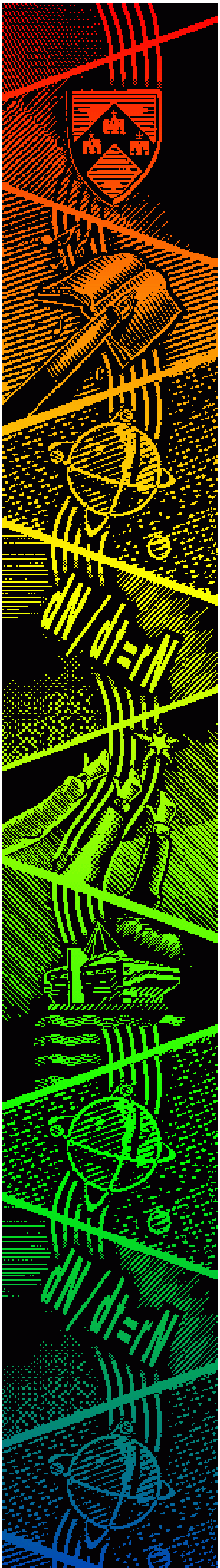

THE UNIVERSITY of York

Discussion Papers in Economics

No. 10/20

Endogenous Financing of Production in General Equilibrium with Incomplete Markets

By

Pascal Stiefenhofer, University of York

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Endogenous Financing of Production in General Equilibrium with Incomplete Markets

Pascal Christian Stiefenhofer*
University of York

July 28, 2010

Abstract

This paper considers the financing of production in a two period general equilibrium model with incomplete markets. This requires a model where the efficient boundary of the production set available to a producer in period two in every state of the world is not independent of the financial activities of the firm in period one. The novelty of the paper is a definition of a class of long run profit maximization objective functions of the firm which is independent of any average utility of the owners of the firm. This generalizes the traditional objective of profit maximization of the Arrow-Debreu model to the case of incomplete markets. The paper shows that equilibrium exists for convex smooth and convex piecewise linear endogenized production sets.

Keywords: Existence, Incomplete Markets, Profit Maximization, Production.
JEL classification: D62, D52, D53

1 Introduction

The classical incomplete markets literature (GEI) has mainly focused on variations of Arrow's seminal two period model with exogenous financial assets [1], [2]. Its extension to production considers a model of the firm with exogenous financial structure and exogenous real structure by assuming fixed production sets available to firms [11], [18], [20]. Imposing a fixed asset structure on the firm introduces the implicit assumption that the real and financial sets available to it are dichotomic. This deeply rooted property of the classical GEI model of production has non-trivial economic consequences for the theory of the firm. (i) Since production sets available to firms are assumed to be fixed, there is no financial activities firms perform on financial markets. All production inputs are financed with the revenue from selling output at given production capacity. Hence, the financing of production capacity is not modeled. (ii) Since firms do not issue stocks or other financial assets in order to finance its capital, the size of its production set is independent of its ability of acquiring cash on financial markets. Hence, the efficient boundary of the production sets available to a firm is independent of the firms financial activities. Consequently, the objective of the firm reduces to the optimization of a real activity similar to the objective of the firm in the classical Arrow-Debreu model [6] with private ownership firms and no financial

*Contact address: Department of Economics and Related Studies University of York. ps515@york.ac.uk.
This paper is part of my PhD Thesis at the University of York, UK.

assets. Many tasks a firm performs in the real world can thus not be modeled in the classical GEI framework.

Problems associated with this economic scenario are well known. Since the gradient vectors of the consumers generally point in different directions when markets are incomplete, they generally evaluate future income streams generated by the firms differently. Hence there is no unanimous agreement of the owners of the firms about the optimal choice of a net activity of each firm. Thus the objective function of the firm is not well defined.

The dominant solution concepts to the problem of the objective function of the firm when partial spanning fails [21] were introduced by Drèze [10] and Grossman and Hart [12]. Each paper proposes a closed form equilibrium definition by assigning some average utility of the (initial/final) stock holders to each firm. Drèze allows for side payments between stockholders in order to achieve unanimous agreement of firm owners on the optimal net activity vector to be implemented by a manager. Grossman and Hart introduce the concept of competitive price perceptions. The main drawback of these solution concepts is the cost of abandoning the decentralization property of the Arrow-Debreu model where the activities of the agents are separated. This property is a consequence of the exogenous two period real asset structure imposed on production sets. Since the two period real asset structure is fixed, the problem of the firm is to chose inputs of production in period one (at fixed production capacity) such that associated period two outputs maximize profits. In order to have a closed form solution of the objective function of the firm, the literature has assigned utilities to firms. The problem with this approach is what utility to assign to each firm, a problem not solved at full satisfaction yet. Many utilities have been suggested in the literature, such as the utility of the manager of the firm, the average utility of the board of directors for example. For a sample of the large literature on utility dependent GEI models with production see [10], [9], [5], [15], [12], [21].

The aim of this paper is to model the financing of production in general equilibrium with incomplete markets while preserving the decentralization property of the Arrow-Debreu model. To attain a coherent model of the firm with this property, it is necessary to extend the concept of the private ownership firm of the Arrow-Debreu model to a concept of the firm which permits firms to take financial and real decisions sequentially. The sequential optimization structure of the firm is a consequence of the assumption of long run profit maximization introduced. It is assumed that firms maximize profits over both periods, hence their long rung activities involve acquiring cash and buying capital in order to build up their production capacity in period one. For that, firms issues stocks and buy capital. In the short run, each firm knows its installed production capacity available to it in each uncertain state of the world. The producer then chooses inputs such that associated outputs maximize profits in each state of the world. Period two short run inputs of production are financed with the revenue generated with the sell of outputs. Consumers want to invest in firms because they want to transfer wealth between future uncertain states of the world. They buy stocks in period one in return for a state dependent dividend payoff in the next period.

The sine qua non of the model is then to show that equilibrium exists. It is shown that, for an endogenized price and technology dependent real asset structure, which is transverse to the reduced rank manifolds, equilibrium exists generically in the endowments by the application of Thom's parametric transversality theorem. A class of well behaved smooth production sets structures, allowing the sequential modeling of the firm is introduced. Finally, the non-smooth convex production set case is considered, where piecewise linear production manifolds are regularized by convolution. Existence then follows from the smooth case. Bottazzi [3] demonstrated generic existence of equilibrium for an exchange economy for price dependent smooth assets. Equilibria exist for more general asset structures.

The model is introduced in section 2. Section 3 shows generic existence for convex smooth production sets. Section 4 considers convex piecewise linear production sets which are regularized by convolution. Section 5 is a conclusion and, section 6 is an appendix.

2 The Economic Model

We consider a two period $t \in T = \{0, 1\}$ model with technological uncertainty in period 1 represented by states of nature. An element in the set of mutually exclusive and exhaustive uncertain events is denoted $s \in \{1, ..., S\}$, where by convention $s = 0$ represents the certain event in period 0. Where no confusion of notation is expected, we sometimes denote S the set of all mutually exclusive uncertain events. We count in total $(S + 1)$ states of nature.

The economic agents are the $j \in \{1, ..., n\}$ producers and $i \in \{1, ..., m\}$ consumers which are characterized by sets of assumptions F and C below. There are $k \in \{1, ..., l\}$ physical commodities and $j \in \{1, ..., n\}$ financial assets, referred to as stocks. In fact, stocks are the only financial assets considered here. This allows for a sufficiently rich structure in order to introduce the benchmark model of production in its simplest form. Physical goods are traded on each of the $(S+1)$ spot markets. Producers issue stocks which are traded at $s = 0$, yielding a payoff in the next period at uncertain state $s \in \{1, ..., S\}$. The quantity of stocks issued by firm $j \in \{1, ..., n\}$ is denoted $z_j \in \mathbb{R}_-$, where $\sum_{j=1}^{n} z_j = \hat{z}$.

There are in total $l(S+1)$ physical goods available for consumption. The consumption bundle of agent $i \in \{1, ..., m\}$ is denoted $x_i = (x_i(0), x_i(s), ..., x_i(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $x_i(s) = (x_i^1(s), ..., x_i^l(s)) \in \mathbb{R}^l_{++}$, and $\sum_{i=1}^{m} x_i = x$. The consumption space for each consumer $i \in \{1, ..., m\}$ is $X_i = \mathbb{R}^{l(S+1)}_{++}$, the strictly positive orthant. The associated price system is a collection of vectors represented by $p = (p(0), p(s), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $p(s) = (p^1(s), ..., p^l(s)) \in \mathbb{R}^l_{++}$, the strictly positive orthant. Each consumer $i \in \{1, ..., m\}$ is endowed with initial resources $\omega_i \in \Omega$, where $\Omega = \mathbb{R}^{lT}_{++}$, and $\omega_i = (\omega_i(0), \omega_i(1))$ a collection of strictly positive vectors. Denote an initial resource vector at time period $t \in T = \{0, 1\}$, $\omega_i(t) = (\omega_i^1(t), ..., \omega_i^l(t)) \in \mathbb{R}^l_{++}$, and the sum of total initial resources, $\sum_{i=1}^{m} \omega_i = \omega$.

There is no aggregate risk in this economy. In total, there are n financial assets traded in period $t = 0$. Denote the quantity vector of stocks purchased by consumer $i \in \{1, ..., m\}$, $z_i = (z_i(1), ..., z_i(n)) \in \mathbb{R}^n_+$, a collection of quantities of stocks purchased from producers $j \in \{1, ..., n\}$, and denote $\sum_{i=1}^{m} z_i = z$, with associated stock price system $q = (q(1), ..., q(n)) \in \mathbb{R}^n_{++}$. Denote producer j's period one vector of capital purchase $y^j(0) \in \mathbb{R}^l_-$, and denote his period two state dependent net activity vector $y_j(s) = (y_j^1(s), ..., y_j^l(s)) \in \mathbb{R}^l$. Let $y_j(t = 1) = (y_j(s), ..., y_j(S)) \in \mathbb{R}^{lS}$ denote the collection of state dependent period t=1 net activity vectors. A period two input of production for every $s \in \{1, ..., S\}$ is by convention denoted $y_j^k(s) < 0$, and a production output in state $s \in \{1, ..., S\}$ satisfies $y_j^k(s) \geq 0$.

2.1 The model of the firm

Each firm $j \in \{1, ..., n\}$ issues stocks z_j at stock price q_j in period one in order to build up production capacity. A firm's total cash acquired via stock market determines the upper bound of the total value of production capacity it can install in the same period. Denote this liquidity constraint $q_j z_j = M_j$, where $M_j \in \mathbb{R}_+$ is a non-negative real number and $z_j \in \mathbb{R}_-$ a feasible financial policy of the firm $j \in \{1,...,n\}$. M_j constraints the quantity of capital $y(0) \in \mathbb{R}^l_-$ a producer j can purchase at spot price system $p(0) \in \mathbb{R}^l_{++}$. The quantity of intermediate goods (capital) $y_j(0)$ purchased by the producer in period one determines a correspondence $\phi_j|_Z$. This correspondence defines the technology of the firm at feasible financial equilibrium policy Z. Let the production set available to each producer $j \in \{1,...,n\}$ in period two be described by this technology, $\phi_j|_Z : \mathbb{R}^m_- \to \mathbb{R}^n_-$, a correspondence defined on the set of period two inputs for each state of nature $s \in S$. Denote the producer's productions set $Y_j|_z \subset \mathbb{R}^l$, where $m + n = l$.

Assumption (P): Each firm $j \in \{1,...,n\}$ maximizes long run profits.

Assumption (P) of long run profit maximization is justified for its economic practicality and its useful consequences. It facilitates the introduction of one period production sets, similar to those of the standard Arrow-Debreu model, with the difference that a period two production set available to a firm is not independent of its financial activities in period one. An immediate implication of assumption (P), together with one period production sets characterized by assumptions (F) below, is associated with the sequential optimization structure of the firm, which implies that firms demand real quantities (capital) in period one and in period two (inputs of production). This consequence is non-trivial, as it eliminates the present value problem of the firm present in classical GEI models of production, where producers choose inputs of production in period one at fixed capital. Consequently, assumption (P) preserves the decentralization property of the standard Arrow-Debreu model when production is modeled under time and uncertainty. This follows from the role capital plays in this model.

Each producer j is further characterized by set of assumptions (F). The characterization of production follows from the expansion of the standard assumptions of Debreu [6] to time and uncertainty, where the financing of production is explicitly modeled, and assumption (P) introduced.

Assumption 1 (F) *(i) For each $j \in \{1,...,n\}$, $Y_j|_z \in \mathbb{R}^{lS}$ is closed, convex, and $(\omega + \sum_{j=1}^n Y_j|_z) \cap \mathbb{R}^{lS}_+$ compact for all $\omega_i \in \mathbb{R}^{lT}_{++}$. $0 \in Y_j|_z \Leftarrow Y_j|_z \supset \mathbb{R}^{lS}_-$. $Y_j|_z \cap \mathbb{R}^{lS}_+ = \{0\}$. (ii) For each $j \in \{1,...,n\}$ denote $\partial Y_j|_z \subset \mathbb{R}^{nS}_+$ a C^∞ manifold. (iii) For each $j \in \{1,...,n\}$, transformation maps $\Phi|_Z(j)$ are non-linear representing decreasing returns to scale technology. (iv) For each $j \in \{1,...,n\}$, endogenized production capacity is bounded above and is characterized by $z_j \in (\sum_{i=1}^m z_i(j), 0)$ in the open interval of feasible financial policies.*

(i) The closedness assumption is introduced for its mathematical convenience. Convexity of the production set implies that no increasing returns to scale technologies are considered, describing the competitive economic environment. For example, it permits constant return to scale or decreasing return to scale technologies further specified by the assumption (iii) on the transformation maps $\Phi|_Z(j)$ for all $j \in \{1,...,n\}$. It is assumed that the total production possibilities of the whole economy are bounded above. Finally a free disposal assumption is introduced, implying the possibility of inaction of the firm. A firm has always the choice of producing no outputs with zero inputs. It is assumed in (ii) that the efficient boundary of the production set is smooth. Here, C^∞ implies differentiability at any order required. The order depending on all transversality arguments employed. Assumption (iv) characterizes the bounds on the level of production capacity $y_j(0) \Rightarrow \Phi|_Z(j)$ accumulated at feasible financial policy Z.

Consider the technical transformation process in period two (independent of the liquidity constraint $M_j \in \mathbb{R}_+$), defined by a map $\phi_j(s) : \mathbb{R}^m_- \to \mathbb{R}^n_+$ in every state $s \in S$. This correspondence determines the production set of firm $j \in \{1, ..., n\}$, denoted $Y_j = \mathbb{R}^l$ for every $s \in S$. In reality, this correspondence is likely to map into $k \neq n$, as there is no reason to expect the same number of goods in each period. This restriction is purely for notational convenience, and changing dimensions will not alter the analysis of this paper.

At $t = 0$, each producer $j \in \{1, ..., n\}$ chooses z_j at spot price q such that equilibrium condition $\sum_{i=1}^{m} z_i(j) = z_j$ is satisfied. This implies that each producer$j$ can acquire cash $\overline{q}z_j = \overline{M}_j$ and buy capital $y_j(0)$ such that $\overline{p}(0) \centerdot \overline{y}_j(0) = \overline{M}_j$ is satisfied. At $t = 1$, it chooses a profit maximizing net activity vector $y_j(s)$ in installed production set $Y_j|_z$ after state $s \in S$ is revealed. A production set $Y_j|_z$ available to a producer $j \in \{1, ..., n\}$ in period two is determined by its technological feasibility map ϕ_j , which is defined by the capital $y_j(0)$ a firm can buy at financial liquidity constraint $\overline{q}z_j = \overline{M}_j$ in period one. More formally, the closed form objective function assigned to each producer $j \in \{1, ..., n\}$ becomes

$$(\bar{z}, \overline{y}(0), \overline{y}(s))_j argmax \left\{ \overline{p}(s) \square y_j(s) \middle| \begin{array}{c} y_j(s) \in Y_j|_z(s) \\ \overline{q}z_j = \overline{p}(0) \centerdot y_j(0) = \overline{M}_j \end{array} \text{ for all } s \in S \right\}, \quad (1)$$

Denote a long run equilibrium output vector associated with the production set boundary $\bar{y}_j \in \partial Y_{j,eff}\,|_z$ .The $t = 1$ maps implied by equation (1), $\pi_j : \mathbb{R}^l_{++} \times \mathbb{R}^l \to \mathbb{R}_+$, for each state $s \in S$ and all producers $j \in \{1, ..., n\}$ define the $(S \times n)$ total long run payoff matrix, a collection of n vectors denoted

$$\Pi(p_1, \phi|_Z) = \begin{bmatrix} p(s) \cdot y_1(s) & \cdots & p(s) \cdot y_n(s) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p(S) \cdot y_n(S) \end{bmatrix}, \quad (2)$$

where $\phi|_Z$ denotes the technology and capacity dependency of the payoff structure.

2.2 The consumers

Each consumer $i \in \{1, ..., m\}$ is characterized by set of assumptions C of smooth economies (Debreu [7]).

Assumptions (C): (i) $u_i : \mathbb{R}^{l(S+1)}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^{l(S+1)}_+$, and C^∞ on $\mathbb{R}^{l(S+1)}_{++}$. $u_i(x_i) = \left\{ x'_i \in \mathbb{R}^{l(S+1)}_+ : u_i(x'_i) \geq u_i(x_i) \right\} \subset \mathbb{R}^{l(S+1)}_{++}$, $\forall x_i \in \mathbb{R}^{l(S+1)}_{++}$. For each $x_i \in \mathbb{R}^{l(S+1)}_{++}$, $Du_i(x_i) \in \mathbb{R}^{l(S+1)}_{++}$, $\forall s$. For each $x_i \in \mathbb{R}^{l(S+1)}_{++}$, $h^T D^2 u_i(x_i) h < 0$, for all nonzero hyperplane h such that $(Du_i(x_i))^T h = 0$. (ii) Each $i \in \{1, ..., m\}$ is endowed with $\omega_i \in \mathbb{R}^{l(S+1)}_{++}$.

Consumers want to transfer wealth between future spot markets and diversify risk. For that, they invest in firms in period $t = 0$, receiving a share of total dividend payoffs which are determined in the next period in return. Denote the sequence of $(S + 1)$ budget constraints

$$B_{z_i} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++},\ z_i \in \mathbb{R}^n_+ : \begin{array}{l} p(0) \cdot (x_i(0) - \omega_i(0)) = -q z_i \\ p(s) \square (x_i(s) - \omega_i(s)) = \Pi(p_1, \phi|_Z) \theta(z_i) \end{array} \right\}, \quad (3)$$

where[1] ownership structure is a $(n \times 1)$ vector defined by the mappings

$$\theta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+, \forall j, \tag{4}$$

where $z_i(j) \in \mathbb{R}_+$ is a non-negative real number for every $j \in \{1, ..., n\}$. $\theta_{ij} = z_i(j) \left[\sum_i z_i(j)\right]^{-1}$ is the proportion of total payoff of financial asset $j \in \{1, ..., n\}$ hold by consumer $i \in \{1, ..., m\}$. In compressed notation, we write

$$B_{z_i} = \quad \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++},\ z_i \in \mathbb{R}^n_+ : p(s)\Box(x_i(s) - \omega_i(s)) \in \hat{\Pi}\left[z_i | \theta(z_i)\right] \right\} \tag{5}$$

where $\hat{\Pi}(p_1, q, y) = \begin{bmatrix} -q_1 & \cdots & -q_n \\ p(1) \cdot y_1(1) & \cdots & p(1) \cdot y_n(1) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p_1(S) \cdot y_n(S) \end{bmatrix}$ represents the full payoff matrix of order $((S+1) \times n)$.

The sequential optimization problem of the consumer $i \in \{1, ..., m\}$ is to invest into firms in period one in order to smooth out future uncertain consumption and to optimize consumption of goods in every $(S+1)$ spot market. For a given price system $p = (p(0), p(1), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$ of consumption goods and price system $q \in \mathbb{R}^n_{++}$ of financial assets, a consumer chooses bundles of consumption goods and quantities of stocks $(x, z)_i \in X_i \times \mathbb{R}^n_{++}$ such that $u_i(x_i; z_i)$ is maximized subject to the sequence of $(S+1)$ constraints in B_{z_i}. Formally

$$(\bar{x}_i; \bar{z}_i) \arg\max \left\{ u_i(x_i; z_i) : \bar{z}_i \in \mathbb{R}^n_{++},\ \bar{x}_i \in B_{\bar{z}_i}(\bar{p}, \bar{q}, \bar{y}; \omega_i),\ \forall i \right\} \tag{6}$$

2.3 Competitive equilibrium with production

We introduce following prize normalization $\mathfrak{S} = \left\{ p \in \mathbb{R}^{l(S+1)}_{++} : \|p\| = \Delta \right\}$ such that the Euclidean norm vector of the spot price system $\|p\|$ is a strictly positive real number $\Delta \in \mathbb{R}_{++}$. A competitive equilibrium of the production economy defined by the endowment vector $\omega \in \Omega$ is a price pair $(\bar{p}, \bar{q}) \in \mathfrak{S} \times \mathbb{R}^n_{++}$ if equality between demand and supply of physical goods and financial assets is satisfied. Its associated competitive equilibrium allocation is a collection of vectors $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm}$ of consumption, production and financial quantities. At financial markets equilibrium with production, agents satisfy equ. (1) and (6). Market clearance conditions are determined by the aggregate excess demands for physical goods, capital and financial assets as expressed by the equilibrium equations:

$$\begin{aligned} \sum\nolimits_i^m (\bar{x}_i(0) - \omega_i(0)) &= \sum\nolimits_j^n \bar{y}_j(0) && (7) \\ \sum\nolimits_i^m (\bar{x}_i(1) - \omega_i(1)) &= \sum\nolimits_j^n \bar{y}_j(1) \\ \sum\nolimits_{j=1}^n \sum\nolimits_{i=1}^m (\bar{z}_i)_j &= 0, \text{ and} \sum\nolimits_{i=1}^m \theta(\bar{z}_i)_j = 1 \forall j. && (8) \end{aligned}$$

[1] $\Box$ denotes the box product. A "s by s" context dependent mathematical operation. For example the s by s inner product.

Equ. (1), (6) and (7) define a closed form incomplete markets general equilibrium with production, denoted (FE)[2]. The next sections establish existence of FE equilibria for convex smooth and piecewise linear production sets.

3 Generic existence of equilibrium

This part of the paper establishes the main existence result for convex, smooth production manifolds. The strategy of the proof is to show that a pseudo equilibrium with production exists, and that every pseudo equilibrium is also a financial markets equilibrium with sequential structure of the firm. The precise relations between pseudo and FE equilibria are the main results of section 3.1. Section 3.2 establishes the class of smooth asset structures for which the existence theorem introduced in the same section guarantees existence. Existence of pseudo equilibria for exchange economies with exogenous financial markets were established by Duffie, Shafer, Geanokopolos, Hirsh, Husseini, and others [16], [5], [17], [22], [19], [3]. Genakopolos et. al. [15] showed that pseudo equilibria exist for an economy with production for the case of exogenous financial markets and where the problem of the firm is to maximize the utility of the average share holder. We show that in a sequential incomplete markets model of the firm with decentralized decisions, pseudo equilibria with endogenously determined production sets exist.

Definition 1 *if $\nexists\ z \in \mathbb{R}^n_{++}$ s.t. $\hat{\Pi}(p_1, q, \phi)\left[z \mid \sum_{i=1}^m \theta(z_i)_{s=1}^S\right] \geq 0$, then $q \in \mathbb{R}^n_{++}$ is a no-arbitrage asset price relative to p_1.*

Lemma 1 *$\exists\ \beta \in \mathbb{R}^S_{++}$ s.t. $q = \sum_{s=1}^S \beta \Box \Pi(p_1, \phi)$.*

Proof. Immediate consequence of the separation theorem for $((S+1) \times n)$ matrices in Gale (1960). It asserts that either $\exists\ z \in \mathbb{R}^n_{++}$ such that $\hat{\Pi} z \geq 0$, or $\exists\ \beta \in \mathbb{R}^{S+1}_{++}$ such that $\beta \hat{\Pi} = 0$. ∎

We can now rescale equilibrium prices without affecting equilibrium allocations, let $P_1 = \beta \Box \bar{p}_1$. The next step is to derive a normalized no arbitrage equilibrium definition [4]. Let $\beta \in \mathbb{R}^S_{++}$ be $\left(\frac{\lambda(s)}{\lambda}\right)_{i=1}$ the gradient vector from the optimization problem of agent 1, called the Arrow-Debreu agent. The Walrasian budget set for the Arrow-Debreu agent is a sequence of constraints denoted

$$B_1 = \left\{ x_1 \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s) \Box (x_i(s) - \omega_i(s)) = \sum_j \theta_{ij} P(s) \Box y_j(s) \end{array} \right\}. \tag{9}$$

For all consumers $i \geq 2$, the no arbitrage budget set consisting of a sequence of $(S+1)$ constraints is denoted

$$B_{i \geq 2} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s) \Box (x_i(s) - \omega_i(s)) \in \langle \Pi(P_1, \phi) \rangle \end{array} \right\}, \tag{10}$$

where $\langle \Pi(P_1, \phi) \rangle$ is the span of the income transfer space of period one. Replace $\langle \Pi(P_1, \phi) \rangle$ with L in $G^n(\mathbb{R}^S)$, where $G^n(\mathbb{R}^S)$ is the Grassmann manifold[3] with its known smooth $(S-n)n$ dimensional structure, and L an n-dimensional affine subspace of $G^n(\mathbb{R}^S)$.

[2] In accordance with the GEI literature, we call such an equilibrium a financial markets equilibrium (FE).

[3] See i.e. Dieudonnè [8] for properties of the Grassmann manifold. See Duffie and Shafer for an exposition of the Grassmann manifold in economics [5].

Denote the pseudo opportunity set $B_i(P, L; \omega_i)$, for each $i \in \{1, ..., m\}$,

$$B_i = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s)\Box(x_i(s) - \omega_i(s)) \subset L \end{array} \right\} . \tag{11}$$

3.1 Pseudo equilibrium and its relational properties with FE

Let $\mathfrak{S}' = \left\{ p \in \mathbb{R}^{l(S+1)}_{++} : p^{0,1} = \Delta \right\}$ be the set of normalized prices, and let $\Delta \in \mathbb{R}_{++}$ be a fixed strictly positive real number. This convenient normalization singles out the first good at the spot $s = 0$ as the numeraire. We introduce following definitions for the long run payoff maps associated with sets $\mathfrak{S}$ and $\mathfrak{S}'$:

Definition 2 *For any $p_1 \in \mathfrak{S}$, such that $\pi : \mathfrak{S} \times \mathbb{R}^l \to \mathcal{A}$, let $\Gamma(P_1, \phi) = \beta\Box \left[\left(proj_\Delta \left(\frac{1}{\beta} \right)^T \Box P_1 \right) \Box y \right]$, where T denotes the transpose, $proj_\Delta(z) = \Delta \left(\frac{z}{\|z\|} \right)$, $\frac{1}{\beta} = \left(\frac{1}{\beta(1)}, ..., \frac{1}{\beta(S)} \right) \in R^S_{++}$, and $\beta = (\beta(1), ..., \beta(S)) \in R^S_{++}$. (ii) For any $p_1 \in \mathfrak{S}'$, such that $\pi : \mathfrak{S}' \times \mathbb{R}^l \to \mathcal{A}$, let $\Gamma(P_1, \phi) = \beta\Box \left[\left(\left(\frac{1}{\beta} \right)^T \Box P_1 \right) \Box y \right]$, where $\mathcal{A}$ is a set of $(S \times n)$ matrices A of order $(S \times n)$.*

Using the no arbitrage result of previous section and above definition leads to the analytically more tractable concept of a pseudo financial markets equilibrium with production for which we will establish existence. The main benefit of a pseudo equilibrium is that it allows to apply transversality arguments. This follows from the two consequences of the normalized gradient vector β of the Arrow-Debreu agent. It gives his (i) standard GE demand satisfying boundary conditions, and (ii) it guarantees independency of aggregate demands, such that Walras law applies [19].

Definition 3 *A pseudo financial markets equilibrium with production $(\bar{x}, \bar{y}), (\bar{P}, \bar{L}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ satisfies:*

(i) $(\bar{x}_1) \arg\ max \left\{ u_1(x_1) \ s.t. \ x_1 \in B_1(\bar{P}, \omega_1) \right\}$ $\quad i = 1$

(ii) $(\bar{x}_i) \arg\ max \left\{ u_i(x_i) \ s.t. \ x_i \in B_i(\bar{P}, \bar{L}, \omega_i) \right\}$ $\quad \forall\ i \geq 2$

(iii) $\left\langle \Gamma(\bar{P}_1, \bar{\phi}) \right\rangle \subset \bar{L}$, *proper if* $\left\langle \Gamma(\bar{P}_1, \bar{\phi}) \right\rangle = \bar{L}$

(iv) $(\bar{y})_j \arg\max \left\{ \bar{p}(s)\Box y_j(s) \,\middle|\, y_j \in Y_j \quad \forall s \in S \right\}$ $\quad \forall\ j$

(v) $\bar{x}_1 + \sum_{i=2}^m \bar{x}_i = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \bar{y}_j$

Lemma 2 *Under assumptions C, demand mappings $f_1(P, w_1)$ and $f_i(P, L, \omega)$ for $i = 2, ..., m$, from argmax (i) and (ii) are C^∞. Under assumptions F, supply mappings $g_j(P)$ for $j = 1, ..., n$, from argmax (iv) are C^∞.*

A proof of this known result is omitted [5]. Smoothness of demand and supply functions follows from the setup of the model for smooth economies. Following results show the relation between pseudo and FE equilibria. They imply that, in order to prove existence of equilibrium it is sufficient to establish existence in the much easier case of a pseudo equilibrium, since every pseudo equilibrium is also a FE equilibrium.

Proposition 1 *For every full rank FE with production $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$, there exists $\beta \in \mathbb{R}^S_{++}$ and a n-dimensional subspace $L \in G^n(\mathbb{R}^S)$ such that $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ is a pseudo FE with production.*

Proof. Trivial. ■

Proposition 2 *If $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ is a pseudo FE with production then for every $\beta \in \mathbb{R}^S_{++}$, there exist financial asset prices $\bar{q} \in \mathbb{R}^n_{++}$ and investment portfolios $\bar{z} = (z(1), ..., z(n)) \in \mathbb{R}^n_{++}$ such that $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ is a $(\bar{x}, \bar{y})$ allocational equivalent FE with production.*

Proof. Using (Definition 3), let the value of the stock prices system q at $t = 0$ be defined by $\bar{q} = \sum_{s=1}^{S} \left(\Gamma(\bar{P}_1, \bar{\phi}) \right)$, let $t = 1$ spot price system p_1 be determined by $\bar{p}_1 = proj \left(\left(\frac{1}{\beta(s)} \right)^T \Box \bar{P}_1(s) \right)$, and let $z_1 = \sum_{i=2}^{m} z_i$. The equivalence of a pseudo equilibrium with production and a financial markets with production then follows from similar arguments as in [19]. ■

3.2 Regular endogenized payoff structures

Long run financial payoffs depend on the technology of the firm, its production capacity installed via financial markets, and on a set of regular prices. Hart [13] illustrated that equilibrium may not exist for some structures of the payoff matrix. He showed that, when price vectors are collinear, the rank of the payoff matrix changes. We will exhibit a class of regular endogenous asset structures for economies with sequential optimization structure of the firm for which equilibria will always exist. Generic existence of equilibrium follows from the application of Thom's parametric transversality theorem.

Definition 4 *Define the rank dependent long run payoff maps $\pi^\rho : \mathbb{R}^l_{++} \times \mathbb{R}^l \to \mathcal{A}^\rho$ for $0 \leq \rho \leq n$. The set of reduced rank matrices A^ρ of order $(S \times n)$ with $rank(A^\rho) = (n - \rho)$ is denoted $\mathcal{A}^\rho$ and is of order $(S \times n)$.*

Proposition 3 *(i) For $1 \leq \rho < n$, A^ρ is a submanifold of A of codimension $(S - n + \rho)\rho$. (ii) for $\rho = n$ the set $\mathcal{A}^\rho = \{\varnothing\}$ is empty, and (iii) for $\rho = 0$, $\mathcal{A}^\rho = \mathcal{A}$ the set of reduced rank matrices is equivalent to the set of full rank matrices.*

Proof. (i) We prove that the set $\mathcal{A}^\rho_{(S \times n)}$ of rank $(n - \rho)$ reduced matrices $A^\rho_{(S \times n)}$, for $1 \leq \rho < n$, is a submanifold of the smooth manifold defined by the full rank n matrices $A_{(S \times n)}$ in the set $\mathcal{A}_{(S \times n)}$. Since $\mathcal{A}_{(S \times n)}$ is homeomorphic to $\mathbb{R}^{Sn}$, for $\mathcal{A}^\rho_{(S \times n)} \subset \mathcal{A}_{(S \times n)}$ the reduced matrices manifold is shown to have codimension $(S - n + \rho)\rho$, for $1 \leq \rho < n$.

Consider the open set U of $(S \times n)$ matrices $\tilde{a} = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right]$ of $rank\ (\tilde{a}) = (n - \rho)$ in the neighborhood of $a = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right]$ such that by invertibility of $\bar{A}$ in a, since $\det\bar{A} \neq 0$, it remains invertible in $\tilde{a}$. Then $\tilde{a}$ has rank $(n - \rho)$ if and only if the last ρ columns of $\tilde{a}$ are spanned by the first $(n - \rho)$ columns in it. This implies that there exists a matrix $b_{(n-\rho)\times\rho}$ such that $\left[\begin{array}{c} \bar{B}_{(n-\rho)\times\rho} \\ \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right] = \left[\begin{array}{c} \bar{A}_{(n-\rho)\times(n-\rho)} \\ \bar{C}_{(S-n+\rho)\times(n-\rho)} \end{array} \right] b_{(n-\rho)\times\rho} \Leftrightarrow$
$b_{(n-\rho)\times\rho} = \bar{A}^{-1}_{(n-\rho)\times(n-\rho)} \bar{B}_{(n-\rho)\times\rho}$, and
$\bar{D}_{(S-n+\rho)\times\rho} = \bar{C}_{(S-n+\rho)\times(n-\rho)} \bar{A}^{-1}_{(n-\rho)\times(n-\rho)} \bar{B}_{(n-\rho)\times\rho}$.
Hence
$$U \cap \mathcal{A}^\rho_{(S \times n)} = \left\{ \tilde{a} = \left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array} \right] \in U : \bar{D} - \bar{C}\bar{A}\bar{B} = 0 \right\}.$$
Then, the map
$U \to \mathbb{R}^{Sn} \simeq \mathbb{R}^{(n-\rho)(n-\rho)} \times \mathbb{R}^{(n-\rho)\rho} \times \mathbb{R}^{(S-n+\rho)(n-\rho)} \times \mathbb{R}^{(S-n+\rho)\rho}$,

$$\left[\begin{array}{c|c} \bar{A}_{(n-\rho)\times(n-\rho)} & \bar{B}_{(n-\rho)\times\rho} \\ \hline \bar{C}_{(S-n+\rho)\times(n-\rho)} & \bar{D}_{(S-n+\rho)\times\rho} \end{array}\right] \mapsto \left(\bar{A},\bar{B},\bar{C},\bar{D}-\bar{C}\bar{A}\bar{B}\right),$$

is locally a diffeomorphism, a chart with the property that the set $U \cap \mathcal{A}^{\rho}_{(S\times n)}$ is the set of points such that the last $(S-n+\rho)\rho$ coordinates vanish, and therefore, the reduced matrices manifold $\mathcal{A}^{\rho}_{(S\times n)}$ has codimension $(S-n+\rho)\rho$. In the cases (ii) and (iii) we look at the corresponding elements in the set of reduced matrices $\mathcal{A}^{\rho}_{(S\times n)}$. In (ii), it is easy to see that there are no linear independent mappings, while in (iii) all mappings are linearly independent, and A is non-singular since A is of full rank. ■

The lemma states that, for $1 \leq \rho < n$, the incomplete income transfer space is rank reduced. Next proposition exhibits a regular asset structure $\mathcal{R}$ for our production economy and shows that, for a map π to the ambient space $\mathcal{A}$ which is transverse to a submanifold $\mathcal{A}^{\rho}$ along all values of the domain of π, $\mathcal{R}$ is big in a topological sense. This follows from the transversality theorem for maps and submanifolds. Since $\mathcal{R}$ is open and dense, it follows that its complement, the set of critical values is closed and of measure zero. Denote the set satisfying $\Gamma \pitchfork \mathcal{A}^{\rho}$, $\mathcal{R}$, and its complement $\bar{\mathcal{R}}$.

Theorem 2 *(i) $\pi \pitchfork \mathcal{A}^{\rho}$ for integers $1 \leq \rho \leq n$. (ii) $\Gamma \pitchfork \mathcal{A}^{\rho}$ for any $\beta \in \mathbb{R}^{S}_{++}$ and integers $1 \leq \rho \leq n$. (iii) $\mathcal{R} = \Gamma^{\rho} \cap \mathcal{A}$ is generic, since it is dense and open.*

Proof. (i) The linear map $D_y\pi$ is surjective everywhere in Y, and $Image\,(D_y\pi) + T_y(\mathcal{A}^{\rho}) = T_y(\mathcal{A})$ is satisfied. (ii) The surjectivity of the push forward map does not change for any scaling $\beta \in \mathbb{R}^{S}_{++}$. (iii) Immediate consequence of the transversality theorem for maps to ambient manifolds and submanifolds [14]. Since each set $\pitchfork$ $(\Gamma, \mathcal{A};\mathcal{A}^{\rho})$ is residual, their intersection is residual. ■

The economic relevance of this result is that it exhibits a class of well defined smooth asset structures for economies with sequential, two argument optimization structure of the firm. For any gneneric production set structure satisfying proposition 3, equilibrium exists by the existence theorem below. Similar to Bottazzi [3], these asset structures are independent of initial endowments and preferences. This is at variance with Duffie and Shafer [5].

Definition 5 *Denote Ψ^{ρ} the vector bundle defined by (i) a basis $P^{\rho} = \left\{P \in \mathbb{R}^{l(S+1)}_{++} : rank(\Gamma(P_1,\phi)) = (n-\rho)\right\}$, and (ii) orthogonal income transfer space $L^{\perp} \subset \langle \Gamma^{\rho}(P_1,\phi)\rangle^{\perp}$,*

$$\Psi^{\rho} = \left\{ \begin{array}{c} (P, \langle\Gamma(P_1,\phi)\rangle^{\perp}, L^{\perp}) \in P^{\rho} \times G^{S-n+\rho}(\mathbb{R}^S) \times G^{S-n}(\mathbb{R}^S) : \\ L^{\perp} \subset \langle\Gamma(P_1,\phi)\rangle^{\perp} \end{array} \right\}. \tag{12}$$

We thus have defined a fiber bundle Ψ^{ρ} of codimension $l(S+1)-1-\rho^2$ containing the spot price system and income transfer space consisting of a base vector P^{ρ} and fiber $G^{S-n}(\mathbb{R}^{S-n+\rho})$. We can now state the main result.

Theorem 3 *There exists a pseudo FE with production $(\bar{x},\bar{y}),(\bar{P},\bar{L}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^{n}(\mathbb{R}^S)$ for generic endowments. Moreover, by the relational propositions, a FE with production $(\bar{x},\bar{y},\bar{z}),(\bar{p},\bar{q}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}^{n}_{++}$ exists for generic endowments.*

Proof. By proposition 3 and using definition 5, define an evaluation map Z^{ρ} on $\Psi^{\rho} \times \mathbb{R}^{l(S+1)m}_{++}$, where denote $\Omega = \mathbb{R}^{l(S+1)m}_{++}$ the set of the economy's total initial endowments, such that the excess demand map $Z^{\rho} : \Psi^{\rho} \times \Omega \to N$.

For the Arrow-Debreu agent have

$$Z_1^{\rho} : \Psi^{\rho} \times \Omega \to N. \tag{13}$$

The evaluation map is a submersion, since $D_{\omega_1} Z_1^{\rho}$ $\forall \omega_1 \in \Omega$ is surjective everywhere. $\exists$ for each $\omega_1 \in \Omega$

$$Z_{1,\omega_1 \in \Omega}^{\rho} : \Psi^{\rho} \to N \pitchfork_{\omega \in \Omega_{\rho}} \{0\}, \tag{14}$$

where $\{0\} \subset N$, and $\rho = 0$. The dimension of the preimage $Z_{1,\omega_1 \in \Omega}^{-1}(\{0\})$ is $l(S+1)-1$. By Thom's parametric transversality theorem[4], it follows that the subset $\Omega_{\rho} \cap \Omega$ is generic since it is open and dense. Equilibria exist. By the equivalence propositions 1 and 2 know that full rank financial markets equilibria with endogenized period two production sets exist.

For all $1 \leq \rho \leq n$ the preimage of the rank reduced evaluation map has dimension $l(S+1)-1-\rho^2$. This implies that for generic endowments $\omega \in \cap_{\rho} (\Omega_{\rho})$, for $\rho = 1, ..., n$, there is no reduced rank equilibrium, since for $Z_1^{\rho}(., \omega)$ the set of $\{0\} = \varnothing$. ■

4 Convex piecewise linear production sets

This part of the paper expands the previous existence result to piecewise linear production sets. A linearity assumption on the transformation map ϕ_j is introduced by replacing assumption (1) in (F) with $F(2)$ below. It is shown that, by similar arguments of the previous section, equilibria exist for regularized production manifolds.

Assumption F(2) $\phi_j(s) : \mathbb{R}_-^{\tilde{m}} \to \mathbb{R}_+^{\tilde{n}}$ piecewise linear for all $s \in S$, and $j \in \{1, ..., n\}$.

Geometrically, each period two production set $Y_j \mid_{\bar{z}}$ is a polyhedral cone, a set generated as a convex hull of a finite number of rays. We apply techinques from regularization theory to production sets[5] in order to smooth out convex, piecewise linear production manifolds $\partial Y_j \mid_{\bar{z}}$ by convolution, and show that these convolutes, denoted Φ_j, are compact and smooth manifolds approximating the piecewise linear production manifolds. For that, we define the state dependent convolute for producer $j \in \{1, ..., n\}$

$$(\lambda_{\sigma} * \phi_j(y))_j (s) = \begin{cases} \int_{\mathbb{R}_-^m} (\lambda_{\sigma}(\zeta)\phi_j(y-\zeta)d\zeta)_j (s) & \text{for } y \in U_{\sigma} \\ 0 & \text{otherwise} \end{cases} \quad \forall s, j \tag{15}$$

where $y \in U_{\sigma}$, and $U_{\sigma} = \{y \in U : B(y, \sigma) \subset U\}$. Continuity of $\phi_j(s)$ implies the existence of a distance $\sigma = \inf_t(\sigma_t)$, where $0 < \sigma < 1$. Associate with measure $\sigma \in [0, 1]$ the manifolds λ_{σ} defined by

$$\lambda_{\sigma}(y)(s) = \frac{1}{\sigma} \lambda \left(\frac{y}{\sigma} \right)(s), \ \forall s \tag{16}$$

Proposition 4 *Each regularized manifold $\partial \tilde{Y}_j \mid_z$ defined by the convolute $\Phi_j(s)$, $\forall s$, is C^{∞} and compact.*

[4]See Thom R. (1954) "Quelques propriétés globales des varìètès differentiables". Comm. Math. Helv. 28, 17-86.

[5]Similar to Chiappori and Rochet (Econometrica, 1987) who applied regularization theory to smooth out utilities .

Proof. For each $j \in \{1, ..., n\}$, denote the state dependent convolute

$$\Phi(s)_j = (\lambda * \phi(y))_j (s) = \int_{\mathbb{R}^m_-} (\phi(y-\zeta)_j \lambda_\sigma(\zeta) d\zeta)_j (s) \tag{17}$$

Can restrict domain of integration to Int supp(λ). See (Dieudonnè [8]). Let $lim_{p\to 0} y^p = -\infty$, and let $lim_{p\to\infty} y^p = 0$. Denote $A = (\{-\infty\}, 0)^m \subseteq \mathbb{R}^m_-$. For any $z \in \mathbb{R}^n_+$, $\exists$ $y\,|_z \in A$. Denote the compact subset associated with any z, $A\,|_z$. $A\,|_z \subseteq A$. Then the image of the continuous map $\Phi : A\,|_z \to \partial\tilde{Y}\,|_z$ is compact by surjectivity of Φ. ■

Proposition 5 *For any $j \in \{1, ..., n\}$ and C^∞ kernel λ, $\lambda*$ is bounded and converges to identity ϕ, it satisfies $\left|(\lambda_\sigma * \phi)_j (s) - \phi(s)\right|_j \leq \varepsilon(s)_j \ \forall s$.*

The proof is in the appendix.

Theorem 4 *For any $\partial\tilde{Y}_j\,|_{\bar{z}}$, there exists a pseudo FE with production $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ $\in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ for generic endowments. Moreover, by the relational propositions a FE with production $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}^n_{++}$ exists for generic endowments.*

Proof. Using propositions 4 and 5, the proof follows from the existence theorem 1 for convex, smooth production manifolds. ■

5 Conclusion

The paper considers endogenous financing of production in a general equilibrium model with incomplete markets. At variance to the literature, capital plays an essential role in determining the firms' production sets. For an endogenous asset structure, where the efficient boundary of the production set is not independent of the activities of the firm on financial markets, equilibrium exists. The model of the firm considered in this paper has two two non-trivial economic properties: (i) the generalization of the profit maximization property of the Arrow-Debreu model to the case of incomplete markets, (ii) hence, the generalization of the decentralization property of the Arrow-Debreu model to incomplete markets. Future work should investigate further properties of this model such as the Modigliani and Miller theorem for example. This is work in progress [23], [24].

6 Appendix

Proof (Proposition 5). Define for every $s \in S$ $diam(\lambda)$ with supp(λ) contained in the unit ball $\mathbb{R}^{m,}_-$. Let $\varepsilon(s) = y(\phi, diam(\lambda))_j (s)$. Now, for any C^∞ kernel λ can define ϕ in $\mathbb{R}^{lS}$ such that for all $s \in S$

$$((\lambda * \phi_j - \phi)(y))_j (s) = \int_{\mathbb{R}^m_-} \left[(\phi(y-\zeta)_j - \phi(y))\, \lambda(\zeta)^{\frac{1}{2}}\, d\zeta\right]_j (s), \tag{18}$$

by Cauchy inequality and Fubini's theorem, and since mass of λ is equal to one, and ζ ranges over its support, we obtain

$$\left(\int_{\mathbb{R}^m_-} |\lambda * \phi_j - \phi)(y)|^2\, dy\right)_j (s) \quad \leq \quad \sup_{\|\zeta\|\leq\sigma} \left(\int_{\mathbb{R}^m_-} |(\phi(y-\zeta)_j - \phi(y))|^2\, dy\right)_j (s) \tag{19}$$

Thus it follows that

$$\left(\int_{\mathbb{R}^m_-} |\lambda * \phi_j - \phi)(y)|\, dy\right)_j (s) \quad \leq \quad \sup_{\|\zeta\|\leq\sigma} \left(\int_{\mathbb{R}^m_-} |(\phi(y-\zeta)_j - \phi(y))|^2\, dy\right)_j^{\frac{1}{2}} (s) \tag{20}$$

denoted $y(\phi, diam(\lambda))_j(s)$. It converges to zero when $diam(\lambda)$ converges to zero. It is bounded above since

$$y(\phi, diam(\lambda)_j(s) \leq c \left(\sum_{k=1}^{m} \left|D^k \phi\,(y)\right|_j^2 (s)\right)^{\frac{1}{2}} \tag{21}$$

where $c = k_1\sigma$. k_1 is a constant of differentiation, and σ a distance. ■

References

[1] K. Arrow. "Le role des valeurs boursières pour la rèpartition la mellieure des risques". *Econometriè, Colloques Internautionaux du Centre National de la Recerche Scientifique*, 40:41–47, 1953.

[2] K. Arrow. "The role of securities in the optimal allocation of risk bearing". *Review of Economic Studies*, 31:91–96, 1964.

[3] J. Bottazzi. "Existence of equilibria with incomplete markets: The case of smooth returns". *Journal of Mathematical Economics*, 24:59–72, 1995.

[4] D. Cass. *"Competitive equilibria with incomplete financial markets"*, 1984.

[5] W. Shafer D. Duffie. "Equilibrium in incomplete markets i: A basic model of generic existence". *Journal of Mathematical Economics*, 14:285–300, 1985.

[6] G. Debreu. *Theory of value.* Wiley, New York, 1959.

[7] G. Debreu. "Smooth preferences". *Econometrica*, 40:603–615, 1972.

[8] J. Dieudonne. *Elements d'analyse.* Gauthier-Villars, Paris, 1972.

[9] J. Drèze. "(Uncertainty) the firm in general equilibrium theory". *The Economic Journal*, 95:1–20, 1985.

[10] J. H. Drèze. *"Investment under private ownership: Optimality, equilibrium and stability"*. Wiley, 1974.

[11] J. Geanokopolos. "An introduction to general equilibrium with incomplete asset markets". *Journal of Mathematical Economics*, 19:1–38, 1990.

[12] S. Grossman and O. Hart. "A theory of competitive equilibrium in stock market economies". *Econometrica*, 47:293–330, 1979.

[13] O. Hart. "On the optimality of equilibrium when the market structure is incomplete". *Journal of Economic Theory*, 11(3):418–443, 1975.

[14] M. Hirsch. *Differential topology.* Springer Verlag, New York, 1972.

[15] M. Quinzii J. Geanakoplos, M. Magill and J. Drèze. "Generic inefficiency of stock market equilibrium when markets are incomplete". *Journal of Mathematical Economics*, 19(1-2):113–151, 1990.

[16] W. Shafer J. Geanokopolos. "Solving a system of simultaneous equations in economics". *Journal of Mathematical Economics*, 19:69–95, 1990.

[17] M. Magill M. Hirsch and A. Mas-Colell. "A geometric approach to a class of equilibrium existence theorems". *Journal of Mathematical Economics*, 19(1-2):95–106, 1990.

[18] M. Magill and M. Quinzii. *Theory of incomplete markets.* MIT press, Massachusetts Institute of Technology, 2002.

[19] M. Magill and W. Shafer. "Characterization of generically complete real asset structures". *Journal of Mathematical Economics*, 19:167–194, 1990.

[20] M. Magill and W. Shafer. Incomplete markets. In W. Hildenbrand and H. Sonnenschein, editors, *Handbook of Mathematical Economics*, volume 4 of *Handbook of Mathematical Economics*, chapter 30, pages 1523–1614. Elsevier, 1991.

[21] R. Wilson S. Ekern. "On the theory of the firm in an economy with incomplete markets". *Bell Journal of Economics and Management Science*, 5:171–180, 1974.

[22] J. Lasry S. Husseini and M. Magill. "Existence of equilibrium with incomplete markets". *Journal of Mathematical Economics*, 19(1-2):39–67, 1990.

[23] P. Stiefenhofer. "A general equilibrium corporate finance theorem: A special case". *Discussion Papers in Economics, University of York*, 11, 2010.

[24] P. Stiefenhofer. "A geometric separation of the activities of the agents: The leading case". *Discussion Papers in Economics, University of York*, 12, 2010.