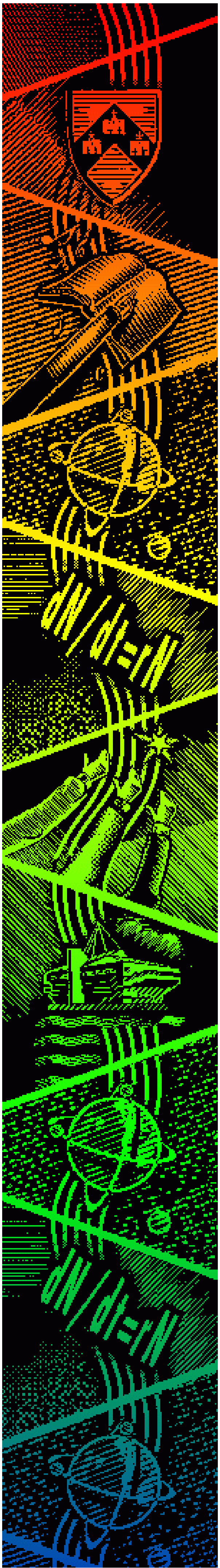

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Stable and Efficient Networks with Farsighted Players: the Largest Consistent Set

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Stable and Efficient Networks with Farsighted Players: the Largest Consistent Set [1]

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Abstract

In this paper we study strategic formation of bilateral networks with farsighted players in the classic framework of Jackson and Wolinsky (1996). We use the largest consistent set (LCS)(Chwe (1994)) as the solution concept for stability. We show that there exists a value function such that for every component balanced and anonymous allocation rule, the corresponding LCS does not contain any strongly efficient network. Using Pareto efficiency, a weaker concept of efficiency, we get a more positive result. However, then also, at least one environment of networks (with a component balanced and anonymous allocation rule) exists for which the largest consistent set does not contain any Pareto efficient network. These confirm that the well-known problem of the incompatibility between the set of stable networks and the set of efficient networks persists even in the environment with farsighted players. Next we study some possibilities of resolving this incompatibility.

JEL Classification: C71, D20.

Keywords: networks, farsighted, largest consistent set.

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1 Introduction

A network is a representation of relations among agents/players in a society or an economy. Formally, a network is a graph which describes the structure of association among the agents. The agents are usually represented by the nodes of the graph and an edge between two nodes represents the existence of some well-defined relation between the two corresponding agents. This relation can be unilateral or bilateral. The corresponding structures of relationship are represented by directed or non-directed networks respectively.

A network is a very powerful tool for describing and analyzing the structure of association among agents as a rich pattern of cooperation among them can be captured in this framework. Among the attractive features of this framework for studying such cooperation, we can highlight that not only this framework is capable of describing pay-off-externalities to a group owing to the formation of other groups but also, it is capable of handling strength of association between agents (see e.g., Bloch and Dutta, 2009; Page and Wooders, 2009). Naturally, these attractive features have stimulated a spate of research in this area of studying strategic network formation.[2]

This study is in the framework of strategic analysis of *bilateral* networks. A player's action is to form link(s) with other players or to sever existing link(s). In this set-up, if a link is to be formed between two players, then consent from both the players is necessary although a player can break a link unilaterally. The now canonical model for analyses in this framework was introduced by Jackson and Wolinsky (1996). In their model, the total pay-off to the players when a network is formed is represented by a *value function* which assigns a real value to every network. This value is distributed to the players according to some *allocation*

[2]See, e.g., the collection of papers in the book edited by Dutta and Jackson (2003) and the survey by Jackson (2006).

rule. For analyzing *strategic formation of networks*, they introduced the notion of pairwise stability. One central result in their work showed that the set of stable networks (with respect to the notion of pairwise stability) and the set of strongly efficient networks, those which are socially optimal, may be disjoint if the allocation rules are intuitively nice.

This impossibility result was followed by a number of studies further exploring this incompatibility between socially optimal states and stable states both within the canonical Jackson-Wolinsky framework and in more specific economic models (see, e.g., the survey by Jackson (2004)). Two important early works within the Jackson-Wolinsky framework where this incompatibility were sought to be resolved are Dutta and Mutuswami (1997) and Currarini and Morelli (2000). However, Dutta and Mutuswami studied the strong and coalition-proof Nash equilibria of a network formation game. Therefore, either only myopic coalitional deviations or only *internal* subcoalitional deviations were considered in their work. Currarini and Morelli do not consider general coalitional moves and nor are the moves endogenous in their model.

For sake of generalized analyses, one important issue is to study the strategic formation of networks and the relation between stable and efficient networks in environments where the players are farsighted, any arbitrary coalition can make a move and such coalitional moves are endogenous. Players are said to be *farsighted* if they anticipate that any action by a group of players may generate a further chain of actions by some other groups. They take this fact into account when computing the final pay-off resulting from their moves (in such models perfect information is assumed). Dutta and Jackson (2003) were possibly the first to emphasize the need for such analyses.

> "Perhaps the most important issue regarding modeling the formation of network is to develop fuller models of networks forming over

> *time,* and in particular allowing players who are *farsighted.* Farsightedness would imply that players' decisions on whether to form a network are not based solely on current pay-offs, but also where they expect the process to go and possibly from emerging steady states or cycles in network formation. ... It is conceivable that, at least in some contexts, farsightedness may help in ensuring the efficiency of the stable state."

(Dutta and Jackson in the "Introduction" of Dutta and Jackson (ed.) (2003), emphasis in the original).

Such studies have emerged over the last few years. Dutta et al. (2005) looks at a rich dynamic model of network formation with farsighted players. However, in their model any arbitrary coalition is not allowed to form. In the study of Herings et al. (2009) and Grandjean et al. (2009) only a pair of players is allowed to be active at any stage. In a recent paper, Page and Wooders (2009), while allowing several rules for network formation (including arbitrary coalitions to form and act), do not look at the issue of the possible incompatibility between the socially optimal states and the stable states in sufficient detail.

The present work is one attempt in this genre. However, we work with a solution notion already well-known and concentrate on the issue of the compatibility between stability and efficiency.

For the present analysis we choose the largest consistent set (LCS) (Chwe (1994)) as the set of stable networks. Several reasons drove this choice. First, this solution concept has gained somewhat a canonical status in the literature on coalitional behaviour with farsighted players. Not only is this set studied for specific situations (see, e.g., Masuda, 2002; Suzuki and Muto, 2005), new solution notions are also routinely compared with this solution (see, e.g., Herings et al., 2009). Secondly, this solution has nice analytical properties: for example, its non-emptiness is ensured in the environment of networks. Moreover, it has been observed (and

Chwe himself noted) that the LCS may be too inclusive. His motivation was:

> "to define a weak concept, one which eliminates with confidence....If Y is consistent and $a \in Y$, the interpretation is not that *a will be* stable but that it is *possible* for a to be stable. If an outcome b is not contained in any consistent Y, the interpretation is that b cannot possibly be stable: there is no consistent story in which b is stable." (Chwe (1994), italics in the original).

Therefore, it is interesting to check whether the above-mentioned incompatibility between the set of stable networks and the set of efficient networks, those which are socially optimal, still survives when the set of stable network becomes avowedly inclusive enough.

We find that in spite of the inclusive nature of the LCS, there exists a value function such that for every component balanced and anonymous allocation rule, the largest consistent set (with respect to the value function and the allocation rule) does not contain any strongly efficient network. This impossibility result provides another corroboration that the well-known incompatibility between stability and efficiency of networks persists even when the players are farsighted. We also show that there exists an environment of networks (with a component balanced and anonymous allocation rule) such that the largest consistent set (with respect to the value function and the allocation rule) does not contain any Pareto efficient network. Next, we study some possibilities of resolving this incompatibility.

Section 2 gives the preliminary definitions. The results are collected in Section 3. The proof of one of the propositions is given in the appendix.

2 Notation and the Preliminary Definitions

The framework and the basic tools for the present analysis were introduced by Jackson and Wolinsky (1996). Below we recall only the essential definitions for completeness. For an elaborate explanation of these concepts and a number of economic examples that fit into this framework, we refer to the comprehensive survey by Jackson (2004).

Networks

Let N be a finite set of players. Given $S \subseteq N$, by g^S we denote the set of all doubleton subsets of S. A *bilateral network* g on N is a subset of g^N. The set of all possible bilateral networks on N is denoted by Z. Given a non-empty network $g \in Z$, an element $\{i, j\} \in g$ (where $i, j \in N$) is a link between players i and j in the network g. We shall often denote the link between i and j simply by ij. The empty network (i.e., the network with no links) is denoted by $\emptyset$.

Players i and j have an *indirect link* between them in a network g if there exist $i_0,\ i_1, \ldots,\ i_m$ in N such that $i_0 = i$, $i_m = j$ and for $k = 0, \ldots,\ m-1$, $i_k i_{k+1} \in g$. Conventionally it is assumed that there is a link between each player and the player itself. A network g induces a partition $\Pi(g)$ of N where two distinct players i and j are in the same element in the partition if and only if there exists an indirect link between them. Given a network g and $i \in N$, by $\Pi_i(g)$ we denote the unique element in $\Pi(g)$ that contains the player i. The *components* of a network g,

$$C(g) = \{g(S) | S \in \Pi(g)\}$$

where $g(S) = g^S \cap g$.

Therefore, for any network g, $g = \cup\{g' |\ g'$ is a component of $g\}$.

Throughout this paper we denote the coalitions of players by S, T etc. and the networks by a, b, g, g' etc..

Value Functions and Allocation Rules

A *value function* $v : Z \mapsto \mathbf{R}$ assigns a real value to every $g \in Z$. This value is generated by some underlying socio-economic process. We follow the standard normalization that $v(\emptyset) = 0$ and the value of a single player is also zero. The set of all such value functions is denoted by V. Recall that given a network g, $C(g)$ denotes the set of the components of g. A value function is *component-additive* if for every $g \in Z$,

$$v(g) = \sum_{g' \in C(g)} v(g').$$

Given a value function $v \in V$, an *allocation rule* $Y : Z \times V \mapsto \mathbf{R}^N$ allocates the value of a network to the players. Given a value function $v \in V$, an allocation rule $Y : Z \times V \mapsto \mathbf{R}^N$ induces a corresponding preference ordering $\succeq_i (v, Y)$ for each $i \in N$ on Z given as follows:

$$\text{for } g, g' \in Z,\ g \succeq_i (v, Y) g' \text{ iff } Y_i(g, v) \geq Y_i(g', v).$$

An allocation rule is *component balanced* if for any *component additive* value function v, $g \in Z$ and $g' \in C(g)$,

$$\sum_{i \in P(g')} Y_i(g, v) = v(g'),$$

where $P(g')$ is the set of players linked in the component g' of g. Given a permutation $\pi : N \mapsto N$, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in Z$. An allocation rule Y is *anonymous* if for every $v \in V$, $g \in Z$ and permutation π,

$$Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$$

for each $i \in N$.

We recall the definitions of two allocation rules which will be useful later. An allocation rule Y^E is said to be *egalitarian* if for every $v \in V$ and $g \in Z$, $Y_i^E(g, v) = v(g)/|N|$. Note that Y^E is anonymous but not component balanced. Given any component additive $v \in V$, the *component-wise egalitarian allocation rule* Y^{CE} is defined by

$$Y_i^{CE}(g, v) = \frac{v(C_i)}{|\Pi_i(g)|}$$

where C_i is the component of g to which the player i belongs. Y^{CE} splits the value equally if the value function is not component additive. Note that Y^{CE} is component balanced as well as anonymous.

Efficient Networks

Given a value function v, a network $g \in Z$ is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \in Z$.

A network $g \in Z$ is *Pareto efficient* relative to a value function v and an allocation rule Y if there does not exist $g' \in Z$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all $i \in N$ with strict inequality for some i.

The Environment of Networks

An *environment of social networks* is represented by $\mathcal{G} = (N,\ Z,\ \{\succeq_i\}_{i \in N},\ \{\rightarrow_S\}_{S \subseteq N})$. Here $\succeq_i$ is the preference relation for $i \in N$ on Z (induced by some underlying value function and allocation rule). For each $i \in N$, $a \succeq_i b$ means that player i weakly prefers network a to network b. The strict part of $\succeq_i$ is denoted by $\succ_i$. The relation $\rightarrow_S$ is the enforcement relation for $S \subseteq N$. For any $a,\ b \in Z$, $a \rightarrow_S b$ implies that the coalition S *can* enforce network b from network a. Formally,

Definition 1 (Jackson and van den Nouweland (2005)) *A coalition S can enforce a network b from a network a if and only if*
(i) a link $ij \in b \setminus a$ implies that $\{i, j\} \subseteq S$ and
(ii) a link $ij \in a \setminus b$ implies that $\{i, j\} \cap S \neq \emptyset$.

For some coalition S and $a,\ b \in Z$, if $a \succ_i b$ for all $i \in S$ then that is written as $a \succ_S b$.

Page and Wooders (2009) also defined such a framework.

Indirect Domination and the Largest Consistent Set (LCS)

Below we give the definitions only. For the motivation and explanation of these concepts we refer to Chwe (1994).

Definition 2 (Chwe (1994)) *For $a,\ b \in Z$, b indirectly dominates a, denoted as $b \gg a$, if there exist $a_0,\ a_1, \ldots,\ a_m$ in Z and coalitions $S_0,\ S_1, \ldots,\ S_{m-1}$ such that $a_0 = a$ and $a_m = b$ and for $j = 0, \ldots,\ m-1$,*
(i) $a_j \rightarrow_{S_j} a_{j+1}$,
(ii) $a_m \succ_{S_j} a_j$.

Definition 3 (Chwe (1994)) *A set $Y \subseteq Z$ is said to be consistent if $Y = \{a \in Z|\ \forall (S, d) \in (2^N \times Z)$ for which $a \rightarrow_S d$, $\exists\ e \in Y$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a\}$. The set $L \subseteq Z$ is said to be the largest consistent set (LCS) if it is consistent and it contains every consistent set.*

By Proposition 2 in Chwe (1994), a non-empty LCS exists for every environment of networks.

3 The Results

First we show an impossibility result which may be viewed as an exact analogue of the impossibility result of Jackson and Wolinsky (1996) in the environment with farsighted players.

Proposition 1 *There exists a value function such that for every component balanced and anonymous allocation rule, the largest consistent set (with respect to the value function and the allocation rule) does not contain any strongly efficient network.*

Proof: The proof is given in the appendix.

We discuss some implications of the result above in the form of the following two remarks.

Remark 1 Results similar in spirit were obtained by Dutta et al. (2005) and Herings et al. (2009) as well. Our result not only reinforce those, but also show that even with an inclusive solution like the LCS, this incompatibility may persist. The notions of stability used in these papers are quite different from that for the LCS. And of course, the notion of pairwise stability, being a concept relevant for myopic players, is quite unrelated to the LCS as well.

Remark 2 Note that if we drop component balance as a requirement, then for every value function, every strongly efficient network is in the LCS with respect to the egalitarian allocation rule Y^E. In the next proposition we discuss the implication of dropping anonymity.

By $\bar{V}$ we denote the class of value functions defined as follows: $\bar{V} = \{v \in V | v(g) > 0$ if and only if g is not totally disconnected$\}$. Dutta and Mutuswami (1997) studied this class of value functions.

Proposition 2 *There exists a component balanced allocation rule such that for every $v \in \bar{V}$, the largest consistent set (with respect to the value function and the allocation rule) contains at least one strongly efficient network.*

Proof: Take a component additive value function $v \in \bar{V}$. Fix $g \in Z$ such that g is strongly efficient with respect to v. We shall define a component balanced (but non-anonymous) allocation rule Y such that $\{g\}$ is internally consistent (see the Lemma in the proof of Proposition 1 (in the appendix) for the definition of internal consistency) with respect to Y. Then, by the Lemma in the appendix we are done.
Case 1: Let there exist $k \in N$ such that k has no link in g with any other player. By the definition of $\bar{V}$ and component additivity, there exists at most one such k. Take $g' \in Z \setminus \{g\}$.
Subcase (a): If k is linked in some component $h_k \in C(g')$ then $Y_k(g', v) = v(h_k)$ and for every other player $j \in \Pi_k(g')$, $Y_j(g', v) = 0$. For every $h \in C(g') \setminus \{h_k\}$ fix some $i_h \in N$ and set $Y_{i_h}(g', v) = v(h)$. For every $j \in N \backslash (\{k\} \cup \{i_h \in N | h \in C(g') \backslash \{h_k\}\})$, $Y_j(g', v) = 0$.
Subcase (b): If k is not linked with any other player in g' then for every $h \in C(g')$ fix some $i_h \in N$ and set $Y_{i_h}(g', v) = v(h)$. For every $j \in N \setminus \{i_h \in N | h \in C(g')\}$, $Y_j(g', v) = 0$.
Case 2: Suppose g is such that every player is linked in g with at least one other player. Take $g' \in Z \setminus \{g\}$. For every $h \in C(g')$ fix some $i_h \in N$ and set $Y_{i_h}(g', v) = v(h)$. For every $j \in N \setminus \{i_h \in N | h \in C(g')\}$, $Y_j(g', v) = 0$.
And in both these cases, $Y_j(g, v) = Y_j^{CE}(g, v)$. So, every player who is linked with some other player in g gets a positive pay-off under Y.

Now take $S \subseteq N$ and $g' \in Z$ such that $g' \neq g$ and $g \rightarrow_S g'$. Suppose we are in Case 1, Subcase (a). Note that then $S \neq \{k\}$. Let $T = N \backslash (\{k\} \cup \{i_h \in N | h \in C(g') \backslash \{h_k\}\})$. Then g indirectly dominates g' according to the following sequence of coalitional moves:

$$g' \rightarrow_T \emptyset \rightarrow_{N \backslash \{k\}} g.$$

Now suppose we are in Case 1, Subcase (b). Then, again, $S \neq \{k\}$. Let $T = N \setminus (\{k\} \cup \{i_h \in N | h \in C(g')\}$. Then, again, g indirectly dominates g' according to the following sequence of coalitional moves:

$$g' \rightarrow_T \emptyset \rightarrow_{N \backslash \{k\}} g.$$

Thus, for Case 1, $\{g\}$ is internally consistent.
Similarly, it can be shown that for Case 2 also, $\{g\}$ is internally consistent. ■

An immediate and interesting question (in the spirit of Dutta and Mutuswami (1997)) is whether Proposition 2 can be strengthened: i.e., whether we can find a component balanced allocation rule such that *every* strongly efficient network is in the largest consistent set or whether the LCS is contained in the set of strongly efficient networks. However, the question, we believe, is still open.

Now, note that the set of Pareto efficient networks with respect to an allocation rule is usually a superset of the set of strongly efficient networks. Even then, the LCS and the set of Pareto efficient networks may be disjoint.

Proposition 3 *There exists an environment of networks (with a component balanced and anonymous allocation rule) such that the largest consistent set (with respect to the value function and the allocation rule) does not contain any Pareto efficient network.*

Proof: Take the following environment which is a slight modification of the one given in the proof of Theorem 1 in Jackson and Wolinsky (1996).

$N = \{1, 2, 3\}$.

For notational convenience we partition Z into the subsets $C1$ to $C4$ such that:

$C1 = \{\{12, 23, 13\}\}$;

$C2 = \{g \in Z |\ g = \{ij, jk\};\ i, j, k \in N\}$;

$C3 = \{g \in Z |\ g = \{ij\};\ i, j \in N\}$;

$C4 = \{\emptyset\}$.

Take the following value function:

$v(\{12, 23, 13\}) = 0$;

for every $g \in C2$, $v(g) = 1 + \epsilon$; where $0 < \epsilon < 0.5$.

for every $i, j \in N$, $v(\{ij\}) = 1$,

$v(\emptyset) = 0$.

Fix the component balanced and anonymous allocation rule Y as follows.

$Y_1(\{12, 23, 13\}, v) = Y_2(\{12, 23, 13\}, v) = Y_3(\{12, 23, 13\}, v) = 0$,

for $i, j, k \in N$, $Y_i(\{ij, jk\}, v) = Y_k(\{ij, jk\}, v) = 0.5,\ Y_j(\{ij, jk\}, v) = \epsilon$;

for $i, j, k \in N$, $Y_i(\{ij\}, v) = Y_j(\{ij\}, v) = 0.5;\ Y_k(\{ij\}, v) = 0$,

$Y_i(\emptyset, v) = Y_j(\emptyset, v) = Y_k(\emptyset, v) = 0$.

For every other value function, $Y = Y^{CE}$. Note that given the value function and the allocation rule, the set of Pareto efficient networks is C2. However, routine calculation (see Chwe (1994)) yields that the LCS for this environment is C3. ■

However, with Pareto efficiency, the incompatibility between socially optimal networks and stable networks is less severe. In the proposition below we show that there is at least one component balanced and anonymous allocation rule, namely the component-wise egalitarian allocation rule, that ensures that the set of Pareto efficient networks has a non-empty intersection with the LCS.

Proposition 4 *For every value function there exists a network that is Pareto efficient as well as an element in the LCS with respect to the component-wise egalitarian allocation rule* Y^{CE}.

Proof: Take some $v \in V$. If v is not component additive, then by the definition of Y^{CE} and Remark 1 we are done.

Now suppose v is component additive. The algorithm described below (this is similar to an algorithm described in Jackson, 2005 (originally due to Banerjee, 1999) selects a network g that is Pareto efficient with respect to Y^{CE} as well as an element in the LCS. First, we introduce a few pieces of notation. For $S \subseteq N$, the set of *components* that can be formed by taking one or more players in S is denoted by $Z(S)$. By $a(h)$ denote the average value of a network h. Also, recall that by $P(h)$ we denote the set of players who are linked in the component h.

The Algorithm:

Step 1: Set $G_1 := N$. Let $A_1 = \{h \in Z(G_1) | a(h) \geq a(h') \text{ for all } h' \in Z(G_1)\}$. Let $C_1 \subseteq A_1$ be a subset of networks satisfying the properties:

a. For any $h, h' \in C_1$, $P(h) \cap P(h') = \emptyset$;

b. For every $C' \subseteq A_1$ for which it is true that for every $h, h' \in C'$, $P(h) \cap P(h') = \emptyset$, $| \cup_{h \in C_1} P(h)| \geq | \cup_{h \in C'} P(h)|$.

That is, in words, C_1 is a collection of components such that each element in C_1 has the highest average value. Additionally, among all such collections of maximal-average-valued components, the components in C_1 together connect the maximum number of players.

Step m: Set $G_m := N \backslash \ (\cup_{h \in C_j; j \in \{1,\ldots,m-1\}} P(h))$. Let $A_m = \{h \in Z(G_m) | a(h) \geq a(h') \text{ for all } h' \in Z(G_m)\}$. Let $C_m \subseteq A_m$ be a subset of networks satisfying the properties:

a. For any $h, h' \in C_m$, $P(h) \cap P(h') = \emptyset$;

b. For every $C' \subseteq A_m$ for which it is true that for every $h, h' \in C'$, $P(h) \cap P(h') = \emptyset$,

$| \cup_{h \in C_m} P(h)| \geq | \cup_{h \in C'} P(h)|$.

Since N is finite, this algorithm terminates in finitely many steps. Let a resulting set of collections of components be $\{C_1, \ldots, C_k\}$. For $j = 1, \ldots, k$, let $N_j = \{i \in N | i \in P(h) \text{ and } h \in C_j\}$. Let the network $g = \cup\{h | h \in C_j \text{ for some } j \in \{1, \ldots, k\}\}$ We show below that g is Pareto efficient with respect to Y^{CE} and it is also in the LCS with respect to Y^{CE}.

Claim 1: g is Pareto efficient with respect to Y^{CE}.

Suppose g is not Pareto efficient with respect to Y^{CE}. This implies that there exists $g' \in Z$ s.t. for all $i \in N$, $Y_i^{CE}(g', v) \geq Y_i^{CE}(g, v)$ and for at least one $j \in N$, $Y_j^{CE}(g', v) > Y_j^{CE}(g, v)$. Fix this j. First, note that $Y_j^{CE}(g', v)$ must be greater than 0 (otherwise, the contradiction is immediate). Now suppose $Y_i^{CE}(g, v) = 0$ for all $i \in N_1$. Then, again, the contradiction is immediate. Therefore, $Y_i^{CE}(g, v) > 0$ for all $i \in N_1$. If $j \in N_1$, then, again, by the definition of N_1 (and the construction of C_1), the contradiction is immediate. Therefore, $j \notin N_1$. Next we show that $Y_j^{CE}(g', v) < Y_i^{CE}(g, v)$ for each $i \in N_1$. Suppose not. Then this contradicts the definition of C_1. Now suppose $Y_j^{CE}(g', v) > Y_i^{CE}(g, v)$ for each $i \in N_2$. Then, this violates the definition of C_2. Therefore, for each $i \in N_2$, $Y_j^{CE}(g', v) \leq Y_i^{CE}(g, v)$. However, it must be true that $Y_j^{CE}(g', v) < Y_i^{CE}(g, v)$ for each $i \in N_2$. Suppose not. Then this contradicts the definition of C_2. Proceeding in this way, we arrive at a desired contradiction. Thus, the claim is proved.

Claim 2: g is in the LCS with respect to Y^{CE}.

Suppose not. Then, by the Proposition 2 in Chwe (1994), there must exist $g' \in Z$ such that $g' \gg g$. We would show that this is impossible. Let, if possible, a sequence of enforcements by which this indirect domination occurs be the following:

$$g(= a_1) \to_{S_1} a_2 \to_{S_2} \ldots \to_{S_{m-1}} a_m \to_{S_m} g',$$

where for each $l \in \{1, \ldots, m\}$, S_l is a coalition, a_l is a network and $a_2 \neq a_1$ without loss of generality. Clearly, under Y^{CE}, $S_1 \cap N_1 = \emptyset$ (otherwise, by the definition of C_1, we would get a contradiction). Then, every component of a_1 (i.e., g) from the collection C_1 also remains a component in a_2. But then, $S_2 \cap N_1 = \emptyset$. Using an identical argument recursively, we get that $N_1 \cap (S_1 \cup \ldots \cup S_m) = \emptyset$ and so, every component of g from the collection C_1 also remains a component of g'. Note that then, under Y^{CE}, $S_1 \cap N_2 = \emptyset$ (otherwise, by the definition of C_2, we would get a contradiction). Using the above argument recursively we find that $N_j \cap (S_1 \cup \ldots \cup S_m) = \emptyset$ for every $j = 1, \ldots, k$. This leads to a contradiction. Thus, the claim is proved. ■

Next, we give a sufficient condition on the value functions which ensures that there exists an allocation rule for which a strongly efficient network is in the LCS.

DEFINITION 4 (Jackson and van den Nouweland (2005)) *A value function $v \in V$ is top-convex if some efficient network also maximizes per-capita value among individuals. Formally, let for coalition S, $p(v, S) = max_{g \in g^S} v(g)/|S|$. The value function is top-convex if $p(v, N) \geq p(v, S)$ for each coalition S.*

We refer to Jackson and van den Nouweland (2005) for a discussion of top-convexity.

Proposition 5 *Suppose a value function v is top-convex. Then every strongly efficient network is in the LCS with respect to the component-wise egalitarian allocation rule Y^{CE}.*

Proof: If v is not component additive, then by the definition of Y^{CE} and Remark 1 we are done.

Suppose v is component additive and let g be a strongly efficient non-empty network. Then the per-capita value of every component of g is $p(v, N)$ (by Jackson and van den Nouweland (2001), section 4). Note that under Y^{CE} the maximum pay-off that any $i \in N$ can get in any network in Z is $p(v, N)$. Now suppose g is not in the LCS. Then, by Proposition 2 in Chwe (1994) there exists g' in the LCS such that $g' \gg g$. Also note that by top convexity, every $i \in N$ is linked with some $j \neq i$ in g. But then there cannot exist g' such that $g' \gg g$. ■

Remark 3 The model studied in Dutta *et al.* (2005) differs slightly from the classical model of Jackson-Wolinsky. In the framework of Dutta *et al.* (2005) the value of a component depends on the structure of the entire network and the value of the *whole network is necessarily the sum of the values of its components.* However, the classical Jackson-Wolinsky model allows the value of the whole network to differ from the sum of the values of its component. As a result, in the model of Dutta *et al.*, component balance has bite even if the underlying value function is not component additive. However, as can be easily seen, our impossibility results (Propositions 1 and 3) are valid in their framework as well.

4 Concluding Remarks

The contribution of this paper is to look at the issue of incompatibility between stability and efficiency in the environment of networks using perhaps the most popular solution concept incorporating farsightedness.

Of course, there still remain a few obvious open questions (about which we have remarked in the body of the paper) that emerge from this work which still remain unanswered.

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5 Appendix: Proof of Proposition 1

Before proceeding to the main body of the proof, for later use we note the following fact in the form of a lemma.

Lemma *Call a set $Y \subseteq Z$ internally consistent if $a \in Y$ implies the following: $\forall (S, d) \in (2^N \times Z)$ for which $a \to_S d$, $\exists\ e \in Y$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a$. If $Y \subseteq Z$ is internally consistent then $Y \subseteq L$.*

Proof of the lemma: (from Chwe (1994)) Let $Y \subseteq Z$ be internally consistent. Define $\Lambda := \cup\{X \subseteq Z |\ X$ is internally consistent$\}$. To prove the lemma, it suffices to show that Λ is consistent. To prove this we need to show that $a \in Z \setminus \Lambda$ implies that there exists $(S, d) \in (2^N \times Z)$ for which $a \to_S d$ and for every $e \in \Lambda$ such that $[e = d$ or $e \gg d]$, $e \succ_S a$. Suppose not, i.e., let there exist $a \in Z \setminus \Lambda$ for which the following is true:

$\forall(S,\ d) \in (2^N \times Z)$ for which $a \rightarrow_S d$, $\exists\, e \in \Lambda$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a$.

Then clearly, $\Lambda \cup \{a\}$ is internally consistent which violates the definition of Λ. So, $Y \subseteq \Lambda \subseteq L$. ■

Now we proceed to the main body of the proof.

Proof of the Proposition 1: Take the following environment which is a slight modification of the one given in the proof of Theorem 2 in Dutta, Ghosal and Ray (2005).

$N = \{1, 2, 3\}$. For notational convenience later, we partition Z into the subsets $C1$ to $C4$ such that:

$C1 = \{\{12, 23, 13\}\}$;

$C2 = \{g \in Z |\ g = \{ij, jk\};\ i, j, k \in N\}$;

$C3 = \{g \in Z |\ g = \{ij\};\ i, j \in N\}$;

$C4 = \{\emptyset\}$.

Take the following value function:

$v(\{12, 23, 13\}) = 9$;

for every $g \in C2$, $v(g) = 0$;

for every $i, j \in N$, $v(\{ij\}) = 8$,

$v(\emptyset) = 0$.

Fix any component balanced and anonymous allocation rule Y. Then, by component balance and anonymity,

$Y_1(\{12, 23, 13\}, v) = Y_2(\{12, 23, 13\}, v) = Y_3(\{12, 23, 13\}, v) = 3$,

for $i, j, k \in N$, $Y_i(\{ij, jk\}, v) = Y_k(\{ij, jk\}, v) = c,\ Y_j(\{ij, jk\}, v) = -2c$, where c is some real number;

for $i, j, k \in N$, $Y_i(\{ij\}, v) = Y_j(\{ij\}, v) = 4;\ Y_k(\{ij\}, v) = 0$,

$Y_i(\emptyset, v) = Y_j(\emptyset, v) = Y_k(\emptyset, v) = 0.$

Here the unique strongly efficient network is $\{12, 23, 13\}$. However, below we show that whatever the value of c, the LCS, $L = \{\{12\}, \{23\}, \{13\}\}$. We consider the following three cases.

Case 1: $c \geq 4$:

First, we show that the set $C3$ is internally consistent. Therefore, we are to show that $a \in C3$ implies the following: $\forall (S, d) \in (2^N \times Z)$ for which $a \rightarrow_S d$, $\exists\, e \in C3$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a$. Take $x \in C3$ and let $x = \{ij\},\ i, j \in N$. Consider $(S, d) \in (2^N \times Z)$ such that $x \rightarrow_S d,\ d \neq x$. Then, by Definition 1, $S \cap \{i, j\} \neq \emptyset$. If $d \in C3$, then set $e = d$. If $d \in C1 \cup C4$ then consider the enforcement $d \rightarrow_{\{i,j\}} x$ and set $e = x$. Suppose $d \in C2$. Then d is either $\{lm, mk\}$ or $\{lk, km\}$ where $l, m \in \{i, j\}, l \neq m,\ k \in N \setminus \{i, j\}$. In the former subcase consider the enforcement $d \rightarrow_{\{m\}} x$ and set $e = x$. In the latter subcase, consider the enforcement $d \rightarrow_{\{k\}} \{lk\}$ and set $e = \{lk\}$. Since $x \succeq_i y$ and $x \succeq_j y$ for every $y \in C3$, we are done. Thus, we show that $C3$ is internally consistent and so, by the lemma, $C3 \subseteq L$.

Next we prove that in fact, $C3 = L$. Suppose not and let some $L \supset C3$ be the LCS. First, we claim that $L \cap C2 = \emptyset$. Take some $x\ (= \{ij, jk\}) \in C2,\ i, j, k \in N$. Consider the enforcement relation $x \rightarrow_{\{j\}} \{ij\}$. Then $\{ij\} \succ_j x$. Moreover, since $y \succ_j x$ for every $y \in Z \setminus \{x\}$, it follows that there does not exist any $e \in L$ such that $[e = \{ij\}$ or $e \gg \{ij\}$ and $e \not\succ_j x]$. Thus, the claim is proved. Next, consider x from $C1 \cup C4$ and the enforcement relation $x \rightarrow_{\{1,2\}} \{12\}$. Note that for any $(S, y) \in (2^N \times Z)$ such that $y \neq \{12\}$, $\{12\} \rightarrow_S y$ implies that $S \cap \{1, 2\} \neq \emptyset$. Moreover, for every $e \in Z \setminus C2$, $\{12\} \succeq_1 e$ and $\{12\} \succeq_2 e$. Therefore, there does not exist $e \in L$ such that $e \gg \{12\}$. Since $\{12\} \succ_{\{1,2\}} x$, $x \notin L$.

Case 2: $-2 < c < 4$:

Note that in this case, for $i, j \in N$, $\{ij\} \succ_{\{i,j\}} g$ for every $g \in Z \setminus C3$ and also, for any $g \in Z$, $\{ij\} \succeq_i g$ and $\{ij\} \succeq_j g$. In this case also, $C3$ is internally consistent, i.e., $a \in C3$ implies the following: $\forall (S, d) \in (2^N \times Z)$ for which $a \rightarrow_S d$, $\exists\ e \in C3$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a$. For proving this, take $x \in C3$ and let $x = \{ij\},\ i, j \in N$. Consider $(S, d) \in (2^N \times Z)$ such that $x \rightarrow_S d$ and $d \neq x$. By Definition 1, $S \cap \{i, j\} \neq \emptyset$. If $d \in C3$, then set $e = d$. If $d \in Z \setminus C3$ then consider the enforcement $d \rightarrow_{\{i,j\}} x$ and set $e = x$. Since $x \succeq_i y$ and $x \succeq_j y$ for every $y \in C3$, we are done. Therefore, by the lemma, $C3 \subseteq L$.

In this case also, $L = C3$. To see this, take $x \in Z \setminus C3$ and consider the enforcement relation $x \rightarrow_{\{1,2\}} \{12\}$. Note that for any $(S, y) \in (2^N \times Z),\ y \neq \{12\}$, $\{12\} \rightarrow_S y$ implies that $S \cap \{1, 2\} \neq \emptyset$. As noted above, for every $e \in Z$, $\{12\} \succeq_1 e$ and $\{12\} \succeq_2 e$. Therefore, there does not exist $e \in L$ such that $e \gg \{12\}$. Since $\{12\} \succ_{\{1,2\}} x$, $x \notin L$.

Case 3: $c \leq -2$:

Again, first we show that $C3$ is internally consistent, i.e., we show that $a \in C3$ implies the following: $\forall (S, d) \in (2^N \times Z)$ for which $a \rightarrow_S d$, $\exists\ e \in C3$ such that $[e = d$ or $e \gg d]$ and $e \not\succ_S a$. Take $x \in C3$ and let $x = \{ij\},\ i, j \in N$. Consider $(S, d) \in (2^N \times Z)$ such that $x \rightarrow_S d$ and $d \neq x$. By Definition 1, $S \cap \{i, j\} \neq \emptyset$. If $d \in C3$, then set $e = d$. If $d \in C1 \cup C4$ then consider the enforcement $d \rightarrow_{\{i,j\}} x$ and set $e = x$. Suppose $d \in C2$. Then, d is either $\{lm, mk\}$ or $\{lk, km\}$ where $l, m \in \{i, j\}, l \neq m$ and $k \in N \setminus \{i, j\}$. In the former subcase consider the enforcement $d \rightarrow_{\{l,k\}} \{lk\}$ and set $e = \{lk\}$. In the latter subcase, consider the enforcement $d \rightarrow_{\{i,j\}} \{ij\}$ and set $e = \{ij\}$. Since $x \succeq_i y$ and $x \succeq_j y$ for every $y \in C3$, we are done. Therefore, $C3$ is internally consistent and so, $C3 \subseteq L$.

Next we prove that once again, in this case also, $L = C3$. Suppose not and let some $L \supset C3$ be the LCS. To begin with, we claim that $L \cap C2 = \emptyset$. Take some $x\ (= \{ij, jk\}) \in C2,\ i, j, k \in N$. Consider the enforcement relation $x \rightarrow_{\{i,k\}} \{ik\}$.

Suppose, there exists $e \in L$ such that $e \gg \{ik\}$. We show below that this is impossible. Let, if possible, a sequence of enforcements by which this indirect domination occurs be the following:

$$\{ik\}(= a_1) \rightarrow_{S_1} a_2 \rightarrow_{S_2} \ldots \rightarrow_{S_{m-1}} a_m \rightarrow_{S_m} e,$$

where for each $l \in \{1, \ldots, m\}$, S_l is a coalition, a_l is a network and $a_2 \neq a_1$ without loss of generality. Then, by the definition of enforcement relation, $S_1 \cap \{i, k\} \neq \emptyset$. We consider two subcases. First take the subcase where $c < -2$. Since $S_1 \cap \{i, k\} \neq \emptyset$ and $e \succ_{S_1} \{ik\}$, e must be either $\{ji, ik\}$ or $\{jk, ki\}$. Therefore, by the definition of enforcement relation, $j \in S_l$ for some $l \in \{1, \ldots, m\}$ and by the definition of indirect domination, $e \succ_{S_l} a_l$. But this is impossible because, for every $g \in Z$, $g \succeq_j \{ji, ik\}$ and $g \succeq_j \{jk, ki\}$. Next, take the subcase where $c = -2$. Then, $\{ik\} \succeq_i g$ and $\{ik\} \succeq_k g$ for every $g \in Z$. Therefore, for this subcase also, there cannot exist $e \in Z$ such that $e \gg \{ik\}$. Since $\{ik\} \succ_{\{i,k\}} \{ij, jk\}$, $\{ij, jk\} \notin L$. Thus, the claim is proved.

Next, take any $x \in C1 \cup C4$ and consider the enforcement relation $x \rightarrow_{\{1,2\}} \{12\}$. Note that for any $(S, y) \in (2^N \times Z)$ such that $y \neq \{12\}$, $\{12\} \rightarrow_S y$ implies that $S \cap \{1, 2\} \neq \emptyset$. By the claim above, $L \cap C2 = \emptyset$ and for every $e \in Z \setminus C2$, $\{12\} \succeq_1 e$ and $\{12\} \succeq_2 e$. Therefore, there does not exist $e \in L$ such that $e \gg \{12\}$. Since $\{12\} \succ_{\{1,2\}} x$, $x \notin L$. ■