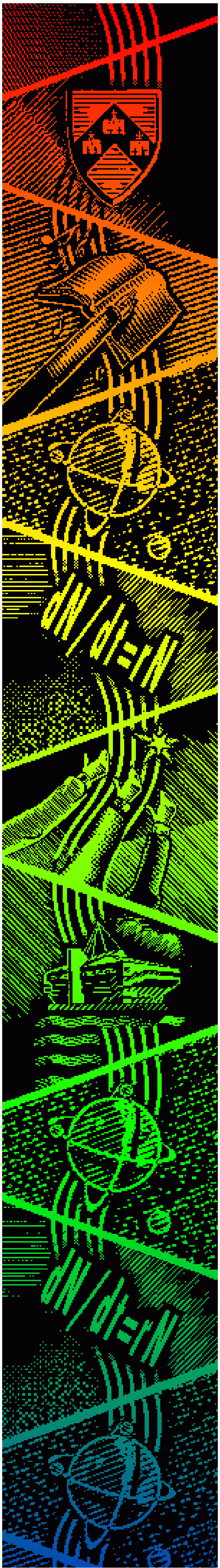


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**On Credible Coalitional Deviations by Prudent Players**

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**On Credible Coalitional Deviations by Prudent Players**[1]

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**Abstract**

In this paper we first explore the predictive power of the solution notion called *conservative stable standard of behaviour* (CSSB), introduced by Greenberg (1990) in environments with farsighted players (as modelled in Xue (1998)) as intuitively it is quite nice. Unfortunately, we find that CSSB has a number of undesirable properties. Therefore, we introduce a refinement of this which we call conservative stable weak predictor. We explore some existence properties of this new solution.

JEL Classification Number: C70, C71, C72.

*Keywords:* Coalitions; stable behaviour; perfect foresight.

# 1 Introduction

The seminal work of Chwe (1994) analyzing coalitional behaviour in a general social environment with players having perfect foresight gave rise to a number of subsequent works. After analyzing the properties of the largest consistent set, the solution notion he introduced, Chwe listed several issues of coalitional behaviour with farsighted players that his solution concept failed to capture. Some of the subsequent literature took up and addressed these problems (at times with drastically different modelling also) with new solution ideas in the set-up of general social environment itself (e.g., Xue (1998), Konishi and Ray (2003), Herings et al. (2004)) and some of these studied specific environments (like Ambrus (2006) in the environment of games in normal form, Bhattacharya (2002) in the environment of voting situations with finite number of outcomes, Duggan and Kalandrakis (2008) and Penn (2009) in the environment of spatial voting situations etc).

One of the most popular ones among these is Xue (1998). In his attempt to improve upon the concept of the largest consistent set (LCS) he correctly recognized that

> ...the inclusiveness of the LCS...stem[s] from the fact that indirect dominance defined on $Z$ [the set of outcomes] fails to capture perfect foresight since it ignores the possible deviation along the way from one alternative (e.g., $a$) to another (e.g., $d$).

So, he tried to incorporate a "credibility" restriction on a coalition's deviations from an outcome. Since the sequence of possible moves by the different coalitions can be of arbitrary length, the framework of *social situations* developed by Greenberg (1990) turned out to be quite useful for his analyses. He obtained his notion of coalitional stability through the concept of a stable standard of behaviour. We consider this idea quite nice and we agree that the CSSB applied to the situation with perfect foresight is an intuitively appealing stability notion.
Toward the end of his paper Xue says that

> In his concluding remarks Chwe recognizes several issues that the notion of LCS fails to address, yet no constructive solution was offered. The notion suggested in this paper resolves most if not all of these issues.

In this paper we start with further examining the usefulness of his idea for different environments. However, we find that unfortunately, in a large class of social environments representable by games in effectivity function form, CSSB has *no predictive power at all* if every path is feasible. This is a serious drawback of the solution concept because this subclass contains very common environments like normal form games, voting situations, games in characteristic function form, social networks (as in the framework of Jackson and Wolinsky (1996)) etc. Furthermore, in Theorem 4.5 of his paper Xue compares CSSB with the LCS and shows that under some assumptions, his stability notion refines the LCS. However, we find that with a somewhat similar, but logically more consistent restriction of feasibility on paths, the reverse is true for a class of social environments which includes the class of voting games with a finite number of outcomes.

Next, we reckon that the failure of CSSB as a predictor may stem from the fact that Xue, following Greenberg, used a "strong" form of domination. Therefore, we weaken the notion of dominance while retaining the main idea behind CSSB. We call the resulting solution conservative stable weak predictor (CSWP). We study some properties of this modified solution in the environment of voting and find that even with this modification the solution does not have nice existence properties.

The following section gives the preliminary definitions and remarks. We study the further properties of CSSB and its problematic features in Section 3. Next we study the properties of CSWP in Section 4. Section 5 concludes.

## 2 Preliminary Definitions and Remarks

We would follow almost all of Xue's definitions and notation.

A *social environment* is represented by $\mathcal{G} = (N,\ Z,\ \{\preceq_i\}_{i\in N},\ \{\underset{S}{\rightarrow}\}_{S\subseteq N})$. Here $N$ is the finite set of players, $Z$ is the set of social states or outcomes, $\preceq_i$ is the preference relation for $i \in N$ on $Z$ and $\underset{S}{\rightarrow}$ is the effectivity relation for $S \subseteq N$. For each $i \in N$, $a \preceq_i b$ means that player $i$ weakly prefers outcome $b$ to outcome $a$. The strict part of $\preceq_i$ is denoted by $\prec_i$ .[2] Thus, for every $i \in N$, $\prec_i$ is irreflexive on $Z$. For any $a,\ b \in Z$, $a \underset{S}{\rightarrow} b$ implies that the coalition $S$ can enforce outcome $b$ from outcome $a$. A number of examples of the games that can be written in this form is provided by Chwe (1994) and Xue (1998).

Definition 1 *Given a social environment $\mathcal{G}$, a path*[3] *is a singleton sequence of the form $\{a_1\}$ or an ordered sequence of the form $\{a_1,\ S_1,\ a_2,\ S_2, \ldots,\ S_{k-1},\ a_k\}$*

[2]In Xue's specification, the individual preferences are assumed to be strict but the results following do not change with this added restriction.

[3]Here we have slightly modified Xue's definition of a path. In our definition of a path, we also specify the coalitions which enforce one outcome from another along the path. This is similar to the definition of history given in Herings *et al.* (2004).

*where for each* $i$*,* $a_i \in Z$*,* $S_i \subseteq N$ *and for every* $j = 1, \ldots, k-1$*,* $a_j \underset{S_j}{\rightarrow} a_{j+1}$*.*

If an outcome $a \in Z$ lies on a path $\alpha$ then that is denoted as $a \in \alpha$. The set $\Pi$ denotes the set of all *possible* paths. For $a \in Z$, $\Pi_a$ denotes the set of all possible paths that originate from $a$, i.e., the possible paths which have $a$ as the first element. Below, some time we shall impose a feasibility restriction and consider the set of *feasible* paths only instead of considering all possible paths. The set of feasible paths is generically denoted by $\Pi^f$. For $a \in Z$, $\Pi^f_a$ denotes the set of all feasible paths that originate from $a$ including $\{a\}$ itself. For a path $\alpha$, $t(\alpha)$ denotes its terminal outcome. Individual preferences are extended on $\Pi$ as follows. For $i \in N$ and for any two paths $\alpha$ and $\beta$, $\alpha \preceq_i \beta$ if and only if $t(\alpha) \preceq_i t(\beta)$ (and similarly for $\prec_i$). This implies that for a sequence of coalitional moves described by a path, the players receive the pay-offs corresponding to the terminal element of the path. For some coalition $S$ and $a,\ b \in Z$, if $a \prec_i b$ for all $i \in S$ then that is written as $a \prec_S b$ and if such a coalition exists, then we also write $a \prec b$. Similarly, for paths $\alpha$ and $\beta$, if $t(\alpha) \prec_S t(\beta)$ then it is also written as $\alpha \prec_S \beta$.

DEFINITION 2 *Suppose* $\Pi^f$ *is given as the set of feasible paths for environment* $\mathcal{G}$*. A standard of behaviour*[4] *(SB)* $\sigma$ *is a map,* $\sigma : Z \mapsto \Pi^f$ *such that for every* $a \in Z$*,* $\sigma(a) \subseteq \Pi^f_a$*.*

DEFINITION 3 *Suppose* $\Pi^f$ *is given as the set of feasible paths for environment* $\mathcal{G}$*. An SB* $\sigma$ *is:*

*(i) A conservative internally stable standard of behaviour (CISSB) for* $\mathcal{G}$ *if for all* $a \in Z$*,* $\alpha \in \sigma(a) \implies$ *there do not exist* $S \subseteq N$*,* $b \in \alpha$ *and* $c \in Z$ *such that* $\{b,\ S,\ c\} \in \Pi^f_b$*,* $\sigma(c) \neq \emptyset$ *and* $\alpha \prec_S \beta$ *for all* $\beta \in \sigma(c)$*;*

[4]The *general* framework of Greenberg (1990) uses concepts like positions, situations, inducement correspondences etc.. A rigorous recasting of the present set-up into Greenberg's framework can be made following Mariotti and Xue (2003).

*(ii) A conservative externally stable standard of behaviour (CESSB) for $\mathcal{G}$ if for all $a \in Z$, $\alpha \in \Pi_a^f \setminus \sigma(a) \implies$ there exist $S \subseteq N$, $b \in \alpha$ and $c \in Z$ such that $\{b,\ S,\ c\} \in \Pi_b^f$, $\sigma(c) \neq \emptyset$ and $\alpha \prec_S \beta$ for all $\beta \in \sigma(c)$.*

*An SB $\sigma$ is a conservative stable standard of behaviour (CSSB) for $\mathcal{G}$ if it is both a CISSB and a CESSB.*

Therefore, for an outcome $x \in Z$, a CSSB specifies the set of "credible" paths from $x$. The underlying idea is that if a coalition makes a feasible deviation from an outcome $x$ to an outcome $y$ then, being farsighted, the players in the coalition examine all the "credible" paths that originate from $y$. A feasible path is not "credible" if some coalition can feasibly deviate from it to another outcome and if its members are strictly better-off at every credible path originating from that outcome. Thus, the intuition behind CSSB is quite appealing.

The corresponding set of stable outcomes is defined as follows.

Definition 4 (Xue (1998)) *Given an environment $\mathcal{G}$, a set $X \subseteq Z$ is said to be a set of stable outcomes under conservatism or Xue-stable if there exists a CSSB $\sigma$ for $\mathcal{G}$ such that $X = \{a \in Z |\ \{a\} \in \sigma(a)\}$.*

Given an environment $\mathcal{G}$, a CSSB $\sigma$ is said to be *non-empty valued* if for every $a \in Z$, $\sigma(a) \neq \emptyset$. Naturally, a stable standard of behaviour should be non-empty valued if it is to make predictions about coalitional behaviour.

# 3 Further Properties of CSSB and Some of its Problems

In this section, first we show that if every path is feasible, then CSSB has no predictive power at all for a large class of environments.

Let us specify the following condition.

**Condition $C$:** An environment $\mathcal{G}$ satisfies Condition $C$ if for every pair $(a, b) \in Z \times Z$, there exists a path $\alpha \in \Pi_a$ such that $t(\alpha) = b$.

That is, an environment $\mathcal{G}$ satisfies Condition $C$ if for any two outcomes in $Z$, there exists a path between them.

A large class of environments satisfies this condition; e.g., games in normal form, voting games, games in characteristic function form, social networks[5] (as in the framework of Jackson and Wolinsky (1996)) etc.

**Theorem 1** *Suppose an environment $\mathcal{G}$ satisfies Condition $C$ and let $\Pi^f$ be $\Pi$ (i.e., every path is feasible). Then, the following SB $\sigma$ is a CSSB:*

$$\textit{for every } a \in Z,\ \sigma(a) = \Pi_a.$$

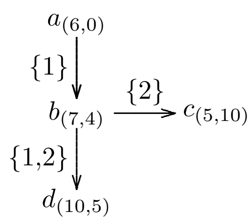
*Therefore, in this case, the entire $Z$ is a Xue-stable set.*

**Proof:** It suffices to show that $\sigma$ as specified in the theorem is conservative internally stable. Suppose otherwise. Then, there exist $a \in Z$ and $\alpha \in \Pi_a$ for which the following is true:
(†) for some $b \in \alpha$, there exist $S \subseteq N$, and $c \in Z$ such that $\{b,\ S,\ c\} \in \Pi_b$, $\Pi(c) \neq \emptyset$ and $\alpha \prec_S \beta$ for every $\beta \in \Pi(c)$.

Therefore, by Condition C, for some $\beta \in \sigma(c)$, $t(\beta) = t(\alpha)$ and by irreflexivity of $\prec_i$, (†) can never hold. ■

[5] In the appendix we have described how the environment of social networks falls within the present formalism.

**Remark 1** Xue has given some examples of environments where CSSB makes some non-trivial predictions. For example, see the game in Figure 2 of Xue (1998) given below. The unique CSSB for this environment is given by $\sigma(b) = \{b, \{2\}, c\}$ and $\sigma(a) = \{a\}, \sigma(c) = \{c\}$ and $\sigma(d) = \{d\}$. Therefore, for this environment the unique Xue-stable set is: $\{a, c, d\}$.




Note, however, that such environments do not satisfy Condition $C$. For example, in this game no outcome can be enforced from outcome $c$ and outcome $d$.

**Remark 2** Recall that the underlying idea behind CSSB is that if a coalition $S$ makes a feasible deviation from an outcome $x$ to an outcome $y$ then, being far-sighted, the players in $S$ examine all the "credible" paths that originate from $y$. A feasible path is not "credible" if some coalition can feasibly deviate from it to another outcome and if its members are strictly better-off at *every* credible path originating from that outcome. Now, this may be too demanding and this seems to be the reason behind the inclusiveness of CSSB. One interesting direction of further study is to relax the requirement of domination in the following way: a feasible path is not "credible" if some coalition can feasibly deviate from it to another outcome and if its members are *weakly* better-off at every credible path originating from that outcome and *strictly better-off at least for one path.* We look at the implications of this modification in the next section.

Next, we state another condition.

**Condition $C'$:** An environment $\mathcal{G}$ satisfies Condition $C'$ if the following holds. For $S \subseteq N$, if there exist $(a, b) \in Z \times Z$, such that $a \underset{S}{\rightarrow} b$, then for every $(c, d) \in Z \times Z$, $c \underset{S}{\rightarrow} d$.

This condition is obeyed by simple games or voting games which we define below (see, e.g., Peleg (1984)).

Definition 5 *The social environment $\mathcal{G}$ is a simple game if there exists a non-empty set $B \subset 2^N$ called the set of winning coalitions such that*
*(i) $T \in B$, $S \supseteq T \Longrightarrow S \in B$.*
*(ii) If $S \in B$ then for any $a, b \in Z$, $a \underset{S}{\rightarrow} b$ and if $S \notin B$ then for no two $a, b \in Z$ is it the case that $a \underset{S}{\rightarrow} b$.*

Below we recapitulate a few concepts which will be useful.

Definition 6 (Chwe (1994)) *For $a,\ b \in Z$, $b$ indirectly dominates $a$, denoted as $a \ll b$, if there exist $a_0,\ a_1, \ldots,\ a_m$ in $Z$ and coalitions $S_0,\ S_1, \ldots,\ S_{m-1}$ such that $a_0 = a$ and $a_m = b$ and for $j = 0, \ldots,\ m-1$,*
*(i) $a_j \underset{S_j}{\rightarrow} a_{j+1}$,*
*(ii) $a_j \prec_{S_j} a_m$.*

Definition 7 (Chwe (1994)) *A set $Y \subseteq Z$ is said to be consistent if $Y = \{a \in Z |\ \forall(S,\ d) \in (2^N \times Z)$ for which $a \underset{S}{\rightarrow} d$, $\exists\ e \in Y$ such that $[e = d$ or $d \ll e]$ and $a \not\prec_S e\}$. The set $L \subseteq Z$ is said to be the largest consistent set (LCS) if it is consistent and it contains every consistent set.*[6]

For our subsequent discussion, we restrict the set of feasible paths as follows.

[6]Chwe (1994) showed that LCS exists for every environment.

Definition 8 *For $a_1 \in Z$, take a path $\alpha \in \Pi_{a_1}$ such that $\alpha = \{a_1,\ S_1, \ldots,\ S_{k-1},\ a_k\}$. The path $\alpha$ is feasible by domination only if for every $j = 1, \ldots, k-1$, $a_j \prec_{S_j} a_k$. Additionally, we assume that every singleton path is feasible by domination.*

Therefore, a non-singleton path from an outcome $a$ is feasible only if the terminal element of this path indirectly dominates $a$ along the path. Xue imposed a somewhat similar feasibility restriction [7] in Theorem 4.5 of his paper for comparing CSSB with the LCS. He showed that under such a restriction, his stability notion refines the LCS. However, we show the following.

**Theorem 2** *Take a social environment $\mathcal{G}$ for which $L \neq \emptyset$[8] and which satisfies Condition $C'$. Let $\Pi^f$ be the set of paths feasible by domination. Then, $\mathcal{G}$ has some Xue-stable set $X$ such that $L \subseteq X$. Moreover, there are some environments for which this inclusion is strict.*

To prove this theorem we first note the following lemmata.

**Lemma 2.1** *Suppose for an environment $\mathcal{G}$, $L \neq \emptyset$ and Condition $C'$ holds. Then, $a \in Z \setminus L$ implies that there exists $b \in L$ such that $a \ll b$.*

**Proof:** Suppose $L \neq \emptyset$ and $a \in Z \setminus L$. Then, by the definition of $L$, there exist $(S,\ d) \in (2^N \times Z)$ for which $a \underset{S}{\rightarrow} d$. Suppose that for no $b \in L$ is it true that $a \ll b$. Take any $b \in L$ and consider the pair $(S, a)$. By $C'$, $b \underset{S}{\rightarrow} a$. Since $a \notin L$ and $\{e \in L |\ a \ll e\} = \emptyset$, $L$ cannot be consistent. But this is a contradiction. ■

**Lemma 2.2** *Suppose there exists a CISSB $\sigma$ for an environment $\mathcal{G}$ such that for*

[7] In Remark 3 below we explain the logical problems of Xue's restrictions which we try to improve here.

[8] This is ensured under quite weak conditions (see, e.g., Chwe (1994), Xue (1997)). For example, every environment for which $Z$ is finite admits a non-empty LCS.

*every* $a \in Z$, $\sigma(a) \neq \emptyset$. *Then there exists a CSSB* $\sigma'$ *for* $\mathcal{G}$ *such that for every* $a \in Z$, $\sigma(a) \subseteq \sigma'(a)$.

**Proof:** The proof is exactly similar to that of Theorem 3.4 in Greenberg *et al.* (1996). ■

**Proof of Theorem 2:** $(i)$ Take an environment $\mathcal{G}$ for which $L \neq \emptyset$ and which satisfies Condition $C'$. Consider an SB $\sigma$ as follows:

$$\text{for every } a \in Z,\ \sigma(a) = \{\alpha \in \Pi^f_a |\ t(\alpha) \in L\}.$$

By Lemma 2.1, for every $a \in Z$, $\sigma(a) \neq \emptyset$.

Next we show that $\sigma$ as defined above is conservative internally stable. Suppose otherwise. Then there exist $a \in Z$ and $\alpha \in \sigma(a)$ for which the following is true:

(‡) for some $b \in \alpha$, there exist $S \subseteq N$ and $c \in Z$ such that $\{b,\ S,\ c\} \in \Pi^f_b$, $\sigma(c) \neq \emptyset$ and $\alpha \prec_S \beta$ for all $\beta \in \sigma(c)$.

Now, consider the pair $(S, c)$ with respect to $t(\alpha)$. By $C'$, $t(\alpha) \underset{S}{\rightarrow} c$ and by (‡) for every $e \in L$, $e = c$ or $c \ll e$ implies that $t(\alpha) \prec_S e$. But then by the definition of LCS, $t(\alpha) \notin L$ which is a contradiction. Therefore, $\sigma$ as defined above is conservative internally stable. Then, by Lemma 2.2, there exists a CSSB $\sigma'$ for $\mathcal{G}$ such that for every $a \in Z$, $\sigma(a) \subseteq \sigma'(a)$. Define the Xue-stable set $X = \{a \in Z |\ \{a\} \in \sigma'(a)\}$. Then, $L \subseteq X$.

$(ii)$ For proving the second part of the theorem, take the following proper [9] simple game, $\mathcal{G}$.

$N = \{1,\ \ldots,\ 7\}$, $Z = \{a, b, c, d, e, f\}$. Let the set of minimal winning coalitions be $W = \{S_1,\ \ldots,\ S_4\}$ where

$$S_1 = \{1, 2, 3\},\ S_2 = \{1, 4, 5\},\ S_3 = \{2, 4, 6\},\ S_4 = \{3, 5, 6, 7\}.$$

[9]A simple game (see Definition 5) is said to be proper if for every $S \in B$, $(N \setminus S) \notin B$.

The players' preferences over $Z$ are the following:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $d$ | $e$ | $b$ | $a$ | $d$ | $b$ | $c$ |
| $c$ | $d$ | $f$ | $e$ | $f$ | $f$ | $d$ |
| $b$ | $c$ | $c$ | $c$ | $c$ | $a$ | $a$ |
| $f$ | $a$ | $d$ | $d$ | $b$ | $e$ | $f$ |
| $a$ | $f$ | $a$ | $f$ | $e$ | $c$ | $e$ |
| $e$ | $b$ | $e$ | $b$ | $a$ | $d$ | $b$ |

This implies the following relations:

$$a \prec_{S_1} c,\ a \prec_{S_1} d,\ f \prec_{S_2} d,\ b \prec_{S_2} c,\ b \prec_{S_2} d,\ c \prec_{S_3} e,\ d \prec_{S_3} e,\ e \prec_{S_4} f.$$

It is easily checked that for no other $x, y \in Z$ and $S \in W$ is it true that $x \prec_S y$.

This environment satisfies the condition $C'$. Let $\Pi^f$ be the set of paths feasible by domination.

In this framework the definition of a consistent set can be simplified as follows (Bhattacharya (2002)): a set $Y \subseteq Z$ is said to be consistent if $Y = \{a \in Z | \ \forall (S,\ d) \in (W \times Z),\ \exists\ e \in Y$ such that $[e = d$ or $d \prec e]$ and $a \not\prec_S e\}$. Then routine computation (see Chwe (1994)) yields that the LCS, $L$, for this game is $\{c, d, e, f\}$. However, $Z$ is a Xue-stable set for this game. To see this, construct an SB $\sigma$ such that for every $x \in Z$, $\sigma(x) = \{\alpha \in \Pi^f_x |\ t(\alpha) \in L\}$. By Lemma 2.2, there exists a CSSB $\sigma'$ for $\mathcal{G}$ such that for every $a \in Z$, $\sigma(a) \subseteq \sigma'(a)$. We claim that $a \in \sigma'(a)$. Suppose otherwise. Then there exist $S \in W$ and $x \in Z$ such that $\{a,\ S,\ x\} \in \Pi^f_a$, $\sigma'(x) \neq \emptyset$ and $a \prec_S t(\beta)$ for all $\beta \in \sigma'(x)$. Check that $\{a,\ S,\ x\}$ is either $\{a,\ S_1,\ c\}$ or $\{a,\ S_1,\ d\}$. But note that $\{c,\ S_3,\ e\} \in \sigma'(c)$ and $\{d,\ S_3,\ e\} \in \sigma'(d)$. Since it is not the case that $a \prec_{S_1} e$, the claim is proved. Similarly it can be shown that $b \in \sigma'(b)$. ■

We obtain the following corollary from the proof of Theorem 2 which may be of independent interest, especially for voting situations.

**Corollary 1** *Take a social environment $\mathcal{G}$ for which $L \neq \emptyset$ and which satisfies Condition $C'$. Let $\Pi^f$ be the set of paths feasible by domination. Then, $\mathcal{G}$ has a non-empty valued largest CSSB.*

**Remark 3** In this remark we point out a few conceptual drawbacks of the feasibility restriction of Xue which we wanted to remove in our feasibility restriction. First, Xue merely requires that for a non-singleton path to be feasible, the terminal element should indirectly dominate the first element of the path but *not necessarily along the path*. Secondly, he does not impose this restriction on the entire set of paths but only on those in a non-empty valued CSSB. However, even with Xue's restriction it can be shown that for every environment we studied in Theorem 2, the LCS is contained in some Xue-stable set. However, the strict inclusion would not be true.

**Remark 4** Theorem 2 shows that CSSB cannot refine the LCS in an important class of environments. However, in such environments the LCS itself suffers from a shortcoming. For the class of simple games with a finite number of outcomes, the LCS contains elements that are stable owing to *only incredible coalitional deviations* (Bhattacharya (2002)).

An immediate question is that whether we can obtain a result like Corollary 1 if we replace the Condition $C'$ by Condition C. In other words, if the set of feasible paths is restricted to be the set of paths set of paths feasible by domination, then can we ensure that a non-empty valued CSSB exists for every social environment? The answer is negative.

**Theorem 3** *There exists a social environment $\mathcal{G}$ for which $L \neq \emptyset$, which satisfies Condition C and for which we take $\Pi^f$ to be the set of paths feasible by domination. However, the unique CSSB for $\mathcal{G}$ is not non-empty valued.*

**Proof:** Take the following environment (somewhat similar to Figure 4 in Xue (1998)) $N = \{1, 2\}$. $Z = \{a, b, c\}$. The players' preferences on $Z$ are given as follows.

$$c \prec_1 a \prec_1 b \text{ and } b \prec_2 a \prec_2 c.$$

The effectivity relations are given as follows.

$a \underset{\{1\}}{\rightarrow} b,\ a \underset{\{1,2\}}{\rightarrow} b,\ a \underset{\{2\}}{\rightarrow} c,\ a \underset{\{1,2\}}{\rightarrow} c;$
$b \underset{\{1\}}{\rightarrow} a,\ b \underset{\{1,2\}}{\rightarrow} a,\ b \underset{\{1,2\}}{\rightarrow} c;$
$c \underset{\{2\}}{\rightarrow} a,\ c \underset{\{1,2\}}{\rightarrow} a,\ c \underset{\{1,2\}}{\rightarrow} b.$
This environment satisfies C.

Note that $x \prec_1 b$ for every $x \in Z \setminus \{b\}$. Since $1 \in S$ for every $(S, x) \in 2^N \times Z$ such that $b \underset{S}{\rightarrow} x$, the only feasible path from $b$ is $\{b\}$ only. Similarly, note that $x \prec_2 c$ for every $x \in Z \setminus \{c\}$. Again, since $2 \in S$ for every $(S, x) \in 2^N \times Z$ such that $c \underset{S}{\rightarrow} x$, the only feasible path from $c$ is $\{c\}$. Now, by using a similar reasoning as above, it can be checked that there are three feasible paths in $\Pi_a^f$, namely, $\{a\}$, $\{a, \{1\}, b\}$ and $\{a, \{2\}, c\}$. Let $\sigma$ be any CSSB for this environment. By definition of a CSSB, $\sigma(b) = \{b\}$ and $\sigma(c) = \{c\}$. Now take, for example, the path $\{a, \{1\}, b\}$ from $a$. Consider the outcome $a$ on this path. Then, $\{a,\ \{2\},\ c\} \in \Pi_a^f$, $\sigma(c) \neq \emptyset$ and $b \prec_2 c$. Therefore, the path $\{a, \{1\}, b\}$ cannot be in $\sigma(a)$. By using a similar reasoning for the other two paths, it can be shown that $\sigma(a) = \emptyset$. ■

# 4 Conservative Stability with Weak Dominance

Recall Remark 2 above where we noted that the underlying idea behind CSSB is that if a coalition $S$ makes a feasible deviation from an outcome $x$ to an outcome $y$ then, being farsighted, the players in $S$ examine all the "credible" paths that originate from $y$. A feasible path is not "credible" if and only if some coalition can feasibly deviate from it to another outcome and if its members are strictly better-off at *every* credible path originating from that outcome. Now, we noted that this may be too demanding and this may be the reason behind the inclusiveness of CSSB. In particular, note that under Condition C, starting from any path with a terminal element $a$ (say) we can reach $a$ again and since an outcome cannot be strictly dominated by itself, the result follows. Below we relax the requirement of domination in the following way: a feasible path is not "credible" if and only if some coalition can feasibly deviate from it to another outcome and if its members are *weakly* better-off at every credible path originating from that outcome and *strictly better-off at least for one path.* Below we express this idea formally.

For some coalition $S$ and $a,\ b \in Z$, if $a \preceq_i b$ for all $i \in S$ then that is written as $a \preceq_S b$. Similarly, for paths $\alpha$ and $\beta$, if $t(\alpha) \preceq_S t(\beta)$ then it is also written as $\alpha \preceq_S \beta$.

Then, the definition of the set of stable paths is altered as follows.

Definition 9 *Suppose $\Pi^f$ is given as the set of feasible paths for environment $\mathcal{G}$. An SB $\sigma$ is:*
*(i) A conservative internally stable weak predictor (CISWP) for $\mathcal{G}$ if for all $a \in Z$, $\alpha \in \sigma(a) \Longrightarrow$ there do not exist $S \subseteq N$, $b \in \alpha$ and $c \in Z$ such that $\{b,\ S,\ c\} \in \Pi^f_b$, $\sigma(c) \neq \emptyset$ and $\alpha \preceq_S \beta$ for all $\beta \in \sigma(c)$ with $\alpha \prec_S \beta$ for* at least *one* $\beta \in \sigma(c)$.
*(ii) A conservative externally stable weak predictor (CESWP) for $\mathcal{G}$ if for all $a \in Z$, $\alpha \in \Pi^f_a \setminus \sigma(a) \Longrightarrow$ there exist $S \subseteq N$, $b \in \alpha$ and $c \in Z$ such that $\{b,\ S,\ c\} \in \Pi^f_b$, $\sigma(c) \neq \emptyset$ and $\alpha \preceq_S \beta$ for all $\beta \in \sigma(c)$ with $\alpha \prec_S \beta$ for* at least *one* $\beta \in \sigma(c)$.

*An SB $\sigma$ is a conservative stable weak predictor (CSWP) for $\mathcal{G}$ if it is both a CISWP and a CESWP.*

Notice that an inclusive result like Theorem 1 will no more hold in general with this stability notion. Also, as we should expect, CSWP can give more precise prediction than CSSB. In Theorem 4 below we show that a non-empty valued CSWP refines at least one (and thus the largest) non-empty valued CSSB in a precise sense.

**Theorem 4** *Suppose $\sigma$ is a non-empty valued CSWP for an environment $\mathcal{G}$. Then there exists a non-empty valued CSSB $\sigma'$ such that for every $a \in Z$, $\sigma(a) \subseteq \sigma'(a)$.*

**Proof:** Note that since $\sigma$ is a CSWP, it is a CISWP. Therefore, by the definitions of CISWP (Definition 9) and that of a CISSB (Definition 3), $\sigma$ is a non-empty valued CISSB. Then, by Lemma 2.2, there exists a CSSB $\sigma'$ for $\mathcal{G}$ such that for every $a \in Z$, $\sigma(a) \subseteq \sigma'(a)$. ■

Below we study some results concerning the existence of the above solution concept for *proper* voting games with the assumption that every path is feasible. Recall that a voting game (see Definition 5 above) is said to be proper if $T \in B$ implies $(N \setminus T) \notin B$.

**Theorem 5** *Suppose $\mathcal{G}$ is a proper voting game.*
*(i) Suppose $Z$ is finite. Also suppose that for every pair $a, b \in Z$, with $a \neq b$ and every winning coalition $S$, either $a \prec_S b$ or $b \prec_S a$.*[10] *Then $\mathcal{G}$ has a non-empty valued CSWP.*
*(ii) Suppose $Z$ is infinite. Then a non-empty valued CSWP may not exist.*
*(iii)Even if $Z$ is finite, but the myopic dominance relation, $\prec$, is not total on $Z$,*

[10]This is true, for example, for majority voting situations with an odd number of players when each player has strict preferences.

*then a non-empty valued CSWP may not exist.*

**Proof:** (i) Construct the following sets recursively.

Step 1: Let $Z_0 = Z$. Pick, if possible, $x_0 \in Z_0$ such that there exists a winning coalition $S$ for which $x_0 \preceq_S z$ for every $z \in Z_0$ and $x_0 \prec_S z$ for at least one $z \in Z_0$. Construct $Z_1 = Z_0 \setminus \{x_0\}$.

Step $m+1$: Take $Z_m$. Pick, if possible, $x_m \in Z_m$ such that there exists a winning coalition $S$ for which $x_m \preceq_S z$ for every $z \in Z_m$ and $x_m \prec_S z$ for at least one $z \in Z_m$. Construct $Z_{m+1} = Z_m \setminus \{x_m\}$.

Since $Z$ is finite there exists a $\bar{Z} \subseteq Z$ such that $Z_r = Z_{r+1} = \cdots = \bar{Z}$.

Consider an SB $\sigma$ as follows:

$$\text{for every } a \in Z,\ \sigma(a) = \{\alpha \in \Pi_a |\ t(\alpha) \in \bar{Z}\}.$$

By our construction of $\bar{Z}$ and the assumption that for every pair $a, b \in Z$, with $a \neq b$ and every winning coalition $S$, either $a \prec_S b$ or $b \prec_S a$, it is obvious that $\sigma(.)$, as specified above, is a CSWP.

(ii) Let $\mathcal{G}$ be the majority voting situation (i.e., a proper voting game where every majority coalition is winning) with $Z = I$, the set of positive integers. For each player $i$, the preference ordering over $Z$ is as follows: $a \prec_i b$ if and only if $a < b$. (This example is the one used by Rubinstein (1980) for showing the possible emptiness of the stability set with infinitely many outcomes).

Although it is obvious that no non-empty valued CSWP exists for this situation, we give a short proof for completeness. Suppose not and let $\sigma$ be a non-empty valued CSWP for this environment. Note that if for every $a \in Z$, the set of terminal outcomes for the paths in $\sigma(a)$ is finite, then $\sigma$ cannot be a CESWP and the contradiction is immediate. Therefore, suppose $a$ be an outcome for which the set of terminal outcomes for the paths in $\sigma(a)$ is infinite. Let $b \in Z$ be such that $b = t(\alpha)$ for some $\alpha \in \sigma(a)$ and for every $c \in Z$ such that $c = t(\alpha)$ for some

$\alpha \in \sigma(a)$, $b < c$. Since $b = t(\alpha)$ for some $\alpha \in \sigma(a)$, the singleton path $\{b\}$ must be in $\sigma(b)$. However, consider the move from $b$ to $a$ by the whole set of players $N$. Since, by the definition of $b$, every player in $N$ strictly prefers every $c \in Z$ such that $c = t(\alpha)$ for some $\alpha \in \sigma(a)$, this violates the assumption that $\sigma$ is a CISWP.

(iii) Consider the following majority rule voting game: $N = \{1, 2, 3\}$, $Z = \{a, b, c\}$. The players' preferences over $Z$ are the following:

| 1 | 2 | 3 |
|---|---|---|
| $b$ | $c$ | $a \sim c$ |
| $a \sim c$ | $b$ | $b$ |
| | $a$ | |

(By the entry $a \sim c$ under column $i$ in the table above we imply that the player $i$ is indifferent between outcome $a$ and outcome $c$.)
Let $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{1, 3\}$.
Suppose $\sigma$ is a CSWP for this environment. We show below that it cannot be non-empty valued. We take the following steps.

Step 1: First we note the following fact. For some $x \in Z$, $y = t(\alpha)$ for some $\alpha \in \sigma(x)$ if and only if the singleton path $\{y\} \in \sigma(y)$. The proof of this fact is exactly similar to that of Lemma 3.5 in Xue (1998) and additionally uses the fact that for a majority rule voting game, any majority coalition can enforce any social state from any other social state.

Step 2: Since for no majority coalition $S$ and $x \in Z$, is it true that $c \prec_S x$, $\{c\} \in \sigma(c)$. Now suppose $\{b\} \notin \sigma(b)$. We show that this leads to a contradiction. With the assumption $\{b\} \notin \sigma(b)$, since $a \not\prec_S c$ for any majority coalition $S$, $\{a\} \in \sigma(a)$. Therefore, by Step 1, for every $x \in Z$, every path starting with $x$ and ending with $a$ must be in $\sigma(x)$. Now consider the path $\{b\}$. Since, for no majority $S$ is it true that $b \preceq_S a$, $\{b\}$ must be in $\sigma(b)$. This is the desired contradiction.

Step 3: So, $\{b\} \in \sigma(b)$. However, by considering the path $\{b, S_2, c\}$ we find that $\{a\}$ must be in $\sigma(a)$ as, otherwise, $\{b\} \notin \sigma(b)$. This implies every possible path is in $\sigma(b)$. But then consider the path $\{a, S_1, b\}$. By this deviation it is seen that $\{a\} \notin \sigma(a)$. This leads to a contradiction. ■

Therefore, although CSWP can refine CSSB, it does not have nice existence properties.

# 5 Conclusion

In this work first we tried to explore some properties, especially in regard to the predictive power, of CSSB in situations with perfect foresight as we considered the idea behind this solution to be intuitively quite nice for such situations. However, we found that the predictive power of this solution is somewhat disappointing; it may be too inclusive. Then we proposed a reasonable refinement of this idea. However, then we find that this modified solution fail to be non-empty valued in reasonably common situations. Thus, to conclude, the solution idea, while intuitively nice, may not be quite useful.

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# 6 Appendix: Social Networks (Jackson and Wolinsky (1996)) in the Present Framework

Let $N$ be the finite set of players and let $g^N$ be the set of all doubleton subsets of $N$. A *bilateral network* $g$ is a subset of $g^N$. Then, $Z = \{a |\ a \subseteq g^N\}$. Given a non-empty network $g \in Z$, an element $\{i, j\} \in g$ (where $i, j \in N$) is a link between players $i$ and $j$ in the network $g$. A value function $v : Z \mapsto \mathbf{R}$ assigns a real value to every network and the set of all value functions are denoted by $V$. Given a value function $v \in V$, an allocation rule $Y : Z \times V \mapsto \mathbf{R}^N$ allocates the value of a network to the players. Given a value function $v \in V$, an allocation rule $Y : Z \times V \mapsto \mathbf{R}^N$ induces a preference ordering $\preceq_i (v, Y)$ for each $i \in N$ on $Z$ given as follows:

$$\text{for } a, b \in Z,\ a \preceq_i (v, Y) b \text{ if and only if } Y_i(a, v) \leq Y_i(b, v) \text{ and}$$

$$\text{for } a, b \in Z,\ a \prec_i (v, Y) b \text{ if and only if } Y_i(a, v) < Y_i(b, v).$$

Given a profile of players' preferences $\{\preceq_i\}_{i\in N}$, we assume that it has been induced by some underlying value function and allocation rule.

The coalitional effectivity relation is specified as follows.

Definition 10 (Jackson and van den Nouweland (2005)) *For $a, b \in Z$, and $S \subseteq N$, $a \underset{S}{\rightarrow} b$ if and only if*
*(i) a link $\{i, j\} \in b \setminus a$ implies that $\{i, j\} \subseteq S$ and*
*(ii) a link $\{i, j\} \in a \setminus b$ implies that $\{i, j\} \cap S \neq \emptyset$.*