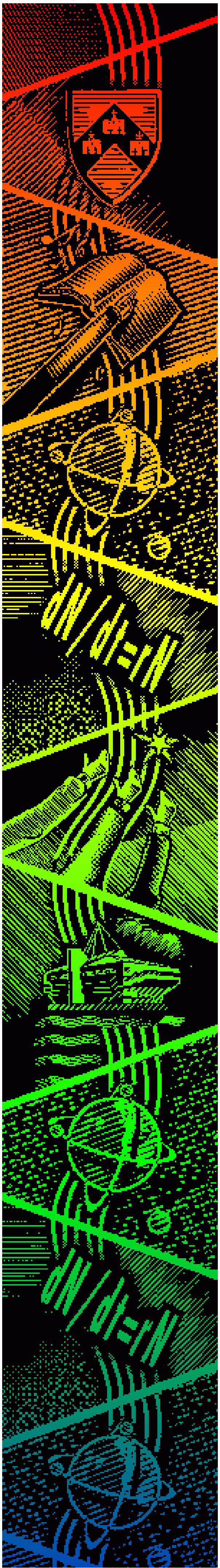

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Decentralized Market Processes to Stable Job Matchings with Competitive Salaries*

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Abstract

We analyze a decentralized trading process in a basic labor market where heterogeneous firms and workers meet directly and randomly, and negotiate salaries with each other over time. Firms and workers may not have a complete picture of the entire market and can thus behave myopically in the process. Our main result establishes that, starting from an arbitrary initial market state, there exists a finite sequence of successive myopic (firm-worker) pair improvements, or bilateral trades, leading to a stable matching between firms and workers with a scheme of competitive salary offers. An important implication of this result is that a general random process where every possible bilateral trade is chosen with a positive probability converges with probability one to a competitive equilibrium of the market.

Keywords: Decentralized market, job matching, random path, competitive salary, stability.

JEL classification: C62, D72.

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1 Introduction

The idea that decentralized market processes where self-interested buyers and sellers make independent decisions freely can settle a market on a competitive equilibrium outcome, can be traced back at least to Adam Smith (1776), who coined the famous term, the Invisible Hand, to describe the self-regulating nature of an uncoordinated market. The objective of this paper is to develop and analyze a decentralized market process for a basic labor market with finitely many heterogeneous firms and workers. This process intends to mimic and reflect the decentralized decision making process in real competitive labor markets, where firms and workers meet directly and randomly, and negotiate salaries with each other over time. Here agents may not have a complete picture of the entire labor market and can thus behave myopically. Using this framework, we investigate the market outcomes of such decentralized and random processes.

The theoretical literature on market processes has predominantly focused on and has also been remarkably successful in analyzing and designing centralized processes for various markets.[1] Nevertheless, many competitive markets, labor markets being a leading example, feature bilateral (job) offers and are typically decentralized (see Roth and Vande Vate (1990), Samuelson and Nordhaus (2010)). Indeed, it is widely observed in labor markets that a worker sequentially works for several employers because a latter employer offers a better salary than a previous employer does; and conversely, a same firm hires different workers over time for the same position as workers who come later may either work more efficiently or demand lower salaries. In addition, it is not uncommon to see that a worker eventually returns to her previous employer but with a different contract. In a labor market where many firms and workers are matched randomly and dynamically, and each agent makes her own decisions independently, possibly by only looking at myopic gains, a natural and important question is then "will such seemingly chaotic, random and dynamic decentralized processes eventually lead the market to an equilibrium state in which a system of competitive salaries exists and simultaneously meets the needs of both firms and workers?"

This paper attempts to resolve the above question in the affirmative. Briefly speaking, we consider a labor market with finite and heterogeneous workers and firms. Each worker can be matched with a firm, generating a joint surplus, which can

[1] See for example, Gale and Shapley (1962) for marriage matching problems; Shapley and Scarf (1974) for housing markets; Crawford and Knoer (1981), Kelso and Crawford (1982), and Crawford (2008) for job matching problems; Demange, Gale and Sotomayor (1986), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2009) for auction markets; Roth (1984), Roth and Sotomayor (1990) for the US medical residency match; Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak and Roth (2005) for school choice problems; and Ostrovsky (2008) for supply chain network. Stock markets and auction markets are typically centralized. AEA annual meeting provides a place for junior economists to meet their future potential academic employers. But this is not a centralized market.

be split freely between the two agents (interpreted as salaries). Given an allocation which consists of a matching between the firms and workers and a scheme of salary offers, a firm and a worker, currently not matched with each other, can block the allocation, resulting in a salary offer which makes both better off and *at least* one of them strictly so. Such a procedure is called a pair improvement of the allocation. A pair improvement can be intuitively regarded as a particular form of bilateral trade arising from the previous allocation. Like a bilateral transaction, a pair improvement, while beneficial for the pair involved, may hurt other agents, resulting in a decrease in total welfare of the market. In this framework, we establish that, starting from an arbitrary initial market state (an allocation), there exists a finite sequence of successive myopic pair improvements leading to a stable matching between firms and workers with a scheme of competitive salary offers. An important implication of this result is that a general random process where every possible pair improvement is chosen with a positive probability converges with probability one to a competitive equilibrium of the labor market.

The general random process is permissive in that it allows for random, chaotic and cyclical bilateral trading scenarios where firms' and workers' behavior might be only myopically oriented, and partnerships between firms and workers can be formed hastily and can also dissolve instantly whenever better opportunities arise. In our opinion, such a random market process presents a satisfactory illustration of the trading behavior in a real uncoordinated market. In addition, myopic pair improvements (bilateral trades) before reaching stability can be simply interpreted as haggling activities where workers retain offers or sellers hold their goods without committing themselves until equilibrium (stability) is reached. Alternatively, these pair improvements can also be regarded as transactions that take place in real-time, where workers move from job to job and firms terminate existing employment relationships and create new job offers.

The current study is most closely related to the seminal work by Crawford and Knoer (1981) and Roth and Vande Vate (1990). Crawford and Knoer (1981) analyzed a labor market with finitely many self-interested and heterogeneous firms and workers. They proposed a centralized market process — a salary adjustment process which always converges to a stable assignment of workers to firms with a scheme of competitive salaries.[2] Our model is similar to theirs but our goal and results differ essentially from theirs in that our process is decentralized and the associated algorithm finds an equilibrium in finitely many steps, whereas theirs is centralized and approaches an equilibrium through a limiting argument.[3]

[2]Kelso and Crawford (1982) and Crawford (2008) generalized and extended both this model and the centralized market process to more complex scenarios where each firm may hire several workers. Demange, Gale and Sotomayor (1986) improved and refined the market process of Crawford and Knoer (1981) so that their new process always leads to an equilibrium in finitely many steps. More recently, several general auction (price adjustment) processes have been proposed by Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2009) among others.

[3]Crawford and Knoer (1981) first proved that their process always converges to a core element

Our decentralized process is in spirit close to Roth and Vande Vate (1990), who reexamined the Gale-Shapley marriage matching model. In this model, Gale and Shapley (1962) proved the existence of a stable marriage matching via a centralized process — the deferred acceptance procedure, while Roth and Vande Vate (1990) proposed a decentralized process which converges to a stable matching.[4] Our study differs from Roth and Vande Vate (1990) in three crucial aspects: first, in the Gale-Shapley model examined by Roth and Vande Vate, money is not explicitly involved and side payment is not allowed; second, their solution of stability corresponds to core, whereas our solution of stability with flexible salaries coincides with both strict core and competitive equilibrium; third, in our model because salaries are flexible and weak pair improvements have to be employed, it is crucial to design an elaborate and novel approach to tackle cycles which typically arise in our decentralized setting.

Our analysis also bears some similarity with the literature on tâtonnement processes, which studies equilibrium stability and how market-clearing prices and efficient allocations are reached with the coordination of a fictitious market maker. Since the first tâtonnement process formulated by Leon Walras in 1874, the study of such processes has been a major issue of economic research. Some of the early contributions on tâtonnement processes include Samuelson (1941), Arrow and Hurwicz (1958) and Scarf (1960, 1973). Compared with tâtonnement processes, our analysis provides an arguably more satisfactory market process toward equilibrium allocations and prices in real economic systems in that our market process is random and decentralized, and that agents could trade on markets sequentially and trade could take place all the time, even at disequilibrium prices. In particular, our market processes do not exclude chaotic and cyclical behavior commonly observed in real economic systems.

In earlier related literature, Koopmans and Beckmann (1957), Shapley and Shubik (1972) examined the existence and structure issues of competitive equilibrium in assignment markets without discussing any market process. Feldman (1974) and Green (1974) studied similar problems and obtained convergent processes for certain subclasses of NTU games. But their approaches do not apply to the labor market or matching models where significant indivisibility is involved.

The rest of the article proceeds as follows. Section 2 presents the model. Section 3 provides preliminary results concerning weak stability. The main results towards stability are established in Section 4. Concluding remarks are provided in Section 5. Omitted proofs are relegated to an Appendix.

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in finitely many steps. Then they showed by a limiting argument that their process approaches a strict core element (i.e., an equilibrium). Notice that their process can find an allocation that is as close to an equilibrium as one wishes in finite time.

[4] Following Roth and Vande Vate (1990), several decentralized processes have been developed for closely related markets; see Chung (2000), Diamantoudi, Miyagawa and Xue (2006) for roommate matching problems; Klaus and Klijn (2007) for matching problems with couples; Kojima and Ünver (2008) for many-to-many matching problems. In all these models, side payment is not allowed, and various notions of stability (but not competitive equilibrium) are used.

2 The Model

Consider a labor market with finitely many heterogeneous firms and workers. Formally, let F and W be two finite disjoint sets of agents, containing $|F|$ firms and $|W|$ workers, respectively. We assume that each firm hires at most one worker and each worker accepts at most one job.[5] A matching μ in the labor market is simply a one-to-one mapping from $F \cup W$ to itself such that (i) $\mu(\mu(x)) = x$ for all $x \in W \cup F$, and (ii) each agent is either self-matched ($\mu(x) = x$), or is matched to a member of the other side (for $x \in W$, $\mu(x) \neq x$ implies $\mu(x) \in F$, and for $x \in F$, $\mu(x) \neq x$ implies $\mu(x) \in W$), in which case $\mu(x)$ is said to be a partner of x.

Denote $V(f,w)$ and $s(f,w)$, respectively, as worker w's productivity and salary at firm f. When a worker w does not work for any firm, his utility is represented by $V(w,w)$, while if a firm f does not hire any worker, her productivity or utility is denoted by $V(f,f)$. [6] For any agent $x \in F \cup W$, value $V(x,x)$ can be alternatively interpreted as agent x's outside options (or, for workers, unemployment benefits) when x is self-matched. Notice that we allow for heterogeneous outside options for the agents. Salaries, together with the parameters $V(f,w)$, $V(w,w)$ and $V(f,f)$, are paid in transferable monetary units. As a result, worker w's total utility at firm f is $V(f,w) - s(f,w)$. We assume that values $V(f,w)$, $V(w,w)$ and $V(f,f)$ are integers for all $f \in F$ and $w \in W$. These values are measured in monetary units and hence are naturally assumed to be integers. We denote this labor market by (F,W,V).

Given a matching μ, let $I(\mu) = \{h \in F \cup W \mid \mu(h) = h\}$ be the set of members who are self-matched at μ. We call the quantity of $\sum_{f \in F \setminus I(\mu)} V(f,\mu(f)) + \sum_{i \in I(\mu)} V(i,i)$ the *market value* associated with the matching μ. Moreover, we say that a matching μ is *efficient* if it holds, for an arbitrary matching ρ, that

$$\sum_{f \in F \setminus I(\mu)} V(f,\mu(f)) + \sum_{i \in I(\mu)} V(i,i) \geq \sum_{f \in F \setminus I(\rho)} V(f,\rho(f)) + \sum_{i \in I(\rho)} V(i,i).$$

If μ is efficient, then we call quantity $\sum_{f \in F \setminus I(\mu)} V(f,\mu(f)) + \sum_{i \in I(\mu)} V(i,i)$ the *efficient market value*, or the efficient value, of the labor market and denote it by $V(F \cup W)$, which is the same for all efficient matchings.

An economic outcome, or simply an *allocation*, of the labor market consists of a matching μ and a payoff vector $u \in \mathbb{R}^{F \cup W}$ such that $u(x) = V(x,x)$ for any $x \in I(\mu)$; and $u(x) + u(\mu(x)) = V(x,\mu(x))$ for any $x \notin I(\mu)$. An allocation (μ,u) is *individually rational* if $u(x) \geq V(x,x)$ for all $x \in F \cup W$. Notice that for each allocation (μ,u) in a labor market (F,W,V), the payoff vector u *uniquely* defines a salary vector s where $s(\mu(w),w) = u(w)$, for matched/employed worker w, and $s(\mu(w),w) = V(w,w)$, i.e., w's unemployment benefit, for self-matched/unemployed worker w. With this

[5] This is the *unit-demand assumption*, which has been employed in Shapley and Shubik (1972), Crawford and Knoer (1981), Demange, Gale and Sotomayor (2006) and others.

[6] When a worker or a firm stays idle, then the worker gets no salary from any firm and the firm pays nothing to any worker.

convention, it is thus sufficient for us to employ the utility vector u in lieu of the description of the salary vector s for each allocation hereafter.

A natural notion of solution for our setting is that of stability. An allocation (μ, u) is *stable or a strict core allocation* if $u(f) + u(w) \geq V(f, w)$ for all $f \in F$, $w \in W$, and $u(x) \geq V(x, x)$ for all $x \in F \cup W$. Namely, an allocation is stable if every worker (firm) has the option of remaining idle and the allocation is not blocked, to be defined shortly, by any pair of firm and worker. It is easy to show that if (μ, u) is a stable allocation and ρ is an efficient matching, then (ρ, u) is stable and μ is also efficient.

We now introduce two notions of blocking pairs: weakly blocking pairs and strongly blocking pairs. A pair (f, w) of firm f and worker w *weakly blocks* an allocation (μ, u) if firm f and worker w are not matched under μ but both can *weakly* improve their well-being by matching with each other and abandoning their partners at μ. Namely, there are $r_f \in \mathbb{R}$ and $r_w \in \mathbb{R}$ such that $r_f + r_w = V(f, w)$ and $r_w \geq u(w)$ and $r_f \geq u(f)$ with at least one strict inequality. For our purpose, we also say that (μ, u) is weakly blocked by a pair of (x, x) if $x \in F \cup W$ is not matched to herself at μ but prefers being single to being matched with $\mu(x)$, i.e., $x \neq \mu(x)$ but $r_x = V(x, x) > u(x)$.

A pair (f, w) is said to *strongly block* an allocation (μ, u) if firm f and worker w are not matched under μ but both can *strictly* improve their well-being by matching with each other and abandoning their partners at μ. Namely, there are $r_f \in \mathbb{R}$ and $r_w \in \mathbb{R}$ such that $r_f + r_w = V(f, w)$ and $r_f > u(f)$ and $r_w > u(w)$. Similarly, we also say (μ, u) is strongly blocked by a pair of (x, x) if $x \in F \cup W$ is not matched to herself at μ but prefers being single to being matched with $\mu(x)$, i.e., $x \neq \mu(x)$ but $r_x = V(x, x) > u(x)$.

Given the definitions of blocking pairs, we can alternatively say that an allocation (μ, u) is *stable* if there is no pair that weakly blocks (μ, u). Similarly, we call an allocation (μ, u) *weakly stable* or *a core allocation* if there is no pair that strongly blocks (μ, u). By definition, a weakly stable allocation is weaker than a stable allocation, in that there might be *efficiency losses* in a weakly stable allocation compared with a stable one.

Evidently, the above model of a labor market can also be regarded as a general assignment market with finitely many buyers and sellers and integral valuations. Here, each seller is in possession of an indivisible good, which is valued possibly differently by the buyers. Given this specification, $V(i, j)$ is then interpreted as the net monetary surplus associated with the partnership of buyer i and seller j, while $s(i, j)$ is simply the price that buyer i is charged for seller j's good.

It is well known in the literature (e.g., Shapley and Shubik (1972), and Crawford and Knoer (1981)) that a job assignment market including the current labor market has at least one competitive equilibrium, and that the set of stable allocations (i.e., strict core) coincides with that of competitive equilibria.[7] An additional important feature of the labor market is that as all values $V(f, w)$, $V(f, f)$ and $V(w, w)$ are

[7]Let $s \in \mathbb{R}^W$ be a salary vector of which s_w is the salary allocated to worker w. For $s \in \mathbb{R}^W$,

integers, and the structure of the market is totally unimodular, the market must have at least one stable outcome with an integral payoff vector $u \in \mathbb{Z}^{F \cup W}$, which implicitly defines an integral salary scheme $s \in \mathbb{Z}^W$. We can therefore restrict ourselves solely to the domain of integer payoffs. Namely, it is sufficient to consider only weakly (strongly) blocking pairs with integer payoffs. Henceforth, all values and salaries to be discussed will be integral.

As a blocking pair may result in multiple allocations, arising from different specifications of wages or surplus division rules, we define an additional basic concept of *pair improvement* so as to fully describe the process from a blocking pair. In general, let (f, w) be a blocking pair of an allocation (μ, u). Introduce a new allocation (μ', u') via the blocking pair (f, w) such that (1) $\mu'(x) = \mu(x)$ and $u'(x) = u(x)$ for any $x \in (F \cup W) \setminus \{f, w, \mu(f), \mu(w)\}$, (2) under μ', f and w are matched, while $\mu(f)$ and $\mu(w)$ are self-matched, and (3) $u'(f) = r_f$ and $u'(w) = r_w$ such that $r_f + r_w = V(f, w)$, while $u'(\mu(f)) = V(\mu(f), \mu(f))$ and $u'(\mu(w)) = V(\mu(w), \mu(w))$. We say that (μ', u') is *a pair improvement* of (μ, u) through the blocking pair (f, w). We also distinguish *weak pair improvements* from *strong pair improvements*, depending on whether the associated blocking pair (f, w) is weak or strong.[8] As described previously, a pair improvement mimics a real transaction between a firm and a worker and can thus be naturally interpreted as a specific form of bilateral trade. Hereafter, we sometimes state a pair improvement as, more intuitively, a bilateral trade in our discussion.

As stated before, our central objective is to analyze the market outcomes of a decentralized and random process where firms and workers meet directly and randomly, and negotiate salaries with each other over time. For this purpose, the first issue we have to deal with is the existence of a finite sequence of successive *bilateral trades* toward a stable allocation from any initial allocation. The following examples demonstrate that an arbitrary sequence of successive weak or strong pair improvements may induce trading cycles.

First consider the case of weak pair improvements. A very simple example of this is a labor market (F, W, V) where $F = \{t\}$, $W = \{x, y\}$, $V(t, x) = V(t, y) = 2$, and $V(i, i) = 0$ for all $i \in F \cup W$. Consider an initial allocation $(\mu^0, u^0) =$

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define the demand set of firm $f \in F$ by

$$D^f(s) = \begin{array}{l} \{w \mid V(f,w) - s_w \geq V(f,f) \text{ and } V(f,w) - s_w \geq V(h,f) - s_h, \forall h \in W\} \\ \cup \{f \mid V(f,f) \geq V(h,f) - s_h, \forall h \in W\}. \end{array}$$

A pair (μ, s) is *a competitive equilibrium* if (1) $\mu(h) \in D^h(s)$ for all $h \in F$; (2) $s_w \geq V(w,w)$ for all $w \in W$, and $\mu(w) = w$ implies $s_w = V(w,w)$ for all $w \in W$.

It is easy to show that (μ, s) is a competitive equilibrium if and only if (μ, u) is stable, where $u(w) = s_w$ for all $w \in W$, $u(f) = V(\mu(f), f) - s_{\mu(f)}$ for all $f \in F$ with $\mu(f) \neq f$, and $u(f) = V(f, f)$ for all $f \in F$ with $\mu(f) = f$.

[8] A pair improvement from a weakly or strongly blocking pair initiated by a single agent can be defined in an analogous way.

$\left\{\left(\underset{1}{t},\underset{1}{x}\right),\left(\underset{0}{y},\underset{0}{y}\right)\right\}$, where we list the agents' (integral) payoffs below the matching. Now (μ^0,u^0) is weakly blocked by the pair (t,y), which when satisfied yields $(\mu^1,u^1)=\left\{\left(\underset{1}{t},\underset{1}{y}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\}$, while (μ^1,u^1) is further weakly blocked by (t,x), which leads to exactly (μ^0,u^0), completing the cycle.

Our next example shows that cycles may also arise under strong pair improvements. Notice in particular that this example also shows that the market value may *not* be monotonic along a path of pair improvements.

Example 1 Consider a labor market (F,W,V) with $F=\{a,b\}$, $W=\{x,y\}$, and $V(i,i)=0$, $\forall i\in F\cup W$, $V(a,i)=4$ and $V(b,i)=5$, $\forall i\in W$.

We start with an initial allocation (μ^0,u^0) with $\mu^0=\{(a,x),\ (b,b)\}$, (y,y) and $u^0(a)=u^0(x)=2$ and $u^0(i)=0$ otherwise.

Choose the first strongly blocking pair to be (b,x), resulting in

$$(\mu^1,u^1)=\left\{\left(\underset{0}{a},\underset{0}{a}\right),\left(\underset{2}{b},\underset{3}{x}\right),\left(\underset{0}{y},\underset{0}{y}\right)\right\}.$$

Now b and y can form the next strongly blocking pair, leading to

$$(\mu^2,u^2)=\left\{\left(\underset{0}{a},\underset{0}{a}\right),\left(\underset{4}{b},\underset{1}{y}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\}.$$

We then choose (a,y) as the next blocking pair, leading to

$$(\mu^3,u^3)=\left\{\left(\underset{1}{a},\underset{3}{y}\right),\left(\underset{0}{b},\underset{0}{b}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\}.$$

Finally, (a,x) is the next blocking pair, which when satisfied, gives

$$(\mu^4,u^4)=\left\{\left(\underset{2}{a},\underset{2}{x}\right),\left(\underset{0}{b},\underset{0}{b}\right),\left(\underset{0}{y},\underset{0}{y}\right)\right\},$$

completing the cycle.

Both examples illustrate the complexity of finding a deterministic path of pair improvements toward (weak) stability in that the choices of both surplus division rules and blocking pairs are important. The examples also demonstrate that an arbitrary decentralized market process does not guarantee convergence to (weak) stability.

3 A Preliminary Result on Weak Stability

To be instructive, we begin with a preliminary result on the existence of a finite path of (strong) pair improvements toward weakly stable allocation, starting from an arbitrary allocation of the labor market. A direct consequence of this result is that the random process where each possible strong pair improvement is chosen with positive probability eventually converges to a weakly stable allocation of the market.[9]

To establish the existence of a finite path toward weakly stable allocation, the key issue is to make sure that the trading process consisting of myopic bilateral trades will not be stuck in some *endless cycles*. To this end, observe first that, as demonstrated in Example 1, each pair improvement *only* results in payoff improvements for the players involved in the blocking pair associated with the pair improvement. This indicates that the market value may decrease after a pair improvement as the abandoned partner's payoff will most likely decrease. Consequently, constructing a finite path of successive strong pair improvements that builds on the monotonicity of the market value is difficult. We instead employ the finiteness of agents and construct a sequence of sets of agents with increasing sizes. For weak stability, such a construction is relatively easy, as both agents involved in a pair are strictly better off after a strong pair improvement, enabling one to invoke certain payoff monotonicity in constructing the sets. For the existence of a finite path of weak pair improvements toward stability, however, the construction is much more involved and demanding so as to precisely deal with trading cycles that arise in the process, as we shall see in the next section.

Specifically, the basic idea of achieving weak stability is to construct a sequence of monotonically increasing sets of firms and workers such that the firms and workers in any such set do not form strongly blocking pairs. Such process proceeds until no strongly blocking pair can be found in the market, establishing weak stability. Roth and Vande Vate (1990) used a similar idea for the marriage matching model where money is absent. Our construction of the monotonically increasing sets, however, also includes proper specifications of wages or surplus division rules for strongly blocking pairs, resulting in different and, in some aspects, more involved arguments than theirs.

Theorem 1 *Consider a labor market (F, W, V) with an arbitrary initial allocation (μ^0, u^0). There exists a finite number of consecutive strong pair improvements which lead to a weakly stable allocation (μ^*, u^*).*

To prove the theorem, we first present the Basic Algorithm, which carries an arbitrary individually rational allocation to a weakly stable allocation in the labor market (F, W, V).

Basic Algorithm

Step 1: If (μ^0, u^0) is weakly stable, stop with output (μ^0, u^0). Otherwise, there is a strongly blocking pair (f^1, w^1) with $V(f^1, w^1) \geq u^0(f^1) + u^0(w^1) + 2$. Match f^1 with w^1. Define $K(1)$ to be $\{(f^1, w^1)\}$ and let the updated allocation be (μ^1, u^1). In addition, let $u^1(f^1) = u^0(f^1) + 1$, $u^1(w^1) = V(f^1, w^1) - u^1(f^1)$; $\mu^0(i)$, $i \in \{f^1, w^1\}$, if any, obtains $u^1(\mu^0(i)) = V(\mu^0(i), \mu^0(i))$; and for all other x, let $u^1(x) = u^0(x)$.[10]

[9]This is an immediate corollary of Proposition 1 in Section 4.

[10]The (initial) wage specification between f_1 and w_1 is inessential: any rule such that both f_1 and w_1 are strictly better off than before will do.

Define an index n and put $n \leftarrow 1$.

Step 2: If (μ^n, u^n) is weakly stable, then stop. Otherwise, there is a blocking pair (f^n, w^n) of (μ^n, u^n) such that $(f^n, w^n) \notin K(n)$.[11] Distinguish three cases:

Case 1. If $w^n \in K(n)$ (hence $f^n \in (F \cup W) \backslash K(n)$), then f^n is the initiator of the next blocking pair, who chooses w^n where $w^n \in \arg\max_{w \in K(n)}[V(f^n, w)$-$u^n(f^n)$-$u^n(w)]$ (choose arbitrarily if there are several such w^n's). Update the allocation to be (μ^{n+1}, u^{n+1}) so that f^n is matched with w^n; $u^{n+1}(w^n) = u^n(w^n) + 1$ and $u^{n+1}(f^n) = V(f^n, w^n) - u^{n+1}(w^n)$; $u^{n+1}(x) = u^n(x)$ for unaffected x and $u^{n+1}(y) = V(y, y)$ for self-matched y. Let $K(n+1) = K(n) \cup \{f^n\}$. Analyze two further sub-cases:

- If $\mu^n(w^n) = w^n$ or if $\mu^n(w^n) \neq w^n$ and $\forall w \in K(n)$, $(\mu^n(w^n), w)$ is not a weakly blocking pair of (μ^{n+1}, u^{n+1}), then return (μ^{n+1}, u^{n+1}) and $K(n+1)$. Put $n \leftarrow n + 1$.
- If $\mu^n(w^n) \neq w^n$ and there is a blocking pair (f^{n+1}, w^{n+1}) of (μ^{n+1}, u^{n+1}) where $(f^{n+1}, w^{n+1}) \in K(n+1)$ and $f^{n+1} = \mu^n(w^n)$. Let f^{n+1} initiate the next blocking pair, choosing w^{n+1} where $w^{n+1} \in \arg\max_{w \in K(n+1)}[V(f^{n+1}, w) - u^{n+1}(f^{n+1}) - u^{n+1}(w)]$. Match f^{n+1} with w^{n+1} and update the allocation to be (μ^{n+2}, u^{n+2}) with $u^{n+2}(w^{n+1}) = u^{n+1}(w^{n+1}) + 1$ and $u^{n+2}(f^{n+1}) = V(f^{n+1}, w^{n+1}) - u^{n+2}(w^{n+1})$. Similarly, $u^{n+2}(y) = V(y, y)$ for unmatched y and $u^{n+2}(x) = u^{n+1}(x)$ for all other x.

Repeat this process until we reach an allocation $\left(\mu^{n+k}, u^{n+k}\right)$ such that $K(n+1)$ contains no strongly blocking pair of $\left(\mu^{n+k}, u^{n+k}\right)$. Rename $\left(\mu^{n+k}, u^{n+k}\right)$ as (μ^{n+1}, u^{n+1}) and put $n \leftarrow n + 1$.

Case 2. $w_n \in (F \cup W) \backslash K(n)$ and $f_n \in K(n)$. This is analyzed similarly as in *Case 1*, with the roles of firms and workers being switched in initiating blocking pairs. This finally yields some (μ^{n+1}, u^{n+1}) where $K(n+1) = K(n) \cup \{w^n\}$ contains no blocking pair of (μ^{n+1}, u^{n+1}). Put $n \leftarrow n + 1$.

Case 3. If every existing strongly blocking pair of (μ^n, u^n) is such that $(f^n, w^n) \in (F \cup W) \backslash K(n)$, or no agent in (f^n, w^n) is in $K(n)$, then construct $K(n+1)$ as $K(n+1) = K(n) \cup \{(f^n, w^n)\}$. Let (μ^{n+1}, u^{n+1}) be the updated allocation so that f^n is matched with w^n; $u^{n+1}(f^n) = u^n(f^n) + 1$, $u^{n+1}(w^n) = V(f^n, w^n) - u^{n+1}(f^n)$; and $u^{n+1}(y) = V(y, y)$ for unmatched y and $u^{n+1}(x) = u^n(x)$ for all other x.[12] Put $n \leftarrow n + 1$.

Step 3: If allocation (μ^n, u^n) contains no strongly blocking pair, then return (μ^n, u^n),

[11] Here, $(f_n, w_n) \notin K(n)$ implies that not both f_n and w_n are in $K(n)$. With some abuse of notation, we denote both a single agent and a pair of agents as elements of $K(n)$ or $F \cup W \backslash K(n)$ hereafter.

[12] The wage specification between f^n and w^n can again be specified arbitrarily here, so long as consistent with the strongly blocking pair (f^n, w^n).

which is weakly stable. Otherwise, go to **Step 2**.

(**End**)

Several remarks are in order for Theorem 1 and Basic Algorithm:

First, Theorem 1 extends Roth-Vande Vate's results to a labor market setting with payoffs, with additional and careful choices of surplus division rules (wages). The Basic Algorithm shows that to design a path toward weak stability, judicious choices of blocking pairs and wage specification are both important. As here we have more degrees of freedom in selecting different wage choices, our algorithm thus contains a somewhat more involved and elaborate design.

Second, in the process of constructing a sequence of strongly blocking pairs to achieve an "internally (weakly) stable" set $K(n)$ (with its members' associated payoffs), we let the newly introduced agent be the initiator of the next blocking pair and we specify the surplus division rule to be "the initiator getting the lion's share of the resulting surplus." Such a specification excludes cases where a single blocking pair breaks multiple existing pairs, ensuring the monotonicity of payoffs of the non-initiators in $K(n)$ along the sequence of strong pair improvements in the process. This payoff monotonicity is crucial for the proof of Theorem 1.

We now present an example. The first part of Example 2 shows that if the surplus division rule is specified differently, then two existing pairs of firms and workers in $K(n)$ can be broken simultaneously by a single blocking pair in the process, disrupting monotonicity. The second part of the example illustrates the Basic Algorithm.

Example 2 Consider a market with $F=\{a,b\}, W=\{x,y\}$, $V(i,i)=0, \forall i\in F\cup W$, and

$$\begin{array}{l} V(a,x)=5, V(b,x)=5, \\ V(a,y)=6, V(b,y)=7. \end{array}$$

Suppose that currently we have $\mu=\{(a,y),(x,x),(b,b)\}$ with $u(a)=4, u(y)=2, u(b)=u(x)=0$ and $K=\{(a,y),(x,x)\}$. Now introduce firm b, who can form a strongly blocking pair with either x or y. Suppose we choose (b,x) and let the initiator firm b obtain an additional payoff of 1 after forming the blocking pair. The resulting allocation is $\mu'=K'=\{(a,y),(b,x)\}$ with $u(a)=4, u(b)=1, u(x)=4, u(y)=2$. The next blocking pair is then (b,y) , which inevitably breaks both (a,y) and (b,x) , and results in a payoff of 0 for worker x, upsetting the monotonicity of workers' payoffs in the process.

We next employ the Basic Algorithm to produce a weakly stable allocation for this labor market, starting with an initial allocation (μ^0,u^0) with $\mu^0=\{(a,a), (b,b), (x,x), (y,y)\}$ and $u^0(i)=0\ \forall i\in F\cup W$.

Pick an arbitrary blocking pair, say, (a,x) , to form $K(1)$, so that

$$(\mu^1,u^1)=\left\{\left(\underset{1}{a},\underset{4}{x}\right),\left(\underset{0}{b},\underset{0}{b}\right),\left(\underset{0}{y},\underset{0}{y}\right)\right\} \text{ and } K(1)=\{(a,x)\}.$$

Introduce worker y, who initiates the next blocking pair (a,y), resulting in

$$\left(\mu^2,u^2\right)=\left\{\left(\underset{2}{a},\underset{4}{y}\right),\left(\underset{0}{b},\underset{0}{b}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\} \text{ and } K\left(2\right)=\left\{\left(a,y\right),\left(x,x\right)\right\}.$$

After two consecutive strong pair improvements from pairs (a,x) and (a,y), we obtain

$$\left(\mu^4,u^4\right)=\left\{\left(\underset{4}{a},\underset{2}{y}\right),\left(\underset{0}{b},\underset{0}{b}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\},$$

and $K\left(2\right)=\left\{\left(a,y\right),\left(x,x\right)\right\}$, which contains no strongly blocking pair of (μ^4,u^4).

Rename (μ^4,u^4) as (μ^2,u^2) and introduce firm b, who can form a strongly blocking pair with either x or y. For future illustration, we analyze the two cases separately.

- The next strongly blocking pair is (b,x). By the algorithm, we have

  $$\left(\mu^3,u^3\right)=\left\{\left(\underset{4}{a},\underset{2}{y}\right),\left(\underset{4}{b},\underset{1}{x}\right)\right\}, \text{ and } K\left(3\right)=\left\{\left(a,y\right),\left(b,x\right)\right\}.$$

  Allocation (μ^3,u^3) is **only** weakly stable with (b,y) being a weakly blocking pair.

- The next strongly blocking pair is (b,y), which, when satisfied, leads to

  $$\left(\mu^3,u^3\right)=\left\{\left(\underset{0}{a},\underset{0}{a}\right)\left(\underset{4}{b},\underset{3}{y}\right),\left(\underset{0}{x},\underset{0}{x}\right)\right\},$$

  and $K\left(3\right)=\left\{\left(a,a\right),\left(b,y\right),\left(x,x\right)\right\}$. Then let firm a initiate the next blocking pair (a,x), resulting in

  $$\left(\mu^4,u^4\right)=\left\{\left(\underset{4}{a},\underset{1}{x}\right),\left(\underset{4}{b},\underset{3}{y}\right)\right\}, \text{ and } K\left(3\right)=\left\{\left(a,x\right),\left(b,y\right)\right\}.$$

  Observe that allocation (μ^4,u^4) is weakly stable, as well as stable.

4 Main Results

In this section we address our central question of whether decentralized and random processes can lead the market to a stable outcome. For this purpose, it is important to show that starting from any initial allocation, there exists a finite sequence of successive myopic bilateral trades toward a stable allocation. This result then implies that the process of choosing each pair improvement with positive probability from any unstable allocation converges to stability with probability one.

We have demonstrated in Section 3 that from any initial allocation there are finite successive *strong* pair improvements leading to a weakly stable allocation in the market. The result is not entirely satisfactory as the market value in a weakly stable allocation can be strictly less than the efficient value, rendering the market in a state of inefficiency and disequilibrium. To attain market efficiency and equilibrium, we now strengthen the previous result to show that given any initial allocation of a labor market, there is a finite sequence of successive *weak* pair improvements that results in a stable allocation, which is also a competitive equilibrium of the market.

Example 2 in the previous section illustrates that the Basic Algorithm may also result in a stable allocation. This might lead to a conjecture that one can probably modify the Basic Algorithm by employing weakly blocking pairs and imposing more detailed surplus specifications in choosing weakly blocking pairs so as to achieve stability. However, a first difficulty of this approach is that the occurrences of multiple blocking pairs and the specification of a "correct" blocking pair are endogenous and typically depend on the status quo configuration and the overall market structure. Moreover, the design of a different set of surplus division rules and the possibility of cycles pose additional challenges. Consequently, the approach of directly generalizing the Basic Algorithm by specifying more detailed choices in blocking pairs is difficult, if not impossible.

We therefore take a different route. The basic idea is to construct an "internally stable" set like $K(n)$ that expands strictly as n increases. The crucial step is to adjust this set to be "internally stable" after the addition of new members. In contrast to the case of weak stability where the Basic Algorithm prevents cycles from happening by maintaining payoff monotonicity of one side of the market during the adjustment, cycles typically arise along a path toward stability. The reason is that with weak pair improvements, we do not have the luxury of always having additional payoffs to make some members in one side of the market strictly better off during the adjustments. We develop a novel and systematic approach to deal with cycles in a way that once the agents arrive in a cycle, we construct a path of successive "bilateral trades " leading the process out of the cycle, with an additional feature that the agents will not enter exactly the same cycle afterwards. In the sequel, to ease exposition, we proceed in two steps: In the first, we consider a simple and "almost stable" market and present an algorithm that generates a finite path of successive weak pair improvements towards stability (Theorem 2). We then use this result to prove a complete paths-toward-stability theorem for the general labor market (Theorem 3).

Consider the following *restricted situation/market*: For an individually rational allocation (μ, u) where u is an integral payoff vector, there exists a worker w^0 such that w^0 is self-matched at μ and such that (μ, u) restricted to $F \cup (W \setminus \{w^0\})$ is stable. We now design an algorithm which finds a finite sequence of weak pair improvements leading (μ, u) to a stable allocation. We start with several key definitions:

Given allocation (μ, u), define, for each $w \in W$,

$$F_w(u) = \{f \in F \mid V(f, w) - u(f) = \max\{V(f', w) - u(f') \mid f' \in F\}\} \qquad (1)$$

and let L_w be a list (linear ordering) of elements of $F_w(u)$. We fix such lists L_w for all $w \in W$ whenever $F_w(u)$ remains the same, and, starting from its first element, each list L_w is used cyclically in the sense that the first element of L_w becomes the next firm when we reach the end of L_w.

An *alternating path* for an allocation (μ, u) from w^0 is an alternating sequence of unmatched and matched firm-worker pairs

$$(f^1, w^0), (f^1, w^1 = \mu(w^1)), (f^2, w^1), (f^2, w^2 = \mu(f^2)), \cdots, (f^{l-1}, w^{l-1} = \mu(f^{l-1})), (f^l, w^{l-1})$$

for an integer $l \geq 1$ such that (i) all the participating agents f^i's and w^i's are distinct, (ii) $f^i \in F_{w^{i-1}}(u) \setminus \{\mu(w^{i-1})\}$ for all $i = 1, \cdots, l$, and (iii) f^i's are not self-matched. For $l \geq 2$, we also call such a sequence with the last pair (f^l, w^{l-1}) being deleted an alternating path. An unmatched pair $\left(f^k, w^{k-1}\right)$, $k \geq 0$, in an alternating path can be interpreted as that if worker w^{k-1} breaks up with her currently matched firm (if any), she would then "point to" firm f^k, indicating her preferences of the next firm she would like to be matched with.

In dealing with weakly blocking pairs, cycles typically arise. The above definitions serve as key tools in treating such cycles in a systematic way. Roughly speaking, $F_w(u)$ serves as a "depository" of firms from which worker w draws a firm to form a weakly blocking pair. L_w serves as an "index", indicating the order w should follow in drawing firms from $F_w(u)$. Finally, an alternating path is a device we use to spin the process out of a cycle when the latter arises so that each adjustment is consistent with weak pair improvements. The specific roles of these tools will be seen more clearly in the Main Algorithm.

Now consider the following algorithm which returns a stable allocation from an initial allocation (μ, u) in the restricted situation as described previously.

Main Algorithm

Step 0: If (μ, u) is stable, then return (μ, u) and stop. Otherwise, put $\mu^0 \leftarrow \mu$ and $u^0 \leftarrow u$. Given (μ^0, u^0), for each matched w under μ^0, reset the list L_w cyclically so that the very first firm in L_w is the one that is matched with w in μ^0. For self-matched w, that is, for w^0, such adjustment is unnecessary and L_{w^0} can be the one constructed initially. Let f^0 be the first element of list L_{w^0}, and put $k \leftarrow 0$.

Step 1: If f^k is self-matched at μ^k, then match f^k with w^k. Let μ^{k+1} be the updated matching, and put $u^{k+1}(w^k) = V(f^k, w^k) - u^k(f^k)$ and $u^{k+1}(x) = u^k(x)$ for other $x \in F \cup W$. Return (μ^{k+1}, u^{k+1}) and stop.

Step 2: If f^k is not self-matched at μ^k, then match f^k with w^k (hence $\mu^k(f^k)$ becomes self-matched). Let μ^{k+1} be the updated matching, and put $u^{k+1}(w^k) \leftarrow V(f^k, w^k) - u^k(f^k)$, $u^{k+1}(\mu^k(f^k)) \leftarrow V(\mu^k(f^k), \mu^k(f^k))$, and $u^{k+1}(x) \leftarrow u^k(x)$ for other $x \in F \cup W$. Also put $w^{k+1} \leftarrow \mu^k(f^k)$.

If (μ^{k+1}, u^{k+1}) is stable, then return (μ^{k+1}, u^{k+1}) and stop. Otherwise, define f^{k+1} as follows:

If list $L_{w^{k+1}}$ is treated for the first time, then let f^{k+1} be the second element of $L_{w^{k+1}}$; otherwise let f^{k+1} be the firm next to the last matched firm in list $L_{w^{k+1}}$. Put $k \leftarrow k + 1$.

Step 3: If $(\mu^k, u^k) = (\mu^{k'}, u^{k'})$ for some integer k' with $0 \le k' < k$, we have got into a cycle and go to **Step 4**; otherwise go to **Step 1**.

Step 4: Let F_Q be the set of firms f whose matched workers $\mu(f)$ change at least once during the cycle according to the updating of μ. Starting from the current allocation (μ^k, u^k), put $F^* \leftarrow \emptyset$ and let w^* be the self-matched worker that appeared when μ^k was updated. **While** $F^* \neq F_Q$ **, execute the following** (*):

(*) Let (μ, u) be the current allocation. Find an alternating path for allocation (μ, u^k) from w^* to a firm $f^* \in F_Q \setminus F^*$ such that all the non-terminal firms belong to F^*.[13]
Carry out **Augment**.
Augment: Proceeding in the reversed order of the alternating path, for each unmatched firm-worker pair (f, w) in the alternating path do the following (1) and (2):

(1) Make f matched to w and let μ' be the updated matching (by construction of the alternating path, $\mu(f)$ becomes self-matched and unless $w = w^*$, $\mu(w)$ also becomes self-matched).

(2) Put $u(w) \leftarrow V(f, w) - u^k(f) - 1$, $u(f) \leftarrow u^k(f) + 1$, $u(\mu(f)) \leftarrow V(\mu(f), \mu(f))$, and unless $w = w^*$, put $u(\mu(w)) \leftarrow V(\mu(w), \mu(w))$.

Proceed until we complete (1) and (2) for the first unmatched pair that involves w^*. Put $w^* \leftarrow \mu(f^*)$, $\mu \leftarrow \mu'$, and $F^* \leftarrow F^* \cup \{f^*\}$.

Step 5: Denote the current allocation by (μ, u) again and let w^0 be the self-matched worker that appeared at the last updating of μ. Update lists L_w of $F_w(u)$ for all $w \in W$. Go to **Step 0**.

(**End**)

We briefly illustrate the essential idea of the Main Algorithm as follows:

Given the restricted market structure, all possible weakly blocking pairs have to involve the current self-matched worker. In Step 1 and Step 2, we always let the current self-matched worker initiate the next weakly blocking pair. Each self-matched worker w chooses, according to the list L_w, a firm that generates the highest net surplus, which is entirely awarded to the worker. Such an arrangement rules out cases where several existing pairs are broken by a single weakly blocking pair, disrupting certain monotonicity property as previously.

[13] Recall that u^k is the payoff vector appearing in Step 3, so that we should find an alternating path with respect to $F_w\left(u^k\right)$. Also, given an alternating path $\left(f^1, w^0 = w^*\right), \left(f^1, w^1 = \mu\left(f^1\right)\right), ..., \left(f^{l-1}, w^{l-1} = \mu\left(f^{l-1}\right)\right), \left(f^l, w^{l-1}\right)$, we call f^l the *terminal* firm and $f^1, f^2, ..., f^{l-1}$ *non-terminal* firms. Notice that if $l = 1$, then f^1 is the terminal firm and $f^1 \in F_Q \setminus F^*$.

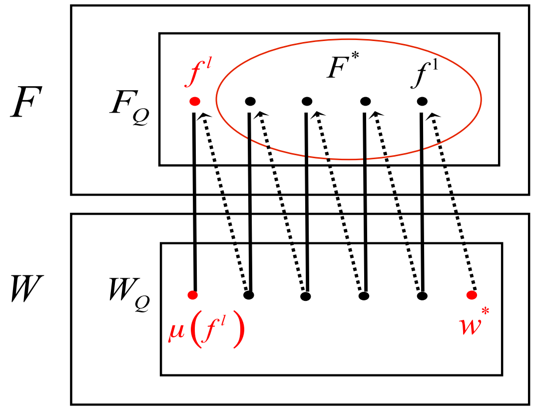

Figure 1. An Alternating Path from w^* to f^l in the Execution of Augment.

Next, such a "greedy" behavior of the self-matched workers raises the possibility of a cycle, where several workers "compete" for the same set of firms. We denote this set of firms in the cycle as F_Q. Intuitively, firms in F_Q are over-demanded by the competing workers in the cycle, which is collected in set W_Q. Observe that by the specification of lists $\{L_w\}_{w \in W_Q}$ in Step 0, at the end of the cycle, workers in W_Q face exactly the same configuration of $\{L_w\}_{w \in W_Q}$ as in μ^0 — that is, every w , $w \in W_Q$, has gone through multiple integer rounds of L_w entirely, and each $w \in W_Q$ has been matched with every firm in L_w at least once during the cycle.

The remaining part of the Main Algorithm serves to spin the process out of the cycle in a consistent way. To this end, we increase the payoffs of the over-demanded firms in F_Q, the adjustment being the smallest increment of 1. This payoff adjustment is completed systematically using alternating paths, as shown in (*) of Step 4. Specifically, starting from the self-matched worker w^* appearing at the end of the cycle, construct an alternating path "$(f^1, w^0 = w^*)$, $(f^1, w^1 = \mu(f^1))$, ..., $\left(f^{l-1}, w^{l-1} = \mu\left(f^{l-1}\right)\right)$, $\left(f^l, w^{l-1}\right)$" such that all firms except f^l are in F^*, a set constructed to temporarily collect firms in F_Q that have already been treated with payoff increases. Augment in Step 4 presents formal procedures to conduct payoff increases of firms in F_Q so that each adjustment is consistent with weak pair improvement. Notice that set F^* is initially empty and hence the very first alternating path has length 1. Alternating paths after the first execution of Augment may, however, have lengths more than 1.[14]

Figure 1 shows an alternating path from w^* to a firm $f^l \in F_Q \backslash F^*$. The dotted arrows connect currently *unmatched* pairs of firms and workers, indicating the next firm a worker would point to if the worker becomes self-matched, while the solid lines connect currently *matched* pairs of firms and workers in the alternating path. In executing Augment, we start with the last unmatched pair $\left(f^l, w^{l-1}\right)$, and match every

[14]Indeed, it is possible that all alternating paths in the execution of (*) may have length 1. We show in Example 4 that for some asymmetric markets, alternating paths with length more than 1 might have to be employed.

pair connected by dotted arrows sequentially until we reach the pair with (f^1, w^*). After one round of Augment, an additional firm f^l is treated with the payoff increase and is then added to set F^*. Moreover, worker $\mu\left(f^l\right)$ becomes the unmatched worker w^*, who initiates the next alternating path for the next round of Augment if the updated F^* is not equal to F_Q.

Intuitively, the complex nature of the Main Algorithm, especially in Augment, is a direct consequence of the constraint that every involved adjustment has to be consistent with weak pair improvements. We are now ready to prove the convergence of the Main Algorithm.

Theorem 2 *For any individually rational allocation (μ, u) that satisfies the restricted situation, the Main Algorithm always finds a stable allocation after a finite number of weak pair improvements.*

Proof. If Step 4 is not executed throughout the algorithm, then it is easy to see the validity of the algorithm. We next prove the validity of Step 4 when cycles arise.

Suppose that Step 4 is executed. Let F_Q (resp., W_Q) be the set of firms f (resp., workers w) whose matched partners change at least once during the cycle according to the updating of μ. We first show that whenever $F^* \neq F_Q$, there exists a desired alternating path from w^* to a firm $f^l \in F_Q \backslash F^*$ and that all other firms involved in the alternating path are in F^*. Suppose by way of contradiction that there does not exist any such alternating path for (μ, u^k) from w^* to $F_Q \setminus F^*$. Let $\hat{F}$ and $\hat{W}$, respectively, be the set of all firms and that of all workers in $F_Q \cup W_Q$ reachable from w^* by alternating paths for (μ, u^k).[15] Notice that by our construction of the cycle and the assumption of no legitimate alternating path from w^* to $F_Q \backslash F^*$, we have $\hat{F} \subseteq F^* \subset F_Q$ and $\hat{W} \subset W_Q$. Namely, all agents reachable from w^* by alternating paths for (μ, u^k) have to be in $F_Q \cup W_Q$.

Given such $\hat{F}$ and $\hat{W}$ and the assumption of non-existence of desired alternating paths, we have

$$|\hat{F}| + 1 = |\hat{W}|, \qquad |F_Q \setminus \hat{F}| = |W_Q \setminus \hat{W}| > 0 \tag{2}$$

and

$$\nexists \text{ matched } (f, w) : w \in \hat{W}, \quad f \in (F_Q \setminus \hat{F}) \cap F_w\left(u^k\right), \tag{3}$$

where observe that F_w is defined with respect to u^k. Condition (2) is derived from the definition of $\hat{F}$ and $\hat{W}$, as well as the fact that we start from the restricted situation with only one self-matched worker and every matching arising in the cycle has only one self-matched worker. According to (3), no worker in $\hat{W}$ can be matched with firms in $(F_Q \setminus \hat{F})$.[16] Notice in particular that (2) and (3) jointly imply that matched

[15] Here, an agent (a firm or a worker) is reachable from w^* if the agent is a member in a legitimate alternating path originates from w^*. By definition, $w^* \in \hat{W}$.

[16] In addition, for unmatched pairs in the alternating paths, all workers in $\hat{W}$ can only "point" to firms in $\hat{F}$ as well. See Figure 2 for a graphical illustration.

pairs in $\hat{F} \cup \hat{W}$ are disjoint from those in $\left(F_Q \setminus \hat{F}\right) \cup \left(W_Q \setminus \hat{W}\right)$ in the sense that no agent in $\hat{F} \cup \hat{W}$ is matched with an agent in $\left(F_Q \setminus \hat{F}\right) \cup \left(W_Q \setminus \hat{W}\right)$.

Now during the cycle, we have a matching that matches all $f \in F_Q$ to workers. Equation (3) and the fact of all agents in $F_Q \cup W_Q$ being involved in the single cycle then jointly imply that there must exist a matched pair (f', w') such that $f' \in \hat{F}$ and $w' \in W_Q \setminus \hat{W}$. (Notice that (i) because the cycle uses lists L_w ($w \in W_Q$) and each L_w has been exhausted entirely at least once during the cycle, we have $F_Q = \cup_{w \in W_Q} F_w\left(u^k\right)$, and (ii) worker $w \in W_Q \setminus \hat{W}$ cannot point to a firm $f \in \hat{F}$ either as this contradicts the fact that $\hat{W}$ is the set of all workers in W_Q reachable from w^* by alternating paths for (μ, u^k).) However, f' cannot be matched to w' since $F_Q \setminus \hat{F}$ must be matched to $W_Q \setminus \hat{W}$ due to (2) and (3).[17] This contradiction establishes the existence of a desired alternating path in the execution of (*).

We next show that every adjustment executed in Augment is consistent with weak pair improvements. Let $(f^1, w^0{=}\, w^*)$, $(f^1, w^1{=}\mu(f^1))$, (f^2, w^1), $(f^2, w^2{=}\mu(f^2))$, $\cdots$, $(f^{l-1}, w^{l-1}{=}\mu(f^{l-1}))$, $(f^l, w^{l-1}))$ be an alternating path found in (*) in Step 4 for some positive integer l. (Note that $\mu(f^l)$ becomes the unique self-matched worker after Augment, which is precisely the w^* for the next round of Augment.)

If $l = 1$, then $(f^1, w^0 = w^*)$ is a weakly blocking pair. (Recall that since $f^1 \in F_{w^0}\left(u^k\right)$, these two agents were matched during the cycle.)

If $l \geq 2$, we have $u(f^l) = u^k(f^l)$, $u(w^{l-1}) = u^k(w^{l-1}) - 1$, and $u^k(f^l) + u^k(w^{l-1}) = V(f^l, w^{l-1})$. Hence $V(f^l, w^{l-1}) - u(f^l) - u(w^{l-1}) = 1$. (Here, recall that by the definition of cycle, for any $f \in F_Q$ the value of $u^i(f)$ remains to be the same for $i = k', \cdots, k$, and that for each $w \in W_Q$ all the values of $V(f, w) - u^i(f)$ for $f \in F_w$ and $i = k', \cdots, k$ are the same. Here parameters k' and k are those appeared in Step 3.) Hence, (f^l, w^{l-1}) is a weakly blocking pair and this validates the operations in (2) of Augment. We make f^l matched to w^{l-1} and update u as (2) in (*) of Step 4. (In effect, $u(f^l)$ is increased by one and $u(w^{l-1})$ remains the same.) Then f^{l-1} becomes self-matched. If $l = 2$, then $(f^1, w^0 = w^*)$ is a weakly blocking pair. (Recall again that these two were matched during the cycle, so that they prefer being matched to being self-matched.)

If $l \geq 3$, we then have $u(f^{l-1}) = V(f^{l-1}, f^{l-1}) < u^k(f^{l-1}) + 1 = V(f^{l-1}, w^{l-2}) - u(w^{l-2})$, so that pair (f^{l-1}, w^{l-2}) becomes a weakly blocking pair. We then perform (1) and (2) of Augment. We repeat this process until we make f^1 matched to $w^0{=}\, w^*$, which completes an execution of Augment.

When finishing the While loop in Step 4, the utility $u(f)$ of each firm f in F_Q is increased by one.

When we go from Step 4 to Step 5, letting w^k be the current self-matched worker, there is no weakly blocking pair within $(F \cup W) \setminus \{w^k\}$. Recall that $F_Q = \cup_{w \in W_Q} F_w\left(u^k\right)$, which implies that for any (f, w) such that $f \in F \setminus F_Q$ and $w \in W_Q$,

[17] Alternatively, if $f' \in \hat{F}$ is matched with $w' \in W_Q \setminus \hat{W}$, we then cannot have that $|F_Q \setminus \hat{F}| = |W_Q \setminus \hat{W}| > 0$ and that firms in $F_Q \setminus \hat{F}$ are exactly matched with workers in $W_Q \setminus \hat{W}$.

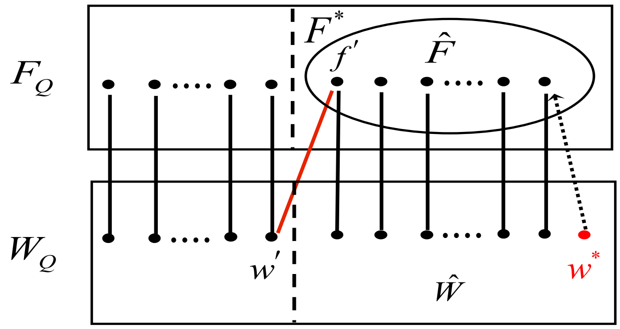

Figure 2. An Illustration on the Existence of Legitimate Alternating Paths.

we have $V(f,w) - u(f) \leq V(\mu(w),w) - u(\mu(w))$, where (μ, u) is the final allocation in Step 4 when Step 4 is completed.

Every time we execute Step 4, at least one value of $u(f)$ $(f \in F)$ increases and $u(f)$ for each $f \in F$ is non-decreasing throughout Main Algorithm (where we neglect the temporary steps in which $u(f)$ becomes $V(f,f)$ during the execution of Augment). Moreover, the set of possible integer values of $u(f)$ $(f \in F)$ is finite. Since the number of all the matchings for fixed $F_w\left(u^k\right)$ $(w \in W)$ is bounded by $|W|!$, after at most $|W|!$ updatings of matching μ, we either get into a cycle, where some $u(f)$ is increased after the execution of Step 4, or the algorithm terminates. We therefore conclude that the algorithm terminates after a finite number of steps. ■

Our proof of the existence of a desired alternating path in (*) can be further illustrated in Figure 2: If there does not exist a desired alternating path from w^* when $F^* \neq F_Q$, then the set of firms (resp., workers) reachable from alternating paths starting with w^* is a strict subset of F_Q, $\hat{F} \subset F_Q$ (resp., a strict subset of W_Q, $\hat{W} \subset W_Q$). This implies that firms in $F_Q \setminus \hat{F}$ are exactly matched with workers in $W_Q \setminus \hat{W}$, and these matched pairs are disjoint from the matched pairs in $\hat{F}$ and $\hat{W} \setminus \{w^*\}$. However, the facts that (i) firms in F_Q and workers in W_Q are involved in a single cycle; (ii) workers in $\hat{W}$ cannot be matched with (and cannot point to) firms outside $\hat{F}$; and (iii) workers in $W_Q \setminus \hat{W}$ cannot point to firms in $\hat{F}$, necessarily imply a link between $\hat{F} \cup \hat{W}$ with the rest of the cycle has to come from a matched pair (f', w'), $f' \in \hat{F}$, $w' \in W_Q \setminus \hat{W}$, which contradicts the established result that firms in $F_Q \setminus \hat{F}$ are exactly matched with $W_Q \setminus \hat{W}$.

We next present two examples to provide further illustration and to facilitate better understanding of the Main Algorithm.

Example 3 Consider a labor market (F, W, V), where

$$F = \{a, b\}, W = \{x, y, z\}$$
$$V(i,j) = 3, \forall\ i \in F \text{ and } j \in W \text{ and } V(k,k) = 0, \forall\ k \in F \cup W.$$

Consider an initial allocation (μ^0, u^0) that satisfies the restricted situation,

$$(\mu^0, u^0) = \left\{ \left(\underset{2}{a}, \underset{1}{x}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}.$$

We have $F_i(u^0) = \{a, b\}\ \forall i \in W$. Let $L_x = ab$, $L_y = ba$, and $L_z = ab$. Notice that the first element of L_x, a (resp., L_y, b), is currently matched with x (resp., y) in μ^0. It can be verified that $(\mu^6, u^6) = (\mu^0, u^0)$, a cycle, where $F_Q = \{a, b\}, W_Q = \{x, y, z\}$. From z, a desired alternating path is (a, z), which is a weakly blocking pair. One round of Augment gives

$$(\mu^6, u^6) = \left\{ \left(\underset{3}{a}, \underset{0}{z}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\},$$

and hence $F^* = \{a\}$. Choose the next alternating path to be (b, x), a weakly blocking pair. An execution of Augment yields a stable allocation

$$(\mu^6, u^6) = \left\{ \left(\underset{3}{a}, \underset{0}{z}\right), \left(\underset{3}{b}, \underset{0}{x}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\}.$$

The setting in Example 3 is somewhat symmetric and we always have the option of constructing every involved alternating path with length 1. We next present an asymmetric example, which shows that without further restrictions on choices of alternating paths and lists L_w, we may have to construct alternating paths with length more than 1 during the execution of Augment.

Example 4 Consider market (F, W, V), where

$$\begin{gathered} F = \{a, b, c, d\}, W = \{x, y, z, t; m\} \\ V(k, k) = 0, \forall\ k \in F \cup W,\ V(i, j) = 3,\ \forall\ i \in F, j \in W \setminus \{m\} \\ V(a, m) = V(b, m) = 2, V(c, m) = V(d, m) = 0 \end{gathered}$$

Consider an initial allocation (μ^0, u^0) that satisfies the restricted situation,

$$(\mu^0, u^0) = \left\{ \left(\underset{1}{a}, \underset{1}{m}\right), \left(\underset{1}{b}, \underset{2}{y}\right), \left(\underset{1}{c}, \underset{2}{z}\right), \left(\underset{1}{d}, \underset{2}{t}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\},$$

Notice that $F_i(u^0) = \{a, b, c, d\}, i \in W \setminus \{m\}$ and $F_w(u^0) = \{a, b\}$. Lists $\{L_w\}_{w \in W}$ are chosen as: $L_x = cdba$, $L_y = bcda$, $L_z = cdba$, $L_t = dabc$, $L_m = ab$. Proceeding as in Step 2, one can verify that a cycle arises such that[18]

$$(\mu^0, u^0) = (\mu^{22}, u^{22}) = \left\{ \left(\underset{1}{a}, \underset{1}{m}\right), \left(\underset{1}{b}, \underset{2}{y}\right), \left(\underset{1}{c}, \underset{2}{z}\right), \left(\underset{1}{d}, \underset{2}{t}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\},$$

[18]The self-matched workers in the 23 allocations can be ordered from μ^0 to μ^{22} as

$$x, z, t, m, y, x, z, m, t, z, m, t, y, x, m, z, t, y, m, x, y, m, x.$$

One can also see that in obtaining this cycle, we have gone through every list L_w entirely at least once.

where $w^* = x, F_Q = \{a, b, c, d\}$ and $W_Q = \{x, y, z, m, t\}$. Rename (μ^{22}, u^{22}) as (μ, u).

The first alternating path can be chosen as (b, x). One round of Augment leads to

$$(\mu, u) = \left\{ \left(\underset{1}{a}, \underset{1}{m}\right), \left(\underset{2}{b}, \underset{1}{x}\right), \left(\underset{1}{c}, \underset{2}{z}\right), \left(\underset{1}{d}, \underset{2}{t}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\},$$

leaving a new $w^* = y$ and $F^* = \{b\}$.

Choose the second alternating path to be (a, y) . An execution of Augment yields

$$(\mu, u) = \left\{ \left(\underset{2}{a}, \underset{0}{y}\right), \left(\underset{2}{b}, \underset{1}{x}\right), \left(\underset{1}{c}, \underset{2}{z}\right), \left(\underset{1}{d}, \underset{2}{t}\right), \left(\underset{0}{m}, \underset{0}{m}\right) \right\},$$

with a new $w^* = m$ and $F^* = \{a, b\}$.

The third alternating path, which can take several other forms but has to have length more than 1, is:

$$\underline{(a, m)}, (a, y), \underline{(b, y)}, (b, x), \underline{(c, x)},$$

where pairs with underscores are currently unmatched. We then proceed from backward and perform (1) and (2) in Augment, resulting in

$$(\mu, u) = \left\{ \left(\underset{2}{a}, \underset{0}{m}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{2}{c}, \underset{1}{x}\right), \left(\underset{1}{d}, \underset{2}{t}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\},$$

with $w^* = z$ and $F^* = \{a, b, c\}$. With (d, z) being the last alternating path, an execution of Augment finally yields

$$(\mu, u) = \left\{ \left(\underset{2}{a}, \underset{0}{m}\right), \left(\underset{2}{b}, \underset{1}{x}\right), \left(\underset{2}{c}, \underset{1}{z}\right), \left(\underset{2}{d}, \underset{1}{y}\right), \left(\underset{0}{t}, \underset{0}{t}\right) \right\}.$$

Consequently, at the start of Step 5, we have allocation (μ, u), which satisfies the restricted situation, with a single self-matched worker $w^0 = t$.

We now prove the existence of a deterministic sequence of finitely many weak pair improvements toward stability for any initial allocation with full generality.

Theorem 3 *Consider a labor market (F, W, V) with an arbitrary initial allocation (μ^0, u^0). Then from (μ^0, u^0) there exists a finite number of weak pair improvements which leads to a stable allocation (μ^*, u^*).*

With the help of the above result, we now develop a decentralized and random process that always results in a stable outcome of the market with probability one. As stated before, this process intends to mimic a natural decentralized decision making process in labor markets where firms and workers meet directly and randomly, and negotiate salaries over time.

First, observe that starting from an arbitrary initial allocation (μ^0, u^0), the set of all allocations, denoted as $\mathcal{A}(\mu^0, u^0)$, that is reachable through a sequence of pair improvements of (μ^0, u^0), is a set with finite elements. To see this, first notice that given a labor market with finite agents, the set of all possible matchings is finite. Secondly, for any allocation $(\mu, u) \in \mathcal{A}(\mu^0, u^0)$, it holds that $u(x) \geq \min\{u^0(x), V(x,x)\}$ for every $x \in F \cup W$.[19] Together with the fact that u is integral, set $\mathcal{A}(\mu^0, u^0)$, the family of all possible allocations (μ, u) satisfying $u \in \mathbb{Z}^{F \cup W}$ and $u(x) \geq \min\{u^0(x), V(x,x)\}$ for every $x \in F \cup W$, is hence a finite set.

We now describe the random market process. Suppose that the market opens with an arbitrary allocation (μ^0, u^0). Consider a Markov process with finite states, where the states are allocations that are reachable via successive pair improvements of (μ^0, u^0), or allocations in $\mathcal{A}(\mu^0, u^0)$. The initial state is allocation (μ^0, u^0). The transition probabilities of states in $\mathcal{A}(\mu^0, u^0)$ are defined in a way such that for every unstable allocation $(\mu, u) \in \mathcal{A}(\mu^0, u^0)$, each pair improvement of (μ, u) is chosen with positive probability. A transition probability may reflect, for example, how likely the agents involved in the corresponding pair improvement would meet and would split the surplus once they meet, as well as the current market structure. Denote the random sequence of allocations generated by the above Markov process starting from the initial allocation (μ^0, u^0) as $\hat{P}(\mu^0, u^0)$. Given Theorem 3 and our specification of the Markov process, it is immediate to show that the random sequence $\hat{P}(\mu^0, u^0)$ converges to a stable allocation in $\mathcal{A}(\mu^0, u^0)$ with probability one.

The above random process is mathematically convenient to describe and interpret. However, from an economic point of view, the random decentralized market process is chaotic and it does not seem plausible that when an unstable allocation (μ, u) is reached again in the middle of the random process, the transition probabilities associated with (μ, u) stay the same as before. We now discuss a similar but more general random process with discrete time, finite states, and possibly time-dependent transition probabilities among states. The market again opens with an arbitrary allocation (μ^0, u^0) at time $t = 0$. For transition probabilities, we assume that *every* (non-stationary) transition probability between two states in $\mathcal{A}(\mu^0, u^0)$ is no less than a fixed number $\varepsilon \in (0, 1)$ at any time. As there are only two classes of states (stable and unstable), it follows that starting from any allocation (μ, u) in $\mathcal{A}(\mu^0, u^0)$ at time t, the random process either finds a stable allocation in $\mathcal{A}(\mu^0, u^0)$ and remains stable afterwards, or continues to move from one unstable allocation to another unstable allocation in $\mathcal{A}(\mu^0, u^0)$. Now observe that the random process always arrives at an allocation in $\mathcal{A}(\mu^0, u^0)$, which contains only finitely many allocations. Suppose the random process does not converge to a stable allocation with probability one in the limit. This necessarily implies that at some point, after reaching an unstable allocation (μ, u) in $\mathcal{A}(\mu^0, u^0)$, the random process jumps among a (finite) set of

[19] Observe that for agent x whose payoff $u(x)$ falls below $V(x,x)$ under (u, μ), her payoff either increases or stays the same after each weak pair improvement. Moreover, agent x's payoff remains individually rational once it becomes so.

unstable allocations (associated with (μ, u)) in $\mathcal{A}(\mu^0, u^0)$ infinitely. As at any t in the random process, each possible pair improvement is chosen with probability no less than ε, it further implies that there is some (μ', u') in $\mathcal{A}(\mu^0, u^0)$ such that no finite path of pair improvements toward stability exists, no matter how one chooses the associated pair improvements. This, however, contradicts Theorem 3. Therefore, the probability of the random process converging to a stable allocation must be one as t goes to infinity.

Let $\hat{P}(\mu^0, u^0; \varepsilon)$ be the random sequence of allocations generated from the initial allocation (μ^0, u^0) as described above. The previous discussion then implies the following:

Proposition 1 *For any initial allocation (μ^0, u^0), the random sequence $\hat{P}(\mu^0, u^0; \varepsilon)$ converges with probability one to a stable allocation.*

5 Concluding Remarks

This paper studies a decentralized labor market where heterogeneous firms and workers meet and negotiate salaries with each other randomly and spontaneously over time. Firms and workers act in an uncoordinated way: They can form a partnership or dissolve their partnership instantly whenever better opportunities present themselves. The key finding of our study is that such a seemingly chaotic, and random dynamic decentralized market process converges with probability one to a competitive equilibrium of the market, provided that every possible bilateral trading arises with positive probability. To establish this result, an essential step is to show that starting from an arbitrary initial market state, there exists a finite sequence of successive myopic bilateral tradings which leads to a stable matching between firms and workers with a scheme of competitive salary offers.

As a natural starting point, we have assumed in our model that each firm hires at most one worker.[20] Of course, such an assumption will not be satisfied in general labor markets where some firm may employ multiple workers. An important and natural direction for further research is hence to study a similar random decentralized market process in labor markets where firms may hire any number of workers. It is well known from Kelso and Crawford (1982) that a competitive equilibrium exists in a labor market where each firm may hire several workers, as long as every firm views all workers as substitutes. As mentioned earlier, they also proposed a centralized adjustment process for this general market. Moreover, Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2006), Sun and Yang (2009) have developed centralized processes for general competitive auction markets. It will be, however, significantly more challenging to examine such general markets from a decentralized perspective. Another interesting question is when agents face a decentralized and random market

[20]This is a standard assumption used in the literature on job matching and assignment markets.

process as the one we developed, do they have incentives to manipulate the process so as to make individual gains in a relatively small labor market? We anticipate developing results in these directions in future work.

Appendix

Proof of Theorem 1. It is sufficient to focus on an individually rational allocation (μ^0, u^0).[21] We now show that the Basic Algorithm returns a weakly stable allocation (μ^*, u^*) after finitely many successive strong pair improvements. As the crux of the Basic Algorithm lies in the inductive construction of $K(n)$, we now show that each update of $K(n+1)$ with the addition of new members to $K(n)$ requires finite successive strong pair improvements so as to reach an allocation (μ^{n+1}, u^{n+1}) such that no strongly blocking pair of (μ^{n+1}, u^{n+1}) is entirely contained in $K(n+1)$:

First, $K(1) = \{(f^1, w^1)\}$ where (f^1, w^1) is a strongly blocking pair of (μ^0, u^0) with a consistent specification of $u^1(f^1)$ and $u^1(w^1)$. By construction, $K(1)$ does not contain a blocking pair of (μ^1, u^1).

Next, in our inductive construction of $K(n+1)$ from $K(n)$ and (μ^n, u^n), the general rule is that whenever there is a blocking pair (f^n, w^n) such that either f^n or w^n is in $K(n)$, then we treat such blocking pairs first.[22] Together with the surplus division rule specified in the Basic Algorithm, such a treatment guarantees that after the introduction of new members into $K(n)$, all firms (or all workers) in $K(n)$ are always at least weakly better off along the sequence of adjustments toward the construction of $K(n+1)$ and (μ^{n+1}, u^{n+1}) (and some strictly better off). We show this separately for the three cases:

Case 1. If $f^n \notin K(n)$, then f^n initiates the next blocking pair with $w^n \in K(n)$. As f^n chooses her best worker w_n to form the blocking pair and she obtains the largest possible payoff that is consistent with the strongly blocking pair, f^n cannot form the next strongly blocking pair (if any) with workers in $K(n)$. Now if w^n is self-matched in $K(n)$, $K(n+1)$ does not entirely contain a blocking pair of (μ^{n+1}, u^{n+1}). If w^n is matched in $K(n)$, only $\mu^n(w^n)$ can initiate the next blocking pair with workers in $K(n)$ as other firms in $K(n)$ are not affected and the payoff vector of workers in $K(n)$ is weakly increased. This validates the operation of letting $\mu^n(w^n)$ to be the next initiator. Finally, as all agents' payoffs are integral and finite, and no worker in $K(n)$ is worse off and at least one worker's payoff strictly increases after each strongly pair improvement, we reach an "internally stable" $K(n+1)$ after finitely many steps.

[21] Otherwise we can start the sequence of strong pair improvements with a string of strong blocking pairs initiated only by individual i such that $u^0(i) < V(i,i)$, resulting in some $\left(\hat{\mu}^0, \hat{u}^0\right)$ where $\hat{u}^0(k) \geq V(k,k), \forall k \in F \cup W$. Then rename $\left(\hat{\mu}^0, \hat{u}^0\right)$ to be $\left(\mu^0, u^0\right)$.

[22] See the specific statement of *Case 3* in the Basic Algorithm.

Case 2. $w^n \notin K(n)$ and $f^n \in K(n)$. The proof for this case is done similarly as in Case 1.

Case 3. Every existing strongly blocking pair is such that $(f_n, w_n) \in (F \cup W) \backslash K(n)$. By the fact that $K(n+1) = K(n) \cup \{(f^n, w^n)\}$ and payoffs of f^n and w^n have both strictly increased, no strongly blocking pair of (μ^{n+1}, u^{n+1}) is hence contained in $K(n+1)$.

Finally, observe that the set $K(n)$ constructed above strictly increases (inclusion-wise) after each execution of Step 2. The execution of Step 2 and Step 3 hence must terminate in finitely many steps until a weakly stable allocation is obtained, as $|F|, |W| < +\infty$ and $K(n)$ can grow no larger than $F \cup W$. ■

Proof of Theorem 3. It is again without loss of generality to consider an individually rational initial allocation. We prove the result by induction on an index $q \geq 1$.

For $q = 1$, we rename (μ^0, u^0) as (μ^1, u^1). If there is a matched pair (x, y) at μ^1, then define $A(1) = \{x, y\}$; otherwise we choose any self-matched agent x and define $A(1) = \{x\}$.

Suppose for an integer $q \geq 1$ that we have

1. an individually rational allocation (μ^q, u^q) such that $u^q(x) = V(x, x)$ for self-matched $x \in F \cup W$, and
2. a non-empty set $A(q) \subseteq F \cup W$ such that there are no weakly blocking pairs within $A(q)$ and such that μ^q does not match any agents in $A(q)$ with agents in $(F \cup W) \setminus A(q)$.

Notice that this is true for $q = 1$. If (μ^q, u^q) is stable, then we are done. Otherwise there exists a weakly blocking pair, and we consider the following three cases.

Case 1: there exists a weakly blocking pair (f', w') with $f' \in A(q) \cap F$,

Case 2: there exists a weakly blocking pair (f', w') with $w' \in A(q) \cap W$, and

Case 3: there exists a weakly blocking pair (f', w') with $f', w' \notin A(q)$ and neither Case 1 nor Case 2 applies.

Here observe that at most one of such f' and w' can belong to $A(q)$.

In Case 1, let

$$A(q+1) = A(q) \cup \{w'\}. \tag{4}$$

We next let w' break up with its partner (if any), or namely, let

$$\begin{aligned} &\mu^q(\mu^q(w')) \leftarrow \mu^q(w') \text{ if } w' \text{ is not self-matched,} \\ &\mu^q(w') \leftarrow w', otherwise. \end{aligned}$$

Denote w' by w^1_{q+1} and the new μ^q by μ^{q+1}_1. Update u^q so that $u^q(x) = V(x, x)$ for all self-matched $x \in F \cup W$. Denote by u^{q+1}_1 the updated u^q.

Now we have the following fact:

$$\text{There exists no weakly blocking pair within } A(q+1) \setminus \{w^1_{q+1}\}. \tag{5}$$

Next notice that by the choice of $w^1_{q+1}(= w')$, there exists a weakly blocking pair within $A(q + 1)$, and that (5) holds due to property (2) of $A(q)$. Apply the Main Algorithm to $A(q+1)$ to obtain an allocation such that there exists no weakly blocking pair within $A(q + 1)$. We then proceed the induction step from q to $q + 1$.

Case 2 is similar to Case 1.

In Case 3, we define μ^{q+1} by making f' matched to w' and also define u^{q+1} similarly as in Case 1, together with some payoffs of f' and w' that are consistent with a weakly blocking pair. Next, define $A(q + 1) = A(q) \cup \{f', w'\}$. It is then clear that there is no blocking pair within $A(q + 1)$.

We eventually obtain a stable allocation by repeating this whole process for at most $|F \cup W|$ times, since we have a strictly increasing (inclusion-wise) sequence $A(q)$, where $q = 1, 2, ...$ ■

References

[1] A. Abdulkadiroğlu, P.A. Pathak, and A.E. Roth (2005), "The New York City high school match," American Economic Review Papers and Proceedings, 95, 2, May, 364–367.

[2] A. Abdulkadirğlu and T. Sönmez (2003), "School choice: A mechanism design approach," American Economic Review, 93, 729–747.

[3] K.J. Arrow and L. Hurwicz (1958), "On the stability of the competitive equilibrium, I, " Econometrica, 26, 522-552.

[4] L. Ausubel (2006), "An efficient dynamic auction for heterogeneous commodities," American Economic Review, 96, 602-629.

[5] K.-S. Chung (2000), "On the existence of stable roommate matchings," Games and Economic Behavior, 33, 206-230.

[6] V.P. Crawford (2008),"The flexible-salary match: a proposal to increase the salary flexibility of the national resident matching program, " Journal of Economic Behavior and Organization, 66, 149-160.

[7] V.P. Crawford and E.M. Knoer (1981), "Job matching with heterogeneous firms and workers," Econometrica, 49, 437-450.

[8] G. Demange, D. Gale and M. Sotomayor (1986), "Multi-item auctions," Journal of Political Economy, 94, 863-872.

[9] E. Diamantoudi, E. Miyagawa and L. Xue (2004), "Random paths to stability in the roommate problem," Games and Economic Behavior, 48, 18-28.

[10] A.M. Feldman (1974), "Recontracting stability," Econometrica, 42, 35-44.

[11] D. Gale, and L. Shapley (1962), "College admissions and the stability of marriage," American Mathematical Monthly, 69, 9-15.

[12] J. Green (1974), "The stability of Edgeworth's recontracting process," Econometrica, 42, 21-34.

[13] F. Gul and E. Stacchetti (2000), "The English auction with differentiated commodities," Journal of Economic Theory, 92, 66-95.

[14] A. Kelso and V.P. Crawford (1982), "Job matching coalition formation and gross substitutes," Econometrica, 50, 1483-1504.

[15] B. Klaus and F. Klijn (2007), "Paths to stability for matching markets with couples," Games and Economic Behavior, 58, 154–171.

[16] F. Kojima and M.U. Ünver (2006), "Random paths to pairwise stability in many-to-many matching problems: A study on market equilibration," International Journal of Game Theory 36, 473-488.

[17] T.C. Koopmans and M. Beckmann (1957), "Assignment problems and the location of economic activities," Econometrica, 25, 53-76.

[18] P. Milgrom (2000), "Putting auction theory to work: the simultaneous ascending auction," Journal of Political Economy, 108, 245-272.

[19] M. Ostrovsky (2008), "Stability in supply chain network, " American Economic Review, 98, 897-923.

[20] A.E. Roth (1984), "The evolution of the labor market for medical interns and residents: A case study in game theory," Journal of Political Economy, 92 (6), 991–1016.

[21] A.E. Roth and M. Sotomayor (1990), "Two-sided matching," Cambridge University Press, New York.

[22] A.E. Roth and J.H. Vande Vate (1990), "Random paths to stability in two-sided matching," Econometrica, 58, 1475-1480.

[23] P.A. Samuelson (1941), "The stability of equilibrium: comparative statics and dynamics, " Econometrica, 9, 97-120.

[24] P.A. Samuelson and W.D. Nordhaus (2010), "Economics," McGraw-Hill, New York.

[25] H. Scarf (1960), "Some examples of global instability of the competitive equilibrium, " International Economic Review, 1 (3), 157-172.

[26] H. Scarf (1973), "The Computation of Economic Equilibria, " Yale University Press, New Haven.

[27] L.S. Shapley and H. Scarf (1974), "On cores and indivisibilities," Journal of Mathematical Economics, 1, 23-37.

[28] L.S. Shapley and M. Shubik (1972), "The assignment game I: the core," International Journal of Game Theory, 1, 111-130.

[29] N. Sun and Z. Yang (2009), "A double-track adjustment process for discrete markets with substitutes and complements," Econometrica, 77, 933-952.