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Situations
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# A Non-empty Core May Not Coincide with the Uncovered Set in Spatial Voting Situations ${ }^{1}$ 

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#### Abstract

In this note it is shown that in contradiction to the well-known claim in Cox [4] (repeated in a number of subsequent works), the uncovered set in a spatial voting situation does not necessarily coincide with the core even when the core is non-empty.


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## 1 Introduction

In this note it is shown that in contradiction to the well-known claim in Cox [4] (repeated in a number of subsequent works), the uncovered set in a spatial voting situation does not necessarily coincide with the core even when the core is nonempty.

In our framework, the set of outcomes or policies under consideration is some convex subset of some finite dimensional Euclidean space and any majority coalition of voters can enforce any outcome from another.

For such an environment Cox made the claim that if individual preferences satisfy a very innocuous symmetry condition then the uncovered set coincides with the core whenever the latter is non-empty. However, he worked with an odd number of voters "for expositional convenience" ([4] p. 409). But his proof used the assumption that the cardinality of the voter set is odd in a non-trivial way. This result has been repeated in subsequent literature. For example, Austen-Smith and Banks [2] state that "the uncovered set coincides with the core when the latter is nonempty and singleton" (p. 274) (though, the definition of the uncovered set they use gives a superset of the uncovered set we have defined here.) A similar remark appears in the recent paper by Penn ([6], p.44).

However, in this note we show that there is a voting situation for which the cardinality of the set of voters is even, the core is non-empty and singleton but for that
situation, the uncovered set does not coincide with the core.
This result of ours is a technical note. We do not see an immediate real-world significance of this result.

The next section gives the preliminary definitions and notation and a few already well-known results useful for our subsequent discussion. Section 3 gives the main result.

## 2 Preliminary Definitions and Notation

Let $Z \subseteq \mathbb{R}^{k}$ be a convex subset of some finite ( $k$-)dimensional Euclidean space. This set, $Z$, is identified to be the feasible set of policies or outcomes on which a voter votes. Let $N=\{1,2 \ldots, n\}$ be the finite set of players or voters. Suppose that the preferences of a player $i$ on $Z$ is represented by a real-valued continuous and strictly concave pay-off function $u_{i} \in C^{0}(Z, \mathbb{R})$ (and thus, every voter's preference ordering is continuous and strictly convex on $Z$ ). The spatial voting situation we consider below is obtained by introducing the method of majority rule voting.

Definition 2.1 (Domination by Majority Rule) Given $x, y \in Z$, the policy $x$ beats/dominates policy $y$ via coalition $S \subseteq N$, if and only if $|S|>|N| / 2$ and $u_{i}(x)>u_{i}(y)$ for each $i \in S$. We denote this as $x \succ_{S} y$. If there exists a majority coalition $S$ via which $x$ dominates $y$, we denote that as $x \succ y$.

The collection $G=\left(Z, N,\left(u_{i}\right)_{i \in N}\right)$ is a spatial voting situation with majority rule. Recall the two well-known solution concepts for such situations that we shall discuss: the core and the uncovered set.

Definition 2.2 (The Core of a Voting Situation) The core of such a voting situation is the following subset

$$
K=\{y \in Z: \nexists z \in Z \text { such that } z \succ y\} .
$$

Definition 2.3 (The Uncovered Set) Let $x, y \in Z$. be two policies. We say that $x$ covers $y$, denoted as $y<_{c} x$ if the following hold:

$$
\begin{aligned}
& x \succ y \\
& z \in Z, \quad z \succ x \Longrightarrow z \succ y .
\end{aligned}
$$

The uncovered set is given by: ${ }^{3}$

$$
U C=\left\{y \in Z: \nexists z \text { such that } y<_{c} z\right\}
$$

Next we recall and collect some more preliminary definitions and concepts useful for our subsequent discussions. We mostly follow [1].

For nonempty set $A \subseteq N$, we say a function $\pi: A \rightarrow A$ is pairing if $\pi$ is one-one and if, for all $i \in A, \pi(\pi(i))=i$.

Definition 2.4 (Plott Condition) Let $A \subseteq N, x \in X$. The set of gradient vectors $\left\{\nabla u_{i}(x)\right\}_{i \in A}$ satisfies the Plott conditions at $x$ if there exists a pairing $\pi: A \rightarrow A$ such that $\nabla u_{i}(x)=-\lambda_{i} \nabla u_{\pi(i)}(x)$ for some $\lambda_{i} \in \mathbb{R}_{++}$.

Recall the following useful result.
Result 2.1 (Sufficient Condition for majority Core) Take a voting situation $G$ and take $x \in Z$. Let $A=\left\{i \in N: \nabla u_{i}(x) \neq 0\right\}$. If $\left\{\nabla u_{i}(x)\right\}_{i \in A}$ satisfies the Plott conditions at $x$ then $x \in K$.

Definition 2.5 For any $x \in Z$ and $y \in \mathbb{R}^{k}$, let

$$
\begin{aligned}
& L_{x}(y)=\left\{z \in R^{k}: z=t x+(1-t) y \text { for some } t \in \mathbb{R}\right\} \\
& \Gamma(x)=\left\{L_{x}(y): y \in Z\right\} .
\end{aligned}
$$

A generic element $L_{x}(y) \in \Gamma(x)$ for some $y \in \mathbb{R}^{k}$ will be simply denoted by $L_{x}$.

[^1]So, $L_{x}$ denotes a straight line passing through $x$ and $\Gamma(x)$ denotes the set of all such straight lines.

Next we recall the following obvious but useful fact.
Lemma 2.1 (Useful Fact) Let $u: Z \rightarrow \mathbb{R}$ be a strictly concave function on $Z$. Let $x \in Z$ and $L_{x} \in \Gamma(x)$. Then, the function $u$ restricted on the subdomain $L_{x} \cap Z$

$$
\left.u\right|_{L_{x} \cap Z}: L_{x} \cap Z \rightarrow \mathbb{R}
$$

is strictly concave on $L_{x} \cap Z$.
From the fact we recall the notion of induced ideal points on a line.
Definition 2.6 (Induced Ideal Point) For any $x \in Z$, the voter $i$ 's induced ideal point on $L_{x} \in \Gamma(x)$ is given by

$$
b_{i}\left(L_{x}\right)=z \in L_{x} \cap Z: \forall y \in L_{x} \cap Z \quad u_{i}(y)<u_{i}(z) .
$$

It represents the unique maximum point of $u_{i}$ on $L_{x} \cap Z$.
Recall that each voter $i$ 's preference ordering is single-peaked on $L_{x} \cap Z$ with $b_{i}\left(L_{x}\right)$ being the peak of player $i([1]$, p. 135).

Next recall that the preferences are said to be Euclidean or circular if for every voter $i \in N$, there exists $\bar{x}_{i} \in Z$ such that for any policy $x \in Z, u_{i}(x)=-\left(x-\bar{x}_{i}\right)^{2}$. For any point $y \in L_{x}$, we can construct two open half-lines $h_{y}^{+}\left(L_{x}\right), h_{y}^{-}\left(L_{x}\right)$. Then, $L^{+}(y)=\left\{i \in N: b_{i}\left(L_{x}\right) \in h_{y}^{+}\left(L_{x}\right)\right\}$ and $L^{-}(y)=\left\{i \in N: b_{i}\left(L_{x}\right) \in h_{y}^{-}\left(L_{x}\right)\right\}$.

Definition 2.7 (Median Points on $L_{x}$ ) For any $x \in Z, L_{x} \in \Gamma(x)$, the set of induced median points in $Z$ on $L_{x}$ is

$$
\left\{z \in Z \cap L_{x}: L^{+}(z), L^{+}(z) \quad \text { are not majority coalitions }\right\}
$$

Then, a complete characterization of the core is given as:
Result 2.2 (A characterization of the core) Take a voting situation $G$. Then, $x \in K$ if and only if $x$ is an induced median point on $L_{x}$ for all $L_{x} \in \Gamma(x)$.

Cox introduces a notion of "limited" asymmetry of preferences as given below.

Definition 2.8 (Limited asymmetry in preferences) We say that preferences are limited in asymmetry by $\alpha<\infty$ if for all $y \in Z, L_{y} \in \Gamma(y), i \in N, r \in \mathbb{R}$

$$
V_{L_{y}}^{i}(r)=\left\{x \in L_{y}: u_{i}(x)=r\right\} \neq \emptyset \Longrightarrow \frac{\max _{x \in V_{L_{y}}^{i}(r)}\left\|x-b^{i}\left(L_{y}\right)\right\|}{\min _{x \in V_{L_{y}}^{i}(r)}\left\|x-b^{i}\left(L_{y}\right)\right\|} \leq \alpha
$$

Circular preferences obviously satisfy this condition.

## 3 The Core and the Uncovered Set

Cox's result is as follows.

Proposition 3.1 [4] Take a voting situation $G$. Suppose the condition of limited asymmetry of preferences holds for every voter. Now suppose $K \neq \emptyset$. Then $K=$ $U C$.

As we mentioned in the introduction, this result has been repeated in a number of subsequent literature.

However, we find the following.
Proposition 3.2 There is a voting situation $G$ for which $|N|$ is even, the core $K$ is non-empty and singleton, and the uncovered set does not coincide with the core.

To prove this proposition we shall use an intermediate result. First recall the definition of a von-Neumann-Morgenstern stable set for such situations.

Definition 3.1 (von-Neumann-Morgenstern Stable Sets) The set $V \subseteq Z$ is a (von-Neumann-Morgenstern) stable set for $G$ if it satisfies

- (internal stability:) there do not exist $x, y \in V$ such that $x \succ y$;
- (external stability) if $x \notin V$ it must be the case that there exists $y \in V$ such that $y \succ x$.

The following proposition is well-known. Although this result is well-known (see, e.g., [5]) for completeness we provide a short proof below.

Proposition 3.3 If a stable set $V$ exists then $K \subseteq V \subseteq U C$.
Proof. If $K \nsubseteq V$ then that violates the external stability of $V$. Let $V$ be a stable set and take, if possible, and $x \in V \backslash U C$. That is, there exists $y \in Z$ such that $x<_{c} y$. This implies that $y \succ x$. Since, $y \notin V$ (otherwise, the internal stability of $V$ is violated), by external stability of $V$, there exists $z \in V$ such that $z \succ y$. But, then, by the definition of the covering relation, $z \succ x$ which again violates the internal stability of $V$.

Proof of the Proposition 3.2: Below we give an example of a situation where the core is singleton, a stable set exists and the stable set does not coincide with the core. Then, by Proposition 3.3 we are done.

Let $N=\{1,2,3,4\}$. The set of outcomes, $Z=\left\{x \in \mathbb{R}^{2} \mid x_{1} \in[-1,1] ; x_{2} \in[-1,1]\right\}$. Each player $i$ has an ideal point $\bar{x}_{i}$ whose coordinates are given as follows. The point $\bar{x}_{1}$ (labelled by $\left.A^{\prime}\right)=(-1,-1) ; \bar{x}_{2}$ (labelled by $\left.B^{\prime}\right)=(1,-1) ; \bar{x}_{3}$ (labelled by $\left.C^{\prime}\right)=(1,1)$ and $\bar{x}_{4}$ (labelled by $\left.D^{\prime}\right)=(-1,1)$. The players' preferences are circular, i.e., for any $i \in N$, and $x \in Z, u_{i}(x)=-\left(x-\bar{x}_{i}\right)^{2}$.
[We refer to the figure given at the very end.]

We show below that the core of this situation is the singleton set containing the point $(0,0)$ (labelled as point $O$ ) while the set $V=\left\{x \in Z \mid x_{1}=0\right.$ or $\left.x_{2}=0\right\}$ is a stable set. For convenience later call the set $\left\{x \in Z \mid x_{1}=0\right\}$ as $V_{1}$ and the set $\left\{x \in Z \mid x_{2}=0\right\}$ as $V_{2}$.

We take the following steps.

Step 1 : Notice that the point $O=(0,0)$ satisfies the Plott condition (Definition 2.5 above) and so, it is in the core of this voting situation.

Step 2: Next we show that no point other than $O$ is in the core. We start with the subset $\Delta_{1}=\left\{x \in Z \mid x_{1}<0, x_{2} \leq 0, x_{1} \geq x_{2}\right\}$. Note that this is a triangle. Choose, without loss of generality, a point $x \in \Delta_{1}$ such that $x_{1}<0, x_{2}<0$ and $x_{1}>x_{2}$. Draw a line of slope $1, L_{x}$, passing through this $x$ and let the line intersect the line $V_{1}$ at the point $y$. It is obvious that $y$ dominates $x$ via coalition $\{2,3,4\}$. Now choose a point $x \in \Delta_{1}$ such that $x_{1}<0$ and $x_{2}=0$, i.e., the point is on a segment of $V_{2}$. Again, draw a line of slope $1, L_{x}$, passing through $x$ and let that intersect the line given by $\left\{x \in Z \mid x_{1}=-x_{2}\right\}$ at a point $y$. Again, it is obvious that $y$ dominates $x$ via coalition $\{2,3,4\}$. Finally, take a point $x \in \Delta_{1}$ such that $x_{1}=x_{2}$. Then it is obvious that $(0,0)$ dominates such a point via $\{2,3,4\}$.

Thus, no point of $\Delta_{1}$ is in the core.
Note that $Z \backslash\{(0,0)\}$ can be partitioned into 8 such triangles like $\Delta_{1}$. Therefore, using the symmetry between these triangles we can show that no point other than $O$ is in the core.

Next we show that $V$ is a stable set.
Step 3 : (External stability of $V$ :) From Step 2 itself we see that for any $x \in Z \backslash V$, there exists an $y \in V$ such that $y \succ x$.

Step 4 : (Internal stability of $V$ :) It is obvious that a point in $V_{1}$ cannot be dominated by another point in $V_{1}$ and a point in $V_{2}$ cannot be dominated by another point in $V_{2}$.

Next we show that a point in $V_{1}$ cannot dominate, nor can be dominated by a point in $V_{2}$.

Let $p$ be a point in $V_{1}$ such that $p_{2}<0$. Let $q$ be a point in $V_{2}$ such that $q_{1}>0$. Let the length of the line segment $O p$ be $0<r \leq 1$ and let the angle $O p q$ be $\theta$. (Please refer to the figure at the end.) Let, without loss of generality, $0<\theta \leq \pi / 4$. Call the line passing through $p$ and $q$, Let
the perpendiculars from $A^{\prime}, B^{\prime}, C^{\prime}$ and $D^{\prime}$ on $L$ be denoted respectively by $A, B, C$ and $D$. Note that if $\theta=\pi / 4$, then the points $B$ and $D$ coincide and then it is obvious that, neither $p$ can dominate $q$, nor $q$ can dominate $p$.

Now suppose $0<\theta<\pi / 4$. It is easy to see that the length of $q D$ is less than that of $p D$. Since $C q$ is certainly less than $C p, p$ cannot dominate $q$. Next we show that the line segment $B p \leq B q$ and thus, $q$ also does not dominate $p$. If $p$ lies between $B$ and $q$, then this is obvious. Now suppose $B$ lies between $p$ and $q$. A little elementary calculation gives that

$$
B q=\cos \theta-\sin \theta+r \tan \theta \sin \theta
$$

whereas $B p=r / \cos \theta-[\cos \theta-\sin \theta+r \tan \theta \sin \theta]$.
(To establish this we take the following steps. Draw a straight line parallel to $p q$ passing through $O$. Let this line intersect the segment $D^{\prime} C^{\prime}$ at the point $l^{\prime \prime}$. Drop perpendiculars from $D^{\prime}$ and $B^{\prime}$ on this straight line and suppose the points of intersection of these perpendiculars with the straight line be $D^{\prime \prime}$ and $B^{\prime \prime}$ respectively. Also, let $O^{\prime}$ be the point on $p q$ at which the perpendicular from $O$ on $p q$ intersects $p q$. The length of the segment $B q$ equals the sum of the the length of the segments $B O^{\prime}$ and $O^{\prime} q$.

Let $B^{\prime} B^{\prime \prime}$ intersect the line $V_{1}$ at the point $m$ and let $D^{\prime} D^{\prime \prime}$ intersect the line $V_{1}$ at the point $m^{\prime \prime}$.

Note that the triangles $O m B^{\prime \prime}$ and $O m^{\prime} D^{\prime \prime}$ are congruent and thus, $B O^{\prime}=$ $B^{\prime \prime} O=O D^{\prime \prime}$.

Next we calculate the length of $O D^{\prime \prime}$ in terms of $\theta$ using the triangles $O m l^{\prime \prime}$ and $D^{\prime} D^{\prime \prime} l^{\prime \prime}$. Note that $O D^{\prime \prime}=O l^{\prime \prime}-D^{\prime \prime} l^{\prime \prime}$. Now, $O l^{\prime \prime}=O m / \cos \theta=1 / \cos \theta$. And $D^{\prime \prime} l^{\prime \prime}=D^{\prime} l^{\prime \prime} \sin \theta=\left(D^{\prime} m+m l^{\prime \prime}\right) \sin \theta=(1+O m \times \tan \theta) \sin \theta=$ $(1+\tan \theta) \sin \theta$. Therefore, $O D^{\prime \prime}=O l^{\prime \prime}-D^{\prime \prime} l^{\prime \prime}=1 / \cos \theta-(1+\tan \theta) \sin \theta=$ $\cos \theta-\sin \theta$.

The length of $O^{\prime} q=O q \times \sin \theta=O p \times \tan \theta \sin \theta=r \sin \theta \tan \theta$. These yield the expression for the length of $B q$. And the length of $B p$ is simply $p q-B q$.)

Therefore,

$$
B q-B p=2[\cos \theta-\sin \theta]-r[\cos \theta-\tan \theta \sin \theta]
$$

and so

$$
\begin{aligned}
& B q-B p \geq 2[\cos \theta-\sin \theta]-[\cos \theta-\tan \theta \sin \theta] \\
& =\cos \theta-\sin \theta-\sin \theta+\tan \theta \sin \theta \\
& =(\cos \theta-\sin \theta)(1-\tan \theta)
\end{aligned}
$$

since $0<\theta<\pi / 4, B q-B p \geq 0$. Since $A p \leq A q, q$ cannot dominate $p$.
From this, using the symmetry of this example, we can show that no point in $V_{1}$ can dominate another point in $V_{2}$ and vice versa.

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The Figure


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    The errors and shortcomings remaining are, of course, ours.
    ${ }^{2}$ Corresponding Author.

[^1]:    ${ }^{3}$ As this note is concerned with Cox ([4]), we are using the definition he has used. However, the solution defined here is actually the set of the maximal elements of the Gillies subrelation rather than the Miller's subrelation. For clarification, we refer to Bordes, Le Breton and Salles [3].

