" as a full examination of location equilibrium conditions is difficult to obtain (Matsumura and Matsushima, 2009: p216). Nonetheless, from a theoretical standpoint, the existence of mixed strategy solutions may explain differences in market structure across seemingly similar markets especially where firms face asym-

[1] d'Aspremont et al. (1979) will henceforth be denoted DGT.

metric costs. Bester *et al.* (1996), using a DGT model, find an infinity of mixed strategy equilibria and argue that, without some coordination mechanism, firms face a strictly positive probability that they would locate at the same point. Thus, where firms face asymmetric production costs, simple coordination failure can be the difference between a high cost firm remaining in the market or being driven out of business. Whilst this result is intuitive, Martin (2001) argues that firms *do not* randomly select a location or product mix but, rather, make such decisions in secret. This generates asymmetric information and inherits the same propensity for coordination failure.

Also pertinent to this discussion are models of competition between firms facing heterogeneous costs within a circular city/Salop (1979) framework. Aghion and Schankerman (2004) and Syverson (2004) adopt a Bayesian set-up, with firms knowing only their own (stochastic) production costs and assume prices are set prior to learning their rivals' locations and costs. Analogous to the linear spatial model results, the (pure strategy) equilibrium breaks down for very large cost differentials as the (low cost) firms' incentive to limit price becomes accute. Interestingly, Alderighi and Piga (2009, 2010), who examine the maximum permissible cost disparity within these models, observe that an Eaton and Lipsey (1978) style "no mill-price undercutting rule" must be satisfied, similar to that derived by Ziss (1993), or "a highly efficient firm's reach could potentially extend beyond its nearest neighbours position" (Alderighi and Piga, 2009: p.3). Furthermore, extending this analysis to allow firms to endogenously select their locations suggests "the distance between two direct competitors is strictly increasing in the average productivity: all else equal, more productive firms are more isolated" (Vogel, 2008: p.450).[2] Therefore, whilst issues arising from using a Hotelling model to examine endogenous cost selection do exist, these problems are comparable to other areas of the spatial competition literature.

[2] To the best of the author's knowledge some papers in this area have examined endogenous cost selection. However, their focus is generally on the entry effects of R&D. See Scalera and Zazzaro (2005) or Ebina and Shimizu (2008).

This paper takes a different approach and, rather than simply examining the effect of cost differentials, analyses *the cause* of cost differentials. That is, we examine the extent to which a firm's ability to innovate - in terms of both efficiency and timing - affects the composition of the final product market. Our results suggest two findings unique to the literature. First, if a firm possesses both an investment timing and efficiency advantage it always becomes the *ex post* dominant firm (**Proposition 3**). Second, where a firm possesses only an investment timing advantage, it can become the *ex post* market leader firm if and only if the *a priori* efficiency gap is not too large (**Summary 1**).These results, whilst consistent with those in the existing literature, suggest that the innovation process plays a crucial part in determining which firm will become the *ex post* market leader. Additionally, whilst it is the *ex post* low cost firm that becomes the market leader, as noted in the literature, it is possible that this firm could be either the *ex ante* efficient or inefficient firm.

Another body of literature to which this paper relates is the work regarding endogenous Stackelberg leadership between quantity setting firms facing asymmetric production costs. Within a two-period Cournot duopoly game, van Damme and Hurkens (1999) observe that a high cost firm finds committing to move first riskier than a low cost firm and so it must be the low cost firm that emerges as the endogenous Stackelberg leader. This assertion backed by Branco (2008) who, within a similar framework, also finds that the low-cost firm becomes the Stackelberg leader. However, both papers disagree as to whether a Cournot equilibrium would obtain if the firms faces symmetric costs.[3]

Whilst this paper does not afford firms the opportunity to commit to becoming a Stackelberg leader, the results presented here contrast with those of both van Damme and Hurkens (1999) and Branco (2008).[4] Whilst the

[3] It is of note that similar models specifying Bertrand competition obtain a reversed results (see van Damme and Hurkens (2004) and Tasnádi (2003)).

[4] In the model presented, the cost reduction stage is akin to that of quantity setting.

results presented here agree that the *ex post* low cost firm would emerge as the market leader, they also suggest that this firm could be either the *ex ante* efficient or inefficient firm. If, as here, the firms are *a priori* heterogeneous with respect to their effectiveness at reducing costs and the timing of their investments then a firm's unit cost is no longer a measure of its efficiency but, instead, a direct consequence of strategic interactions *given* the firms' relative efficiencies (investment timing and cost reduction effectiveness). Therefore, the van Damme and Hurkens (1999) and Blanco (2008) results ignore the possibility that this firm could be either the *ex ante* efficient or inefficient firm. That is, an *a priori* inefficient firm may, in fact, possess an incentive to commit to becoming the Stackelberg leader if the *ex ante* efficiency gap is not too large.

The rest of the paper proceeds as follows: section 2 decsribes the model's specification; section 3 analsyses the model including the equilibrium conditions for prices, locations and cost reduction and; section 4 concludes.

# 2 Model

Consider a three-stage duopoly model. In the first stage, the firms compete in cost reduction sequentially before making symmetric location and price choices in the second and third periods respectively.

In the second and third stages, firms compete in (linear) spatial- and then price competition respectively; moving simultaneously in each case. As in Hotelling's (1929) seminal work, two firms, $A$ and $B$, supply a physically homogenous good from different locations on the real axis. The location of firm $N$ is denoted by $n \in \mathbb{R}$ and, consequently, the location of each firm is measured from zero. This implies that firms are free to locate anywhere on the real axis and, without loss of generality, we assume $a < b$.[5] Firms maximise profits at each stage.

[5] It is trivial to derive the results for the case $a > b$.

Consumers are uniformly distributed across a linear city of unit length and are assumed to have a density of one. For simplicity the city is defined by $[0,1] \in \mathbb{R}$ which ensures that any $a, b \notin [0,1]$ simply implies that a firm is locating outside of the city's boundaries. It is also assumed that consumers face unit demands and consume either zero or one units of production. Consequently, a consumer, located at $x \in [0,1]$, would only purchase a good from firm $N$, located at $n$, if and only if

$$U_x = s - p_N - t(x-n)^2 \geq 0$$

where $s$ is a fixed utility of consuming a good, $p_N$ is the price charged by firm $N$, $t$ is a measure of consumer heterogeneity and $t(x-n)^2$ is a quadratic transport cost incured by a consumer having to move a distance of $|x-n|$ to consumre good $N$.[6] In order to ensure that total demand is equal to one, or that each consumer purchases a good, $s$ is assumed to be large enough that $U_x \geq 0$ for all $x \in [0,1]$ for at least one of the firms' products and consumers only purchase the good that maximises their utility.

In the first stage the firms sequentially invest in cost reduction, at a cost, to reduce their *ex ante* production cost. At the beginning of the first stage, we assume that both firms face a symmetric production cost, $c$, but these initial production costs can be reduced by $\varphi_N(I_N)$ at a cost of $C_N(I_N)$; where $I_N$ is the investment level of firm $N$. Furthermore, it is assumed that the cost reduction schedule and investment cost functions are linear and quadratic in investment respectively.[7] More formally, the cost reduction and investment

[6] The assumption of quadratic transport costs is to ensure pure strategy Nash equilirbia in the location and pricing stages (see d'Aspremont *et al* (1979)).

[7] Whilst these functional form assumptions are not ideal, they are necessary for two reasons. First, to make the mathematics more tractible. Attempts were made to generalise the model using a convex investment cost schedule but these rendered the model intractable.

Second, to keep the model specification in line with other cost selection models within the linear sptial competition literature.This linear cost reduction schedule is assumed across a broad range of the literature: georgraphic spillover effects (see Mai and Peng (1999) and cost schedules are given by:

$$\begin{aligned}\varphi_N(I_N) &= m_N I_N;\\ C_N(I_N) &= \frac{1}{2}I_N^2;\end{aligned}$$

where $m_N > 0$ represents the ability of firm $N$ to implement the cost reducing technology or the (constant) marginal cost reduction per unit of investment. The final unit cost of firm $N$, used in the price and location stages, is given by

$$c_N = c - \varphi_N(I_N)$$

In specifying the first stage in this manner it is possible to allow firms to differ with respect to both their investment timing and efficiency. In the case of investment timing this is obvious, as firms move sequentially and this can be thought of as the firm's differing in their abilties to spotting new investment opportunities. However, the cost reduction schedule allows for firm heterogeneity with respect to their investment efficiencies if $m_A \neq m_B$. Without loss of generality, the remainder of this paper assumes $m_A > m_B \geq 0$ or, more simply, that firm $A$ is more effective at implementing cost reduction dollar-for-dollar than firm $B$. With both of these assumptions in place it is possible to examine the relative importance of investment efficiency and timing advantages.

The game is solved through backward induction and at each stage firms maximise their profits. Consequently, only pure strategy Nash equilibria are examined in this paper. The game proceeds as follows. In the first stage, firms select their investment levels, $I_N \in [0, \infty)$, sequentially; both cases where firms $A$ and $B$ move first are examined. In the second stage, each firm selects its location, $n \in \mathbb{R} \forall N \in \{A, B\}$, simultaneously. Finally, in the third

Piga and Poyago-Theotoky (2005)); licensing (see Matsumura and Matsushima (2004 and 2010a)) and; public vs. private firms (see Matsumura and Matsushima(2010b)).

stage, firms select their prices, $p_N \in [c_N, \infty)$, simultaneously.

# 3 Analysis

## 3.1 Price Stage

Recall that consumers *i*) maximise utility and; *ii*) can only purchase one unit of production. In this instance, a consumer, located at $x$, is indifferent between purchasing either good $A$ or good $B$ if and only if

$$s - p_A - t(x-a)^2 = s - p_B - t(b-x)^2 \tag{1}$$

As $a < b$ by assymption, the total demand for good $A$ is given by all consumers located to the left of $x$ and demand for good $B$ is the residual demand, $1-x$. Solving equation (1) yields the demand functions:

$$\begin{aligned} D_A &= x = \frac{p_B - p_A}{2t(b-a)} + \frac{(a+b)}{2} \\ D_B &= 1 - x = \frac{p_A - p_B}{2t(b-a)} + \frac{(2-a-b)}{2} \end{aligned}$$

Taking these demand functions as given, firm $N$'s profits are given by the expression:[8]

$$\pi_N = (p_N - c + \varphi_N)D_N - C_N$$

Firms simultanesouly select prices, $p_N \in [c_N, \infty)$, to maximise profits, taking location and investment choices as given. The relevenant first order condi-

[8]For notational purposes, as $I_N$ is known at this stage we abbreviate $\varphi_N(I_N)$ and $C_N(I_N)$ to $\varphi_N$ and $C_N$ respectively.

tions are:

$$\begin{aligned}
\frac{\partial \pi_A}{\partial p_A} &= \frac{p_B - 2p_A + c - \varphi_A}{2t(b-a)} + \frac{(a+b)}{2} = 0 \\
\frac{\partial \pi_B}{\partial p_B} &= \frac{p_A - 2p_B + c - \varphi_B}{2t(b-a)} + \frac{(2-a-b)}{2} = 0
\end{aligned}$$

The second order conditions are given by $\frac{\partial^2 \pi_N}{\partial p_N^2} = -\frac{1}{b-a} < 0 \forall N$ and are, therefore, met for all price pairs, $(p_A, p_B)$. Solving these equations simultaneously yields the equilbrium prices:

$$p_A = \frac{3c - 2\varphi_A - \varphi_B + t(b-a)(2+a+b)}{3} \tag{2}$$

$$p_B = \frac{3c - \varphi_A - 2\varphi_B + t(b-a)(4-a-b)}{3} \tag{3}$$

From these conditions, it is possible to observe that a subgame perfect Nash equilibrium in prices always exists, regardless of the locations and the *ex post* cost disparity between the two firms. Furthermore, the signs of the components within the equilbrium price functions are as one would expect, with prices increasing in the *a priori* unit cost and decreasing in the cost reduction efforts of both firms. However, assuming $\varphi_A = \varphi_B$, it is notable that the equilbrium prices of both firms are affected more by changes to their own *ex post* unit cost than to changes to a rival's *ex post* unit cost.

## 3.2 Location Stage

With the equilibrium prices given by equations (2) and (3), the relevant profit functions for both firms are:

$$\pi_A = \frac{[\varphi_A - \varphi_B + t(b-a)(2+a+b)]^2}{18t(b-a)} - C_A$$

$$\pi_B = \frac{[\varphi_B - \varphi_A + t(b-a)(4-a-b)]^2}{18t(b-a)} - C_B$$

Each firm, $N$, simultaneously selects its location, $n \in \mathbb{R}$, in order to maximise profits taking investment decisions and equilibrium prices as given. The first order conditions are given by

$$\begin{aligned}\frac{\partial \pi_A}{\partial a} &= 0 = -\frac{2}{9}\frac{t(1+a)[t(b-a)(2+a+b)+\varphi_A-\varphi_B]}{t(b-a)} \\ &\quad + \frac{1}{18}\frac{[t(b-a)(2+a+b)+\varphi_A-\varphi_B]^2}{t(b-a)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial \pi_B}{\partial b} &= 0 = \frac{2}{9}\frac{t(2-b)[t(b-a)(4-a-b)-\varphi_A+\varphi_B]}{t(b-a)} \\ &\quad + \frac{1}{18}\frac{[t(b-a)(4-a-b)-\varphi_A+\varphi_B]^2}{t(b-a)^2}\end{aligned}$$

Solving these equations simultaneosuly for $a$ and $b$ yields:

$$a = \frac{(\varphi_A - \varphi_B)}{3t} - \frac{1}{4}$$

$$b = \frac{(\varphi_A - \varphi_B)}{3t} + \frac{5}{4}$$

The relevant second order conditions are given by:

$$\frac{\partial^2 \pi_A}{\partial a^2} = \frac{1}{486}\frac{(4(\varphi_A-\varphi_B)+9t)(4(\varphi_A-\varphi_B)-27t)}{t}$$

$$\frac{\partial^2 \pi_B}{\partial b^2} = \frac{1}{486}\frac{(4(\varphi_A-\varphi_B)-9t)(4(\varphi_A-\varphi_B)+27t)}{t}$$

As these equations must be strictly negative, it is trivial to show $\varphi_A - \varphi_B \in (-\frac{9}{4}t, \frac{9}{4}t)$ must hold for both second order conditions to be met, or that the *ex post* cost disparity between the two firms is not too large relative to the transport cost.

This additional condition ensures that a pure strategy Nash equilibrium in locations exists but, unlike the pricing game, only where the *ex post* production costs are not too different. This is because, in allowing for cost heterogeneity, a highly efficient firm may have an incentive to drive its inefficient rival from the market. A similar argument is made by Alderighi and Piga (2008). Recalling the demand functions and substituting in the equilibrium price and location choices yields:

$$D_A = \frac{1}{18}\frac{(4(\varphi_A - \varphi_B) + 9t)}{t}$$

$$D_B = -\frac{1}{18}\frac{(4(\varphi_A - \varphi_B) - 9t)}{t}$$

Therefore, if $\varphi_A - \varphi_B \geq \frac{9}{4}t$ then the *ex post* high cost firm, firm $B$, would be driven out of the market whilst the low cost firm, firm $A$, would serve all demand.[9] It is for this reason that large *ex post* cost differentials undermine the existence of a pure strategy Nash equilibrium in the location stage given that, for all $\varphi_A - \varphi_B \geq \frac{9}{4}t$, firm $B$'s demand and profits are driven to zero. Thus, there no longer exists a unique optimal location for firm $B$ as, no matter where it locates, it would surely be driven from the market. Given that firm $B$ no longer has a unique optimal location, a pure strategy Nash equilibrium cannot exist. Consequently, we restrict our focus to cases in which $|\varphi_A - \varphi_B| < \frac{9}{4}t$, or a pure strategy Nash equilibria exist in the location stage.[10]

With this restriction placed on the *ex post* cost differential, it is possible to say something of the location choice of each firm. First, where $\varphi_A = \varphi_B$, both firms locate symmetrically outside of the market (at $a = -\frac{1}{4}$ and $b = \frac{5}{4}$ respectively). As the quadratic transport costs make price competition increasingly fierce, firms prefer to locate beyond the boundaries of the city

[9] This is because as $D_A \geq 1$ and $D_B \leq 0$.

[10] See Matsumura and Matsushima (2009) for a discussion on the existence mixed strategy Nash equilibria in a similar case.

to mitigate the price competition effects on their profits (Lambertini, 1997). Second, the structure of the firms locations implies that the distance between the two firms is fixed at $\frac{3}{2}$, regardless of the size of the cost disparity. Second, where $\varphi_A \neq \varphi_B$, it is the firm with lower unit costs that locates closer to the centre of the market, with the inefficient firm moving further from the city to shield itself from aggresive price competition. Additionally, as $|\varphi_A - \varphi_B| \rightarrow \frac{9}{4}t$ the efficient firm's location converges to $\frac{1}{2}$, the centre of the market. Therefore, it is only an efficient firm that is able to locate within the city's boundaries at the expense of its inefficient rival who, fearful of limit pricing, is driven away from the centre of the market.

### 3.3 Cost Reduction Stage

In the first stage of the model, the firms select investment levels, $I_N \in [0, \infty) \forall N$, *sequentially*. This sequential investment assumption obviously affords one firm a strategic timing advantage, but can also be thought of as another way in which the firms' relative investment abilities differ. For example, an investment timing advantage in this case may reflect a difference in the firms' abilities to spot new investment opportunities; be a consequence of a disparity in the skills of the firms respective R&D departments or; more simply, be luck.

Recalling the equilibrium locations, prices and $m_A > m_B \geq 0$, the first stage profit functions are given by:

$$\begin{aligned}\pi_A &= \frac{1}{108}\frac{(4(m_A I_A - m_B I_B) + 9t)^2}{t} - \frac{1}{2}I_A^2 \\ \pi_B &= \frac{1}{108}\frac{(4(m_A I_A - m_B I_B) - 9t)^2}{t} - \frac{1}{2}I_B^2\end{aligned}$$

In the following sections, the equilibrium investment levels are determined where $(i)$ the *ex ante* efficient firm, firm $A$, moves first and; $(ii)$ the *ex ante* inefficient firm, firm $B$, moves first in sections 3.3.1 and 3.3.2 respectively.

### 3.3.1 Efficienct Firm Moves First

In this case firm $A$ has a Stackelberg leadership advantage in cost reduction and its investment decision is made taking firm $B$'s optimal response as given. Assuming $R_B(I_A)$, the solution to $\frac{\partial \pi_B}{\partial I_B} = 0$, is the best responce function of firm $B$ to any the investment choice of firm $A$, the equilibrium investment decision of firm $A$ is given by:

$$\arg\max_{I_A} \pi_A(I_A, R_B(I_A))$$

Solving this equation yields firm $A$'s equilibrium investment level:

$$I_A = \frac{18tm_A(16m_B^2 - 27t)}{216tm_A^2 - (8m_B^2 - 27t)^2} \tag{4}$$

Substituting $I_A$ into $R_B(I_A) = \frac{\partial \pi_B}{\partial I_B} = 0$ obtains:

$$I_B = \frac{18tm_B(16m_A^2 + 8m_B^2 - 27t)}{216tm_A^2 - (8m_B^2 - 27t)^2} \tag{5}$$

In order to obtain a reasonable equilibrium in the cost reduction stage two conditions must be satisfied: $(i)$ equilibrium investment levels must be non-negative and; $(ii)$ the second order conditions must be satisfied. However, in order for the game to yield a subgame perfect Nash equilibrium it is also necessary for the pure strategy equilbria in the cost reduction stage to be consistent with those in the price and location stages. This occurs if $\varphi_A(I_A) - \varphi_B(I_B) = m_A I_A - m_B I_B \in (-\frac{9}{4}t, \frac{9}{4}t)$. This leads to our first proposition.

**Proposition 1** *Equilibrium investment levels are non-negative, the second order conditions are satisfied and equilbrium investment in the first stage is consistent with a subgame perfect Nash equilibrium at all stages if and only* *if*

$$
\begin{aligned}
m_A^2 &\in (m_B^2, \frac{27}{16}t - \frac{m_B^2}{2}) \qquad (6) \\
m_B^2 &\in [0, \min\{m_A^2, \frac{27}{8}t - 2m_A^2\}) \qquad (7)
\end{aligned}
$$

**Proof.** In Appendix 5.1 ■

Equations (6) and (7) ensure that a reasonable equilbrium is not only ensured in the investment stage, but also that equilbrium investment decisions do not undermine the existence of a pure staregy Nash equilibrium for the entire game. However, they also imply that, for an equilibrium to exist, the *a priori* heterogeneity between the two firms cannot be too large. In fact, given the simplicity of (6) and (7) is not too difficult to formalise the maximum level of heterogeneity permissable in the model. This implies:

**Corollary 1** *Given conditions* (6) *and* (7) *and taking* $m_B^2$ *as given, the maximum a priori heterogeneity between the firms' cost reducing efficiency is given by*

$$\max\{m_A^2\} - m_B^2 = \frac{27}{16}t - \frac{3}{2}m_B^2 = \Upsilon$$

*If this holds, then:*
*(i)* $m_A^2 \in (0, \frac{27}{16}t)$ *and* $m_B^2 \in [0, \frac{27}{24}t)$ *and;*
*(ii) As* $m_B^2 \rightarrow \frac{27}{24}t$*,* $\Upsilon \rightarrow 0$

All of the above results can be derived very simply from equations (6) and (7) assuming $m_A^2 > m_B^2 \geq 0$ and so a formal proof is omitted. However, the importance of these results comes from their implications for the model with regards to ensuring a pure strategy Nash equilibrium. Given firm $A$ possesses both an investment timing and efficiency advantage at the begining of the game, **Corollary 1** simply states that the efficiency advantage cannot be too great or firm $A$'s equilibrium actions would undermine the stability of a (pure strategy) equilbrium solution. Quite simply, then, firm $A$'s effectiveness at cost reduction must be capped relative to that of firm $B$. If this were not the case, and the efficiency gap is greater than or equal to $\Upsilon$, then either: $(i)$ the equilibrium investment levels would be negative; $(ii)$ the second order conditions are violated; $(iii)$ the location stage has no pure strategy Nash equilibrium or; $(iv)$ some combination of $(i) - (iii)$ occurs.

Assuming that conditions (6) and (7) are met, it is then possible to make a number of observations regarding the equilibrium investment levels of the firms.

**Proposition 2** *For all $m_A^2$ and $m_B^2$ as defined in* (6) *and* (7)*:*

$(i)$ *$I_A$ and $I_B$ are strictly positive;*

$(ii)$ $\frac{\partial I_A}{\partial m_A} > 0$, $\frac{\partial I_B}{\partial m_A} < 0 \ \forall \ m_B^2 \in [0, \frac{27}{24}t)$;

$(iii)$ *$I_B \rightarrow 0$ as the a priori efficiency gap tends to $\Upsilon$ and;*

$(iv)$ $I_A > I_B$

**Proof.** In Appendix 5.2 ■

**Proposition 2** makes four observations regarding the equilibrium investment levels of the firms if conditions (6) and (7) are met. The first, that the equilibrium investment levels of both are strictly positive, rules out the posibility that one, or both, firms remain passive in the cost reduction stage. Moreover, it goes further than the assumption made to derive **Proposition 1**, that equilibrium investment levels simply be non-negative, and rules out any case in which one firm would simply "give up" and exit the market.

The second implies that the equilibrium investment decision of firm $A$ $(B)$ increases (decreases) as firm $A$ becomes more efficient relative to firm $B$. This occurs because, as cost reduction in this model is a strategic substitute, firm $A$ is able to use its investment timing advantage to temper its rival's cost reducing investment.[11] Therefore, as firm $A$ becomes a stronger competitor,

[11] Here:

$$R_B'(I_A) = \frac{8m_Am_B}{8m_B^2 - 27t} < 0$$

relative to firm $B$, it is better able to take advantage of its timing and efficiency advantages by investing more heavily in cost reduction and forcing firm $B$'s equilbrium investment to contract.

The third states that, as the efficiency gap converges to $\Upsilon$, firm $A$ becomes sufficiently aggressive that firm $B$'s optimal investment decision is to invest *nothing*. However, from **Corollary 1**, the maximum level of heterogenity is decreasing in $m_B^2$ and, as $m_B^2$ increases, the maximum *a priori* efficiency gap, $\Upsilon$, converges to zero. Therefore, as the weaker firm becomes relatively stronger ($m_B^2$ increases), the size of the efficiency gap required to drive $I_B$ to zero becomes smaller.[12] Therefore, it must be that the *ex ante* efficient firm becomes more aggresive when competing against relatively tougher rivals.

The final observation simply notes that the equilibrium investment levels of firm $A$ are always strictly larger than those of firm $B$. As cost reduction in this model acts as strategic substitute, an efficient firm has an incentive to invest heavily in cost reduction in the first stage to limit the cost reduction of its weaker rival. Therefore, as firm $A$ also possesses an investment timing advantage, even where the firms are (almost) symmetric, it will use this advantage to cement its dominance in the final product market through aggressive and preemptive investment. Thus, it is intuitive that $I_A > I_B$.

Despite all of these observations, both (6) and (7) ensure that, whilst firm $B$'s optimal investment strategy is restricted as its rival becomes more aggressive, it is always optimal to invest. In fact, firm $B$'s optimal investment decision has to satisfy two contrasting incentives: $(i)$ investing in cost reduction allows the firm to retain a small, but positive, market share but; $(ii)$ cost reductions exacerbate fierce price competition in the final product market. Which of these incentives dominates depends on the relative strengths of the two firms but, in general, it is optimal to make a small investment to protect a niche in the market whilst ensuring the *ex post* market is not too fiercely

[12] Observe $\frac{\partial \Upsilon}{\partial m_B^2} < 0$

competitive.

Knowing $I_A > I_B$ and $m_A^2 > m_B^2 \geq 0$ also implies that the *ex post* cost reduction of firm $A$ is larger than that of firm $B$; as the cost reduction form is simply given by

$$\varphi_i(I_i) = m_i I_i \ \forall \ i \in \{A, B\}$$

Intuitively, then, where $I_A > I_B$ and $m_A^2 > m_B^2 \geq 0$ it must hold that $\varphi_A(I_A) > \varphi_B(I_B)$. Therefore, a firm with an investment timing and efficiency advantage is able, through an aggressive, preemptive investment strategy, to maintain these competitive advantages into the location and prices stages. For firm $A$, the benefits of possessing a lower unit cost and serving a larger proportion of the market are the impetus for it to act aggressively in cost reduction as it is able to cement its position in the final product market as the dominant firm.

These equilibrium investment levels yield profits for each firm given by:

$$\pi_A = -\frac{3}{4}\frac{t(16m_B^2 - 27t)^2}{216tm_A^2 - (8m_B^2 - 27t)^2} \tag{8}$$

$$\pi_B = -\frac{81}{4}\frac{(16m_A^2 + 8m_B^2 - 27t)^2(8m_B^2 - 27t)}{[216tm_A^2 - (8m_B^2 - 27t)^2]^2} \tag{9}$$

Examining these profit functions leads to one observations: the profits obtained by both firms are strictly positive. Once again, this is a direct consequence of **Proposition 1**. Of course, if this were not the case then it would be impossible for an equilbirum to be sustained as one firm would have no unique optimal location.

All this leads to:

**Proposition 3** *If $m_A$ and $m_B$ are defined by* (6) *and* (7) *and firm $A$ has a Stackelberg leadership advantage in the cost reduction stage then:*

*(i)* $I_A > I_B > 0$*;*

*(ii)* $\varphi_A > \varphi_B$*;*

*(iii)* $\pi_A > \pi_B$ *and;*

*(iv) a pure strategy Nash equilibrium is enured across all stages of the game.*

**Proof.** Follows directly from **Propositions 1, 2** and Appendix 5.3 ■

The third observation, that firm $A$ obtains larger profits than firm $B$, should not come as a surprise given the previous propositions. As firm $A$ aggressively invests in preemptive cost reduction to become the *ex post* low cost firm and, because of this, is able to limit the market share of firm $B$ can attain. Quite simply, the efficient firm marginalises the inefficient firm in the final product market. Furthermore, as firm $A$ becomes relatively stronger, the *ex post* cost differential becomes larger until $D_A \rightarrow 1$ and $D_B \rightarrow 0$. Obviously, as the demand for firm $B$ becomes smaller and the cost differential incrases then the profits associated with competition are pushed towards zero.

The implications of this result are clear. Where a firm possesses both an investment timing and efficiency advantage, the *ex ante* efficient firm invests aggresively in cost reduction in order to cement its place as the dominant firm in the final product market. The efficient firm's investment generates lower *ex post* unit costs, serves a greater proportion of demand and yields larger profits than its *a priori* (and, indeed, *ex post*) inefficient rival. Nevertheless, whilst the *ex ante* efficient firm invests to cement its position as the market leader, the *ex ante* inefficient firm still possesses an incentive to invest. However, the size of the investment must always remain relatively small for two reasons: first, investment is used in order to maintain the weaker firm's (niche) market position by reducing the size of the *ex post* cost asymmetry and; second, the investment is kept relatively small in order to mitigate the effects of increased price competition.

### 3.3.2 Inefficient Firm Moves First

In this case firm $B$ has a Stackelberg leadership advantage in cost reduction and makes its investment decision taking firm $A$'s optimal response as given. Assuming $R_A(I_B)$, the solution to $\frac{\partial \pi_A}{\partial I_A} = 0$, is the best response function of firm $A$ to the investment choices of firm $B$, the equilibrium investment decision of firm $B$ is given by:

$$\arg\max_{I_B} \pi_B(R_A(I_B), I_B)$$

Solving this equation yields firm $B$'s equilibrium investment level:

$$I_B = \frac{18tm_B(16m_A^2 - 27t)}{216tm_B^2 - (8m_A^2 - 27t)^2} \tag{10}$$

Substituting $I_A$ into $R_B(I_A) = \frac{\partial \pi_B}{\partial I_B} = 0$ obtains:

$$I_A = \frac{18tm_A(8m_A^2 + 16m_B^2 - 27t)}{216tm_B^2 - (8m_A^2 - 27t)^2} \tag{11}$$

As in the previous case, a reasonable (pure strategy) equilibrium exists in the cost reduction stage if two conditions are satisfied: $(i)$ equilibrium investment levels must be non-negative and; $(ii)$ the second order conditions must be satisfied. However, in order for the game to yield a subgame perfect Nash equilibrium it is also necessary for the pure strategy equilbria in the cost reduction stage to be consistent with those in the price and location stages. Again, this occurs if $\varphi_A(I_A) - \varphi_B(I_B) = m_A I_A - m_B I_B \in (-\frac{9}{4}t, \frac{9}{4}t)$. This leads to:

**Proposition 4** *Equilibrium investment levels are non-negative, the second order conditions are satisfied and equilbrium investment levels in the first stage are consistent with a subgame perfect Nash equilibrium at all stages if* *and only if*

$$m_A^2 \in (m_B^2, \min\{\frac{27}{16}t, \frac{27}{8}t - 2m_B^2\}) \quad (12)$$

$$m_B^2 \in [0, \min\{m_A^2, \frac{27}{16}t - m_B^2\}) \quad (13)$$

The conditions imposed in equations (12) and (13) ensure that a reasonable equilibrium exists in the cost reduction stage that does not undermine a subgame perfect Nash equilbrium. However, given the form of these restrictions it no longer so simple to generalise the maximum level of heterogeneity supported within the model, but we can still infer from these conditions that the heterogeneity between the firms cannot be too large. If this the efficiency gap is too large then either: $(i)$ the equilibrium investment levels are negative; $(ii)$ the second order conditions are violated; $(iii)$ the location stage has no pure strategy Nash equilibrium or; $(iv)$ some combination of $(i) - (iii)$ occurs.

Keeping this result in mind, it is possible to make some observations regarding the equilibrium investment levels of both firms. However, before this it is necessary to define a critical value of $m_B^2$, denoted $m_B^I$, such that, for any relevant $m_A^2$, the equilibrium investment levels of the firms are equal if $m_B^2 = m_B^I$.[13] For all other values of $m_B^2$, $I_A \neq I_B$. With this in mind, we obtain:

**Proposition 5** *For all $m_A^2$ and $m_B^2$ as defined in* (12) *and* (13)*:*

$(i)$ *$I_A$ and $I_B$ are strictly positive;*

$(ii)$ $\frac{\partial I_A}{\partial m_B} < 0$, $\frac{\partial I_B}{\partial m_B} > 0 \; \forall \; m_A^2 \in (0, \frac{27}{16}t)$ *and;*

$(iii)$ $I_A > I_B$ *if,* $\forall \; m_A^2 \in (0, \frac{27}{16}t)$, $m_B^2 < m_B^I \in [0, \max\{m_B^2\})$

**Proof.** In appendix 5.5 ■

**Proposition 5** makes three observations about the equilibrium investment levels where the inefficient firm possesses a Stackelberg leadership ad-

[13] A formal definition and derivation of $m_B^I$ can be found in Appendix 5.5.

vantage in the cost reduction stage and conditions (12) and (13) are met. The first, that equilibrium investment levels are strictly positive, again rules out that the posibility that one, or both, firms remain passive in the cost reduction stage. As in the previous case, this observation goes beyond the assumption that required equilibrium investment levels be non-negative, which was a key assumption made to derive **Proposition 4**.

The second implies either firm can be rendered (almost) passive during the investment stage. That is, because the equilibrium investment levels of firm $A$ ($B$) to decrease (increase) as firm $B$ becomes relatively more efficient, the intial level of heterogeneity plays an important role in determing the equilibrium investment levels. As in the previous case, firm $B$ remains passive if the initial level of heterogeneity is very large. In contrast, firm $A$ is only rendered passive where the initial level of heterogeneity between the firms is sufficiently small. Simply, where the firms' initial parameters are more "equal," a timing advantage enables the *a priori* weaker firm to preemptively invest in cost reduction and restrict the investment level of firm $A$. Thus, it can become a more fierce competitor when the firms are more symmetric.

The third observation extends this analysis further. It argues that, for a given $m_A^2$, if $m_B^2 \leq m_B^I$ the inefficient firm is very weak relative to its rival and, consequently, its ability to preemptively invest in cost reduction is weak also. Thus, even with a first mover advantage in the cost reduction stage, the *ex ante* weaker firm is unable and unwilling to act aggresively (or at least reasonably) to restrict its rival's investment and, consequently, the equlibrium investment level of firm $B$ is less than or equal to that of firm $A$. However, once $m_B^2 > m_B^I$, the inefficient firm becomes sufficiently strong, relative to its rival, to be able to take advantage of its timing advantage and cement a position in the market; beyond simply filling a niche. In this case, firm $B$ becomes more effective at preemptively investing and, therefore, invests more. In turn, this drives down the equilibrium investment decisions of firm $A$ and firm $B$ is induced to invest sufficiently to drive $I_A$ below $I_B$.

Therefore, if the *a priori* efficiency gap is not too large then the *a priori* weaker firm will invest **more** than its *ex ante* efficient rival.

The equilbrium investment observations imply firm $B$ can become the "investment leader," but are not sufficient to ensure that firm $B$ will become the *ex post* low cost firm too. Rather, because $m_A^2 > m_B^2 > 0$, ensuring $I_B > I_A$ does not directly imply $m_B I_B > m_A I_A$. Instead, the cost reduction of firm $B$ will only be larger if its investment levels, relative to firm $A$'s, are sufficiently large to overcome this relative inefficiency. Comparing the firms' cost reductions yields:

$$\varphi_A - \varphi_B = \frac{18tm_A(8m_A^4 - 27t(m_A - m_B)(m_A + m_B))}{216tm_B^2 - (8m_A^2 - 27t)^2}$$

Again, it is necessary to define a critical value of $m_B^2$, denoted $m_B^\varphi$, such that, for any relevant $m_A^2$, the equilibrium cost reduction levels of the firms are equal if $m_B^2 = m_B^\varphi$.[14] For all other values of $m_B^2$, $\varphi_A \neq \varphi_B$. Consequently, one observes:

**Proposition 6** *For all $m_A^2$ and $m_B^2$ as defined in* (12) *and* (13)*:*
*(i)* $\varphi_A > \varphi_B$ *if* $\forall\ m_A^2 \in (0, \frac{27}{16}t)$, $m_B^2 < m_B^\varphi \in [0, \max\{m_B^2\})$
*(ii)* $m_B^\varphi > m_B^I$

**Proof.** In appendix 5.6 ■

Both observations in **Proposition 6** imply that an *ex ante* inefficient firm can become the *ex post* low cost firm if: (*i*) it is not too inefficient relative to its rival and; (*ii*) it has a Stackelberg leadership advantage in the cost reduction stage. Recall from **Proposition 5** that *ceteris paribus* the equilibrium investment level of firm $B$ ($A$) increases (decreases) as firm $B$ becomes relatively more efficient and, once $m_B^2 > m_B^I$, firm $B$'s equilibrium investment is larger than that of firm $A$. However, as firm $B$ is relatively

[14] A formal definition and derivation of $m_B^\varphi$ can be found in Appendix 5.6.

inefficient, for it to become the *ex post* low cost firm it is not sufficient for firm $B$ to simply invest more than firm $A$ but, rather, it must invest **enough** to overcome this disadvantage. This occurs once $m_B^2 > m_B^\varphi > m_B^I$. That is, once difference between $I_A$ and $I_B$ is sufficiently in firm $B$'s favour, or the *a priori* efficiency gap sufficiently small, firm $B$ is induced to reduce its costs to such an extent that it can overcome its initial inefficiency and become the low cost firm.

Finally, taking into account the equilibrium investment levels, the corresponding profit functions are given by:

$$\pi_A = -\frac{81}{4}\frac{t^2(8m_A^2 - 27t)(8m_A^2 + 16m_B^2 - 27)^2}{[216tm_B^2 - (8m_A^2 - 27t)^2]^2} \tag{14}$$

$$\pi_B = -\frac{3}{4}\frac{t(16m_A^2 - 27t)^2}{216tm_B^2 - (8m_A^2 - 27t)^2} \tag{15}$$

Similar to investment and cost reduction, it is possible to make some observations regarding the profit levels of the firms. Before proceeding, it is necessary to define a final critial value of $m_B^2$, denoted $m_B^\pi$, such that, for any relevant $m_A^2$, the equilibrium profit levels of the firms are equal if $m_B^2 = m_B^\pi$.[15] For all other values of $m_B^2$, $\pi_A \neq \pi_B$. This leads to:

**Proposition 7** *For all* $m_A^2$ *and* $m_B^2$ *as defined in* (12) *and* (13)*:*
*(i)* $\pi_A,\ \pi_B > 0$
*(ii)* $\pi_A > \pi_B$ *if* $\forall\ m_A^2 \in (0, \frac{27}{16}t)$, $m_B^2 < m_B^\pi \in [0, \max\{m_B^2\})$
*(iii)* $m_B^\pi > m_B^\varphi > m_B^I$

**Proof.** In appendix 5.7 ■

The first observation of **Proposition 7** states that the equilibrium profit levels of both firms are strictly positive if $m_A^2$ and $m_B^2$ are defined as in (12) and (13). Of course, were this not the case then it would be impossible

[15] A formal definition and derivation of $m_B^\pi$ can be found in Appendix 5.7.

to sustain an (pure strategy) equilbirum as one firm would have no unique optimal location. Consequently, **Proposition 4** ensures this holds by the underlysing assumption that a pure strategy Nash equilibrium is ensured in the location and price stages.

The second and third observations imply an *ex ante* inefficient firm with a Stackelberg advantage in the cost reduction stage can only become the *ex post* dominant firm, in all respects, if and only if $m_B^2 > m_B^\pi$. Thus, only when the *ex ante* inefficient firm is not too inefficient relative to its rival is it able to fully take advantage of its timing advantage and become the market leader. The rationale behind this result is analogous to that behind **Proposition 6**. As firm $B$ becomes relatively more efficient, the equilibrium investment levels of firms $A$ and $B$ decrease and increase respectively. Therefore, as the *a priori* efficiency gap becomes smaller, the *ex ante* inefficient firm becomes better able to preemptively invest in cost reduction. Once the initial efficiency gap is small enough, $m_B^2 > m_B^\pi$, firm $B$ is able to use its timing advantage to overcome its initial disadvantage and become the market leader in terms of costs, demand and profits.

An additional, and unstated, result can be found within **Proposition 7**. For all $m_A^2$, there exists a possible range of $m_B^2$ such that an *a priori* inefficient firm can become the *ex post* low cost firm but earn *lower* profits in the final product market. Thus, if $m_B^2 \in (m_B^\varphi, m_B^\pi)$, firm $B$ invests more, has lower *ex post* production costs and serves a larger proportion of the market but yields smaller profits. Over this range, the costs incurred by firm $B$ in becoming the *ex post* low cost firm are are sufficiently large to outweigh the cost and demand benefits gained by the *ex post* efficient firm. Whilst the inefficient firm can obtain a larger market share and possesses lower unit costs, in doing so, the additional investment costs prevent it from yielding larger profits than its *ex ante* efficient rival.

Considering all of these propositions together, it is possible to state:

**Summary 1** *If $m_A^2$ and $m_B^2$ are defined by* (12) *and* (13), *$m_A^2 > m_B^2$ and* *firm B possesses a Stackelberg leadership in cost reduction, four potential equilibria may obtain for a given* $m_A^2 \in (0, \frac{27}{16}t)$*:*

1. $m_B^2 \in [0, m_B^I)$: $I_A > I_B$, $m_A I_A > m_B I_B$ and $\pi_A > \pi_B$

2. $m_B^2 \in [m_B^I, m_B^\varphi)$: $I_A \leq I_B$, $m_A I_A > m_B I_B$ and $\pi_A > \pi_B$

3. $m_B^2 \in [m_B^\varphi, m_B^\pi)$: $I_A < I_B$, $m_A I_A \leq m_B I_B$ and $\pi_A > \pi_B$

4. $m_B^2 \in [m_B^\pi, m_A^2)$: $I_A < I_B$, $m_A I_A < m_B I_B$ and $\pi_A \leq \pi_B$

Finally, a pure strategy Nash equilibrium is ensured across all stages of the game.

**Proof.** Follows directly from **Propositions 4, 5, 6 and 7** ■

This result has a number of interesting implications. However, the key result is simply, as the efficiency gap becomes smaller, an *ex ante* inefficient firm becomes a tougher competitor if it has a Stackelberg leadership advantage in the cost reduction stage. The reason for this is simple. When the efficiency gap is sufficiently large the inefficient firm is always either unable or unwilling to invest aggressively in cost reduction. As the initial efficiency gap becomes smaller, the inefficient firm is better able to take advantage of its first mover advantage in the cost reduction stage and adopts an increasingly aggressive investment strategy. Therefore, by moving first it is able to compensate its inefficiency by restricting its rival's investment decision and manipulating a better *ex post* situation for itself. Consequently, the initial heterogenity between the firms plays an important role in determing the market outcome and, in general, relatively weak firms "give up" whilst stronger firms "fight."

# 4 Conclusion

This paper examines the effects of endogenous cost selection on a firm's product and pricing decisions where the firms are heterogeneous with respect to their *ex ante* investment efficiencies and investment timing. In doing so, two results, unique to the literature, are obtained.

First, where an *ex ante* efficient firm possesses Stackelberg leadership in cost reduction, they generate lower costs, greater demand and yield larger profits ex post than their *ex ante* inefficient rival. This result suggests, where a firm possesses both an investment timing and efficiency advantage, the *ex ante* efficient firm invests aggresively in cost reduction in order to cement its place as the dominant firm in the final product market. Thus, the impetus for the efficient firm to invest are the benefits of being the dominant firm in the final product market. In contrast, the *ex ante* inefficient firm's incentive to invest is, ultimately, an attempt to balance two contrasting incentives: $(i)$ to increase investment in order to maintain its market position by reducing the size of the *ex post* cost asymmetry and; $(ii)$ to reduce investment so as to mitigate increased price competition in the final product market. In general, the latter incentive dominates as the *a priori* efficient firm is able to preemptively invest in cost reduction and force its rival to act "soft" in order to prevent itself undermining its profits through tough price compeition.

Second, if an *ex ante* inefficient firm is the Stackelberg leader in the cost-reduction stage, there are a four potential equilibrium outcomes that depend on the relative efficiencies of the two firms. If the *a priori* efficiency gap is large, then the *ex ante* inefficient firm invests simply to protect some market share. This result obtains because, where the *ex ante* firm is very weak and unable to make the most of its timing advantage, investment serves only to increase price competition in the final product market and undermine its profits. Therefore, for a large efficiency gap, the *ex ante* inefficient firm "gives up." However, as the initial efficiency gap becomes smaller, the inefficient firm becomes better able to preemptively invest in cost reduction and temper the investment decision of its rival firm. Consequently, the firm begins to "fight." However, it is only when the gap between the two firms is sufficiently small that the inefficient firm is able to overcome this efficiency disadvantage, using its timing advantage to either invest more than its *a priori* efficient rival; become the *ex post* low cost firm; or become the dominant firm in the final product market. Therefore, the initial heterogenity between the firms plays an important role in determing the market outcome and, in general, relatively weak firms "give up" whilst stronger firms "fight."

These results are unique in the literature and suggest that the firms' ability to innovate - in terms of timing and efficiency - play a crucial role in determining the composition of the final product market. Furthermore, whilst the *ex post* low cost firm does, in general, become the market leader it is not *a priori* obvious whether this will be an *ex ante* efficient or inefficient firm. Consequently, simply examining the structure of a market in which firms face asymmetric unit costs may miss some crucial underlying mechanisms.

It would be interesting to examine this model within the framework of an endogenous leadership game and examine whether this "stronger" firm's incentives to preemptively invest are upheld. That is, despite the possibility that the *ex ante* inefficient firm can become the *ex post* market leader if it preemptively invests, this only obtains if the *a priori* efficiency gap is sufficiently small. In contrast, as with the literature regarding endogenous Stackelberg leaderhip, the *ex ante* efficient firm always finds it optimal to invest first. This, however, is left for future research.

# 5 Appendix

## 5.1 Proof of Proposition 1

**Proof.** The second order conditions are given by:

$$\frac{\partial^2 \pi_A}{\partial I_A^2} = \frac{216tm_A^2}{(8m_B^2 - 27t)^2} - 1 < 0 \tag{A1}$$

$$\frac{\partial^2 \pi_B}{\partial I_B^2} = \frac{8m_B^2}{27t} - 1 < 0 \tag{A2}$$

From equation $(A1)$ it is possible to observe that the denominators of equations (4) and (5) are strictly negative. Given this, it is necessary that the numerators of (4) and (5) must be non-positive for $I_A$ and $I_B \geq 0$ to hold. Given that $m_A > m_B > 0$ and $t > 0$, this implies:

$$\begin{aligned} 16m_B^2 - 27t &\leq 0 \\ 16m_A^2 + 8m_B^2 - 27t &\leq 0 \end{aligned}$$

Rearraning these inequalities, noting $m_A > m_B > 0$ and $t > 0$, yields:

$$m_A^2 \in (m_B^2, \frac{27}{16}t - \frac{m_B^2}{2}] \tag{A3}$$

$$m_B^2 \in [0, \min\{m_A^2, \frac{27}{8}t - 2m_A^2\}) \tag{A4}$$

The restrictions in $(A3)$ and $(A4)$ ensure that equilibrium investment is non-negative, but it also necessary to check that they are compatible with the second order conditions. This requires:

$$m_A^2 < \frac{(8m_B^2 - 27t)^2}{216t} \tag{A5}$$

$$m_B^2 < \frac{27}{8}t \tag{A6}$$

Equations $(A5)$ and $(A6)$ offer the maximum *ex ante* cost reducing efficiencies for each firm that are consistent with the second order conditions. It is most obvious that equation $(A6)$ is always met as it is fairly trivial to show, from $(A4)$, and assuming $m_A = m_B$, that $\max\{m_B^2\} = \frac{27}{24}t < \frac{27}{8}t$. Knowing that $m_B^2 \in [0, \frac{27}{24}t)$ makes it easier to demonstrate that equation $(A5)$ must always hold too. Taking $m_B^2 \in [0, \frac{27}{24}t)$ as given it is possible to show

$$\max\{m_A^2\} - \frac{(8m_B^2 - 27t)^2}{216t} = -\frac{1}{432}\frac{(8m_B^2 - 27t)(16m_B^2 - 27t)}{t}$$

which is strictly negative for all $m_A^2$ and $m_B^2$ pairs as defined by $(A3)$ and $(A4)$. Therefore, the second conditions hold under these restrictions.

Finally it is necessary to evaluate whether the restirctions in (6) and (7) yield equilibrium investment levels consistent with subgame perfect Nash equilibrium for the entire model. For this to occur we require $m_A I_A - m_B I_B \in (-\frac{9}{4}t, \frac{9}{4}t)$. The *ex post* cost differential is given by

$$m_A I_A - m_B I_B = \frac{18t[27t(m_B - m_A)(m_A + m_B) - 8m_B^4]}{216tm_A^2 - (8m_B^2 - 27t)^2}$$

which is strictly positive given the conditions in $(A3)$ and $(A4)$. Therefore, it is necessary that $m_A I_A - m_B I_B < \frac{9}{4}t$, or

$$-\frac{243}{4}\frac{t^2(16m_A^2 + 8m_B^2 - 27t)}{216tm_A^2 - (8m_B^2 - 27t)^2} < 0$$

Of course, holds if and only if $m_A^2 < \frac{27}{16}t - \frac{m_B^2}{2}$. This implies that the restrictions derived from the second order and non-negative investment conditions are generally consistent with $m_A I_A - m_B I_B \in (-\frac{9}{4}t, \frac{9}{4}t)$. However, it is necessary to remove a single case from $(A3)$, where $m_A^2 = \frac{27}{16}t - \frac{m_B^2}{2}$, which yields (6), and (7) remains identical to $(A4)$. $\blacksquare$

## 5.2 Proof of Proposition 2

**Proof.** $(i)$ It is trivial to show that the equilibrium investment levels are strictly positive as this follows directly from the proof of **Proposition 1**.

$(ii)$ Taking $m_A^2$ and $m_B^2$ defined as in (6) and (7):

$$\begin{aligned}
\frac{\partial I_A}{\partial m_A} &= \frac{-18t(16m_B^2 - 27t)(216tm_A^2 + (8m_B^2 - 27t)^2)}{[216tm_A^2 + (8m_B^2 - 27t)^2]^2} > 0 \\
\frac{\partial I_B}{\partial m_A} &= \frac{-288tm_A m_B(8m_B^2 - 27t)(16m_B^2 - 27t)}{[216tm_A^2 + (8m_B^2 - 27t)^2]^2} < 0
\end{aligned}$$

$(iii)$ Recall firm $B$'s investment function given in equation (5) and assume that the efficiency gap is given by

$$\Upsilon = \frac{27}{8}t - \frac{3}{2}m_B^2$$

If this is the case then $I_B = 0$. However, it is known that $I_B > 0$ for all other levels of the efficiency gap (**Proposition 1**) and that, for any $m_B^2$, $\frac{\partial I_B}{\partial m_A} < 0$. Therefore, taking $m_B^2$ as given, it must be that increasing the size of the efficiency gap reduces $I_B$ until it equals zero.

$(iv)$ Where equations (6) and (7) hold:

$$I_A - I_B = \frac{-18t[(m_A - m_B)(16m_A m_B + 27t) + 8m_B^3]}{216tm_A^2 + (8m_B^2 - 27t)^2} > 0$$

This is strictly positive for all relevant $m_A^2$ and $m_B^2$ pairs which implies that $I_A > I_B$ in all relevant cases. ■

## 5.3 Proof of Proposition 3 ($\pi_A > \pi_B$)

**Proof.** $(iii)$Taking $m_B^2 \in [0, \frac{27}{24}t)$ as given, we note that $\pi_A - \pi_B = 0$ if and only if $m_A^2 = \underline{m_A^2}$ or $\overline{m_A^2}$ where

$$\underline{m_A^2} = \frac{1}{576}\frac{243t^2(32m_B^2 - 81t) + \Lambda}{t(8m_B^2 - 27t)}$$
$$\overline{m_A^2} = \frac{1}{576}\frac{243t^2(32m_B^2 - 81t) - \Lambda}{t(8m_B^2 - 27t)}$$

and

$$\Lambda = \sqrt{-3t(16m_B^2 - 81t)(32m_B^2 - 81t)(16m_B^2 - 27t)^3}$$

It is obvious that the domains of both $\underline{m_A^2}$ and $\overline{m_A^2}$ are given by $m_B^2 \in [0, \frac{27}{16}t] \cap [\frac{81}{32}t, \frac{27}{8}t) \cap (\frac{27}{8}t, \frac{81}{16}t]$. However, from **Proposition 1** and **Corollary 1** it is know that $m_B^2 \in [0, \frac{27}{24}t)$, which is strictly contained in this domain and so this does not pose a problem.

As $\pi_A - \pi_B = 0$ only if $m_A^2 = \underline{m_A^2}$ or $\overline{m_A^2}$ it must be that $\pi_A - \pi_B$ has a constant for all $m_A^2 \in (\underline{m_A^2}, \overline{m_A^2})$ and so we examine if: $(i)$ $\underline{m_A^2} < \min\{m_A^2\}$ and; $(ii)$ $\max\{m_A^2\} < \overline{m_A^2}$. Simply, that $m_A^2 \in (\underline{m_A^2}, \overline{m_A^2})$.

First, $\underline{m_A^2} = 0$ if and only if $m_B^2 \in \{0, \frac{81}{32}t\}$ and implies that $\underline{m_A^2}$ is continuous, with the same sign, over $m_B^2 \in [0, \frac{27}{24}t)$. Therefore, we can evaluate

$$\min\{m_A^2\} - \underline{m_A^2} = m_B^2 - \underline{m_A^2}$$

for all $m_B^2 \in [0, \frac{27}{24}t)$. Doing so yields

$$\min\{m_A^2\} - \underline{m_A^2} = \frac{1}{576}\frac{288tm_B^2(16m_B^2 - 81t)^2 + 19683t^3 - \Lambda}{t(8m_B^2 - 27t)} \tag{A7}$$

Equation $(A7)$ is zero if and only if $m_B^2 = 0, \frac{189}{64}t \pm \frac{27}{64}t\sqrt{17}$. Therefore, it is quite trivial to check that this $\min\{m_A^2\} \geq \underline{m_A^2}$ for all $m_B^2 \in [0, \frac{27}{24}t)$. Hence, for all $m_B^2$ defined as in equation (7), $m_A^2$ is strictly larger than $\underline{m_A^2}$.

Second, $\overline{m_A^2} = 0$ if and only if $m_B^2 = \frac{81}{32}t$. Again, this implies that $\overline{m_A^2}$ is continuous over $m_B^2 \in [0, \frac{27}{24}t)$. This only leaves us to evaluate

$$\max\{m_A^2\} - \overline{m_A^2} = \frac{27}{16}t - \frac{m_B^2}{t} - \overline{m_A^2}$$

for all $m_B^2 \in [0, \frac{27}{24}t)$. With some manipulation

$$\max\{m_A^2\} - \overline{m_A^2} = \frac{1}{576} \frac{-9t(16m_B^2 - 27t)^2 + \Lambda}{t(8m_B^2 - 27t)} \tag{A8}$$

Equation $(A8)$ is zero if and only if $m_B^2 = \frac{27}{16}t$ which lies outside the relevant range of $m_B^2$. In this case, it is trivial to check that this is strictly negative. Hence, for all $m_B^2$ defined as in equation (7), $m_A^2$ is strictly smaller than $\overline{m_A^2}$.

The implication of equations $(A7)$ and $(A8)$ being positive and negative respectively, for all relevant $m_A^2$ and $m_B^2$ pairs, is that $m_A^2 \in (\underline{m_A^2}, \overline{m_A^2})$. However, it is already known that between these critical values $\pi_A - \pi_B$ must have a constant sign. Thus, taking $m_A^2 = t$ and $m_B^2 = 0$ we obtain

$$\pi_A - \pi_B = \frac{294}{361}t > 0$$

Naturally, if $\pi_A - \pi_B$ is strictly positive for these value of $m_A^2$ and $m_B^2$, it must hold from the signs of equations $(A7)$ and $(A8)$ that this holds for all relevant values of $m_A^2$ and $m_B^2$. Thus, $\pi_A - \pi_B > 0$ holds for all $m_A^2$ and $m_B^2$ as defined in equations (6) and (7). ■

## 5.4 Proof of Proposition 4

**Proof.** The second order conditions are given by:

$$\frac{\partial^2 \pi_A}{\partial I_A^2} = \frac{8m_A^2}{27t} - 1 < 0 \tag{A9}$$

$$\frac{\partial^2 \pi_B}{\partial I_B^2} = \frac{216tm_B^2}{(8m_A^2 - 27t)^2} - 1 < 0 \tag{A10}$$

From equation $(A9)$ it is possible to observe that the denominators of equations (10) and (11) are strictly negative. Given this, it is necessary that the numerators of (10) and (11) must be non-positive for $I_A$ and $I_B \geq 0$ to hold. Given that $m_A > m_B > 0$ and $t > 0$, this implies:

$$\begin{aligned} 16m_A^2 - 27t &\leq 0 \\ 8m_A^2 + 16m_B^2 - 27t &\leq 0 \end{aligned}$$

Rearraning these inequalities, noting $m_A > m_B > 0$ and $t > 0$, yields:

$$m_A^2 \in (m_B^2, \min\{\frac{27}{16}t, \frac{27}{8}t - 2m_B^2\}] \tag{A11}$$

$$m_B^2 \in [0, \min\{m_A^2, \frac{27}{16}t - m_B^2\}) \tag{A12}$$

The restrictions in $(A11)$ and $(A12)$ ensure that equilibrium investment is non-negative, but it also necessary to check that they are compatible with the second order conditions. This requires:

$$m_A^2 < \frac{27}{8}t \tag{A13}$$

$$m_B^2 < \frac{(8m_A^2 - 27t)^2}{216t} = \widetilde{m_B} \tag{A14}$$

Equations $(A13)$ and $(A14)$ offer the maximum *ex ante* cost reducing efficiencies for each firm that are consistent with the second order conditions. It is most obvious that equation $(A13)$ is always met as it is fairly trivial to show $\max\{m_A^2\} = \frac{27}{16}t$. It is then trivial to note that $\widetilde{m_B}$ is strictly positive and, consequently, that $\widetilde{m_B} > \min\{m_B^2\} = 0$. Thus, it is then a case of simply ensuring that $\max\{m_B^2\} < \widetilde{m_B}$.

First, for $m_A^2 \in (0, \frac{27}{24}t]$, $\max\{m_B^2\} = m_A^2$. Thus

$$\max\{m_B^2\} - \widetilde{m_B} = -\frac{1}{216}\frac{64m_A^4 - 81t(8m_A^2 - 9t)}{t}$$

This equals zero where $m_A^2 = \frac{81}{16}t \pm \frac{27}{16}t\sqrt{5}$ both of which are strictly larger than $\frac{27}{24}t = \max\{m_B^2\}$. Therefore, for all $m_A^2 \in (0, \frac{27}{24}t]$ and over all $m_B^2 \in (0, \frac{27}{24}t)$ it must hold that $\max\{m_B^2\} - \widetilde{m_B}$ has the same sign. It is then easy to show $\max\{m_B^2\} - \widetilde{m_B} < 0$ for all $m_A^2 \in (0, \frac{27}{24}t]$.

Second, where $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t]$, $\max\{m_B^2\} = \frac{27}{16}t - \frac{m_A^2}{2}$. This implies

$$\max\{m_B^2\} - \widetilde{m_B} = -\frac{1}{432}\frac{(8m_A^2 - 27t)(16m_A^2 - 27t)}{t}$$

This is negative for all $m_A^2 \in (0, \frac{27}{16}t)$ and equal to zero if and only if $m_A^2 = \frac{27}{16}t$. However, all possible $m_B^2$ are strictly less than $\max\{m_B^2\}$, which doesn't cause a problem. Therefore, this implies that the second order conditions are met for all $m_A^2$ and $m_B^2$ pairs.

Finally, it is necessary to check that $\varphi_A - \varphi_B = m_A I_A - m_B I_B \in (-\frac{9}{4}t, \frac{9}{4}t)$ also holds, or that the *ex post* cost asymmetry allows for a subgame perfect Nash equilibrium. This is true if and only if

$$m_A I_A - m_B I_B > -\frac{9}{4}t \quad \text{(A15)}$$

$$m_A I_A - m_B I_B < \frac{9}{4}t \quad \text{(A16)}$$

From equation $(A15)$ we obtain

$$\frac{243}{4}\frac{t^2(8m_A^2 + 16m_B^2 - 27t)}{216tm_B^2 - (8m_A^2 - 27t)^2} > 0$$

which implies $m_A^2 + 2m_B^2 < \frac{27}{8}t$. Equation $(A16)$ on the other hand requires

$$\frac{9}{4}\frac{t(8m_A^2 - 27t)(16m_A^2 - 27t)}{216tm_B^2 - (8m_A^2 - 27t)^2} < 0$$

This equation implies that $m_A^2 < \frac{27}{16}t$.

Combining this final elements with $(A11)$ and $(A12)$ the non-negative investment and second order conditions it is apparent that we must remove a single case (where $m_A^2 = \frac{27}{16}t$). Doing this obtains the restriction in equations (12) and (13). ■

## 5.5 Proof of Proposition 5

**Proof.** $(i)$ it is trivial to demonstrate that $I_A$ and $I_B$ are strictly positive as this follows directly from **Proposition 4**.

$(ii)$Taking $m_A^2 \in (m_B^2, \frac{27}{16}t)$ as given and $m_B^2$ defined as in (13):

$$\begin{aligned}\frac{\partial I_A}{\partial m_B} &= \frac{-288tm_Am_B(8m_A^2 - 27t)(16m_A^2 - 27t)}{[216tm_B^2 + (8m_A^2 - 27t)^2]^2} < 0 \\ \frac{\partial I_B}{\partial m_B} &= \frac{-18t(16m_A^2 - 27t)(216tm_B^2 + (8m_A^2 - 27t)^2)}{[216tm_B^2 + (8m_A^2 - 27t)^2]^2} > 0\end{aligned}$$

Consequently, it is easy to see that, taking $m_A^2$ as given, an increase to $m_B^2$ decreases (increases) the equilibrium investment levels of firm $B$ ($A$).

$(iii)$Where equations (12) and (13) hold

$$I_A - I_B = -\frac{18t[(m_B - m_A)(16m_Am_B + 27t) + 8m_B^3]}{216tm_B^2 + (8m_A^2 - 27t)^2}$$

This function is equal to zero if and only if $m_B^2 = m_B^I$ where

$$m_B^I = \frac{1}{1024}\frac{[16m_A^2 - 27t \pm \sqrt{\Omega}]^2}{m_A^2}$$

and

$$\Omega = 729t^2 + 864tm_A^2 - 256m_A^4$$

Of course, for the function to be continuous we require that $m_A^2 \neq 0$ and $\Omega \geq 0$. The first of these follows from **Proposition 4**, but $\Omega \geq 0$ if and only if

$$m_A^2 \in [\frac{27}{16}t(1-\sqrt{2}), \frac{27}{16}t(1-\sqrt{2})]$$

This clearly contains all relevant $m_A^2$ and so is not an issue.

Note that there are two critical values of $m_B^2$ which we will denote as

$$\begin{aligned} m_B^{I+} &= \frac{1}{1024}\frac{[16m_A^2 - 27t + \sqrt{\Omega}]^2}{m_A^2} \\ m_B^{I-} &= \frac{1}{1024}\frac{[16m_A^2 - 27t - \sqrt{\Omega}]^2}{m_A^2} \end{aligned}$$

It is possible to observe that $m_B^{I+} > 0$. This is because $m_B^{I+} = 0$ could only occur if $m_A^2 = 0$ but, if this were the case, $m_B^{I+} = \frac{0}{0}$ which is not defined. Therefore, $m_B^{I+}$ is continuous over $m_A^2 \in (0, \frac{27}{16}t)$ and it is trivial to demonstrate that it is always positive over this range. In contrast, $m_B^{I-} > 0$ and if $|m_A^2| = \frac{27}{8}t$ and is, therefore, continuous over all $m_A^2 \in (0, \frac{27}{16}t)$ also. However, it is possible to demonstrate that $m_B^{I-}$ is always positive over this range.

Given that both $m_B^{I+}$ and $m_B^{I-}$ are strictly positive over the relevant range, it is useful to observe

$$m_B^{I-} - m_B^{I+} = -\frac{1}{256}\frac{(16m_A^2 - 27t)\sqrt{\Omega}}{m_A^2} > 0$$

It is also important to note that the second order conditions are violated if[16]

$$m_B^2 > \frac{(8m_A^2 - 27t)^2}{216t} = \Gamma$$

Now

$$m_B^{I+} - \Gamma = \frac{1}{13824}\frac{(16m_A^2 - 27t)(-16m_A^2(16m_A^2 - 81t) - 729t^2 + 27t\sqrt{\Omega})}{tm_A^2}$$

[16] As this value of $m_B^2$ is derived from the second order conditions, it must be that all relevant $m_B^2$ - contained in equation (13) - meet this criteria. Therefore

$$\max\{m_B^2\} < \Gamma$$

which equals zero if and only if $m_A^2 = \frac{27}{16}t$. Therefore, for $m_A^2 \in (0, \frac{27}{16}t)$ the equation has the same sign. In this case, it is trivial to show that it is always negative. Likewise

$$m_B^{I-} - \Gamma = -\frac{1}{13824}\frac{(16m_A^2 - 27t)(16m_A^2(16m_A^2 - 81t) + 729t^2 + 27t\sqrt{\Omega})}{tm_A^2}$$

This function equals zero if and only if $m_A^2 = \frac{27}{16}t$ or $\frac{27}{8}t$ which implies it has the same sign for all $m_A^2 \in (0, \frac{27}{16}t)$. In this case, it is trivial to check that it must be positive.

Consequently, we can ignore $m_B^{I-}$ (as it would violate the SOCs) and simply use $m_B^{I+} = m_B^I$. This implies that for all $m_B^2 \in (m_B^I, \Gamma)$ it must hold that $I_A - I_B$ has the same sign also. Therefore, we must examine

$$\max\{m_B^2\} - m_B^I = m_A^2 - m_B^I \;\forall\; m_A^2 \in (0, \frac{27}{24}t] \tag{A17}$$

$$\max\{m_B^2\} - m_B^I = \frac{27}{16}t - \frac{m_A^2}{2} - m_B^I \;\forall\; m_A^2 \in (\frac{27}{24}t, \frac{27}{16}t] \tag{A18}$$

From equation $(A17)$ we observe

$$m_A^2 - m_B^I = -\frac{1}{512}\frac{(16m_A^2 - 27t)\sqrt{\Omega} + 729t^2 - 512m_A^4}{m_A^2}$$

which is never equal to zero. Thus, it is continuous over all $m_A^2 \in (0, \frac{27}{24}t]$ and positive over this range. Similarly, from equation $(A18)$ we observe

$$\frac{27}{16}t - \frac{m_A^2}{2} - m_B^I = -\frac{1}{512}\frac{(16m_A^2 - 27t)(16m_A^2 - 27t + \sqrt{\Omega})}{m_A^2}$$

which equals zero if $m_A^2 = \frac{27}{16}t$. However, as this is not a potential value for $m_A^2$ it must be that it is continuous over all $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$ and it is trivial to demonstrate that this, too, is positive.

Therefore, it must be that, for all $m_A^2 \in (0, \frac{27}{16}t)$, $m_B^I \in (0, \max\{m_B^2\})$. ∎

## 5.6 Proof of Proposition 6

**Proof.** $(i)$The *ex post* cost differential is given by

$$m_A I_A - m_B I_B = \frac{18tm_A m_B(8m_A^4 - 27t(m_A - m_B)(m_A + m_B))}{216tm_B^2 + (8m_A^2 - 27t)^2}$$

Furthermore, this equation is equal to zero if and only if $m_B^2 = m_B^\varphi$ where

$$m_B^\varphi = -\frac{1}{27}\frac{m_A^2(8m_A^2 - 27t)}{t}$$

Naturally, this is strictly positive and implies that it must be larger than the minimum potential value of $m_B^2$. Therefore, for $m_A^2 \in (0, \frac{27}{24}t]$ we find

$$\max\{m_B^2\} - m_B^\varphi = m_A^2 - m_B^\varphi = \frac{8}{27}\frac{m_A^4}{t} > 0$$

which implies that $\max\{m_B^2\} > m_B^\varphi$ for all $m_A^2 \in (0, \frac{27}{24}t]$. In addition, for all $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$

$$\max\{m_B^2\} - m_B^\varphi = \frac{27}{16}t - \frac{m_A^2}{2} - m_B^\varphi = \frac{1}{432}\frac{(8m_A^2 - 27t)(16m_A^2 - 27t)}{t} > 0$$

which implies $\max\{m_B^2\} > m_B^\varphi$ for all $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$.

Together, these imply that $\max\{m_B^2\} > m_B^\varphi$ for all $m_A^2 \in (0, \frac{27}{16}t]$. Moreover, as $m_A I_A - m_B I_B = 0$ if and only if $m_B^2 = m_B^\varphi$ we can observe that the sign of the equation is the constant on either side of $m_B^\varphi$. Thus, it is trivial to check, for a given $m_A^2$, if $m_B^2 < m_B^\varphi$, $m_A I_A - m_B I_B > 0$. Therefore, if $m_B^2 > m_B^\varphi$ then $m_A I_A - m_B I_B < 0$.

$(ii)$ It is easy to check the relationship between $m_B^\varphi$ and $m_B^I$ by observing

$$m_B^\varphi - m_B^I = -\frac{1}{13824}\frac{(16m_A^2 - 27t)(16m_A^2(16m_A^2 - 27t) - 729t^2 + 27t\sqrt{\Omega})}{tm_A^2}$$

This equation is zero if and only if $m_A^2 = \frac{27}{16}t$. Therefore, over all $m_A^2 \in (0, \frac{27}{16}t]$ this must have the same sign. Therefore, it is trivial to then check, for any $m_A^2$, that $m_B^\varphi > m_B^I$. ■

## 5.7 Proof of Proposition 7

**Proof.** $(i)$ Taking $m_A^2$ and $m_B^2$ as defined in (12) and (13) is is trivial to demonstrate that the profits for both firms are strictly positive.

$(ii)$ The difference in firm profits is given by

$$\pi_A - \pi_B = \frac{6t[m_A^2(81t - 32m_A^2)(8m_A^2 - 27t)^2 + 729t^2 m_B^2(32m_B^2 - 81t) - 864m_A^2 m_B^2(8m_B^2 - 27t)]}{[216tm_B^2 - (8m_A^2 - 27t)^2]^2}$$

Of course, it is not obvious whether this function is positive or negative. However, we can observe that $\pi_A - \pi_B = 0$ if and only if $m_B^2 = m_B^\pi$ where

$$m_B^\pi = \frac{1}{576}\frac{243t^2(32m_A^2 - 81t) \pm \sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$

where

$$\Sigma = -3t(16m_A^2 - 81t)(32m_A^2 - 81t)(16m_A^2 - 27t)^3$$

Of course, we require that $\Sigma \geq 0$. This occurs for all $m_A^2 \in [0, \frac{27}{16}t] \cap [\frac{81}{32}t, \frac{81}{16}t]$ which is contained in the feasible set of $m_A^2$.

However, this does imply that there are two critical values for which $\pi_A - \pi_B = 0$ given by

$$\begin{aligned} m_B^{\pi+} &= \frac{1}{576}\frac{243t^2(32m_A^2 - 81t) + \sqrt{\Sigma}}{t(8m_A^2 - 27t)} \\ m_B^{\pi-} &= \frac{1}{576}\frac{243t^2(32m_A^2 - 81t) - \sqrt{\Sigma}}{t(8m_A^2 - 27t)} \end{aligned}$$

First, over the relevant range of $m_A^2 \in (0, \frac{27}{16}t)$, $m_B^{\pi+} = 0$ if and only if $m_A^2 = 0$; which is not contained in this set. Therefore, for all $m_A^2 \in (0, \frac{27}{16}t]$, over which the function is continuous, the sign will be the same. It is trivial, then, to check that this is positive. Second, over the relevant range of $m_A^2 \in (0, \frac{27}{16}t]$, $m_B^{\pi-} = 0$ if and only if $m_A^2 = 0$ also. Consequently, the sign on this is constant for all $m_A^2 \in (0, \frac{27}{16}t]$ and it is trvial to observe that this is positive.

Therefore, we know that both $m_B^{\pi+}$ and $m_B^{\pi-}$ are positive. However, it is fairly simple to observe which is larger by checking

$$m_B^{\pi+} - m_B^{\pi-} = \frac{1}{288}\frac{\sqrt{\Sigma}}{(8m_A^2 - 27t)}$$

For all $m_A^2$ as defined as in equation (12) it must be that this is negative, or $m_B^{\pi+} < m_B^{\pi-}$.

However, we must also check that they meet the necessary SOCs. This requires

$$\begin{aligned}
m_B^{\pi+} - \Gamma &= \frac{1}{1728}\frac{-(16m_A^2 - 27t)(256m_A^4 - 2160tm_A^2 + 3645t^2) + 3\sqrt{\Sigma}}{t(8m_A^2 - 27t)} < 0 \\
m_B^{\pi-} - \Gamma &= \frac{1}{1728}\frac{-(16m_A^2 - 27t)(256m_A^4 - 2160tm_A^2 + 3645t^2) - 3\sqrt{\Sigma}}{t(8m_A^2 - 27t)} < 0
\end{aligned}$$

Interestingly, both of these equations only equal zero where $m_A^2 = \frac{27}{16}t$. Thus, for all $m_A^2 \in (0, \frac{27}{16}t)$ these equations have the same sign and it is not difficult to check that these must be negative. This means that both of these critical values are below the upper bound set by the SOCs. Consequently, it is possible to argue that for all $m_B^2 \in (m_B^{\pi+}, m_B^{\pi-})$ the difference in firm profits has the same sign.

Therefore, it is only left to check that $m_B^{\pi+}$ or $m_B^{\pi-}$ are contained within the feasible set of $m_B^2$. Obviously, as both of these are strictly possitive, it must be that they are above the lower bound. However, we must check whether they are under the upper bound. First, for all $m_A^2 \in (0, \frac{27}{24}t]$ we can check

$$m_B^{\pi+} - \max\{m_B^2\} = m_B^{\pi+} - m_A^2 \tag{A19}$$
$$= \frac{1}{576}\frac{-288tm_A^2(16m_A^2 - 81t) - 19683t^3 + \sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$
$$m_B^{\pi+} - \max\{m_B^2\} = m_B^{\pi-} - m_A^2 \tag{A20}$$
$$= \frac{1}{576}\frac{-288tm_A^2(16m_A^2 - 81t) - 19683t^3 - \sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$

Equation $(A19)$ equals zero if and only if $m_A^2 = 0$. Therefore, for all $m_A^2 \in (0, \frac{27}{24}t]$ it must have the same sign (and is continuous). It is trivial then to show that this is negative. Equation $(A20)$ is zero if and only if $m_A^2 = \frac{27}{64}t(7 \pm \sqrt{17}) > \frac{27}{24}t$. Therefore, this too has the same sign over the relevant range but, in this instance, the sign is positive. Consequently, for all $m_A^2 \in (0, \frac{27}{24}t]$ only $m_B^{\pi+}$ is contained within the set of possible $m_B^2$.

Second, we must check, for $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$

$$m_B^{\pi+} - \max\{m_B^2\} = m_B^{\pi+} - (\frac{27}{16}t - \frac{m_A^2}{2}) \tag{A21}$$
$$= \frac{1}{576}\frac{6561t^3 - 7776t^2m_A^2 + 2304tm_A^4 + \sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$
$$m_B^{\pi+} - \max\{m_B^2\} = m_B^{\pi-} - (\frac{27}{16}t - \frac{m_A^2}{2}) \tag{A22}$$
$$= \frac{1}{576}\frac{6561t^3 + 7776t^2m_A^2 - 2304tm_A^4 - \sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$

Equation $(A21)$ is equal to zero only where $m_A^2 = \frac{27}{16}t$. Thus, for all $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$, this equation has the same sign and it is simply to sho that this is negative. The second equation, $(A22)$, is zero if and only if $m_A^2 = \frac{27}{16}t$. Therefore, this too has the same sign over the relevant range. However, in this case, the sign is positive. Consequently, for all $m_A^2 \in [\frac{27}{24}t, \frac{27}{16}t)$ only $m_B^{\pi+}$ is contained within the set of possible $m_B^2$.

Therefore, given that the sign of $\pi_A - \pi_B$ is constant for all $m_B^2 \in$

$(m_B^{\pi+}, m_B^{\pi-})$, $m_B^{\pi+} < \max\{m_B^2\}$ and $m_B^{\pi-} > \max\{m_B^2\}$ it must be that for all $m_B^2 \in (m_B^{\pi+}, \max\{m_B^2\})$ the sign of $\pi_A - \pi_B$ is constant. We shall rename $m_B^{\pi+} = m_B^{\pi}$ as it is contained in the relevant set and it is then simple to demonstrate that, for all $m_A^2 \in (0, \frac{27}{16}t)$, if $m_B^2 > m_B^{\pi}$ then $\pi_A < \pi_B$.

$(iii)$ Finally

$$m_B^{\pi} - m_B^{\varphi} = \frac{1}{1728}\frac{(16m_A^2 - 27t)(246m_A^4 - 1296tm_A^2 + 2187t^2) + 3\sqrt{\Sigma}}{t(8m_A^2 - 27t)}$$

This equals zero if and only if $m_A^2 = 0$ or $\frac{27}{16}t$ and is continuous between. This means that $m_B^{\pi} - m_B^{\varphi}$ has the same sign for all $m_A^2 \in (0, \frac{27}{16}t)$ and it is quite easy to show that this is positive.

Thus, $m_B^{\pi} > m_B^{\varphi} > m_B^{I}$ over the relevant range. ■

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