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Conditional and Unconditional Multiple Equilibria with Strategic Complementarities

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Abstract. We take a general model of externalities matching the Cooper & John framework with identical agents. If each agent's payoff depends on a parameter interpreted as the favourableness of the environment, we explore how the number of Nash equilibria varies with this parameter, especially in the cases in which the reaction curves are either concave or convex. In many examples the environmental conditions are themselves endogenous because either market or regulatory forces interact with agents' Nash equilibrium actions. This gives the idea of a simultaneous equilibrium in the environment and players' symmetric actions. We analyse how this generalised equilibrium behaves as a function of some additional parameters conditioning the environmental response to players actions. We show that generally there is a fold bifurcation in these equilibria.

We illustrate the principles with two examples from industrial economics (cost spillovers between firms and demand spillovers under imperfect competition).

Keywords: cost spillovers, Nash and Market equilibrium, coordination failure

JEL Nos: C62, C72, D43, D62

1. Introduction

There is a huge literature on models in which the setting naturally involves strategic complementarity and non-cooperative behaviour. This originates with the synthesising paradigm of Cooper & John (1988) and has applications in both macroeconomic (King and Wolman, 2004) and microeconomic (Echenique and Sabarwal, 2003) areas. There are various key results in this area: typically there will be multiple non-cooperative equilibria, in most applications some of these are more socially desirable than others. It follows that the system envisaged can get stuck at an undesirable noncooperative equilibrium. In Cooper and John and the subsequent applications often the environment is defined by some exogenous parameter θ-for any given value of θ there are likely to be these multiple conditional equilibria (conditional on a fixed value of θ). Usually θ is interpreted as reflecting the favourableness of the environment to each player. So higher values of θ induce all the players to choose higher levels of actions, given what the other players choose, and then, with strategic complementarity, multiplier effects will arise, with the aggregate response exceeding each individual response (Cooper & John,(1988)).

However there are unanswered questions. How does the configuration of conditional noncooperative equilibria vary with θ, for example do the number of non-cooperative equilibria systematically vary with θ and how can we characterise the different equilibria that arise beyond saying that they differ in their social desirability? Also is θ really exogenous or are there forces which will interact with the noncooperative behaviour of players to induce changes in θ? If so then we would have an unconditional joint (θ,action) equilibrium. In many applications this is the case, eg in market situations θ may be the price and players actions are demands or supplies in which case the requirements of market equilibrium naturally induce an interdependence between players actions and θ. Alternatively θ may be a regulatory parameter set in response to players actions. The questions are important: first if there are multiple equilibria we cannot predict the behaviour of the system in terms of either its long run position or its response to parameter changes. Secondly the appropriate policy control measures can vary with the equilibrium that we are trying to attain. But if there is another layer of adjustment through θ then endogenous variation in θ could eliminate the indeterminacy eg if the only unconditional equilibrium values of θ generate a unique non-cooperative conditional equilibrium in actions. If this happens then multiplicity of equilibria with strategic complementarity is not fundamental-embedding the conditional equilibria in a more general equilibrium model eliminates the indeterminacy.

Cooper & John focus on symmetric Nash equilibria in actions for a given θ, ie with mutual best responses all players choose the same action x given θ. Vives (2005) also uses this approach. Given our assumptions (which mirror but add to Cooper & John) all Nash equilibria will also be symmetric and will be characterised by a common action x.

We impose some additional structure on players best response functions (in particular taking them to be concave or convex in the average action of rivals) which allows us to characterise the number and nature of noncooperative equilibria for a given θ. However variations in θ can lead to qualitative shifts in the equilibrium configurations, we therefore examine these shifts. Three aspects of the externality matter: whether it reduces or

increases a player's action (a beneficial or detrimental externality); whether it increases or reduces the marginal return to a player's action (complementary or substitutable externalities) and finally whether the curvature of the reaction function is positive (convex reaction curve) or negative (concave reaction curve). These correspond to the signs of different order derivatives of the players' payoff functions. The qualitative shifts occur because the Nash equilibrium correspondence (giving the equilibrium non-cooperative action x as a function of θ) is multivalued for intermediate θ, due to the multiplicity of Nash equilibria. For example, with three Nash equilibria existing for a certain range of intermediate θ, agents can end up at any of the three.

Next we introduce a broader idea of equilibrium in which θ and the actions of the players are simultaneously and endogenously determined. We call this an unconditional $\theta - x$ equilibrium. In this broader equilibrium not only are players in Nash equilibrium given θ but also the Nash equilibrium action x and θ are linked through an additional general function $\theta = \Theta(x)$. For example θ could reflect external market or regulatory conditions and the value of θ adjusts with the action x of the agents. We analyse the nature of $\theta - x$ equilibria for different types of positive spillovers and different forms of $\Theta(x)$. In fact, for the sake of transparency, we take $\Theta(x) = A - Bx$ (but the principles are general). We focus on how the equilibrium manifold $\{\theta, x\} = H(A, B)$ varies with the intercept and slope of $\Theta(x)$. As $\Theta(x)$ varies we encounter critical equilibria- marginal changes in the function $\Theta(x)$ can lead to qualitative changes in the equilibrium configurations, e.g. changing the finite number of $\theta - x$ equilibria between one, two and three depending on the nature of the spillovers. This is important, it means that we can explain sudden occasional structural shifts in individual actions as a response to just marginal environmental changes. On the other hand it also means that the additional equilibrating mechanism through θ will not generally help in improving determinacy of the system. So the multiplicity of equilibrium which arises with strategic complementarity is fundamental in the sytems we study.

We use the Cooper & John framework to illustrate the principles: I identical agents

with an action externality between them, increasing and concave (in own action) payoff functions, multiplier effects arising from a common factor θ and add to it a downward sloping $\Theta(x)$ function. In the final section we analyse some examples: first a case of price taking firms with a cost externality (here θ is identified with the real output price). Second a case of imperfectly competitive firms with product differentiation where the spillover comes through the demand for a firm's output. An interesting feature of this example is that the reaction function will not necessarily be monotonic so that in this case asymmetric Nash equilibria may also arise.

2. Conditional Nash Equilibria in games with strategic complementarities.

Cooper and John,(1988), have a fixed number I of agents, with payoff functions given by

$$V(x_i, k_i, \theta) \qquad i = 1, .., I$$

where $x_i \geq 0$ denotes the action of each individual player, $i = 1, .., I$. k_i is an aggregate index denoting the average action of all the other players

$$k_i = \Sigma_{j \neq i} x_j / (I - 1) \qquad i = 1, .., I$$

and θ is a parameter common to all players' payoff functions. The functions $V(\cdot)$ exhibit the following properties:

A1 (i) $V_x(x, k, \theta) > 0$, (ii) $V_{xx}(x, k, \theta) < 0$, (iii) $V_{xk}(x, k, \theta) > 0$, (iv) $V_{x\theta}(x, k, \theta) > 0$

Assumptions (i) and (ii) simply mean that the functions $V(\cdot)$ are increasing and concave in the agent's own choice; assumption (iii) implies that the marginal payoff of any individual i increases with k_i. In this context, the behavior of each agent will depend on how he/she expects every other agent will act on average. Assumption (iv) implies that larger values of θ tend to increase each individual agent's action given the actions of others.

Since payoff functions are identical, the reaction functions are identical as well for all agents and are defined by[1]:

$$x^*(k_i, \theta) = \arg\max_{x_i}\{V(x_i, k_i, \theta) \,|x_i \geq 0\} \qquad i = 1, .., I$$

Given the assumptions on the $V(\cdot)$ functions, the game exhibits strategic complementarities. The reaction curves (RCs henceforth) of the players are positively sloped, with slope (at interior best responses) given by

$$\frac{\partial x^*(k_i, \theta)}{\partial k_i} = -\frac{V_{xk}}{V_{xx}} > 0$$

and since $V_{x\theta} > 0$, the RCs shift in a direction of increasing x_i as θ increases given k_i:

$$\frac{\partial x^*(k_i, \theta)}{\partial \theta} = -\frac{V_{x\theta}}{V_{xx}} > 0$$

Given these assumptions on the payoff functions the Nash equilibria must be symmetric (see Appendix A1). Cooper and John,(1988), prove that the presence of strategic complementarity is necessary for multiple symmetric equilibria to arise (see Figure 1). If the RCs always had nonpositive slope there could be at most one Nash equilibrium.

[1] The best response x_i of i is

$$\begin{aligned} x_i \text{ solving } V_x(x_i, k_i, \theta) &= 0 \text{ if } V_x(0, k_i, \theta) \geq 0 \\ x_i &= 0 \text{ if } V_x(0, k_i, \theta) < 0 \end{aligned}$$

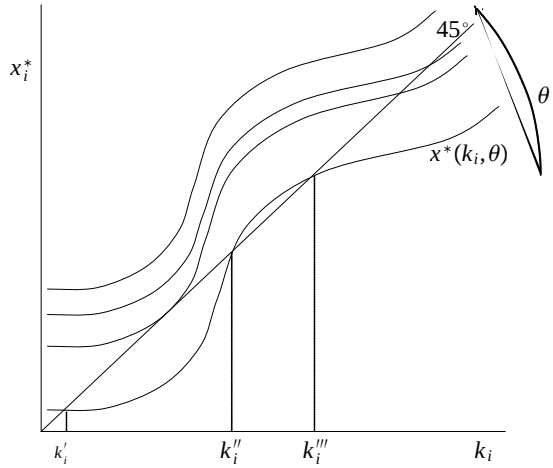

Figure 1: Multiple NE with strategic complementarities

Figure 1 shows some alternative possible reaction functions with strategic complementarity and different equilibrium patterns. It follows that to characterise the Nash equilibria here we need to impose more structure on the payoff functions.

Also Figure 1 is drawn for a given value of parameter θ. Since $V_{x\theta} > 0$, changes in θ will cause the reaction functions to shift in a direction of increasing x_i as θ rises with k_i constant. Thus as θ changes, the number of equilibria may change. In the extreme there will be no Nash equilibria if θ is such that the reaction function is always on one side of the 45^0 line.

We add some fairly weak assumptions

A2 $V(0, k_i, \theta) = 0, V(x_i, k_i, \theta) > 0$ for $x_i > 0, k_i \geq 0$

$$\frac{\partial V(0,0,\theta)}{\partial x_i} \text{ is finite for any finite } \theta$$

i.e., without the presence of the other agents, agent i has a finite positive marginal payoff when his/her action is zero.

The curvature of the reaction curve is[2]

$$\frac{\partial^2 x^*}{\partial k_i^2} = \frac{2V_{xx}V_{xk}V_{xkx} - V_{xx}^2 V_{xkk} - V_{xk}^2 V_{xxx}}{V_{xx}^3} \tag{1}$$

This is of ambiguous sign. We call the reaction function concave if $\partial^2 x^*/\partial k_i^2 < 0$ everywhere and similarly convex if $\partial^2 x^*/\partial k_i^2 > 0$ everywhere.

What are the possible NE at fixed values of the parameter θ?

To define the switch between activity and inactivity, it is useful to define an intermediate θ^* such that

$$\theta^* \text{ solves } V_x\left(0, 0, \theta^*\right) = 0$$

and the reaction curve for $\theta = \theta^*$ goes through the origin. At θ^* if the other agents are choosing $x_j = 0$ for $j \neq i$ then agent i wishes to choose $x_i = 0$. We assume that θ^* exists and is finite. For $\theta < \theta^*$ (the environment is less favourable), agent i will still stay inactive if the others are inactive (there is a value of $\overline{k}$ strictly positive at which $V_x\left(0, \overline{k}, \theta\right) = 0$ so that i would be inactive at any $k \leq \overline{k}$). However, for more favourable environments ($\theta > \theta^*$), the intercept of the reaction curve is positive and individual i takes a positive action even if all others have zero action.

2.1. Characterising NE With Concave Reaction Functions. Here the reaction curve for agent i is concave to the 45^0 line and shifts with θ as shown in Figure 2.

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[2]

$$\begin{aligned}
\frac{\partial}{\partial k_i}\frac{\partial x_i^*}{\partial k_i} &= -\frac{V_{xkx}\left(\partial x_i^*/\partial k_i\right) + V_{xkk}}{V_{xx}} + V_{xk}\frac{V_{xxx}\left(\partial x_i^*/\partial k_i\right) + V_{xxk}}{V_{xx}^2} \\
&= \frac{V_{xx}V_{xk}V_{xkx} - V_{xx}^2 V_{xkk} - V_{xk}^2 V_{xxx} + V_{xk}V_{xxk}V_{xx}}{V_{xx}^3} \\
&= \frac{2V_{xx}V_{xk}V_{xkx} - V_{xx}^2 V_{xkk} - V_{xk}^2 V_{xxx}}{V_{xx}^3}
\end{aligned}$$

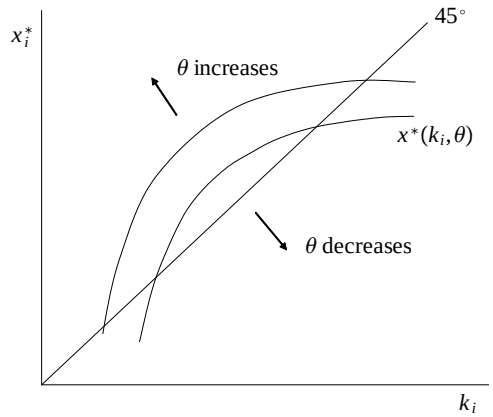

Figure 2: Concave reaction functions

To see the nature of NE in this case we make an additional assumption on the payoff functions:

A3 for any $\theta \geq \theta^*$ there is a point (x, k) with $k > x$ such that $x = x^*(k, \theta)$.

This condition implies that there must exist a point on the reaction curve below the 45^0 line for any $\theta \geq \theta^*$. We can then characterise the NE with concave reaction functions.

1. First suppose that $\partial x^*(0, \theta^*)/\partial k_i > 1$; this means that near the origin the reaction function must be on the upper side of the 45^0 line at $\theta = \theta^*$.

(i) When $\theta = \theta^*$ there are two NE at θ^*, one with $x_i = k_i = 0$ and the other with positive actions, as in Figure 3.

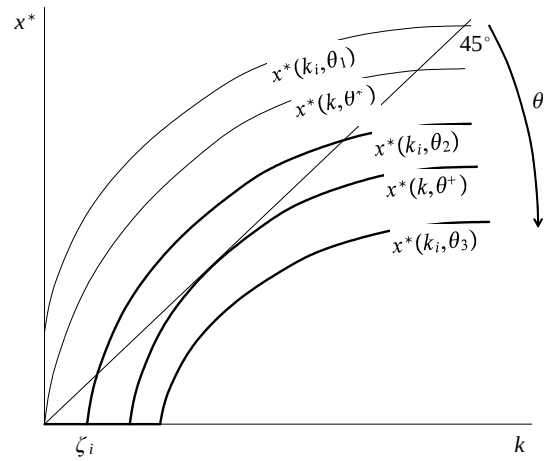

Figure 3: NE at fixed values of θ,
$\partial x^* (0, \theta^*) / \partial k_i > 1$

(ii) When $\partial x^*(0, \theta^*)/\partial k_i > 1$, we can define θ^+ and x_i^+ as the solution to

$$\frac{\partial x^*}{\partial k_i} = 1 \Rightarrow V_{xk}\left(x_i, k_i, \theta^+\right) = -V_{xx}\left(x_i, k_i, \theta^+\right)$$
$$V_x\left(x_i, k_i, \theta^+\right) = 0 \qquad i = 1, .., I$$

This defines a point on the 45^0 line at which there is a reaction curve corresponding to a parameter θ^+ just tangent to the 45^0 line. Parameter θ^+ must always exist (see Appendix A2).

Then when $\theta = \theta^+$ there is a Nash equilibrium on the 45^0 line with the reaction function of firm i just tangential to the 45^0 line. There is also another Nash equilibrium at $x_i = k_i = 0$. This is because $i's$ best response is zero for any k_i below the horizontal intercept ς_i, as illustrated in Figure 3.

(iii) When $\theta^+ < \theta < \theta^*$ we have three NE, one at 0 and the other two with positive agents' choices, as shown in Figure 3 for $\theta = \theta_2$.

(iv) If $\theta < \theta^+$ there is a unique Nash equilibrium at 0, as in Figure 3 when $\theta = \theta_3$.

(v) if $\theta > \theta^*$ there is a unique Nash equilibrium with positive choices, (as in Figure 3 for $\theta = \theta_1$), since we have assumed that the RC must cross the 45^0 line.

2. If $\partial x^*(0,\theta^*)/\partial k_i \leq 1$, then the reaction curve is always below the 45^0 line. In this scenario, there is a unique Nash equilibrium at θ^* with choices equal to 0, as Figure 4 illustrates.

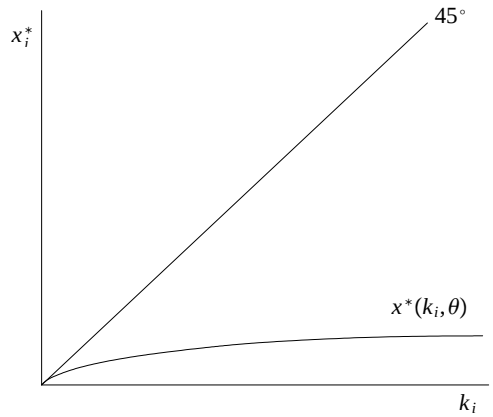

Figure 4: $\theta = \theta^*$, $\partial x^*(0,\theta^*)/\partial k_i \leq 1$

As θ increases above θ^* the reaction curve shifts vertically upwards, since, by assumption, as parameter θ increases, the RCs can never cross. Hence the intermediate parameter θ^+ does not exist if $\partial x^*(0,\theta^*)/\partial k_i < 1$, but if $\partial x^*(0,\theta^*)/\partial k_i = 1$ then $\theta^+ = \theta^*$.

When $\theta > \theta^*$ there is a unique Nash equilibrium with positive choices similar to Figure 3 for the case of $\theta = \theta_1$. On the other hand, when $\theta < \theta^*$ there is a unique Nash equilibrium at the origin similar to Figure 3 for the case of $\theta = \theta_3$.

Thus, with concave reaction functions, depending on θ the number of Nash equilibria are:

	$\frac{\partial x^*(0,\theta^*)}{\partial k_i} > 1$		$\frac{\partial x^*(0,\theta^*)}{\partial k_i} \leq 1$
$\theta > \theta^*$	$1, x_i > 0$	$\theta > \theta^*$	$1, x_i > 0$
$\theta = \theta^*$	$2, x_i > 0, x_i = 0$	$\theta = \theta^*$	$1, x_i = 0$
$\theta^* > \theta > \theta^+$	$3, x_i^1, x_i^2 > 0, x_i^3 = 0$	$\theta < \theta^*$	$1, x_i = 0$
$\theta = \theta^+$	$2, x_i^1 > 0, x_i^2 = 0$		
$\theta^+ > \theta$	$1, x_i = 0$		

Table 1: Number of NE with Concave RCs, $\theta^* > \theta^+$

2.2. Characterising NE with Convex Reaction Functions. If (1) is positive at all x, k, then the reaction curve for firm i is convex to the 45^0 line and shifts with θ as shown in Figure 5.

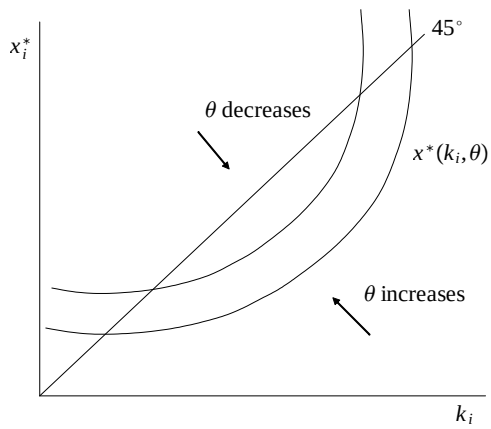

Figure 5: Convex reaction functions

Again there are two cases depending on whether the slope of the reaction curve at the origin (at θ^*) is greater or less than unity. In either case we assume that eventually for high k there is a point on the reaction curve above the 45^0 line.

A4 For any θ there is a $k < x$ such that $x = x^*(k, \theta)$

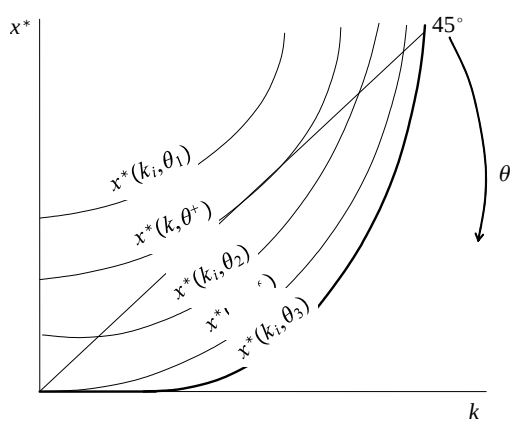

Figure 6: NE at fixed values of θ,
$\partial x^* (0, \theta^*) / \partial k_i < 1$

1. If the slope of the reaction curve at the origin is less than unity ($\partial x^* (0, \theta^*) / \partial k < 1$) the reaction curve initially lies below the 45^0 line. Then at θ^* there are two Nash equilibria, one with zero choices and the other with positive choices on the 45^0 line. In this case parameter θ^+ always exists (from an argument similar to that in Appendix A2) and $\theta^* < \theta^+$. At fixed values of parameter θ, the patterns of possible NE (as shown in Figure 6) are:

(*i*) if $\theta > \theta^+$ there is no Nash equilibrium

(*ii*) if $\theta = \theta^+$ there is a unique Nash equilibrium with positive choices

(*iii*) if $\theta^* < \theta < \theta^+$ there are two NE with positive choices

(*iv*) if $\theta < \theta^*$ there are two NE, one with positive choices and the other at 0

2. If the slope of the reaction curve at the origin is greater than or equal to unity at θ^* ($\partial x^* (0, \theta^*) / \partial k \geq 1$) then at θ^* the reaction curve is everywhere above the 45^0 line and at θ^* the unique Nash equilibrium has choices 0. At $\theta > \theta^*$ there is no Nash equilibrium, all agents would have an incentive to continually expand their own choice. At $\theta < \theta^*$ the whole reaction curve must shift downwards so there is a $\overline{k_i} > 0$ at which the best response by i is to choose $x_i = 0$; i drops out of the market whenever $k_i \leq \overline{k_i}$. Then there are two NE one with choices 0 and the other with positive choices.

	$\frac{\partial x^*(0,\theta^*)}{\partial k_i} < 1$		$\frac{\partial x^*(0,\theta^*)}{\partial k_i} \geq 1$
$\theta > \theta^+$	no equilibrium	$\theta > \theta^*$	no equilibrium
$\theta = \theta^+$	$1, x_i > 0$	$\theta = \theta^*$	$1, x_i = 0$
$\theta^+ > \theta > \theta^*$	$2, x_i^1, x_i^2 > 0$	$\theta < \theta^*$	$2, x_i^1 > 0, x_i^2 = 0$
$\theta = \theta^*$	$2, x_i^1 > 0, x_i^2 = 0$		
$\theta^* > \theta$	$2, x_i^1 > 0, x_i^2 = 0$		

Table 2: Number of NE with Convex RCs, $\theta^+ > \theta^*$

2.3. The Conditional Nash Equilibrium Correspondence. The Nash equilibrium correspondence (NEC henceforth) shows how the Nash equilibrium action varies with θ. Figure 3 gives the Nash equilibria for varying θ in the concave reaction curve case, when $\partial x^*(0,\theta^*)/\partial k > 1$. Here starting from a high value θ^* we continuously reduce parameter θ and the RC shifts downwards to the south east. The two positive Nash equilibrium choices get closer together converging to a single value at θ^+. After that further reductions in θ result in the unique Nash equilibrium choices being at the origin. This yields the generic NEC shown in Figures $7-8$. In Figure 7 below θ^+ and above θ^* this is single valued, but between these values, it is a correspondence with three possible choices at each θ.

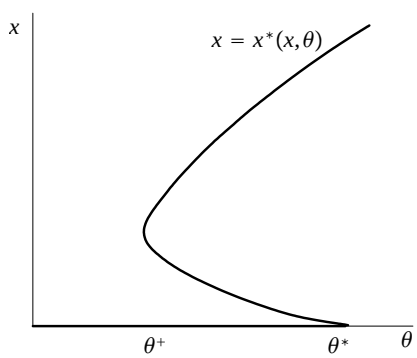

Fig. 7: NEC, $\partial x^*(0,\theta^*)/\partial k > 1$, concave RCs

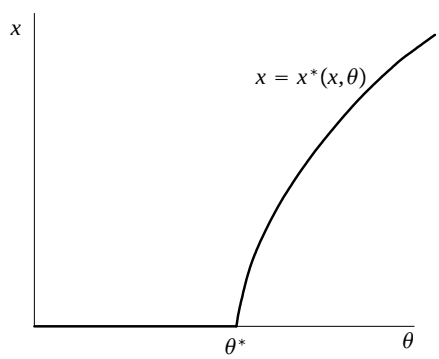

Fig. 8: NEC, $\partial x^*(0,\theta^*)/\partial k \leq 1$, concave RCs

Figure 8 takes the case in which the reaction functions are concave and have a slope smaller than unity at the origin.

It is important to realise that the type of NEC shown here is global and generic depending only on the technological assumptions **A1-A3** that we have made: that is in general no action will be undertaken for a range of low values of parameter θ, then there are multiple possible aggregate choices for a given θ, with a low positive action level decreasing with θ and a high positive action level increasing with the same parameter.

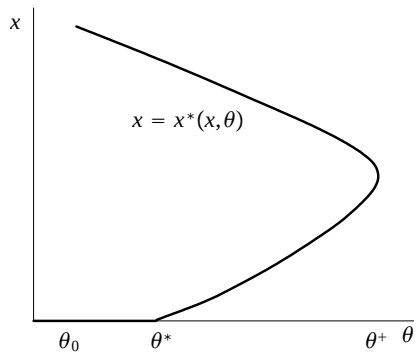

Fig. 9: NEC, $\partial x^*(0,\theta^*)/\partial k < 1$, convex RCs

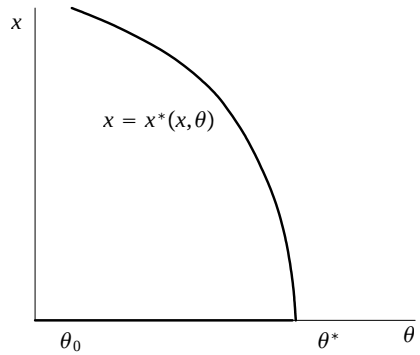

Fig. 10: NEC, $\partial x^*(0,\theta^*)/\partial k \geq 1$, convex RCs

A similar construction for convex RCs gives the NEC for the case of convex reaction functions. In Figure 6 we have shown how the Nash equilibrium choices vary with θ when $\partial x^*(0,\theta^*)/\partial k < 1$. Starting from θ^+ and continuously reducing θ, the RCs shift downwards to the south east, thus yielding the generic form of the NEC as illustrated in Figure 9. When instead $\partial x^*(0,\theta^*)/\partial k \geq 1$, continuous reductions in θ yield the NEC illustrated in Figure 10.

3. Endogenising parameter θ : Unconditional $\theta - x$ equilibrium

In many applications there is also an equilibrating process on θ so that x, θ are simultaneously determined. This requirement is defined by some function $\theta = \Theta(x)$ eg θ may be a regulatory parameter which adjusts the favourableness of the environment in response to the actions x. For instance x reflects the behaviour of the private sector and θ is a control parameter set by a public sector agency. Another example would have firms choosing quantity x while the market auctioneer sets the price θ. We assume that

A5 $\Theta(0) > 0; \Theta(x) \geq 0$ for all x, $\Theta^{'}(x) < 0, \Theta(\overline{x}) = 0$ for some high $\overline{x}$. $x^*(k,0) = 0$ for any k.

Apart from normalisations on θ, the substance of this assumption is that θ decreases with x and becomes zero at some finite $\overline{x}$. Similar analysis follows if θ is increasing in x.

Now a symmetric $\theta - x$ equilibrium requires x, θ to solve both

$$\begin{aligned} x &= x^*(x, \theta) \\ \theta &= \Theta(x) \end{aligned}$$

3.1. Characterising the Number of $\theta - x$ Equilibria and their Regularity. From Figures $7 - 10$ it is clear that $\theta - x$ equilibrium is not generally unique but we can apply the idea of regular and critical equilibria which is associated with study of the equilibrium manifold (Balasko,(1992), Echenique and Sabarwal, (2003)). The NEC has particular generic and global properties. It is well defined but non-monotonic. With concave RCs the NEC generally has an S-shape but what we might call a rotated V-shape in the convex case. These patterns are generic under our technological assumptions. Exactly what shape the segments of the S-shape or rotated V-shape have depend on the precise functional form of the payoff functions, there may be local wiggles within segments. The $\theta - x$ equilibrium combines the NEC with the function $\Theta(x)$. The basic shape of the NEC allows us to find lower bounds to the number of $\theta - x$ equilibria and their regularity.

To get simple clear configurations we take $\Theta(x)$ to be a stylised linear function

$$\Theta(x) = A - Bx \qquad A, B > 0$$

Concave Reaction Curves. Various positions of a linear function are shown, assuming $\Theta(x) = 0$ at a finite x. The result is that for relatively inelastic $\Theta(x)$ there can be either multiple $\theta - x$ equilibria (if A is relatively small) or a unique equilibrium with θ above θ^* (for relatively high values of A). When the RCs are concave but have a slope at the origin below unity it is somewhat simpler: from inspection of Figure 8 any linear function $\Theta(x)$ will cut the NEC at most once, either at a positive or zero level of action.

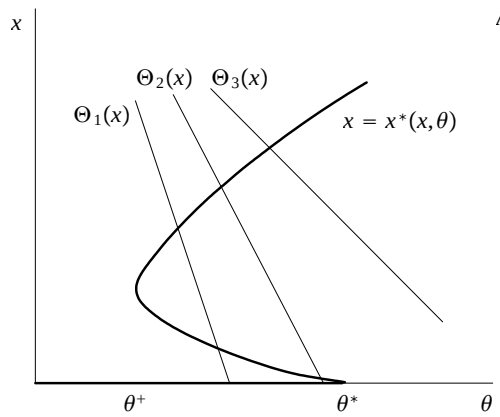

Figure 11: $\theta - x$ equilibria, concave RCs

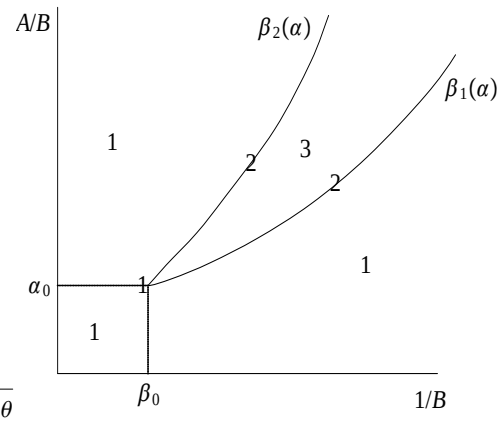

Figure 12: Concave RCs, $\frac{\partial x^*(0,\theta^*)}{\partial k_i} > 1$

We can use the $\Theta(x)$ linearisations to see how the number of $\theta - x$ equilibria varies with the parameters of $\Theta(x)$. In fact Appendix A3 shows that we can find a complete characterisation of the equilibrium set in the $(A/B, 1/B)$ space as shown in Figure 12. There are critical values of the vertical intercept $A/B = \alpha_0$ and the slope $1/B = \beta_0$ defined by the tangent of NEC at θ^*, which divide the space of all $\Theta(x)$ functions into regions with a given number of equilibria. The numbers in Figure 12 refer to the number of equilibria within a region. For example for any α above α_0 there is a line segment between α and θ^* and also a tangent to the NEC passing through α. These two lines define slopes $\beta_1(\alpha), \beta_2(\alpha)$ between which there are three equilibria. If the slope is equal to either $\beta_1(\alpha)$ or $\beta_2(\alpha)$ we lose the third equilibrium. For slopes outside this range or intercepts below α_0 there is a single equilibrium. For a given intercept above α_0 the nature of the equilibrium set suddenly changes discontinuously as the slope increases from a single equilibrium to two equilibria (at $\beta_2(\alpha)$), then to three equilibria (between $\beta_2(\alpha)$ and $\beta_1(\alpha)$), to two equilibria again (at $\beta_1(\alpha)$) and finally to a unique equilibrium (above $\beta_1(\alpha)$); the system is structurally unstable and exhibits a fold bifurcation, see Figs 15-16 below (Strogatz,(1994)).

With concave reaction functions but a slope less than unity at the origin the irregularity of equilibrium does not arise: since $x = x^*(x, \theta)$ is always nondecreasing and $\Theta(x)$ strictly

decreasing, there is always at most one equilibrium.

Convex Reaction Curves. If the reaction functions are convex Figure A2 of Appendix A4 shows that the tangent at $\theta = 0$ with its slope β_0 and intercept α_0 divide the space of $\Theta(x)$ functions into areas with different equilibrium configurations. At α_0, β_0 there is a unique equilibrium at $\theta = 0$ and α_0. For intercepts α of $\Theta(x)$ above α_0 the tangent to NEC $\beta(\alpha)$ defines another unique equilibrium. For slopes of $\Theta(x)$ that are greater than $\beta(\alpha)$ there are two equilibria while for slopes less than $\beta(\alpha)$ there are no equilibria. For intercepts of $\Theta(x)$ below α_0 for any slope there is a unique equilibrium which may involve $x > 0$ or inactivity. Figure 13 shows the $\theta - x$ equilibrium and Figure 14 shows how the number of equilibria varies with the parameters of $\Theta(x)$.

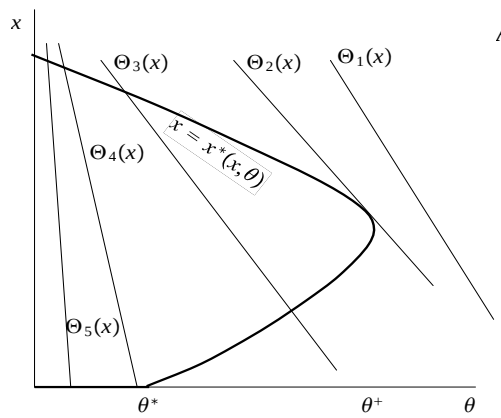

Figure 13: Convex RCs, $\frac{\partial x^*(0,\theta^*)}{\partial k_i} < 1$

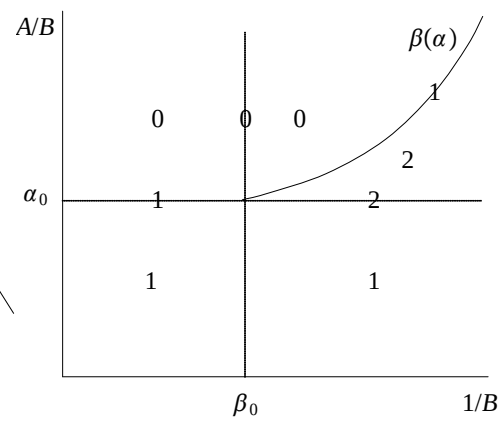

Figure 14: $\theta - x$ equilibrium, Convex RCs

Again the nature of the equilibrium set shifts discontinuously with the slope and intercept of $\Theta(x)$. For example take a fixed slope above β_0 and gradually increase the intercept from an initial value below α_0. First there is a unique equilibrium with inactivity which moves continuously to a unique equilibrium with positive activity. But when the intercept moves past α_0 there are suddenly two equilibria. This pattern of two equilibria increases until the curve $\beta(\alpha)$ is reached at which point there is only a single equilibrium. With further increases in the intercept we lose even this equilibrium.

The Significance of these Results. When there are multiple equilibria, comparative statics are generally ambiguous unless a movement from any equilibrium in the original set gives directions of change which are identical for all equilibria in the new equilibrium set. For example in the original position suppose E_1, E_2, E_3 are all $\theta - x$ equilibria; in the new situation $E_1^{'}, E_2^{'}, E_3^{'}$ are equilibria. Comparative statics are unambiguous (strong) only if a movement from any of E_1, E_2, E_3 to any of $E_1^{'}, E_2^{'}, E_3^{'}$ gives the same directions of change. If θ is treated as parametric as in Figures 2-6 with convex or concave reaction curves, the comparative statics are ambiguous (movement starting from a high level equilibrium is not in the same direction as movement starting from a low level equilibrium). Comparative statics with respect to $\Theta(x)$ functions are also ambiguous when the reaction curves are convex or concave. For example in Figure 11 within the region with three equilibria, falls in B lead to a fall in x (possibly discontinuously) so long as we start from a high action equilibrium, whether the move is to a high or a low action equilibrium, but the effect on θ is ambiguous. Starting instead from a low action equilibrium, the fall in B leads to a rise in x and fall in θ, whether the movement is to a high or low action new equilibrium. Hence the nature of the comparative static effects depends on the starting point. Once B reaches the point at which one of the equilibria is at θ^+ the two positive action equilibria merge together and vanish for further falls in B. Analogous arguments apply with convex reaction curves in the case of Figure 13. Our model violates the sufficient condition for strong comparative statics in Echenique and Sabarwal (2003). As in their model comparative statics are problematic at critical equilibria: for some directions of change, even locally equilibria cease to exist.

Figures 11-14 have been drawn with a single curvature to NEC. If the curvature of NEC changes over its length there may be further equilibria.

4. Examples

We apply the preceding arguments to two cases of industrial equilibrium.

4.1. Cost Externalities in Perfect Competition. There are I identical price-taking firms with a cost externality between them, decreasing returns to scale and downward sloping market demand. Homogeneous output x is produced and sold in a perfectly competitive market. Total cost of each firm i is affected by the average output level of other firms in the market, k_i. Each firm has cost function $C(x_i, k_i)$. The cost function satisfies

A6 $C_x > 0, C_{xx} > 0, C_k < 0, C_{xk} < 0, C(0,k) = 0, C > 0$ if $x > 0, k \geq 0, C_x(k,k)$ finite for any k

i.e., marginal cost is positive and increasing but finite on the 45° line. Total and marginal cost fall with increases in the average output of other firms. There are no fixed costs.

Individual profit is

$$\pi_i = Px_i - C(x_i, k_i)$$

where P is the output price (in terms of our general notation we can identify P with θ, and π with V). The best responses solve the first order condition for profit maximisation

$$P = C_x(x_i, k_i) \qquad \text{if } P - C_x(0, k_i) > 0$$

$$x_i = 0 \qquad \text{if } P - C_x(0, k_i) \leq 0$$

Hence the optimal output of the individual firm solves

$$x^*(k_i, P) = \arg\max_{x_i}\{\pi(x_i, k_i, P) | x_i \geq 0\} \qquad i = 1, .., I$$

The slope of the reaction curve is

$$\frac{\partial x^*}{\partial k_i} = -\frac{C_{xk}}{C_{xx}} > 0 \tag{2}$$

when the externalities are beneficial and there are strategic complementarities. Similarly

$$\frac{\partial x^*}{\partial P} = \frac{1}{C_{xx}} > 0$$

and the reaction curve shifts in a direction of increasing x_i as P rises with k_i constant.

The curvature of the reaction curve depends on third order derivatives of the cost function

$$\frac{\partial^2 x_i}{\partial k_i^2} = \frac{2C_{xx}C_{xxk}C_{xk} - C_{xx}^2 C_{xkk} - C_{xk}^2 C_{xxx}}{C_{xx}^3} \tag{3}$$

If negative this gives us concave RCs, if positive convex RCs.

The equivalent of θ^* is an intermediate price P^* defined by

$$P^* = C_x(0,0)$$

At P^* if other firms are producing zero output then firm i wishes to produce zero. We know that P^* is finite.

Concave Reaction Functions. If (3) is negative at all x, k, then the reaction curve for firm i is concave to the 45^0 line and shifts with P as shown in Figure 2.

At P^* i will wish to produce less than k_i if $P^* < C_x(k_i, k_i)$. Hence for high k_i the reaction curve will fall below the 45 degree line if

$$\lim_{k_i \to \infty} C_x(k_i, k_i) > P^* = C_x(0,0)$$

ie eventually increasing marginal cost offsets the cost reduction due to the externality. In effect for sufficiently high k at P^* the reaction curve is below the 45^0 line. It is also then below the 45^0 line at $P < P^*$. For sufficiently high average outputs of other firms, i's best response is a lower level of output than this (which implies that the reaction curve eventually passes below the 45^0 line for all prices). This replaces assumption **A3**.

The pattern of NE then follows those of Table 1. With concave reaction functions, there will tend to be multiple NE, but a unique Nash equilibrium if the slope of the reaction curve at the origin is below unity.

The Number of NE with Convex Reaction Functions. If (3) is positive at all x, k, then the reaction curve for firm i is convex to the 45^0 line (this is similar to the

framework of Orjasniemi et al., (2008), who have oligopolistic rather than price taking output market). Here for a sufficiently high real output price the reaction curve is wholly above the 45^0 line for any k

$$P > C_x(k,k) \text{ for any } k \text{ for sufficiently high } P$$

Then at sufficiently high real output prices the reaction curve must lie wholly above the 45^0 line.

Applying the standard analysis with convex RCs, there may be no Nash equilibrium at all or either one or two equilibria depending on the level of P and the slope of the RC at the origin. This gives us a pattern of NE as in Table 2.

Market/Strategic Equilibrium. The price itself, P, is generally endogenous being determined jointly with k from the interaction between the market demand curve and the aggregate supply curve. Taking the market demand curve as linear

$$D = A - BP \qquad A, B > 0 \tag{4}$$

gives a number of equilibria which vary with the slope and position of the demand curve and the convexity/concavity of the RCs. With concave RCs, the exact number and type of equilibria shift discontinuously as the demand curve changes slope or intercept, exactly as in Figure 12.

With convex RCs this is also true except that now the largest number of possible equilibria is two as the slope and intercept of demand varies as in Figure 14.

An Explicit Example. Suppose the cost function is

$$C(x,k) = [abx(k+A_1)^{-b} + A_2^a]^{1/a} - A_2^a \quad i = 1,..,I; 0 < a < 1; b > 0$$

The reaction curves are

$$x_i = \frac{(k_i + A_1)^b}{ab}\left[\left(P\frac{(k_i + A_1)^b}{b}\right)^{\frac{a}{1-a}} - A_2^a\right] \text{ if } P\frac{(k_i + A_1)^b}{b} > 1 \tag{5}$$

$$x_i = 0 \text{ if } P\frac{(k_i + A_1)^b}{b} \leq 1 \tag{6}$$

The reaction curve is upward sloping and its second order derivative is

$$\frac{\partial^2 x_i}{\partial k_i^2} = \frac{(k_i + A_1)^{b-2}}{a(1-a)^2}\left[(a+b-1)\left(P\frac{(k_i + A_1)^b}{b}\right)^{\frac{a}{1-a}} + (a-1)^2(1-b)A_2^a\right]$$

which can be either positive or negative depending on the strength of the externality effects. For $b = 1$, $\partial^2 x_i/\partial k_i^2 > 0$ and the reaction function is convex.

We use the output reaction functions as defined by (5) and the demand curve as defined by (4), to show that, the exact number and type of market-strategic equilibria,(ME), shift discontinuously as the demand curve changes slope or intercept (see Figures 15 and 16 which are constructed for the technological parameters in Tables 3,4). In Figure 15 there is one critical equilibrium for values of A, B which gives x equal to 2.3 (near this equilibrium the slope of the equilibrium manifold in Figure 15 becomes unbounded). There is another apparent critical equilibrium when $x = 0$ which corresponds to losing the origin as a ME equilibrium. Figure 16 looks similar but here there is the generic critical point when the equilibrium x is 0.23. However there is another apparent critical point when the equilibrium x is about 1.7 which arises due to a local wiggle in the NEC.

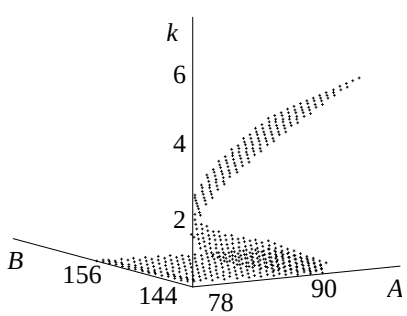

Figure 15: ME with concave RC's

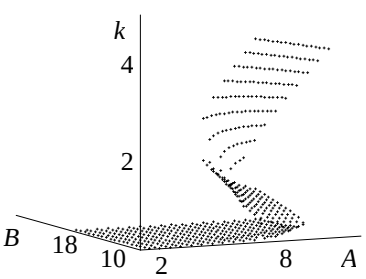

Figure 16: ME, convex RC's

	Concave$^{(1)}$	Concave$^{(2)}$
	$\frac{\partial RC(P^*,0)}{\partial k} > 1$	$\frac{\partial RC(P^*,0)}{\partial k} < 1$
P^*	0.61	0.44
P^+	0.51	$-$
x^1, P^1	$0.00, 0.58,\ \frac{\partial RC(P,k)}{\partial k} > 1$	$5.33, 0.50,\ \frac{\partial RC(P,k)}{\partial k} < 1$
x^2, P^2	$0.29, 0.57,\ \frac{\partial RC(P,k)}{\partial k} > 1$	
x^3, P^3	$3.81, 0.52,\ \frac{\partial RC(P,k)}{\partial k} < 1$	
(1) $a = 0.2,\ b = 0.3,\ A_1 = 0.5,\ A_2 = 1.9,\ B = 151,\ A = 87$ (2) $a = 0.2,\ b = 0.3,\ A_1 = 1.5,\ A_2 = 1.9,\ B = 151,\ A = 87$		
Table 3- Alternative market equilibria - Concave reaction curves		

	Convex$^{(3)}$	Convex$^{(4)}$
	$\frac{\partial RC(P^*,0)}{\partial k} < 1$	$\frac{\partial RC(P^*,0)}{\partial k} > 1$
P^*	0.10	0.77
P^+	0.72	$-$
x^1, P^1	$0.12, 0.64,\ \frac{\partial RC(P,k)}{\partial k} < 1$	$0.00, 0.65,\ \frac{\partial RC(P,k)}{\partial k} > 1$
x^2, P^2	$0.52, 0.58,\ \frac{\partial RC(P,k)}{\partial k} > 1$	$0.44, 0.59,\ \frac{\partial RC(P,k)}{\partial k} > 1$
x^3, P^3	$4.42, 0.04,\ \frac{\partial RC(P,k)}{\partial k} > 1$	$3.65, 0.15,\ \frac{\partial RC(P,k)}{\partial k} > 1$
(3) $a = 0.5,\ b = 1.5,\ A_1 = 0.5,\ A_2 = .0005, B = 14.5,\ A = 9.5$ (4) $a = 0.5,\ b = 1.5,\ A_1 = 1.5,\ A_2 = .9, B = 14.5,\ A = 9.5$		
Table 4- Alternative market equilibria - Convex reaction curves		

Note that in the convex reaction curve case in Table 4 we find an additional $P - x$ equilibrium due to the nonlinearity of the NEC correspondence.

4.2. Imperfect Competition with Demand Externalities. Another application is to a downstream duopoly with product differentiation in which the inverse demand function for each firm depends on the output level of its rival. Firm i has inverse demand function

$$p_i = P(x_i, x_j)$$

We allow the rival's output to have two different effects on the market inverse demand: on the one hand the higher x_j the greater the willingness to pay for x_i due to complementarities between the goods. On the other hand the higher the total outputs (x_i, x_j) the more saturated is the market and the lower the willingness to pay for additional units. In general the marginal market saturation effect may be different for the two goods-an increase in x_i may generate a larger or smaller fall in p_i than an increase in x_j. There is

only one factor of production, labour, and each unit of labour is supplied at a price $1/\theta$. Output is produced according to a linear production function where one unit of labour is required to produce one unit of output. Hence the payoff function for firm i is

$$V(x_i, x_j) = P(x_i, x_j)x_i - (1/\theta)\, x_i$$

We assume that $V_{xx} < 0$ (marginal net revenue is falling).

The best output response of i is defined by

$$\begin{aligned} x_i \text{ solves } V_x(x_i, x_j, \theta) &= 0 \text{ if } V_x(0, x_j, \theta) > 0 \\ x_i &= 0 \text{ if } V_x(0, x_j, \theta) \leq 0 \end{aligned}$$

which we can write in our usual way as

$$x_i = x^*(x_j, \theta)$$

The slope $\partial x^*(x_j, \theta)/\partial x_j$ is of ambiguous sign depending on the sign of V_{xk} at any point and $V_{x\theta} > 0$. Similarly there may be convex or concave RCs.

Note that the reaction function is identical for both the firms. But even so there may be asymmetric NE as well as symmetric ones-essentially because we no longer have monotonic RCs. It is easiest to see this in the context of a specific example

$$p_i = A + \beta(A_1 + x_j)^\alpha - x_i - bx_j$$

where $\beta, \alpha, A, A_1, b > 0$. The best response for firm i is

$$x_i = (1/2)A + (1/2)\beta(A_1 + x_j)^\alpha - (1/2)bx_j - (1/2\theta) \tag{7}$$

so long as $V_1(0, x_j, \theta) = A + \beta(A_1 + x_j)^\alpha - bx_j - 1/\theta > 0$ but $x_i = 0$ if $V_1(0, x_j, \theta) \leq 0$. The slope of the reaction curve for interior best responses is set by

$$\partial x^*/\partial x_j = (1/2)\alpha\beta(A_1 + x_j)^{\alpha-1} - (1/2)b$$

The concavity or convexity of the reaction curve is given by the sign of

$$\partial^2 x^*/\partial x_j^2 = (1/2)\alpha(\alpha - 1)\beta(A_1 + x_j)^{\alpha-2}$$

so that the reaction curve is concave if $\alpha < 1$, convex if $\alpha > 1$.

To see some specific examples set $A = 1$, $b = 7.45$, $A_1 = 0.5$, $\beta = 3.5$, $\alpha = 2.75$. There are four NE, two symmetric and two asymmetric exactly as in Figure 17. The loci ab and cd show the high and low output symmetric equilibria respectively as functions of θ. For values of θ below $\overline{\theta}$ there are also two asymmetric equilibria lying on the two loci ge and fe.

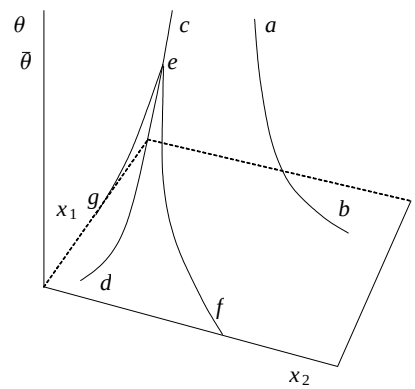

Figure 17: NEC, $A = 1$, $b = 7.45$, $A_1 = 0.5$, $\beta = 3.5$, $\alpha = 2.75$

This example has a high value of b so that the market saturation effect of the rival's output is much higher than the own output effect. Typically with $b < 1$ there will not be any asymmetric equilibria, but two symmetric equilibria. In this case, the equivalent of θ^* is an intermediate wage defined by

$$\theta^* = 1/\left(A + \beta A_1^{\alpha}\right)$$

which ensures that the reaction curve passes through the origin. The equivalent of θ^+ and x^+ are

$$\begin{aligned}
\theta^+ &= \alpha\left[(b+2)(1-\alpha)\left(\frac{b+2}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} + \alpha A_1(b+2) + \alpha A\right]^{-1} \\
x^+ &= \left(\frac{b+2}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} - A_1
\end{aligned}$$

At θ^* if the other firm is producing zero output then firm i wishes to use no labour and the level of output then corresponds to $0 = x^*(0, \theta^*)$. For fixed input prices, we can use the output reaction functions as defined by (7), to show that, for specific values of the parameters, it is possible to derive all the Nash equilibrium configurations analysed in the previous sections. For instance, with $\partial x^*(0,\theta)/\partial x_j = (1/2)\alpha\beta {A_1}^{\alpha-1} - (1/2)b > 1$ (< 1) and $\alpha < 1$ $(\alpha > 1)$, changes in θ yield all the Nash equilibrium configurations for the concave (convex) case when θ is exogenous. If $b > 2$, then the reaction functions will no longer be monotonic, and changes in θ can generate asymmetric Nash equilibria. In this context, we can identify a critical θ and x, $\overline{\theta}$ and $\overline{x}$, such that only symmetric equilibria exist for $\theta = \overline{\theta}$. The critical $\theta - x$ pair solves

$$\begin{aligned} \frac{\partial x^*\left(\overline{x}, \overline{\theta}\right)}{\partial x_j} &= -1 \\ x^*\left(\overline{x}, \overline{\theta}\right) &= \overline{x} \end{aligned}$$

and is given by

$$\begin{aligned} \overline{\theta} &= \left[\beta\left(\frac{b-2}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}} + (b+2)\left(A_1 - \left(\frac{b-2}{\alpha\beta}\right)^{\frac{1}{\alpha-1}}\right) + A\right]^{-1} \\ \overline{x} &= \left(\frac{b-2}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} - A_1 \end{aligned}$$

Then for $0 < \theta < \overline{\theta}$, there are two symmetric NE and two asymmetric NE, as in Figure 17. Further reductions of θ lead to lower outputs in the low symmetric NE but to higher outputs in the high symmetric NE; in the asymmetric equilibria, any decrease in θ leads the high output firm to produce more and the low output firm to decrease its production levels. Eventually for sufficiently low θs, the low output firm is driven out of the market, as illustrated in Figure 17.

Suppose now that $b \leq 1$, so that changes in θ only generate symmetric NE and the NEC is exactly as in Figures $7-8$ when $\alpha < 1$ or as in Figures $9-10$, when $\alpha > 1$. Suppose also that θ is determined endogenously, according to the following linear function, which positively relates $1/\theta$ to the total level of output:

$$1/\theta = B + B_1\left(x_1 + x_2\right) \quad B, B_1 > 0$$

Then with concave RCs, the exact number and type of equilibria shift discontinuously as the wage function changes slope or intercept, similarly to Figure 12. With convex RCs this is also true except that now the largest number of possible equilibria is two as the slope and intercept of the wage curve vary, similarly to Figure 14.

5. Conclusions

In the strategic complementarity paradigm with identical players, the common outcome is that there may be multiple non-cooperative symmetric equilibria which can often be Pareto ranked-the system can get stuck at a low level equilibrium in which all players are uniformly badly off as compared with their outcomes in an alternative equilibrium. It has been used to explain many features of social situations: Keynesian type phenomena, undesirable monetary/inflation equilibria, social security/effort structures.

With identical increasing reaction functions the equilibria are all symmetric. In this paper we add some structure to the payoff functions which allows us to determine their number. The crucial idea that we add is that of concavity or convexity of reaction functions. With reaction curves satisfying one of these assumptions we can tie down the number of symmetric equilibria and they range from $0-3$. We add an exogenous parameter θ to the reaction curves to delineate the number of non-cooperative equilibria as a function of θ. The Nash equilibria are conditional on θ. Here θ has the interpretation of setting the favourableness of the environment, both payoffs and best responses increase with θ. For some values of θ the equilibrium is unique but for other values there are several alternative non-cooperative equilibria. This allows us to characterise the θ equilibrium correspondence, showing the set of conditional non-cooperative equilibrium best choices for each value θ.

It also allows us to embed the strategic complementarity equilibrium model in a broader equilibrium approach in which there are some additional equilibrating forces on θ which leads to the idea of an unconditional $\theta - x$ equilibrium. In this, x is a non-cooperative equilibrium given a particular value $\overline{\theta}$ of θ, but $\overline{\theta}$ itself is an equilibrium value of θ given x. For simple characterisations of the additional equilibrating forces on θ (a

$\theta - x$ equilibrium essentially adds a new equilibrium relation $\theta = \Theta(x)$), we can then determine which of the non-cooperative equilibria can appear as $\theta - x$ equilibria and especially under what circumstances the system can remain at a low level Nash equilibrium. Unfortunately this approach does not generally eliminate multiplicity of equilibria despite the endogeneity of θ. It suggests that a more direct hands on approach to equilibrium selection of θ is required to eliminate the indeterminacy eg Stackleberg leadership in the choice of θ.

We also add some exogenous parameters to the function $\Theta(x)$ and show that as these vary the set of $\theta - x$ equilibria can discretely shift. Basically these shifts in $\Theta(x)$ can generate a fold bifurcation in the $\theta - x$ equilibrium set.

To motivate the abstract discussion, we analyse some examples. We take an industry with a large number of price-taking firms who enjoy technological spillovers in costs-marginal and total costs of any firm for a given output level are lower, the higher is the average output level of all other firms. Each firm chooses its output taking the real output price P and the average output of other firms as parametric. If P is fixed we have the standard strategic complementarity paradigm. Here P plays the role of the parameter θ. At industry level P may be determined by the interaction of market demand and the aggregate output of firms. Market equilibrium determines both P and the output of each firm x in a $P-x$ equilibrium. The function $\Theta(x)$ derives from the market demand function and the output supply correspondence of firms. We show that the number and nature of $P - x$ equilibria vary with the slope and intercept of the market demand curve, and that variations in these demand parameters can yield a fold bifurcation in the $P-x$ equilibria.

A second example highlights the lack of robustness of the usual strategic complementarity results to non-monotonicity of the reaction curve. This is a duopoly with product differentiation and identical constant marginal costs where the inverse demand curve facing one firm is not monotonic in the output of the second firm. On the one hand there is a product differentiation/complementarity effect which increases the demand for one firm as the output of the second rises. On the other hand there is a market saturation effect

which reduces the demand for one firm when the output of the other rises. This can generate non-monotonic reaction curves, and, as a result, multiple symmetric or asymmetric Nash equilibria. Taking the common marginal cost as the parameter $1/\theta$, we explore how the equilibrium output levels vary with θ.

The strategic complementarity paradigm has been a rich stimulus generating new reasons why equilibria may fail to be attractive normatively. It has been applied in both static and dynamic contexts. The present paper adds some general structure to allow more precise characterisation of the possible outcomes and also embeds the approach in a model in which additional different equilibrating forces are at work, eg. prices in markets, to test whether the common welfare consequences of the paradigm hold up.

A. Appendix

A.1. Symmetric Nash equilibria.

Proposition 1. *Any Nash equilibrium of the game must be a symmetric Nash equilibrium. At a symmetric Nash equilibrium $x^*(x_i, \theta) = x_i$, so any Nash equilibrium is on the 45^0 line.*

Proof. *This follows from the detailed assumptions on $V(\cdot)$. With identical payoff functions, the reaction functions will be identical and given by:*

$$x_i = x^*(\Sigma_{j \neq i} x_j/(I-1), \theta)$$

Any inequality between agents' actions (for instance $x_1 > x_2 > x_3 ...$) means it is possible to identify a highest action, say $x_i = \max_k x_k$, and a lowest action, $x_j = \min_k x_k$, for any $k = 1...I$ and $i \neq j$. But since we are assuming $V_{xk}(x, k, \theta) > 0$ (the RCs have a positive slope), this would also imply

$$k_i < k_j$$

and hence

$$x_i = x^*(k_i, \theta) < x^*(k_j, \theta) = x_j$$

which is a contradiction. So any Nash equilibrium is on the 45^0 line[3]. ■

[3]Note that we cannot have a Nash equilibrium in which one agent chooses $x_i = 0$ and all the other

A.2. With Concave RCs θ^+ Exists. Since at θ^* the RC has slope greater than unity at the origin but also crosses the 45^0 line, there must be a point on the RC above the 45^0 line at which it has slope unity. From this point, consider the locus of points at which the slope of successive RCs is unity as θ falls. This locus cannot pass through the origin or cut the vertical axis at $x_i > 0$: if it did, it would have to meet or cross the RC for θ^*, which means that two RCs for different prices must cross, contradicting the fact that $\partial x^*/\partial\theta > 0$. Hence the locus must cross the 45^0 line which defines θ^+. The identical argument holds for convex RCs.

A.3. Generic number of equilibria with Concave Reaction Functions. To explore the regularity of these equilibrium patterns for the concave case in which the slope of the RCs exceeds unity at the origin, Figure $A1$ displays the basic setting.

In Figure $A1$ the tangent at θ^* defines α_0, β_0. For intercepts above α_0 there are three equilibria when the slope is in between $\beta_1(\alpha)$ and $\beta_2(\alpha)$. For slopes below $\beta_2(\alpha)$ there is a single equilibrium with positive activity but for slopes below $\beta_1(\alpha)$ the single equilibrium has inactivity.

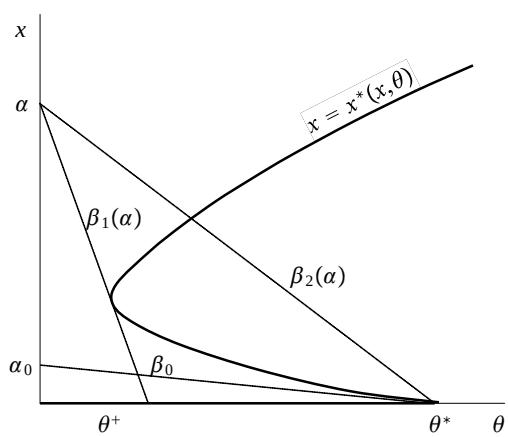

Figure $A1$: Concave RCs,
$\partial x^*(0, \theta^*)/\partial k_i > 1$

agents choose $x_j > 0$, with $i \neq j$. Since payoff functions are increasing in the average action of all the other players, then the agents choosing $x_j > 0$, with $i \neq j$, would face a lower k than the agents choosing $x_i = 0$ and so should choose $x_j = 0$ as well.

A.4. Generic number of equilibria with Convex Reaction Functions. In Figure $A2$, α_0, β_0 are defined by the intercept and slope of the tangent to the NEC at $\theta = 0$, here there is a unique equilibrium. If the intercept of the $\Theta(x)$ functions is higher than this there is a tangent to the NEC defined by the slope $\beta(\alpha)$. This also defines a $\Theta(x)$ function giving a unique equilibrium. For $\Theta(x)$ intercepts above α_0 and slopes below β_0 there is no equilibrium while for slopes above β_0 there are always two equilibria. If the $\Theta(x)$ intercept is exactly α_0 but the slope is steeper than β_0 there are two equilibria. If the slope is flatter than β_0 and the intercept is α_0 there is a unique equilibrium. For intercepts below α_0 for any slope the equilibrium is always unique but may involve inactivity.

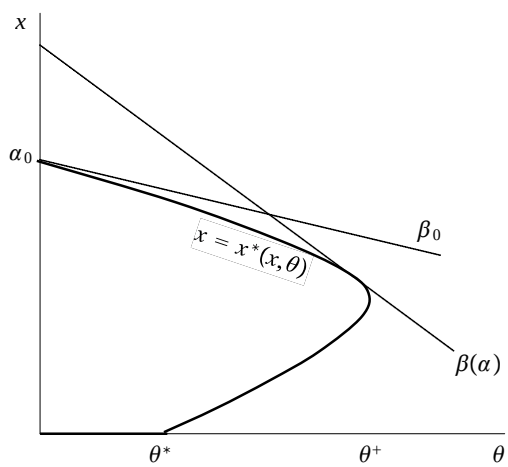

Figure $A2$: Convex RCs,
$\partial x^*(0, \theta^*) / \partial k_i < 1$

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