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Production in General Equilibrium with Incomplete Markets

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Abstract

Short and long run production is introduced in a two period general equilibrium model with incomplete markets, where firms are profit maximizers. They maximize profits in the long run, which implies profit maximization over both periods. The sequential structure of the model is such that, firms issue shares in the short run in order to build up long run production capacity. Long run production takes place in the second period subject to long run technological feasibility and installed capacity constraints. It is shown that equilibrium exists generically.

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JEL classification: D62, D52, D53.

1 Introduction

This paper considers production in a general equilibrium model with incomplete markets and technological uncertainty, where financial markets consist of the stocks issued by profit maximizing firms [1, 2]. The stock market provides the role of separating ownership and control, and permits agents the sharing of risks but only incompletely. Incomplete markets are a consequence of technological uncertainty, where there are more uncertain states of nature than securities at long run equilibrium. Short and long run production sets are considered in a two period set up, where firms maximize profits over both periods. In the short run, firms issue stocks in order to build up long run production capacity. Once capacity is installed, production takes place subject to long run technological feasibility and capacity constraints. Profits are then distributed to stock holders.

Following seminal work by Debreu [8], the systematic discussion about production under uncertainty began with the influential paper by Diamond [10]. Beyond this, a sample of contributions attempting to generalize production to incomplete markets is represented by Ekern, Wilson, Drèze, Grossman, Hart, Duffie, Shafer, Geanokopolos and others [15, 12, 19, 14, 18]. The vast literature on production assigns arbitrarily

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utilities to firms, such as i.e. the Drèze or Grossman-Hart criterion, and have in common the nature of a public good problem, where a group of stockholders decide about the optimal activity of the firm. This is at variance with the model presented in this paper, where the objective of the firm is profit maximization.

To the present, general equilibrium models with incomplete markets have implicitly assumed that stockholders automatically provide the money to finance production activities. In these models the stock market is exogenously determined and independent of the efficient boundary of the production set. In fact, however, firms do not only obtain cash through retained earnings but issue stocks and other financial assets, or borrow from banks in order to raise money for capital expenditures.

This paper sheds light on this imperfection by showing that this dichotomy, namely the independence of the real and financial sector, is not tenable. Firms raise money for capital expenditures through the endogenously determined stock market, where the objective of the firm, profit maximization, links the efficient boundary of the short run production set with the sphere of the stock market. Moreover, the model accommodates the objective function of profit maximization. This follows from the fact that firms build up long run production capacity in the first period by issuing shares. The sine qua non of the model is then to show that equilibrium exists. It is shown that, for an endogenized price and technology dependent real asset structure, which is transvers to the reduced rank manifolds, equilibrium exists generically in the endowments by the application of Thom's parametric transversality theorem. Finally, the non-smooth convex production set case is considered, where the piecewise linear production manifolds are regularized by convolution. Existence then follows from the smooth case. Bottazzi [6] demonstrated generic existence of equilibrium for an exchange economy for price dependent smooth assets. Equilibria exist for more general asset structures.

The model is introduced in section 2. Section 3 shows generic existence for convex smooth production manifolds. In the next section, convex piecewise linear production manifolds are regularized by convolution. Section 5 is an appendix.

2 The model

We consider a two period $t \in \{0, 1\}$ model with uncertainty in period 1. An element in the set of mutually exclusive and exhaustive uncertain events is denoted $s \in S$, where by convention $s = 0$ represents the certain event in period 0, and S denotes the set of all mutually exclusive uncertain events. For every production set Y_j, there exists a set of states of nature $S_j = \{1, ..., \mathbb{S}\}$, where $\mathbb{S} \geq 2$, for all S_j. Denote $\mathcal{S} = \{S_1, ..., S_j, ..., S_n\}$, where $S \subseteq \mathcal{S}$. We count a total of $(S + 1)$ possible states of the world. The economic agents are the $j \in \{1, ..., n\}$ producers and $i \in \{1, ..., m\}$ consumers which are characterized by sets of assumptions F and C.

There are $k \in \{1, ..., l\}$ physical commodities (goods) and $j \in \{1, ..., n\}$ financial assets, referred to as stocks. Goods are traded on each of the $(S + 1)$ spot markets. Firms issue stocks which are traded at $s = 0$, yielding a payoff in the next period at uncertain state $s \in \{1, ..., S\}$. The quantity vector of stocks issued by firm j is denoted $z_j \in \mathbb{R}_-$. Other assets such as bonds or options can be introduced without any further difficulties. There are total $l(S + 1)$ goods. The consumption of agent i is denoted $x_i = (x_i(0), x_i(s), ..., x_i(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $x_i(s) = \left(x_i^1(s), ..., x_i^l(s)\right) \in \mathbb{R}^l_{++}$, and $\sum_{i=1}^m x_i = x$. The consumption space for each i is

$X_i = \mathbb{R}^{l(S+1)}_{++}$. The associated price system is a collection of vectors represented by $p = (p(0), p(s), ..., p(S)) \in \mathbb{R}^{l(S+1)}_{++}$, with $p(s) = \left(p^1(s), ..., p^l(s)\right) \in \mathbb{R}^l_{++}$. There are n financial assets traded in period 0. Denote the quantity vector of stocks purchased by consumer i, $z_i = (z_i(1), ..., z_i(j), ..., z_i(n)) \in \mathbb{R}^n_+$, and denote $\sum_{i=1}^m z_i = z$, with associated spot price system $q = (q(1), ..., q(j), ..., q(n)) \in \mathbb{R}^n_{++}$. A period 0 net activity vector for firm j is denoted $y_j(0) = \left(y_j^1(0), ..., y_j^k(0), ..., y_j^l(0)\right) \in \mathbb{R}^l$, where $y_j^k(0) < 0$ denotes a short run input, and $y_j^k(0) > 0$ denotes a long run capacity element to be installed in period 0. Denote a long run state dependent net activity vector $y_j(s) = \left(y_j^m(1) \times y_j^n(s), ..., y_j^m(1) \times y_j^n(S)\right) \in \mathbb{R}^{lS}$, where $y_j^m(1) \in \mathbb{R}^m_-$ is the long run input defined via profit maximization in period 0. Let $\sum_{j=1}^n y_j = y$ denote the long run net activity vectors.

We assume $l(S+1)$ complete commodity markets and hypothesize technological uncertainty. Incomplete financial markets is a consequence of this hypothesis.

The producer: Consider the sequential structure of the optimization problem of the firm. Firms build up long run production capacity in the first period, for that, they issue stocks. Once capacity is installed, production activities take place subject to long run production sets in the second period. Uncertainty in production is introduced by a random variable $s \in S_j$ for every j.

Assumption (T): For every production set $Y_j(s)$, $s \in S_j \geq 2$.

Assumption (P): Firms maximize long run profits.

Denote a $t = 0$ production set $Y_j(0) \subset \mathbb{R}^l$. The objective of long run profit maximization implies profit maximization over both periods. For that, each firm issues the quantity of stocks $z(j) \in \mathbb{R}_-$ in the short run. At $t = 0$, the firm's problem is

$$(\bar{y}_j(0); \bar{z}(j)) \arg\max \left\{\bar{p}(0) \cdot y_j(0) : y_j(0) \in Y_j(0)\right\}. \tag{1}$$

Production takes place in the second period, once capacity is installed. At $t = 1$ firms maximize profits in every state s subject to long run technological feasibility and capacity constraints. Denote the long run production set $Y_j|_z$. At $t = 1$, the firm's problem is

$$(\bar{y}_j) \arg\max \left\{\bar{p}(s) \Box y_j(s) : y_j \in Y_j|_z\right\}. \tag{2}$$

Denote a long run equilibrium output vector associated with the production set boundary $\bar{y}_j \in \partial Y_{j,eff}|_z$. Each firm j is characterized by set of assumptions F. (Debreu [8])

Assumptions F (i) For each j, $Y_j|_z \subset \mathbb{R}^{lS}$ is closed, convex, and $\left(\omega + \sum_{j=1}^n Y_j|_z\right) \cap \mathbb{R}^{lS}_+$ compact $\forall \omega_i \in \mathbb{R}^{lS}_{++}$. $0 \in Y_j|_z \Longleftarrow Y_j|_z \supset \mathbb{R}^{lS}_-$. $Y_j|_z \cap \mathbb{R}^{lS}_+ = \{0\}$. (ii) For each j, denote $\partial Y_j|_z \subset \mathbb{R}^{nS}$ a C^∞ manifold for transformation maps (1) $\phi_j : \mathbb{R}^m_- \times S_j \to \mathbb{R}^{nS}_+$ non-linear.[1]

[1] Here, C^∞ implies differentiability at any order required. The order depends on all transversality arguments employed. m denotes the inputs and n the output elements of the production set.

The $t=1$ maps implied by equation (2), $\pi_j : \mathbb{R}^l_{++} \times \mathbb{R}^l \times S_j \to \mathbb{R}^S_+$, $\forall j$, define the $(S \times n)$ total long run payoff matrix, a collection of n vectors denoted

$$\Pi(p_1, \phi) = \begin{bmatrix} p(s) \cdot y_1(s) & \cdots & p(s) \cdot y_n(s) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p(S) \cdot y_n(S) \end{bmatrix}. \tag{3}$$

The consumer: Each consumer $i \in \{1, ..., m\}$ is characterized by set of assumptions C. (Debreu [9])

Assumptions C: (i) $u_i : \mathbb{R}^{l(S+1)}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^{l(S+1)}_+$, and C^∞ on $\mathbb{R}^{l(S+1)}_{++}$. $u_i(x_i) = \left\{ x_i' \in \mathbb{R}^{l(S+1)}_+ : u_i(x_i') \geq u_i(x_i) \right\} \subset \mathbb{R}^{l(S+1)}_{++}$, $\forall x_i \in \mathbb{R}^{l(S+1)}_{++}$. For each $x_i \in \mathbb{R}^{l(S+1)}_{++}$, $Du_i(x_i) \in \mathbb{R}^{l(S+1)}_{++}$, $\forall s$. For each $x_i \in \mathbb{R}^{l(S+1)}_{++}$, $h^T D^2 u_i(x_i) h < 0$, for all nonzero hyperplane h such that $(Du_i(x_i))^T h = 0$. (ii) Each i is endowed with $\omega_i \in \mathbb{R}^{l(S+1)}_{++}$.

Consumers want to transfer wealth between future spot markets. For that, they invest in firms in period $t = 0$, receiving a share of total dividend payoffs which are determined in the next period in return. Denote the sequence of $(S+1)$ budget constraints

$$B_{z_i} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++},\ z_i \in \mathbb{R}^n_+ : \begin{array}{l} p(0) \cdot (x_i(0) - \omega_i(0)) = -q z_i \\ p(s) \Box (x_i(s) - \omega_i(s)) = \Pi(p_1, \phi) \theta(z_i) \end{array} \right\}, \tag{4}$$

where[2] ownership structure is a $(n \times S)$ matrix denoted $\left[\theta(z_i)^S_{s=1}\right] = \begin{bmatrix} [\theta(z_i)_1](1) & \cdots & [\theta(z_i)_1](S) \\ \vdots & [\theta(z_i)_j](s) & \vdots \\ [\theta(z_i)_n](1) & \cdots & [\theta(z_i)_n](S) \end{bmatrix}$ defined by the mappings

$$\theta_{ij} : \mathbb{R}_+ \times S \to \mathbb{R}^S_+, \forall j, \tag{5}$$

where $z_i(j) \in \mathbb{R}_+$ is a positive real number for every $j = 1, ..., n$. In compressed notation, we write

$$B_{z_i} = \quad \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++},\ z_i \in \mathbb{R}^n_+ : p(s) \Box (x_i(s) - \omega_i(s)) \in \hat{\Pi} \left[z_i | \theta(z_i)^S_{s=1} \right] \right\} \tag{6}$$

where $\hat{\Pi}(p_1, q, y) = \begin{bmatrix} -q_1 & \cdots & -q_n \\ p(1) \cdot y_1(1) & \cdots & p(1) \cdot y_n(1) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p_1(S) \cdot y_n(S) \end{bmatrix}$ represents the full payoff matrix of order $((S+1) \times n)$, and $\left[z_i | \theta(z_i)^S_{s=1} \right] =$

[2] $\Box$ denotes the box product. A "s by s" context dependent mathematical operation. For example the s by s inner product.

$$\left[\begin{array}{c|ccc} z_i(1) & [\theta(z_i)_1](1) & \cdots & [\theta(z_i)_1](S) \\ \vdots & \vdots & [\theta(z_i)_j](s) & \vdots \\ z_i(n) & [\theta(z_i)_n](1) & \cdots & [\theta(z_i)_n](S) \end{array}\right]$$ denotes the ownership structure, a partitioned matrix of order $(n \times (S+1))$.

We introduce following prize normalization $\mathfrak{S} = \left\{p \in \mathbb{R}^{l(S+1)}_{++} : \|p\| = \Delta\right\}$ such that the Euclidean norm vector of the spot price system $\|p\|$ is a strictly positive real number $\Delta \in \mathbb{R}_{++}$.

Definition 1 *A financial markets equilibrium with production* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}^{n}_{++}$ *satisfies:*

(i)	$(\bar{x}_i; \bar{z}_i) \arg\max \{u_i(x_i; z_i) : \bar{x}_i \in B_{\bar{z}_i}(\bar{p}, \bar{q}, \bar{y}; \omega_i)\}$	$\forall i$
(ii)	$(\bar{y}_j(0); \bar{z}(j)) \arg\max \{\bar{p}(0) \cdot y_j(0) : y_j(0) \in Y_j(0)\}$	$\forall j$
	$(\bar{y}_j) \arg\max \{(\bar{p}(s) \Box y_j(s)) : y_j \in Y_j \mid_{\bar{z}}\}$	$\forall j$
(iii)	$\sum_i^m (\bar{x}_i - \omega_i) = \sum_j^n \bar{y}$	
(iv)	$\sum_{i=1}^m \theta(\bar{z}_i)_j = 1 \; \forall j,$ *and* $\sum_{j=1}^n \sum_{i=1}^m (\bar{z}_i)_j = 0.$	

We show that incomplete markets is a consequence of technological uncertainty, and that the ownership structure is independent from control of production vector y_j.

Proposition 1 $n < S \Longleftarrow Y_j \mid_{\bar{z}}$ *for all* j, *and* $S_j \geq 2$.

Proof. Let $S_j = 1$ for every j, and $\sum_j S_j = \mathcal{S}$. Then long run profit prospects $\pi(p) > 0$ imply long run capacity adjustment and market entrance until $n = S$. Let $S > 1$ for every j, and $\sum_j S_j = \mathcal{S}$. Then $\pi(p) > 0$ implies market entrance and the issue of new securities such that in the limit as $\pi(p) \to 0$ the number of firms $j \to n < S$ by assumption (T). Similar for $\pi(p) < 0$, firms exit the market. ∎

Theorem 1 $y_j \in Y_j \mid_{\bar{z}}$ *independent of* $i = 1, ..., m$ *for every* j.

Proof. Note that $\theta_{ij} : z_i(j) \times S(j) \to \mathbb{R}^S_+, \forall j, i$, independent of $\bar{y}_j(i), \forall i$. ∎

3 Generic existence for convex smooth production manifolds

In this section we show existence of equilibrium. It is known that a pseudo equilibrium exists. See Duffie, Shafer, Geanokopolos, Hirsh, Husseini, and others [2, 6, 17, 22, 23]. We show that a pseudo equilibrium for a more general asset structure, permitting the modeling of production, exists. We apply the Cass trick [7], which says that, if isolate the Arrow-Debreu agent (unconstraint budget set) then any solution of $\bar{x}_1 + \sum_{i=2}^m \bar{x}_i = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \bar{y}_j$ is also a solution of the standard general equilibrium model.

Definition 2 *if* $\nexists \; z \in \mathbb{R}^n_{++}$ *s.t.*
$\hat{\Pi}(p_1, q, \phi) \left[z \mid \sum_{i=1}^m \theta(z_i)^S_{s=1}\right] \geq 0,$ *then* $q \in \mathbb{R}^n_{++}$ *is a no-arbitrage asset price relative to* p_1.

Lemma 1 $\exists \; \beta \in \mathbb{R}^S_{++}$ *s.t.* $q = \sum_{s=1}^S \beta \Box \Pi(p_1, \phi)$.

Proof. Immediate consequence of the separation theorem for $((S+1)\times n)$ matrices in Gale (1960). It asserts that either $\exists\ z \in \mathbb{R}^n_{++}$ such that $\hat{\Pi}z \geq 0$, or $\exists\ \beta \in \mathbb{R}^{S+1}_{++}$ such that $\beta\hat{\Pi} = 0$. ■

We can rescale equilibrium prices without affecting equilibrium allocations, $P_1 = \beta\Box\bar{p}_1$. The next step is to derive a normalized no arbitrage equilibrium definition. Let $\beta \in \mathbb{R}^S_{++}$ be $\left(\frac{\lambda(s)}{\lambda}\right)_{i=1}$ the gradient vector from the optimization problem of agent 1, called the Arrow-Debreu agent. The Walrasian budget set for the Arrow-Debreu agent is a sequence of constraints denoted

$$B_1 = \left\{ x_1 \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P\cdot(x_i-\omega_i) = \sum_j \theta_{ij}P\cdot y_j(0) \\ P(s)\Box(x_i(s)-\omega_i(s)) = \sum_j \theta_{ij}P(s)\Box y_j(s) \end{array} \right\}. \tag{7}$$

For all consumers $i \geq 2$, denote the no arbitrage budget set by the sequence of $(S+1)$ constraints

$$B_{i\geq 2} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P\cdot(x_i-\omega_i) = \sum_j \theta_{ij}P\cdot y_j(0) \\ P(s)\Box(x_i(s)-\omega_i(s)) \in \langle\Pi(P_1,\phi)\rangle \end{array} \right\}, \tag{8}$$

where $\langle\Pi(P_1,\phi)\rangle$ is the span of the income transfer space. Replace $\langle\Pi(P_1,\phi)\rangle$ with L in $G^n(\mathbb{R}^S)$, where $G^n(\mathbb{R}^S)$ is the Grassmanian manifold[3] with its known smooth $(S-n)n$ dimensional structure, and L an n-dimensional affine subspace of $G^n(\mathbb{R}^S)$.

Denote the pseudo opportunity set $B_i(P,L;\omega_i)$, for each i,

$$B_i = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P\cdot(x_i-\omega_i) = \sum_j \theta_{ij}P\cdot y_j(0) \\ P(s)\Box(x_i(s)-\omega_i(s)) \subset L \end{array} \right\}. \tag{9}$$

Let $\mathfrak{S}' = \left\{ p \in \mathbb{R}^{l(S+1)}_{++} : p^{0,1} = \Delta \right\}$ be the set of normalized prices, and let $\Delta \in \mathbb{R}_{++}$ be a fixed strictly positive real number. This convenient normalization singles out the first good at the spot $s=0$ as the numeraire.

We introduce following definitions for the long run payoff maps associated with sets $\mathfrak{S}$ and $\mathfrak{S}'$: For any $p_1 \in \mathfrak{S}$, such that $\pi : \mathfrak{S}\times\mathbb{R}^l \times S \to \mathcal{A}$, let $\Gamma(P_1,\phi) = \beta\Box\left[\left(proj_\Delta\left(\frac{1}{\beta}\right)^T\Box P_1\right)\Box y\right]$, where T denotes the transpose, $proj_\Delta(z) = \Delta\left(\frac{z}{\|z\|}\right)$, $\frac{1}{\beta} = \left(\frac{1}{\beta(1)},...,\frac{1}{\beta(S)}\right) \in R^S_{++}$, and $\beta = (\beta(1),...,\beta(S)) \in R^S_{++}$. (ii) For any $p_1 \in \mathfrak{S}'$, such that $\pi : \mathfrak{S}'\times\mathbb{R}^l\times S \to \mathcal{A}$, let $\Gamma(P_1,\phi) = \beta\Box\left[\left(\left(\frac{1}{\beta}\right)^T\Box P_1\right)\Box y\right]$, where $\mathcal{A}$ is a set of $(S\times n)$ matrices A of order $(S\times n)$.

We can now define the pseudo financial markets equilibrium with production. We then state the relational propositions between a full rank FE with production and a pseudo FE with production.

Definition 3 *A pseudo financial markets equilibrium with production* $(\bar{x},\bar{y}),(\bar{P},\bar{L}) \in \mathbb{R}^{l(S+1)m}_{++}\times\mathbb{R}^{l(S+1)n}_{+}\times\mathfrak{S}'\times G^n(\mathbb{R}^S)$ *satisfies:*

[3] See i.e. Dieudonnè [11]. See Duffie and Shafer for an exposition of the Grassmanian manifold in economics [13].

$$\begin{array}{lll}
(i) & (\bar{x}_1) \arg\ max \left\{u_1(x_1)\ s.t.\ x_1 \in B_1(\bar{P}, \omega_1)\right\} & i=1 \\
(ii) & (\bar{x}_i) \arg\ max \left\{u_i(x_i)\ s.t.\ x_i \in B_i(\bar{P}, \bar{L}, \omega_i)\right\} & \forall\ i \geq 2 \\
(iii) & \left\langle \Gamma(\bar{P}_1, \bar{\phi})\right\rangle \subset \bar{L},\ proper\ if \left\langle \Gamma(\bar{P}_1, \bar{\phi})\right\rangle = \bar{L} & \\
(iv) & (\bar{y}_j(0)) \arg\ max \left\{\bar{P} \cdot y_j : y_j \in Y_j(0)\right\} & \forall\ j \\
 & (\bar{y}_j(s)) \arg\ max \left\{\bar{P}_1(s)\Box y_j(s)) : y_j \in Y_j\left|_{\bar{z}}\right.\right\} & \forall\ j \\
(v) & \bar{x}_1 + \sum_{i=2}^m \bar{x}_i = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \bar{y}_j &
\end{array}$$

Lemma 2 *Under assumptions C, demand mappings $f_1(P, w_1)$ and $f_i(P, L, \omega)$ for $i = 2, ..., m$, from argmax (i) and (ii) are C^∞. Under assumptions F, supply mappings $g_j(P)$ for $j = 1, ..., n$, from argmax (iv) are C^∞.*

A proof of this known result is omitted [13].

Proposition 2 *For every full rank FE with production $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$, there exists $\beta \in \mathbb{R}^S_{++}$ and a n-dimensional subspace $L \in G^n(\mathbb{R}^S)$ such that $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ is a pseudo FE with production.*

Proof. Trivial. ■

Proposition 3 *If $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ is a pseudo FE with production then for every $\beta \in \mathbb{R}^S_{++}$, there exist financial asset prices $\bar{q} \in \mathbb{R}^n_{++}$ and investment portfolios $\bar{z} = (z(1), ..., z(n)) \in \mathbb{R}^n_{++}$ such that $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ is a $(\bar{x}, \bar{y})$ allocational equivalent FE with production.*

Proof. Using (Definition 3), let $\bar{q} = \sum_{s=1}^S \left(\Gamma(\bar{P}_1, \bar{\phi})\right)$, let $\bar{p}_1 = proj\left(\left(\frac{1}{\beta(s)}\right)^T \Box \bar{P}_1(s)\right)$, and let $z_1 = \sum_{i=2}^m z_i$. See [24]. ■

Long run payoffs depend on the technology of the firm, its production capacity installed, and on a set of regular prices. Equilibrium does not exist for critical prices. The next step is then to introduce rank dependant maps, and to exhibit a class of transverse price and technology dependent maps. We will show that equilibria exists for this smooth rank dependent real asset structure, denoted π^ρ.

Definition 4 *Define the rank dependent maps $\pi^\rho : \mathbb{R}^l_{++} \times \mathbb{R}^l \times S \rightarrow \mathcal{A}^\rho$ for $0 \leq \rho \leq n$. The set of reduced rank matrices A^ρ of order $(S \times n)$ with $rank(A^\rho) = (n - \rho)$ is denoted $\mathcal{A}^\rho$ and is of order $(S \times n)$.*

Lemma 3 *(i) For $1 \leq \rho < n$, A^ρ is a submanifold of A of codimension $(S - n + \rho)\rho$. (ii) for $\rho = n$ the set $\mathcal{A}^\rho = \{\varnothing\}$ is empty, and (iii) for $\rho = 0$, $\mathcal{A}^\rho = \mathcal{A}$ the set of reduced rank matrices is equivalent to the set of full rank matrices.*

Proof. Consider the open set U of $(S \times n)$ matrices $\tilde{a} = \left[\begin{array}{c|c} A_{(n-\rho)\times(n-\rho)} & B_{(n-\rho)\times\rho} \\ \hline C_{(S-n+\rho)\times(n-\rho)} & D_{(S-n+\rho)\times\rho} \end{array}\right]$ of $rank\,(\tilde{a}) = (n - \rho)$ since $\det\bar{A} \neq 0$. There exists a matrix $b_{(n-\rho)\times\rho}$ such that $\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} b \Leftrightarrow b = A^{-1}B$, and $D = CA^{-1}B$.[4] ■

The lemma states that, for $1 \leq \rho < n$, the incomplete income transfer space is rank reduced. The rank dependent endogenized long run asset structure has following properties.

[4] It is known that $\mathcal{A}^\rho$ constitutes a submanifold complex of $\mathcal{A}$. See Hirsch [21]

Proposition 4 *(i) $\pi^{\rho} \pitchfork \mathcal{A}^{\rho}$ for integers $1 \leq \rho \leq n$. (ii) $\Gamma^{\rho} \pitchfork \mathcal{A}^{\rho}$ for any $\beta \in \mathbb{R}^{S}_{++}$ and integers $1 \leq \rho \leq n$. (iii) $\Gamma^{\rho} \cap \mathcal{A}$ is generic, since it is dense and open.*

Proof. (i) The linear map $D_y \pi^{\rho}$ is surjective everywhere in Y. (ii) This property does not change for any $\beta \in \mathbb{R}^{S}_{++}$. (iii) Immediate consequence of the transversality theorem for maps. Since each set $\pitchfork(\Gamma^{\rho}, \mathcal{A}; \mathcal{A}^{\rho})$ is residual, their intersection is residual. ■

Definition 5 *Denote Ψ^{ρ} the vector bundle defined by (i) a basis $P^{\rho} = \left\{ P \in \mathbb{R}^{l(S+1)}_{++} : rank(\Gamma^{\rho}(P_1, \phi)) = (n - \rho) \right\}$, and (ii) orthogonal income transfer space $L^{\perp} \subset \langle \Gamma^{\rho}(P_1, \phi) \rangle^{\perp}$,*

$$\Psi^{\rho} = \left\{ \begin{array}{c} (P, \langle \Gamma^{\rho}(P_1, \phi) \rangle^{\perp}, L^{\perp}) \in P^{\rho} \times G^{S-n+\rho}(\mathbb{R}^S) \times G^{S-n}(\mathbb{R}^S) : \\ L^{\perp} \subset \langle \Gamma^{\rho}(P_1, \phi) \rangle^{\perp} \end{array} \right\}. \tag{10}$$

We thus have defined a fiber bundle Ψ^{ρ} of codimension $l(S+1) - 1 - \rho^2$ containing the spot price system and income transfer space consisting of a base vector P^{ρ} and fiber $G^{S-n}(\mathbb{R}^{S-n+\rho})$. We can now state the main result.

Theorem 2 *There exists a pseudo FE with production $(\bar{x}, \bar{y}), (\bar{P}, \bar{L}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ for generic endowments. Moreover, by the relational propositions, a FE with production $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}^{n}_{++}$ exists for generic endowments.*

Proof. By (Proposition 4) and using (Definition 6) define an evaluation map Z^{ρ} on $\Psi^{\rho} \times \mathbb{R}^{l(S+1)m}_{++}$, where denote $\Omega = \mathbb{R}^{l(S+1)m}_{++}$ the set of the economy's total initial endowments, such that the excess demand map $Z^{\rho} : \Psi^{\rho} \times \Omega \to N$.

For the Arrow-Debreu agent have

$$Z^{\rho}_1 : \Psi^{\rho} \times \Omega \to N \tag{11}$$

The evaluation map is a submersion, since $D_{\omega_1} Z^{\rho}_1 \ \forall \omega_1 \in \Omega$ is surjective everywhere. $\exists$ for each $\omega_1 \in \Omega$

$$Z^{\rho}_{1, \omega_1 \in \Omega} : \Psi^{\rho} \to N \pitchfork_{\omega \in \Omega_{\rho}} \{0\}, \tag{12}$$

where $\{0\} \subset N$, and $\rho = 0$. The dimension of the preimage $Z^{-1}_{1, \omega_1 \in \Omega}(\{0\})$ is $l(S + 1) - 1$. By Thom's parametric transversality argument[5], it follows that the subset $\Omega_{\rho} \cap \Omega$ is generic since it is open and dense. Equilibria exist. By the equivalence propositions know that full rank financial markets equilibria with production exist.

For all $1 \leq \rho \leq n$ the preimage of the rank reduced evaluation map has dimension $l(S+1) - 1 - \rho^2$. This implies that for generic endowments $\omega \in \cap_{\rho} (\Omega_{\rho})$, for $\rho = 1, ..., n$, there is no reduced rank equilibrium, since for $Z^{\rho}_1(., \omega)$ the set of $\{0\} = \varnothing$. ■

4 The case of convex piecewise linear production manifolds

We replace the non-linearity assumption in F(1) with F(2)

[5] See i.e. Hirsch for an exposition of Thom's parametric transversality theorem [21]. For more on transversality see R. Abraham and J. Robbin (1967), Transversal Mappings and Flows. (W.A. Benjamin).

Assumption F(2) $\phi_j : \mathbb{R}^m_- \times S \to \mathbb{R}^{nS}_+$ piecewise linear $\forall j$.

We regularize the long run convex non-smooth production manifolds $Y_j\,|_{\bar{z}} = \mathbb{R}^{nS}$ by convolution and show that these convolutes, denoted Φ_j, are compact and smooth manifolds approximating the piecewise linear production manifolds. Define the state dependent convolute for firm j

$$\left(\lambda_\sigma * \phi_j(y)\right)_j(s) = \begin{cases} \int_{\mathbb{R}^m_-} \left(\lambda_\sigma(\zeta)\phi_j(y-\zeta)d\zeta\right)_j(s) & \text{for } y \in U_\sigma \\ 0 & \text{otherwise} \end{cases} \quad \forall s, j \tag{13}$$

where $y \in U_\sigma$, and $U_\sigma = \{y \in U : B(y,\sigma) \subset U\}$. Continuity of $\phi_j(s)$ implies the existence of a distance $\sigma = \inf_t(\sigma_t)$, where $0 < \sigma < 1$. Associate with measure $\sigma \in [0,1]$ the manifolds λ_σ defined by

$$\lambda_\sigma(y)(s) = \frac{1}{\sigma}\lambda\left(\frac{y}{\sigma}\right)(s)\,,\ \forall s \tag{14}$$

Proposition 5 *Each regularized manifold $\partial\tilde{Y}_j\,|_z$ defined by the convolute $\Phi_j(s)$, $\forall s$, is C^∞ and compact.*

Proof. For each j, denote the state dependent convolute

$$\Phi(s)_j = (\lambda * \phi(y))_j(s) = \int_{\mathbb{R}^m_-} (\phi(y-\zeta)_j \lambda_\sigma(\zeta)d\zeta)_j(s) \tag{15}$$

Can restrict domain of integration to Int supp(λ). See (Dieudonnè [11]). Let $lim_{p\to 0} y^p = -\infty$, and let $lim_{p\to\infty} y^p = 0$. Denote $A = (\{-\infty\}, 0)^m \subseteq \mathbb{R}^m_-$. For any $z \in \mathbb{R}^n_+$, $\exists$ $y\,|_z \in A$. Denote the compact subset associated with any z, $A\,|_z$. $A\,|_z \subseteq A$. Then the image of the continuous map $\Phi : A\,|_z \to \partial\tilde{Y}\,|_z$ is compact by surjectivity of Φ. ■

Proposition 6 *For any j and C^∞ kernel λ, $\lambda*$ is bounded and converges to identity ϕ, it satisfies $\left|(\lambda_\sigma * \phi)_j(s) - \phi(s)\right|_j \leq \varepsilon(s)_j\ \forall s$.*

The proof is in the appendix.

Corollary 1 *For any $\partial\tilde{Y}_j\,|_{\bar{z}}$, there exists a pseudo FE with production $(\bar{x},\bar{y}),(\bar{P},\bar{L})$ $\in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ for generic endowments. Moreover, by the relational propositions a FE with production $(\bar{x},\bar{y},\bar{z}),(\bar{p},\bar{q}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}^n_{++}$ exists for generic endowments.*

Proof. Obvious. ■

5 Appendix

Proof (Proposition 7). Define for every $s \in S$ $diam(\lambda)$ with supp(λ) contained in the unit ball $\mathbb{R}^{m,}_-$. Let $\varepsilon(s) = y(\phi, diam(\lambda))_j(s)$. Now, for any C^∞ kernel λ can define ϕ in $\mathbb{R}^{lS}$ such that for all $s \in S$

$$\left((\lambda * \phi_j - \phi)(y)\right)_j(s) = \int_{\mathbb{R}^m_-} \left[(\phi(y-\zeta)_j - \phi(y))\,\lambda(\zeta)^{\frac{1}{2}}\,d\zeta\right]_j(s)\,, \tag{16}$$

by Cauchy inequality and Fubini's theorem, and since mass of λ is equal to one, and ζ ranges over its support, we obtain

$$\left(\int_{\mathbb{R}^m_-} \left|\lambda * \phi_j - \phi)(y)\right|^2 dy\right)_j (s) \quad \leq \quad \sup_{\|\zeta\| \leq \sigma} \left(\int_{\mathbb{R}^m_-} \left|(\phi(y-\zeta)_j - \phi(y))\right|^2 dy\right)_j (s) \tag{17}$$

Thus it follows that

$$\left(\int_{\mathbb{R}^m_-} \left|\lambda * \phi_j - \phi)(y)\right| dy\right)_j (s) \quad \leq \quad \sup_{\|\zeta\| \leq \sigma} \left(\int_{\mathbb{R}^m_-} \left|(\phi(y-\zeta)_j - \phi(y))\right|^2 dy\right)_j^{\frac{1}{2}} (s) \tag{18}$$

denoted $y(\phi, diam(\lambda))_j(s)$. It converges to zero when $diam(\lambda)$ converges to zero. It is bounded above since

$$y(\phi, diam(\lambda)_j(s) \leq c \left(\sum_{k=1}^{m} \left|D^k \phi\,(y)\right|_j^2 (s)\right)^{\frac{1}{2}} \tag{19}$$

where $c = k_1\sigma$. k_1 is a constant of differentiation, and σ a distance. ■

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