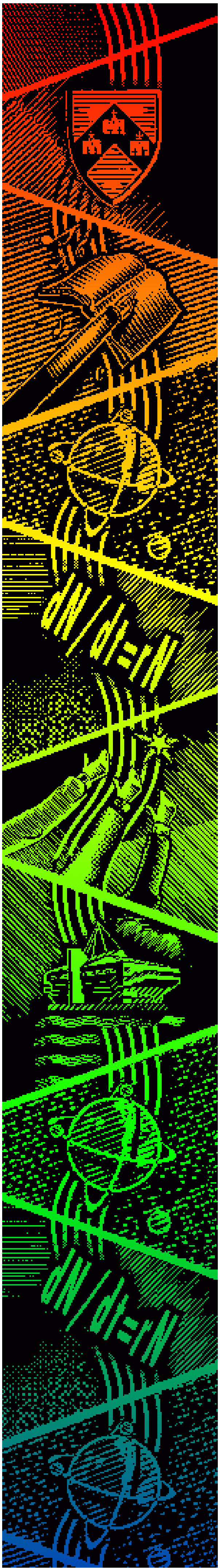

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Finiteness of the Number of Equilibria in a Production Economy with Uncertainty

By

Pascal Stiefenhofer

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

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Pascal Christian Stiefenhofer*
University of York

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Abstract

This paper shows that the inverse image of the natural projection defines a ramified covering with finite layers. Finiteness of equilibria for regular two period production economies with uncertainty follows from this property.

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JEL classification: D20, D50, D51

1 Introduction

The number of equilibria for smooth regular exchange economies is originally studied by Debreu [3]. Debreu's result of finiteness of equilibria is expanded to smooth private ownership production economies by Fuchs [6]. Balasko [2] reformulates the problem of finiteness of equilibria in the context of the natural projection approach for smooth exchange economies. This paper expands the natural projection approach introduced by Balasko [1] to the problem of the number of equilibria of a two period private ownership production model with uncertainty ([4], Chapter 7). It shows that the property of finiteness of the number of equilibria of the static exchange model carries over to more general dynamic models including production and uncertainty. Section two introduces the model and the main results. Section three is a conclusion.

2 Two period private ownership production model with uncertainty

We consider a two period version of the private ownership production model introduced in Debreu ([4]), chapter,7. Uncertainty is denoted by a realization of a random variable s in the set of mutually exclusive and exhaustive states of nature denoted by $s \in \{1, ..., S\}$. There are $i \in \{1, ..., m\}$ consumers, $j \in \{1, ..., n\}$ producers, and $k \in \{1, ..., l\}$ physical goods. For all consumers $i \in \{1, ..., m\}$, a consumption bundle is a collection of vectors $x_i = (x_i(0), ..., x_i(s), ..., x_i(S)) \in X_i = \mathbb{R}^{l(S+1)}$, where consumption in a particular state $s \in \{0, 1, ..., S\}$ is a vector $x_i(s) = (x_i^1(s), ..., x_i^l(s)) \in \mathbb{R}^l$. Associated with physical commodities is a set of normalized prices $\mathbf{S} = \{p \in \mathbb{R}_{++}^{l(S+1)} : p^l(s) = 1, \forall s \in \{0, 1, ..., S\}\}$. For a particular realization $s \in \{0, 1, ..., S\}$ have $p(s) \in \mathbf{S} = \mathbb{R}_{++}^{(l-1)} \times \{1\}$. Consumers are

*Contact address: Department of Mathematics University of York. pascal.stiefenhofer@york.ac.uk.

further endowed with a fraction θ_{ij} representing the exogenously determined ownership structure of the private ownership production economy. θ_{ij} satisfies for each $j \in \{1, ..., n\}$ and $i \in \{1, ..., m\}$ $0 \leq \theta_{ij} \leq 1$, and $\sum_i \theta_{ij} = 1$. Denote the set of ownership structures $\Theta = \{\theta_{ij} \in \mathbb{R}_+^{nm} : \sum_i \theta_{ij} = 1, \forall i \in \{1, ..., m\}\}$.

Consumers and producers satisfy assumptions for smooth economies, Debreu [5]. Consumers are endowed with initial resources $\omega_i = (\omega_i(0), ..., \omega_i(s), ..., \omega_i(S)) \in \Omega = \mathbb{R}_{++}^{l(S+1)}$, where initial endowments in a particular state $s \in \{0, 1, ..., S\}$ is a vector $\omega_i(s) = (\omega_i^1(s), ..., \omega_i^l(s)) \in \mathbb{R}_{++}^l$. Consumer $i \in \{1, ..., m\}$ is further characterized by a smooth Marschallian demand functions $f_i : \mathbf{S} \times \mathbb{R}^{(S+1)} \rightarrow \mathbb{R}^{l(S+1)}$, where $f_i(p, w_i)$ is defined for price vector $p \in \mathbf{S}$, and wealth level $w_i \in \mathbb{R}^{(S+1)}$.

Producers are characterized by production sets and their smooth supply functions. An activity y_j is a collection of vectors $y_j = (y_j(0), ..., y_j(s), ..., y_j(S)) \in \mathbb{R}^{l(S+1)}$, where an activity in state $s = 0$ is a vector of inputs $y_j(0) = (y_j^1(0), ..., y_j^l(0)) \in \mathbb{R}_-^l$, and an activity $y_j(s) = (y_j^1(s), ..., y_j^l(s)) \in \mathbb{R}_+^l$ in a particular state $s \in \{1, ..., S\}$ is the associated vector of $g_j : \mathbf{S} \rightarrow \mathbb{R}^{l(S+1)}$, where $g_j(p)$ is a smooth supply function defined on the set of normalized prices. Standard assumptions of smooth production economies introduced in [2] hold for each production set Y_j. In particular Y_j is convex, inactivity 0 is an element in Y_j, and the efficient boundary of Y_j has a strictly positive Gaussian curvature.

2.1 Equilibrium of the private ownership production model

Each consumer $i \in \{1, ..., m\}$ chooses a utility maximizing consumption bundle $x_i \in X_i$ at fixed $\omega_i \in \Omega$ and $\theta_{ij} \in \Theta$. Each producer $j \in \{1, ..., n\}$ chooses profit maximizing net activities $y_j \in Y_j$.

Definition 1 *An equilibrium of the two period private ownership production model is a price vector $p \in \boldsymbol{S}$, at given pair (ω, θ) if for utility maximizing consumers $i \in \{1, ..., m\}$ and profit maximizing producers $j \in \{1, ..., n\}$ equilibrium between aggregate demand and aggregate supply is satisfied:*

$$\sum_i f_i(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot g_j(p)) = \sum_i \omega_i + \sum_j g_j(p). \tag{1}$$

$\mathcal{E} = \{(p, \omega) \in \mathbf{S} \times \Omega : \sum_i f_i(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot g_j(p)) - (\sum_i \omega_i + \sum_j g_j(p)) = 0\}$ is a smooth manifold embedded in $\mathbf{S} \times \Omega$. The restriction of a projection mapping $\mathcal{E} \longrightarrow \Omega$ is a smooth proper map defining an open covering of its set of regular values, called the natural projection. The set of singular values Σ of the map $\pi : \mathcal{E} \longrightarrow \Omega$ is therefore closed by properness of π and is a set of measure zero in Ω by Sard's theorem. Its complement, the set of regular values $\mathcal{R} = \Omega \setminus \Sigma$ is open and dense[1]. We show that the number of equilibria associated with $\omega \in \mathcal{R}$ of the dynamic production model is finite and locally constant.

Theorem 1 π^{-1} *is a finite covering of the two period production economy with uncertainty for every $\omega \in \mathcal{R}$.*

Proof. Let $\{p\}$ consist of a single element of $\pi^{-1}(\omega)$. Consider the tangent map of elements of $\mathcal{E}$ not contained in the set of critical points, $p \notin \mathcal{E}_c$. Then as a non critical point in $\mathcal{E}$ there exists a bijective map $D\pi_p$ which by the inverse function theorem implies that $\pi : \mathcal{E} \longrightarrow \Omega$ is locally a diffeomorphism. By the inverse function theorem there exists an open set U of $\omega \in \mathcal{R}$ and an open set V of $p \in \mathcal{E}$ such that the restriction of the natural projection to V, $\pi \mid_V : V \rightarrow U$ is a diffeomorphism. It follows from the one-to-one

[1] See Balasko [2] for these properties of the equilibrium manifold.

property of this map that $\pi^{-1}(\omega) \cap V = \{p\}$. Since V is open in $\mathcal{E}$ it follows from the definition of open sets of $\pi^{-1}(p)$ as intersections with $\pi^{-1}(\omega)$ of open sets of $\mathcal{E}$ that the subset $\{p\}$ is open in $\pi^{-1}(p)$. The union of all open subsets $\{p\} \in \pi^{-1}(\omega)$ define an open covering $\mathcal{P}$ of $\{p\} \in \pi^{-1}(\omega)$. Compactness of the set $\pi^{-1}(\omega)$ follows from compactness of the preimage of a compact set $\{\omega\}$ by the proper mapping $\pi : \mathcal{E} \longrightarrow \Omega$. It follows from compactness of $\pi^{-1}(\omega)$ that the open covering has a finite subcovering defined by the unique element of $\pi^{-1}(\omega)$. The union of a finite number of elements defines the set $\pi^{-1}(\omega)$ which is therefore a finite set. This proves finiteness of the number of equilibria. ■

Theorem 2 *For every regular two period production economy $\omega \in \mathcal{R}$ there exists an open neighborhood $U \subset \mathcal{R}$ of ω. For every nonempty $\pi^{-1}(\omega)$, $\pi^{-1}(U)$ is the union of a finite number pairwise disjoint open sets $V_1, ..., V_n$ and the restriction of the map π defined by $\pi_k : V_k \to U$ being a diffeomorphism for $k \in \{1, ..., n\}$.*

Proof. By theorem (1) have a nonempty finite set of elements defined by $\pi^{-1}(\omega)$. Let $p_1, ..., p_n$ be all elements of the inverse image of $\pi : \mathcal{E} \longrightarrow \Omega$ defined by $\pi^{-1}(\omega)$ for every $\omega \in \mathcal{R}$. Provided that all open sets are small enough, it is always possible to consider open disjoint unions $\bar{U}_1, ..., \bar{U}_n$ in $\mathcal{E}$ of $p_1, ..., p_n$ such that $\pi \mid_{U_i}$ where $U_i = \pi(\bar{U}_i)$ is a diffeomorphism. $\mathcal{E} \setminus (\bar{U}_1 \cup, ..., \cup, \bar{U}_n)$ is closed in $\mathcal{E}$ and its image by properness of π is closed in Ω. Let $U = (U_1 \cap, ..., \cap, U_n)\pi(\mathcal{E} \setminus (\bar{U}_1 \cup, ..., \cup, \bar{U}_n))$. Obviously, U is open in Ω. We need to show that $\omega \in U$ follows from $\pi^{-1}(\omega) \subset \bar{U}_1 \cup, ..., \cup, \bar{U}_n$ implying that $\omega \in U$ does not belong to $\pi(\mathcal{E} \setminus (\bar{U}_1 \cup, ..., \cup, \bar{U}_n))$. Let $V_n = \bar{U}_n \cap \pi^{-1}(U)$. Then for all $k \in \{1, ..., n\}$, $\pi_k \mid_{V_k}$ obviously determines a diffeomorphism between V_n and $\pi(V_n)$. It only remains to prove that $\pi^{-1}(U)$ is equal to the union of all V_n. This follows by contradiction. Let $\{p\} \in \pi^{-1}(U)$. Assume that $\{p\}$ does not belong to any V_n. Then $\{p\}$ must belong to $\mathcal{E} \setminus (\bar{U}_1 \cup, ..., \cup, \bar{U}_n)$, implying that $\omega = \pi(p) \in \pi(\mathcal{E} \setminus (\bar{U}_1 \cup, ..., \cup, \bar{U}_n))$ and ω does therefore not belong to U. A contradiction. ■

3 Conclusion

This paper shows finiteness of the number of long run equilibria of a model where the center object of study is production in a dynamic set up. It shows via natural projection approach that finiteness of the number of equilibria of regular economies of the Arrow-Debreu exchange model carry over to more general models including production and time.

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