THE UNIVERSITY of York

Discussion Papers in Economics

No. 10/12

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June 7, 2010

Abstract

The paper considers production in a simple two period general equilibrium model with incomplete markets. It shows, by application of convex sets analysis, the separation of economic activities of the agents. The paper improves on Stiefenhofer (2010) by taking a geometric approach to the study of the decentralization theorem. This theorem separates the economic activities of the agents, hence generalizes the objective function of the firm of the Arrow-Debreu model to the case of incomplete markets, where firms are profit maximizers.

1 Introduction

Drèze [2], and Grossman and Hart [5] introduce objective functions of firms which are not independent of the utilities of their owners into the analysis of general equilibrium with incomplete markets. Thus, in their economic scenarios, firms are utility maximizers, and therefore, the economic activities of the agents centralized. This result is a consequence of the exogenous asset structure considered in their models.

Stiefenhofer [7] adds more structure to the economic model by endogenizing the firms' production sets. The novelty of this model is the generalization of the objective function of the firm of the Arrow-Debreu model to incomplete markets. This objective function is independent of the utility of the owners of the firm, hence, the economic decisions of the agents separated. This objective function has a nice property, it rehabilitates the profits maximization criterion of the Arrow-Debreu model [1].

This paper considers a geometric approach to the the separation theorem of economic activities of the agents in incomplete markets introduced in [7]. We apply convex sets analysis [3] to show the decentralization of the objective function of the firm of the model introduced in Stiefenhofer [6]. The asset structure of the leading case model is sufficiently rich to prove the result in its simplest form.

Part 2 of the paper introduces the model. Part three states the main result, and part four is a conclusion.

2 The Model

This paper considers a variation of the model of the firm introduced in [6] and [7] for the case that financial activities are explicitly modeled. This requires the introduction of an

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extensive form model of the firm. We consider a simple model with sufficient structure to highlight the main properties of interest. Let the budget constraints of the single agent as a consumer be

$$\begin{aligned} p(0) \cdot x(0) &= p(0) \cdot \omega(0) - qz \\ p(s) \cdot x(s) &= p(s) \cdot \omega(1) + R(\bar{y}, s)z. \end{aligned}$$

A consumption bundle $x = (x(0), x(s))$ is a collection of vectors defined on the strictly positive orthant $\mathbb{R}^{l(S+1)}_{++}$ with associated strictly positive price system $p = (p(0), p(s))$ in $\mathbb{R}^{l(S+1)}_{++}$. A financial quantity z (number of stocks) is a strictly positive real number $\mathbb{R}_{++}$ with associated price system q in $\mathbb{R}_{++}$. We denote the initial resources of this economy $\omega = (\omega(0), \omega(1))$ in $\mathbb{R}^{2l}_{++}$. Note that there is no aggregate uncertainty in this economy, instead we consider firm specific risk. An uncertain state of nature is an element denoted s in the exhaustive set of mutually exclusive elements S. $R(\bar{y}, s)z$ denotes the return of investment into the firm. The consumer invests into the firm in order to transfer wealth across time and between uncertain states of nature.

In a one agent model the agent also performs the role of the producer, and therefore, adds following variables to his constraints

$$\begin{aligned} p(0) \cdot x(0) &= p(0) \cdot \omega(0) - qz + qb - p(0) \cdot \bar{k}(0) \\ p(s) \cdot x(s) &= p(s) \cdot \omega(1) + R(\bar{y}, s)z + p(s) \cdot y(s) \end{aligned},$$

where $\bar{k}(0)$ denotes the capital purchased. Let aside the modeling of financing production for a while, therefore, let $\xi = z + \hat{b}$, where $\hat{b}$ is a feasible financial policy of the firm such that $\hat{b} \Rightarrow Y|_{\hat{b}}$. Here, take production set $Y|_{\hat{b}}$ as given. This production set is available to the firm in period two. The consumer's budget set is defined by

$$B_\xi = \left\{ (x; \xi, y) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R} \times \mathbb{R}^{lS} : \begin{array}{l} p(0) \cdot x(0) = p(0) \cdot \omega(0) - q\xi - p(0) \cdot \bar{k}(0) \\ p(s) \cdot x(s) = p(s) \cdot \omega(1) + R(\bar{y}, s)\xi + p(s) \cdot y(s) \end{array} \right\}. \tag{1}$$

The agent 's control problem is to choose $(x; \xi, y)$ such that utility of consumption of goods is maximized. By reduced form, we mean a model where financial policies are not explicitly modeled and decisions of the agents not fully separated. We formally introduce the reduced form model below.

Definition 1 *A reduced form equilibrium $(\bar{p}, \bar{q})$ with associated equilibrium allocations $\left(\bar{x}; \bar{\xi}, \bar{y}\right)$ for generic initial resources $\omega \in \Omega$ satisfies:*

$$\begin{array}{ll} (i) & \left(\bar{x}; \bar{\xi}, \bar{y}\right) \arg\max \left\{ u(x) : x \in B_\xi \right\} \\ (ii) & \bar{\xi} = 0 \\ & \bar{x}(0) = \omega(0) + \bar{k}(0) \\ & \bar{x}(s) = \omega(1) + \bar{y}(s) \quad \text{for all } s \in \{1, ..., S\}. \end{array} \tag{2}$$

Proposition 1 *Let $Y|_{\bar{\xi}}$ and Ξ be two nonempty convex sets. Then $(\bar{y}, \bar{\xi})$ is a geometric solution of the reduced form problem (Def.1)*

$$\left(\bar{x}; \bar{y}, \bar{\xi}\right) \arg\max \left\{ u(y + \xi) : y \in Y|_{\bar{\xi}},\ \xi \in \Xi \right\} \tag{3}$$

if and only if

$$\nabla u(\bar{y} + \bar{\xi}) \in N_{Y|_{\bar{\xi}}}(\bar{y}) \cap N_\Xi(\bar{\xi}). \tag{4}$$

Proof 1 (proposition 1) *Let* $v : \mathbb{R}^{S+1} \times \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $v(y, \xi) = u(y + \xi)$. *Then the two variable control problem above is equivalent to*

$$(\bar{y}, \bar{\xi}) \arg\max \left\{ v(y, \xi) : (y, \xi) \in Y|_{\bar{\xi}} \times \Xi \right\} \tag{5}$$

By application of the separation theorem for convex sets $(\bar{y}, \bar{\xi})$ *is a solution of this control problem if and only if*

$$\left(\nabla_y v(\bar{y}, \bar{\xi}), \nabla_\xi(\bar{y}, \bar{\xi}) \right) \in N_{Y|_{\bar{\xi}} \times \Xi}(\bar{y}, \bar{\xi}) \tag{6}$$

where ∇_y *denotes the gradient of* v *with respect to* y, *and* ∇_ξ *denotes the gradient of* v *with respect to* ξ. *From the definition of a normal cone (appendix) it follows that*

$$N_{Y|_{\bar{\xi}} \times \Xi}(\bar{y}, \bar{\xi}) = N_{Y|_{\bar{\xi}}}(\bar{y}) \times N_\Xi(\bar{\xi}) \tag{7}$$

and from the definition of the function v *that*

$$\nabla_y v(\bar{y}, \bar{\xi}) = \nabla_\xi v(\bar{y}, \bar{\xi}) = \nabla u(\bar{y} + \bar{\xi}) \tag{8}$$

so that $\left(\nabla_y v(\bar{y}, \bar{\xi}), \nabla_\xi(\bar{y}, \bar{\xi}) \right) \in N_{Y|_{\bar{\xi}} \times \Xi}(\bar{y}, \bar{\xi})$ *reduces to* $\nabla u(\bar{y} + \bar{\xi}) \in N_{Y|_{\bar{\xi}}}(\bar{y}) \cap N_\Xi(\bar{\xi})$.

We now introduce the extensive form model, where decisions are fully decentralized and financial policies explicitly modeled. Consider the consumer's constraints

$$\begin{aligned} p(0) \cdot x(0) &= p(0) \cdot \omega(0) - qz \\ p(s) \cdot x(s) &= p(s) \cdot \omega(1) + \theta(\bar{z}) R(\bar{y}, s), \end{aligned}$$

where qz is the value the consumer is willing to invest into the firm. $\theta(\bar{z})R(\bar{y})$ denotes the proportional share of total dividend payoff the consumer receives in the next period. Denote the consumers budget set

$$B_z = \left\{ (x; \xi, y) \in \mathbb{R}^{l(S+1)}_{++} \times \mathbb{R} \times \mathbb{R}^{lS} : \begin{array}{l} p(0) \cdot x(0) = p(0) \cdot \omega(0) - qz \\ p(s) \cdot x(s) = p(s) \cdot \omega(1) + \theta(\bar{z}) R(\bar{y}, s) \end{array} \right\}. \tag{9}$$

As a producer, the agent issues stocks b satisfying $qb = qz$, buys capital $k(0)$ such that income from selling stocks is equal to his expenditure on capital consumption, therefore, $qb = p(0) \cdot k(0)$. At $t = 0$, the producer's long run problem is to

$$\left(\bar{k}(0); \bar{b} \right) \arg\max \left\{ \bar{q} b : \bar{q}\bar{z} = \bar{q} b = \bar{p}(0) \cdot k(0) \right\}, \tag{10}$$

where the level of capital, $\bar{k}(0)$, implies total production capacity available to the firm, a correspondence $\Phi|_{\hat{b}}$. This correspondence in turn determines the production set available to the firm, denoted $Y|_{\hat{b}}$. Given this production set, and the set of states of nature, the producer's $t = 1$ short run problem is to

$$(\bar{y}(s)) \arg\max \left\{ \bar{p}(s) \cdot y(s) : y(s) \in Y|_{\hat{b}}(s), \forall s \in S \right\}. \tag{11}$$

Inputs of production are financed with sells from outputs. The level of revenue a firm can generate in each state $s \in \{1, ..., S\}$ depends on the available production set determined in the certain state of the world. We formally introduce the equilibrium definition of this model.

Definition 2 *$(\bar{p},\bar{q})$ is a decentralized objective function extensive form equilibrium with associated equilibrium allocations $((\bar{x},\bar{z}),(\bar{y},\bar{b}))$ for generic initial resources $\omega\in\Omega$, if for any feasible $\hat{b}\leqslant\bar{z}$ following conditions are satisfied:*

$$
\begin{array}{ll}
\textit{(i)} & (\bar{x};\bar{z})\arg\max\left\{u(x):x\in B_z\right\} \\
\textit{(ii)} & \underset{(\bar{y},\hat{b};(\bar{k}(0)))}{\arg\max}\left\{\bar{q}b+\sum_{s=1}^{S}\bar{p}(s)\cdot y(s): \begin{array}{l}\bar{q}\bar{z}\geq\bar{q}b=\bar{p}(0)\cdot k(0)\\ y(s)\in Y|_{\hat{b}}(s)\end{array} \quad s\in S\right\} \\
\textit{(iii)} & \bar{z}+\hat{b}=0 \quad \theta(\bar{z})=1 \\
 & \bar{x}(0)=\omega(0)+\bar{k}(0) \\
 & \bar{x}(s)=\omega(1)+\bar{y}(s) \textit{ for all } s\in\{1,..,S\}.
\end{array}
\tag{12}
$$

Proposition 2 *$\bar{x}|_{\bar{z}}$ is a solution of*

$$
max\left\{u(x;z):x\in B\right\} \tag{13}
$$

if and only if, $\bar{x}|_{\bar{z}}\in B$, and

$$
\partial u(\bar{x}|_{\bar{z}})\cap N_B(\bar{x}|_{\bar{z}})\neq\{0\} \tag{14}
$$

is satisfied.

Proof 2 (proposition 2) *(i) $\bar{x}|_{\bar{z}}$ is a solution of utility max (2) if and only if $\bar{x}|_{\bar{z}}\in B$ and*

$$
intU_{\bar{x}|_{\bar{z}}}\cap B=\varnothing.
$$

By the separation theorem for convex sets (appendix), there exists $P=\beta_i p\in R^n, P\neq 0$ such that

$$
H_P^-=\{x\in R^n: P\cdot x\leq P\cdot x',\forall x\in B,\forall x'\in intU_{\bar{x}}\}
$$

since $\bar{x}|_{\bar{z}}\in B$,

$$
H_P^-=\left\{x\in R^n: P\cdot\bar{x}|_{\bar{z}}\leq P\cdot x',\forall x'\in intU_{\bar{x}|_{\bar{z}}}\right\}.
$$

By continuity of utility, $intU_{\bar{x}}=U_{\bar{x}}$, and by continuity of the scalar product,

$$
\begin{array}{lcl}
H_P^+=\left\{\forall x'\in U_{\bar{x}|_{\bar{z}}}: P\cdot\bar{x}\leq P\cdot x',\forall x'\in U_{\bar{x}|_{\bar{z}}}\right\} & \Leftrightarrow & P\in\partial u(\bar{x}|_{\bar{z}}) \\
H_P^-=\left\{\forall x\in B: P\cdot x\leq P\cdot\bar{x}|_{\bar{z}},\right\} & \Leftrightarrow & P\in N_B(\bar{x}|_{\bar{z}})
\end{array}
$$

hence, there exists p such that $\partial u(\bar{x}|_{\bar{z}})\cap N_B(\bar{x}|_{\bar{z}})\neq\{0\}$ is satisfied.

(ii) Suppose that $\bar{x}|_{\bar{z}}\in B$, and there exists $P\in\partial u(\bar{x}|_{\bar{z}})\cap N_B(\bar{x}|_{\bar{z}})$, $P\neq 0$. If $\bar{x}|_{\bar{z}}$ is not a solution of the utility maximization problem (2) then there exists x'$\in intU_{\bar{x}|_{\bar{z}}}\cap B$. Since $P\in\partial u(\bar{x}|_{\bar{z}})$, we have

$$
P\cdot x\prime>P\cdot\bar{x}|_{\bar{z}}
$$

But since $P\in N_B(\bar{x}|_{\bar{z}})$ and $x\prime\in B$, it follows that $P\cdot x\prime\leq P\cdot\bar{x}|_{\bar{z}}$ which contradicts that $x\prime$ is preferred to $\bar{x}|_{\bar{z}}$.

Proposition 3 *$\bar{y}|_{\bar{z}}$ is a solution of*

$$
max\left\{\Pi(p;z):y\in Y|_{\bar{z}}\right\} \tag{15}
$$

if and only if, $\bar{y}|_{\bar{z}}\in Y$, and

$$
\partial u(\bar{y})\cap N_Y(\bar{y})\neq\{0\} \tag{16}
$$

is satisfied.

Proof 3 (proposition 3) *(i) $\bar{y}|_{\bar{z}}$ is a solution of profit max in (3) if and only if $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$ and*

$$int\Pi_{\bar{y}} \cap Y|_{\bar{z}} = \varnothing.$$

By the separation theorem for convex sets (appendix), there exists $p \in R^n, p \neq 0$ such that

$$H_p^- = \left\{y \in R^n : p \cdot y \leq p \cdot y', \forall y \in Y, \forall y' \in int\Pi_{\bar{y}|_{\bar{z}}}\right\}$$

since $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$,

$$H_p^- = \left\{y \in R^n : p \cdot \bar{y}|_{\bar{z}} \leq p \cdot y', \forall y' \in int\Pi_{\bar{y}|_{\bar{z}}}\right\}.$$

By continuity of utility, $int\Pi_{\bar{y}|_{\bar{z}}} = \Pi_{\bar{y}|_{\bar{z}}}$, and by continuity of the scalar product,

$$\begin{array}{lcl} H_p^+ = \left\{\forall y' \in U_{\bar{y}} : p \cdot \bar{y} \leq p \cdot y', \forall y' \in \Pi_{\bar{y}|_{\bar{z}}}\right\} & \Leftrightarrow & p \in \partial\Pi(\bar{y}|_{\bar{z}}) \\ H_p^- = \left\{\forall y \in Y : p \cdot y \leq p \cdot \bar{y}|_{\bar{z}}, \right\} & \Leftrightarrow & p \in N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}}) \end{array}$$

hence, there exists p such that $\partial u(\bar{y}|_{\bar{z}}) \cap N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}}) \neq \{0\}$ is satisfied.

(ii) Suppose that $\bar{y}|_{\bar{z}} \in Y|_{\bar{z}}$, and there exists $p \in \partial\Pi(\bar{y}|_{\bar{z}}) \cap N_Y(\bar{y}|_{\bar{z}}), p \neq 0$. If $\bar{y}|_{\bar{z}}$ is not a solution of the profit maximization problem in (3), then there exists $y\prime \in int\Pi_{\bar{y}|_{\bar{z}}} \cap Y|_{\bar{z}}$. Since $p \in \partial\Pi(\bar{y}|_{\bar{z}})$, we have

$$p \cdot y\prime > p \cdot \bar{y}|_{\bar{z}}$$

But since $p \in N_{Y|_{\bar{z}}}(\bar{y}|_{\bar{z}})$ and $y\prime \in Y|_{\bar{z}}$, it follows that $p \cdot y\prime \leq p \cdot \bar{y}|_{\bar{z}}$ which contradicts that $y\prime$ is preferred to $\bar{y}|_{\bar{z}}$.

3 Result

The result below separates the activities of the agent as a consumer and as a producer. This follows from the separation of the objective function of the firm from the utility of the owners of the firm. This in turn is a consequence of the endogenous asset structure considered in this example.

Theorem 1 *$(\bar{p}, \bar{q})$ is a geometrically reinterpreted extensive form equilibrium with associated equilibrium allocations $\left((\bar{x}, \bar{z}), (\bar{y}, \bar{b})\right)$ for generic initial resources $\omega \in \Omega$ with decentralized objective function of the firm if for any feasible $\hat{b} \leqslant \bar{z}$ following conditions are satisfied:*

$$\begin{array}{ll} (i) & (\bar{x}; \bar{z}) \arg\max\left\{u(x) : x \in B_z\right\} \\ (ii) & \underset{\left(\bar{y}, \hat{b}; (\bar{k}(0))\right)}{\arg\max} \left\{\bar{q}b + \sum_{s=1}^{S} \bar{p}(s) \cdot y(s) : \begin{array}{l} \bar{q}\bar{z} \geq \bar{q}b = \bar{p}(0) \cdot k(0) \\ y(s) \in Y|_{\hat{b}}(s) \end{array} \quad s \in S\right\} \\ (iii) & \bar{z} + \hat{b} = 0 \quad \theta(\bar{z}) = 1 \\ & \bar{x}(0) = \omega(0) + \bar{k}(0) \\ & \bar{x}(s) = \omega(1) + \bar{y}(s) \text{ for all } s \in \{1, .., S\}. \end{array} \tag{17}$$

Proof 4 (Theorem 1) *By proposition (1) $((\bar{p}, \bar{q}), (\bar{x}, \bar{\xi})))$ is a reduced form equilibrium satisfying (i) of the extensive form model with decentralized activities if and only if the geometric first order conditions of proposition (2) hold. The profit maximization problem (ii) of the extensive form model with decentralized activities $(\bar{y}, \hat{b})$ is satisfied if and only if the geometric first order condition of proposition (3) holds. Since using proposition (1) $(\bar{x}, \bar{z}), (\bar{y}, \hat{b})$ satisfies (i) of the (centralized) reduced form model if and only if both geometric first order conditions hold proposition (2), proposition (3), $((\bar{p}, \bar{q}), ((\bar{x}, \bar{z}), (\bar{y}, b)))$ is a geometric extensive form with decentralized activities equilibrium.*

4 Conclusion

The separation result introduced in this paper generalizes the profits maximization criterion of the Arrow-Debreu model to the case of incomplete markets. The novelty of this result is the independency of the objective function of the firm from the utility of the stock holders. This improves on the theory of the firm in incomplete markets where firms are considered to be utility maximizers [4]. In this paper we take a geometric approach to the separation theorem introduced in [7] by using convex analysis.

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