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By

E Randon, University of Bologna and P Simmons, University of York

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Strategic Consumption Complementarities: Can Price Flexibility Eliminate Inefficiencies and Instability?*

E. Randon†and P. Simmons‡

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Abstract

Generally, two facts occur with strategic complementarities and fixed prices: i) the equilibria are multiple, and ii) if the complementarities are strong, the law of demand is violated and the equilibrium is unstable. In this paper, we analyse the effect of price flexibility on these features as well as on market welfare properties. Assuming an exchange economy with H agents consuming two goods with one strategic complement, we show that flexibility of prices may remove both the multiplicity of the equilibria and the instability of behaviour when the externalities are strong. The equilibrium with beneficial externality is shown to be Pareto optimal while the equilibrium with detrimental externality requires corrections.

Keywords: Externalities, Strategic Interaction, Stability

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In certain markets, price and quantity rise over time although there is no real shortage of the good in question. Examples are markets for fashion goods, holiday resorts, gold, some financial products. One possible explanation is that the aggregate demand curve is rising over time whilst supply factors are unchanged. Another one is that, from period to period, the system is not in equilibrium and the variation of prices and quantities is just the dynamic adjustment. For the markets we cite above, the more likely rationale is a form of herd behaviour. This is essentially a disequilibrium phenomenon: for example, you go on holiday to Costa Rica and this makes we want to go too. The individual demands shift and hence so does the aggregate market equilibrium. The externality effect of herd behaviour causes both a quantity and a price adjustment. If in the process of quantity adjustment at the current price, individual demand is increasing, then price adjustment should be considered. Excess aggregate demand should lead to an increase in price which will choke off the increasing excess demand caused by the herd behaviour/externality effects. The current literature does not integrate these two effects. On the one hand, the ultimate outcome of herd behaviour effects is explored with constant prices (Cornes and Homma (1979), Cornes (1980)) and typically the equilibrium with strong herding effects is shown to be unstable-eventually everyone goes to Costa Rica on holiday, or with negative herding eventually no one goes to Costa Rica. On the other hand, if there is no externality effect and prices adjust according to Walrasian tatonnement process, then stability depends typically on forms of assumptions which exclude complementarities (e.g. gross substitutability, the aggregate form of the weak axiom of revealed preference). In this paper we try to combine the price and quantity effects considering both the equilibria and the dynamics of disequilibrium. In particular we show that although at

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†Department of Economics, University of Bologna. E-mail: emanuela.randon@unibo.it

‡Department of Economics and Related Studies, University of York. E-mail: ps1@york.ac.uk

fixed prices there is spiralling instability of the equilibrium with strong externality, once we allow for price adjustment in a traditional way, this equilibrium becomes stable.

It has long been known that, with strategic complementarities with fixed prices, there may be multiple equilibria with varying efficiency properties (Cooper and John (1988)). If the strategic effects are very strong, due to strong interdependence of payoffs, then the equilibria may exhibit perverse comparative statics and instability. Indeed, Cornes and Homma (1979), Cornes (1980) and Yeung (2002) have noticed that with strong interdependence of consumption the law of demand may be violated-with a higher price, quantity demanded may increase. In fact this is true whether the externalities are beneficial (agent i's utility increases with the activity of agent j) or detrimental (agent i's utility decreases with the activity of agent j). This anomalous behaviour occurs because of the feedback effects between individuals. For example, if the price of a good rises, then one individual will decrease consumption given the consumption of others, but as others adjust reducing their consumption, with detrimental externalities, the first individual is induced to actually increase consumption in response.

The welfare consequences of the anomalous behaviour are relevant. Generally, with strong strategic complementarities aggregate demand for a good at a given price is lower in a market equilibrium than in a Pareto optimum, if the externalities are detrimental, or it is higher, if the externalities are beneficial. There is under-consumption of the good in the private market solution with detrimental externalities and over-consumption with beneficial externalities. However, the economic relevance of a Nash equilibrium with strong interactions is limited since the equilibrium tends to be unstable under the usual Cournot adjustment mechanism if prices are held fixed. Indeed for this reason Cooper and John confine their attention to stable (and symmetric) Nash equilibria ruling out the equilibria with bad behaviour. Cornes and Homma (1979) examine a dynamic system in three variables: the consumption of a good subject to external effects by each of two individuals and its relative price. Taking the price as constant, they find the instability result referred to above.

In this paper we mainly analyse the market equilibrium features and the welfare properties of strategic interactions when both prices and quantities adjust. In this scenario, there are three questions to be answered: first, what are the possible stationary points, second, what are their stability properties and third, what is the proper welfare analysis. We focus on externalities generating strategic complements. Preferences are homogenous and quasi linear but individual heterogeneity is allowed through different individual endowments. In such a context we find that there is a unique Nash-market equilibrium: market equilibrium effects eliminate the multiplicity of equilibria typical of the partial equilibrium theory with fixed prices. Moreover, if, for each good, price adjusts to give zero excess demand, and, in goods affected by strong externality effects, individuals are making mutual best responses, then the equilibrium may be stable even if it is unstable when prices are fixed. Thus, although the strategic interaction between individuals gives forces pushing towards multiple equilibria and instability, we find that these same forces are dissipated once sufficient price adjustment through tatonnement is included.

Perhaps more surprisingly, we find that with beneficial externalities the Nash-market solution is Pareto optimal despite the presence of externalities in the market. So, the combination of mutual best responses and price flexibility can overcome the usual market failure due to externalities in the scenario considered. But if there are detrimental externalities it may be necessary to destroy endowment to ensure that the Nash-market solution is Pareto optimal.

We analyse the dynamics in the context of an exchange economy with H individuals and two commodities, only one of which causes mutual externalities. The assumption of quasi linear preferences puts all the income effects on the non externality inducing good and so one might expect unusually well-behaved and strong results in this case. For similar reasons, this assumption has been introduced by Diamond and Mirrlees (1973). In a partial equilibrium framework, even with this restriction, there are multiple equilibria (corner solutions arise) giving unstable behaviour when strong interactions occur. Instead we show that with price flexibility, uniqueness of the equilibrium occurs in a general equilibrium context. Only one equilibrium arises with equal consumption of the externality inducing good for each individual. Other equilibria that would result in a partial equilibrium scenario are ruled out because they lead to excess demand or excess supply. Next we look at the stability properties and we find conditions on the speed of adjustment of prices and quantities which will sustain an equilibrium displaying anomalous behaviour. The features of the adjustment process required for long run behaviour with strong strategic interactions are different in small and large markets, respectively. With price flexibility there is stability more likely in large than in small markets. Finally we consider the welfare properties of the outcome and we show the Pareto optimality of the equilibrium selected.

The plan of the paper is to introduce the framework in Section 1. In the following sections, we focus on strategic complementarities and we define them in Section 2. Next we analyse the nature of equilibria with strong and weak strategic complementarities (Section 3) and their stability properties (Section 4). In the following section, we consider the local comparative statics of the equilibrium. The welfare properties are presented in Section 6. In Section 7, we analyse qualitatively the implications of strategic substitutes for the nature of equilibrium. We argue that in this case the uniqueness of equilibrium is not guaranteed. In particular we show that anomalous behaviour never occurs with strategic substitutes, justifying our focus on the analysis of strategic complementarities. An example is then considered together with some numerical simulations. Final considerations conclude the paper.

1 Notation

To clarify the notation, let individuals be labelled $h = 1, ..., H$, let good x cause the externality and the second good be labelled y so that preferences for h are given by $u(x_h, y_h, \Sigma_{k \neq h} x_k/(H-1))$, where for example x_k denotes the quantity of good x consumed by individual k. Preferences are strongly monotone in own consumption of each good and strictly quasiconcave in all arguments. Individual consumption patterns are affected by the average consumption level in the market. For example, individuals' preferences for a fashion good are affected by the proportion of other individuals who consume the good. In fact, congestion cases, where the total rather than average consumption of others affects the preferences of a given individual, can also be included by a simple redefinition of utility.

To simplify the notation, we put $\Sigma_{k \neq h} x_k/(H-1) = z_h$. The relative price of good x is p (the price of y is normalised to unity). There are aggregate initial endowments X, Y of the two goods with individual endowments X_h, Y_h for individual h. Consumer h has wealth $m_h = pX_h + Y_h$ in units of good y.

The individual best responses or reaction curves are defined by

$$\begin{aligned} & \{RC_X(p, X_h, Y_h, z_h), RC_Y(p, X_h, Y_h, z_h)\} \\ = \ & \arg\max_{x_h, y_h} \{u(x_h, y_h, z_h) | px_h + y_h \leq m_h, x_h \geq 0, y_h \geq 0\}, \end{aligned}$$

or substituting out the budget constraint

$$\begin{aligned} & RC_X(p, X_h, Y_h, z_h) \\ = \ & \arg\max_{x_h} \{u(x_h, p(X_h - x_h) + Y_h, z_h) | pX_h + Y_h \geq x_h \geq 0\} \ with \ h = 1, ..., H. \end{aligned}$$

Note that with identical preferences the reaction curves are identical functions for all individuals.

An equilibrium of the system is p, x_h such that

$$\begin{aligned} x_h &= RC_X(p, X_h, Y_h, z_h) \qquad \text{for all } h \\ X &= \Sigma_h RC_X(p, X_h, Y_h, z_h). \end{aligned}$$

That is there are mutual best responses (in both goods) and zero excess demand for x (by Walras Law this then implies no excess demand for y).

It is very difficult to characterise equilibrium outcomes generally under preference and endowment heterogeneity. For example the Sonnenschein-Mantel-Debreu type of results indicates that if individual consumer demands satisfy only the basic properties of rationality then market equilibrium may take almost any form. Moreover, to ensure that there is a common reaction curve, as in the Cooper and John case in a general equilibrium framework, restrictions must be imposed on the possible heterogeneity between individuals. Assuming identity of both preferences and endowments means that there is little reason to trade in equilibrium. One case which allows some heterogeneity but retains identical reaction curves is that of identical preferences, quasilinear in the nonexternality inducing good, explaining why we focus on this case.

Indeed, to have a common reaction curve for all individuals when there is endowment heterogeneity, we should assume that the reaction curve is independent of the initial endowment. So, with a slight abuse of notation we require

$$RC_X(p, X_h, Y_h, z_h) = RC_X(p, z_h).$$

Since the reaction curve solves the equation

$$\frac{\partial u(x_h, p(X_h - x_h) + Y_h, z_h)}{\partial x_h} = p \frac{\partial u(x_h, p(X_h - x_h) + Y_h, z_h)}{\partial y_h},$$

for x_h to be independent of the initial endowment requires the mrs to be independent of y_h, and hence quasilinear preferences of the form

$$u(x_h, p(X_h - x_h) + Y_h, z_h) = U(x_h, z_h) + b(p(X_h - x_h) + Y_h).$$

Such forms of preferences are used in the literature (see for example Diamond and Mirrlees (1973)).

Often we use numerical subscripts for partial derivatives so e.g. $U_1(\cdot)$ is the first partial derivative of $U(\cdot)$ with respect to its first argument, $U_{12}(\cdot)$ the second order cross partial, etc.

With identical quasilinear preferences, the conditions for equilibrium reduce to

$$\begin{aligned} x_h &= RC_X(p, z_h) \qquad \text{for all } h \\ X &= \Sigma_h RC_X(p, z_h). \end{aligned}$$

Notice that this implies that if $p, x_1, ..., x_H$ is an equilibrium and $s_1, ..., s_H$ is any permutation of $x_1, ..., x_H$ then so is $p, s_1, ..., s_H$. That is at given prices, either equilibria are symmetric with $x_h = X/H$ or come in $H!$ tuples. For example if $H = 3$ then if x_1, x_2, x_3 satisfies

$$\begin{aligned} x_1 &= RC_X(p, (x_2 + x_3)/2) \\ x_2 &= RC_X(p, (x_1 + x_3)/2) \\ x_3 &= RC_X(p, (x_1 + x_2)/2) \\ X &= x_1 + x_2 + x_3 \end{aligned}$$

any permutation such as $s_1 = x_2, s_2 = x_1, s_3 = x_3$ is also an equilibrium. Also if $p, x_1 = x_2 = x_3 = X/3$ satisfies

$$X/3 = RC_X(p, X/3),$$

then $p, x_1 = x_2 = x_3 = X/3$ is an equilibrium. Generally this symmetric equilibrium will exist and there will be a unique p yielding the symmetric equilibrium.[1]

2 Strategic complementarities

In the rest of the paper we concentrate on strategic complementarities. This is the most interesting case in terms of uniqueness of equilibrium, stability behaviour and welfare properties. Anomalous behaviour never occurs with strategic substitutes. This fact justifies concentrating on complementarities.

We define

Definition 1: Strategic complementarities for good x to be the case in which

$$\partial RC_X(p, z_h)/\partial x_l > 0 \qquad l \neq h.$$

Cooper and John (1988) take a single variable x_h affecting each individuals payoff $u_h(x_h, x_k, \theta_h)$ where θ_h is an exogenous factor and restrict attention to identical preferences and symmetric Nash equilibria which are stable under the usual Cournot adjustment process. They do not consider an underlying market equilibrium process. In terms of the strategic interaction between individuals, our model generalises theirs in "endogenising" θ_h; we define $\theta_h = p$ and

$$u(x_h, z_h, p) = U(x_h, z_h) + b(p(X_h - x_h) + Y_h).$$

Our definitions of the nature of strategic interaction follow theirs[2]. Basically with complementarity, at fixed prices, the reaction curves have a positive slope.

Definition 2: Strong strategic complementarities for good x hold if

$$\partial RC_X(p, z_h)/\partial z_h > 1 \qquad \text{for any } p, z_h.$$

[1] For quasi concavity of quasilinear pereferences we need $U_{11} < 0$. But the reaction curve solves

$$bp = U_1(x_h, z_h)$$

from which

$$\frac{\partial RC}{\partial p} = \frac{b}{U_{11}} < 0$$

So there is at most one solution for p. With appropriate corner conditions on utility (eg both goods will always have positive consumption) there is a unique solution for p.

[2] Cooper and John consider a utility function, $u_h(x_h, x_k, \theta_h)$, with the usual well-behaved properties. They define the following cases: i) positive (negative) spillovers: $\partial u_h/\partial x_k > 0$ ($\partial u_h/\partial x_k < 0$); ii) strategic complementarity: $\partial^2 u_h/\partial x_h \partial x_k > 0$; iii) strategic substitutability: $\partial^2 u_h/\partial x_h \partial x_k < 0$

In our terms in their model the reaction curve solves

$$\frac{\partial u_h(x_h, x_k, \theta_h)}{\partial x_h} = 0 \tag{1}$$

The slope of the reaction curve is

$$\frac{\partial x_h}{\partial x_k} = -\frac{\partial^2 u_h}{\partial x_h \partial x_k} / \frac{\partial^2 u_h}{\partial x_h^2}$$

which is positive with strategic complementarities, negative with strategic substitutability. We have directly identified the positive slope with the case of strategic complementarities.

Note that $\partial RC_X(p, z_h)/\partial x_l = [\partial RC_X(p, z_h)/\partial z_h]/(H-1)$ so we can equivalently define strong complementarities as

$$\partial RC_X(p, z_h)/\partial z_h = (H-1)\partial RC_X(p, z_h)/\partial x_l > 1 \Rightarrow \partial RC_X(p, z_h)/\partial x_l > \frac{1}{H-1}.$$

Effectively there are strong complementarities if the marginal effect of a unit increase in the average consumption of others exceeds unity. Note that even if there are strong complementarities in this sense, it is quite possible for i to react to a unit increase in consumption by l less than proportionally if the consumption of all others than i, l remains fixed. In particular in a large economy (H high) the influence of any one individual can become small.

Externalities are weak if the opposite inequality holds. Strong complementarities restrict the relative slopes of the reaction curves for constant prices.

We also want to distinguish cases in which the externality is beneficial or detrimental.

Definition 3 Externalities are beneficial if $\partial U(\cdot)/\partial z_h > 0$ and are detrimental if $\partial U(\cdot)/\partial z_h < 0$.

3 Nature of Equilibrium with Strategic Complementarities

We can characterise the equilibrium points with strategic complementarities:

Proposition 1 *With identical preferences quasilinear in y and with strategic complementarity, there is a unique equilibrium with $x_h = X/H$ for each h; p is equal to the marginal utility of x_h at $x_h = X/H, z_h = X/H$.*

(1) With a common reaction curve in x and complementarities, any equilibrium point has identical consumption of good x for each individual (see Appendix A1).

(2) $x_h = 0$ cannot be an equilibrium since as $X > 0$ there is excess supply of x. Similarly an equilibrium cannot involve $y_h = 0$ since then there is excess supply of y.

(3) With either weak or strong complementarity, the unique equilibrium point has $x_h = X/H$. Since an equilibrium must have equal consumption of good x by each individual, the only zero excess demand position has them each consuming an equal share of the aggregate endowment of x.

The equilibrium prices are set by

$$U_1(X/H, X/H) = bp$$

where the subscript 1 refers to the first partial of $U(\cdot)$. And equilibrium quantities of y are

$$y_h = p(X_h - X/H) + Y_h.$$

On the other hand if endowments are identical there is no trade in the unique equilibrium which has

$$x_h = X/H = X_h, y_h = Y/H = Y_h.$$

Note that in a partial equilibrium with quasi linear preferences, multiple equilibria may arise. In addition to the interior equilibrium, corner solutions may occur. However if we allow for price flexibility, only the interior equilibrium exists, since the corner consumption will involve excess demand or excess supply.

4 Stability Properties

We also want to analyse the dynamic behaviour of the system allowing for price flexibility. In a two good world, because of the budget constraint and homogeneity of degree zero, we can normalise the price of y to unity and look at tatonnement in the first market. So, there is an $H+1$ equation system:

$$\begin{aligned}\frac{dx_h}{dt} &= \delta[RC_X(p, z_h) - x_h] \qquad \text{all } h \\ \frac{dp}{dt} &= \delta_p\left[\Sigma_h RC_X(p, z_h) - X\right].\end{aligned}$$

The $\delta's$ are positive and have the interpretation of speeds of adjustment in quantities. These are equal for all individuals, but adjustment in prices can be faster or slower[3]. Implicit in the quantity adjustment rule is the idea that each individual expects all other individuals to hold their quantities constant. Each individual partially adjusts their consumption to the disequilibrium between the best response assuming quantities of all others are constant and their own current consumption. So each individual is expecting extreme inertia by all others. Of course many other expectations could be used. A rational individual might respond just with his equilibrium consumption (either his Nash equilibrium consumption at fixed prices, or he may anticipate the price adjustment and respond with Nash equilibrium consumption at equilibrium prices).[4] One justification for making the Cournot adjustment assumption comes from experimental evidence. Cheung and Friedman (1997) devise a variety of one-shot games and try to assess how subjects in experimental play form expectations about rivals reactions. In terms of statistical testing they find that three forms of learning are prevalent: the Cournot form used here, a long memory form in which the whole history of past play is important and an adaptive form. However in terms of point estimates the modal pattern is close to Cournot. One way of justifying this is to argue that implicitly individuals are acting with bounded rationality or with high discount rates or that in the background there is a risk that the market will end (Fudenberg and Maskin (1986)).

To see the local dynamics we can approximate the system by

$$\begin{bmatrix} dx_1/dt \\ . \\ . \\ dx_H/dt \\ dp/dt \end{bmatrix} = \begin{bmatrix} -\delta & \delta\partial RC_X/\partial x & . & \delta\partial RC_X/\partial x & \delta\partial RC/\partial p \\ \delta\partial RC_X/\partial x & -\delta & & & \delta\partial RC/\partial p \\ . & . & & & . \\ \delta\partial RC_X/\partial x & . & & -\delta & . \\ \delta_p\partial RC_X/\partial x & . & . & \delta_p\partial RC_X/\partial x & \delta_p\Sigma_h\partial RC_X/\partial p \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ x_H \\ p \end{bmatrix}.$$

Note that at a non-symmetric equilibrium the terms $\partial RC_X/\partial x$ may be evaluated at different points for each individual and so may have different values in each of the first H rows but have identical values within a row, and in the last row may have different values in each column.

In our analysis we know that any equilibrium must involve equal consumption of X by all individuals. In this symmetric equilibrium case the Jacobian has the form

$$\begin{bmatrix} -\delta & \delta a/(H-1) & . & \delta a/(H-1) & \delta c \\ \delta a/(H-1) & -\delta & . & \delta a/(H-1) & \delta c \\ . & . & . & . & . \\ \delta a/(H-1) & . & \delta a/(H-1) & -\delta & \delta c \\ \delta_p a & \delta_p a & . & \delta_p a & \delta_p Hc \end{bmatrix} \tag{2}$$

[3]Notice that the speed of adjustmnet in prices is not independent of the units in which quantities are measured.

[4]We have explored relaxations of these assumptions. For example we have considered the following cases: i) different deltas for the quantity adjustment process; ii) different speed in the reaction curve adjustment process; iii) proportional reaction curves and price adjustment process. But in each case, the simplicity of the local dynamics is lost.

where $a = \partial RC_X/\partial x$ and $c = \partial RC_X/\partial p$. This has $H-1$ repeated eigenvalues of $-\delta(1+a/(H-1))$ and a pair of roots

$$\lambda_{1,2} = \frac{(a-1)\delta}{2} + \frac{H\delta_p c}{2} \pm \frac{\{(a-1)\delta + H\delta_p c]^2 + 4Hc\delta\delta_p\}^{0.5}}{2}. \tag{3}$$

Compare these roots with those that would emerge when prices are fixed (the partial equilibrium case). If $H = 2$ we have 2 real roots

$$\delta(a-1), -\delta(1+a). \tag{4}$$

If $H > 2$ we have roots

$$\delta(a-1) \text{ and } H-1 \text{ roots of } -\delta[1+a/(H-1)].$$

So as H increases we always keep the root $\delta(a-1)$ and add extra roots of $-\delta[1+a/(H-1)]$. What does this mean? The basic dynamic system with fixed prices has individuals in a symmetric position. But locally the strategic interaction between them has asymmetric forces. When $H = 2$ applying the linearisation for local stability analysis and so transforming from the original space of individual consumptions to the space spanned by the eigenvectors, we create two new "local" consumers who are asymmetric in their dynamics. Locally one moves according to $\exp(\delta(a-1)t)$ and the other according to $\exp(-\delta(1+a)t)$. Moreover adding further individuals, still with fixed prices, the additional consumers just replicate the second type of consumer: the first local type still has the same local dynamics $\exp(\delta(a-1)t)$ but each of the other types have local dynamics $\exp(-\delta[1+a/(H-1)]t)$.

Note also that if $a = 0$ for any H we have H repeated roots of $-\delta$, that is when $a = 0$ and the reaction curves display no effective strategic interaction, the "local " consumers just coincide with the original consumers and symmetry between the individuals for all H is preserved.

In the general equilibrium system with price adjustment, if $H = 2$ we have three roots

$$-\delta(1+a), \frac{\delta(a-1)}{2} + c \pm \frac{(\delta^2 a^2 - 2\delta^2 a - 4\delta ac + \delta^2 + 4\delta c + 4c^2 + 8c\delta_p a)^{0.5}}{2}. \tag{5}$$

Thus with price adjustment we keep one real root but the "dominant local consumer" becomes intermixed and cyclical with the price adjustment-the real root $\delta(a-1)$ becomes complex. Adding consumers just adds to the repeated real roots but keeps the complex dominant consumer.

If $\delta_p = 0$ (in which case prices are fixed) the non repeated eigenvalues in (5) become

$$\lambda_{1,2} = \frac{(a-1)\delta}{2} \pm \frac{\delta(a-1)}{2}$$

or:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \delta(a-1) \end{aligned}$$

and we recover the two local consumers with real roots as in (4) and a zero root for the price adjustment.

When $\delta_p = 0$ local stability requires that $0 < a < 1$. This result generalises the stability condition cited by Cornes and Homma to an arbitrary number of individuals. Note that without price flexibility ($\delta_p = 0$), the equilibrium with anomalous behaviour ($a > 1$) is always unstable. Allowing for price flexibility:

Proposition 2 [5] *With identical preferences quasilinear in y*

(1) with weak strategic complementarity the unique equilibrium is locally stable

(2) with strong strategic complementarity if $[(a-1)\delta + H\delta_p c] < 0$ the unique equilibrium is locally stable

In a small market, stable long run anomalous behaviour depends principally on the relative speed of the two adjustment mechanisms. If the adjustment speed of price is sufficiently faster than the adjustment of quantities, we would expect to see anomalous behaviour persist in the long run. But in a large market stable anomalous behaviour is very likely to occur, whatever the relative speed of the two adjustment processes. This arises because when H is high the stabilising effects of the tatonnement process outweigh the unstable effects of strategic interaction.

5 Local Comparative Static Effects of Price and Income Changes

With strategic complementarities, there is a unique Nash-market equilibrium. If either these externalities are weak, or price adjustment is sufficiently strong or the market is large, the equilibrium is locally stable. In these circumstances there is interest in the comparative statics of the equilibrium wrt the aggregate initial endowment X.

With no externalities individual preferences are $U(x_h)+by_h$ and at the market/Nash equilibrium $x_h = X/H, bp = U_1(X/H)$. Without externalities, an increase in the aggregate endowment leads to a fall in the price.

With externalities the equilibrium price is defined by

$$RC_X(p, \frac{X}{H}) = \frac{X}{H}.$$

Variations in the endowment lead to equilibrium price variations according to

$$\frac{\partial RC_X}{\partial X/H} dX/H + \frac{\partial RC_X}{\partial p} dp = dX/H$$

$$\frac{dp}{dX/H} = \frac{1 - \partial RC_X/\partial(X/H)}{\partial RC_X/\partial p}. \tag{6}$$

With strong complementarities the numerator and denominator are both negative so an increase in the endowment leads to an increase in the equilibrium price. This result confirms that strong complementarity leads to anomalous behaviour. With weak complementarities the numerator is positive so then the equilibrium price falls as the endowment rises.

Proposition 3 *With strong complementarities, an increase in the aggregate endowment of x leads to a rise in its equilibrium price.*

On the other hand due to the quasi linearity, the equilibrium price in all cases is unaffected by the aggregate endowment of y.

[5] See Appendix A2 for a demonstration of this proposition.

6 Welfare Effects

First notice that the Nash-market equilibrium solution with beneficial externalities is actually Pareto optimal despite the presence of external effects. A Pareto optimal outcome will solve

$$
\begin{aligned}
& \max U(x_1, \Sigma_{k\neq 1} x_k/(H-1)) + by_h \\
st \qquad & U(x_k, \Sigma_{h\neq k} x_h/(H-1)) + by_k \geq \overline{u}_k \quad all\ k \neq 1 \\
& x_1 + ... + x_H \leq X \\
& y_1 + ... + y_H \leq Y.
\end{aligned}
$$

With beneficial externalities, the constraints are binding. Thus, we can use the resource constraints to eliminate x_H, y_H and to introduce multipliers μ_k

$$
\begin{aligned}
& \max_{x_{-H}, y_{-H}} U(x_1, (X-x_1)/(H-1)) + by_1 + \sum_{k=2}^{H-1} \mu_k [U(x_k, (X-x_k)/(H-1))) + by_k - \overline{u}_k] \\
& + \mu_H [U(X - \Sigma_{h=1}^{H-1} x_h, \Sigma_{h=1}^{H-1} x_h/(H-1)) + b(Y - \Sigma_{h=1}^{H-1} y_h) - \overline{u}_H].
\end{aligned}
$$

The foc's wrt the nonexternality inducing good y imply that there is a common value of the marginal utility of income $\mu_k = 1$:

$$
\begin{aligned}
b &= b\mu_H \\
\mu_k b &= b\mu_H \qquad k = 2, ..., H-1.
\end{aligned}
$$

Then for consumptions of the externality good

$$
U_1(x_k, (X-x_k)/(H-1)) - U_2(x_k, (X-x_k)/(H-1))/(H-1) =
$$

$$
= U_1(X - \Sigma_{h=1}^{H-1} x_h, \Sigma_{h=1}^{H-1} x_h/(H-1)) - U_2(X - \Sigma_{h=1}^{H-1} x_h, \Sigma_{h=1}^{H-1} x_h/(H-1))/(H-1)
$$

for $k = 1, ..., H-1$.

However at the symmetric Nash-market equilibrium values $x_h = X/H$, $\Sigma_{k\neq h} x_k/(H-1) = X/H$. The above becomes

$$
U_1(\frac{X}{H}, \frac{X}{H}) - U_2(\frac{X}{H}, \frac{X}{H})/(H-1) = U_1(\frac{X}{H}, \frac{X}{H}) - U_2(\frac{X}{H}, \frac{X}{H})/(H-1)
$$

which is certainly satisfied. Then by nonsatiation and the strict quasiconcavity of preferences, the first order conditions for Pareto optimality are sufficient, and so the Nash-market equilibrium is a Pareto optimum. Thus in this scenario the combination of mutual best responses and perfectly competitive price adjustment overcomes the usual inefficiency of markets in the presence of externalities.

This is in sharp contrast to the results of Cooper and John[6]. It arises because we allow for general equilibrium adjustment through prices-we allow for the trade-offs between the externality

[6] They define a Pareto optimum by

$$
\max_{x's} u_h(x_h, x_k) + u_k(x_k, x_h)
$$

without any constraint. So the FOC for a Pareto optimum is

$$
\frac{\partial u_h(x_h, x_k)}{\partial x_h} + \frac{\partial u_k(x_k, x_h)}{\partial x_h} = 0
$$

But with spillovers the NE solves $\frac{\partial u_h(x_h, x_k)}{\partial x_h} = 0$ and comparing these two equations the solutions must differ.

good and other goods. Nevertheless it is a striking result that by embedding their approach in a wider context, the efficiency results are reversed. Note that this is not the only case in which externalities lead to no efficiency loss-altruism of various kinds can also achieve this e.g. the rotten kid theorem (Becker (1974) or Mas-Colell (exercise 16.C.4, (1995)). Of course in our case the quasilinear preferences are important for the result.

In the market solution how does welfare change if the aggregate endowments change? If individuals have identical preferences and endowments with no externalities ($u_h(x_h, y_h)$), in equilibrium utility is $u_h(X/H, Y/H)$. An increase in the aggregate endowment of x equally shared between the individuals raises the utility of each. Similarly with quasilinear preferences but different endowments (X_h, Y_h) and $u_h(x_h, y_h) = U(x_h) + by_h$, in equilibrium the individuals have equal consumption of good x and p is set at the marginal rate of substitution between x and $y : bp = U'(X/H)$. Individual h has equilibrium utility of $u_h = U(X/H) + b[p(X_h - X/H) + Y_h]$. If the endowment of x changes, equilibrium utility changes by

$$\begin{aligned}\frac{du_h}{dX/H} &= U'(X/H) + b[p'(X_h - X/H) - p] \\ &= bp'(X_h - X/H).\end{aligned}$$

Since $p' < 0$ the individual with higher endowment of x on the average loses and the individual with lower endowment of x gains from the price fall.

What happens in these cases with externalities? Take the case in which the externality is beneficial i.e. $u_2 > 0$. With identical preferences and endowments in the equilibrium with identical consumptions for all individuals, equilibrium utility is $u_h(X/H, y_h, X/H)$ (where $y_h = Y_h$ since endowments are identical) so an increase in the aggregate endowment of x changes equilibrium utility by

$$\frac{du_h}{dX/H} = \frac{du_h(X/H, Y_h, X/H)}{dx_h} + \frac{du_h(X/H, Y_h, X/H)}{dz_h}, \tag{7}$$

whose sign is positive in the case of a beneficial externality.

With quasilinear preferences, $u_h(x_h, y_h, z_h) = U(x_h, z_h) + by_h$, and differing endowments, equilibrium utility is $u_h = U(X/H, X/H) + bp(X_h - X/H) + bY_h$ where $bp = U_1(X/H, X/H)$. Suppose the average endowment changes by $dX/H = \Sigma dX_h/H$ with the change dX being shared amongst individuals according to the terms $dX_h \geq 0$.Then

$$\begin{aligned}\frac{du_h}{dX/H} &= U_1(X/H, X/H) + U_2(X/H, X/H) + bp(\frac{dX_h}{dX/H} - 1) + bp'(X_h - X/H) \\ &= U_2(X/H, X/H) + bp\frac{dX_h}{dX/H} + bp'(X_h - X/H).\end{aligned}$$

With weak strategic complementarity $p' < 0$ and so in these cases the last term is negative for individuals with high endowment of x and positive for low endowment individuals. With a beneficial externality the first two terms are non-negative and so then individuals who start poorer in x gain from an increase in the aggregate endowment of x. However those who are plentifully endowed with x may lose from an increase in its aggregate availability. This is because those rich in x are selling x and those poor in x are buying but its price has fallen. On the other hand with strong complementarity $p' > 0$ and then from the last term the rich will tend to gain and poor to lose from the increased endowment.

The welfare effects of endowment changes when the externality is detrimental ($\partial u_h/\partial z_h < 0$) are interesting. With identical endowments we recall (7), the marginal utility in equilibrium

$$\frac{du_h}{dX/H} = \frac{du_h(X/H, Y_h, X/H)}{dx_h} + \frac{du_h(X/H, Y_h, X/H)}{dz_h}.$$

With detrimental externalities, the first term is positive but the second term is negative. Hence the effect of the endowment increase is ambiguous. Individuals are worse off in equilibrium if the increase in consumption of x by others causes more harm than the gain from increased own consumption of x. If this occurs it implies that Pareto optimality in this case is satisfied only if the endowment restrictions are slack. In that sense it may lead to increased efficiency to throw endowment of x away. With quasi linear preferences and varying endowments, (7) becomes

$$\frac{du_h}{dX/H} = U_2(X/H, X/H) + bp\frac{dX_h}{dX} + bp'(X_h - X/H). \tag{8}$$

With detrimental externalities, once account is taken of price flexibility, overall both individuals may still lose from an increase in X if the first term in (8) is sufficiently negative. In this case, it may be that small is beautiful, a social planner could gain from destroying some endowment of x. Note also that if utility were fully interpersonally comparable, then in the individual welfare change, the term $U_2(X/H, X/H)$ is equal for each individual. Then with weak strategic complementarity, an increase in the aggregate endowment of x reduces the spread of equilibrium utilities, whereas with strong complementarity it increases the spread. In brief:

Proposition 4 *With identical preferences quasilinear in y*

(1) with strategic complementarity and beneficial externalities the equilibrium is Pareto optimal

(2) with strategic complementarity and detrimental externalities there is over-consumption of the good in the market solution.

For the sake of completeness, we qualitatively outline the case of strategic substitutes in the next section. We show that stable anomalous behaviour can never occur, justifying our focus on strategic complementarities.

7 The Effects of Strategic Substitutes

We define strategic substitutability to be the case in which

$$\partial RC_X(p, z_h)/\partial x_l < 0 \qquad l \neq h$$

and say that substitutability is strong if

$$\partial RC_X(p, z_h)/\partial x_l < -1/(H-1) \qquad l \neq h.$$

In terms of preferences with strategic substitutability, the marginal utility of one's own consumption falls as the consumption of others increases.

With strategic substitutability Nash-market equilibria still come in $H!$-tuples so that any permutation of equilibrium consumptions is also an equilibrium. However even with the strong preference assumptions we are making there may be asymmetric equilibria in addition to the symmetric equilibrium of identical consumption of x for all individuals. Since there may be multiple equilibria, the local stability and comparative statics of equilibrium may be of less interest. Nevertheless around the symmetric equilibrium the local stability analysis yields the same formulae for the eigenvalues as in the complements case. To explore the dynamics, with strong strategic substitutes $a < -1$ and so $\lambda_1 = -\delta(1 + a/(H-1))$ is real and positive. The conjugate roots have a negative real part but since the repeated roots are positive, the equilibrium is locally unstable.

With weak strategic substitutes $-1 < a < 0$. Then the repeated root is negative. The conjugate roots have again a negative real part and so with weak substitutes the symmetric equilibrium is locally stable.

As far as the comparative statics is concerned, if p increases the Nash equilibrium quantity in the symmetric equilibrium always falls so that anomalous behaviour can never arise. This is why we have concentrated on complementarity as previously stated.

Again with substitutability an increase in the aggregate endowment of x reduces the spread of equilibrium utilities in the interpersonally comparable utility case.

8 An Example

We consider a two individual case. Take individual utility (identical for the two individuals):

$$u_h = x_h^{\beta}(A + a_1 x_k)^{\alpha}/\beta + y_h, 0 < \beta < 1, \alpha > 0$$

and budget constraint

$$px_h + y_h = m = pX_h + Y_h$$

where X_h, Y_h are the individual initial endowments. Preferences are defined for $(A + a_1 x_k)^{\alpha} > 0$.

The reaction curve for h (RC_h) is

$$x_h = p^{1/(\beta-1)}(A + a_1 x_k)^{\alpha/(1-\beta)}.$$

Since

$$\frac{\partial RC_h}{\partial x_k} = \frac{a_1 \alpha p^{1/(\beta-1)}(A + a_1 x_k)^{(\alpha+\beta-1)/(1-\beta)}}{1-\beta},$$

externalities are positive if $a_1 > 0$ and otherwise are negative. The interactions are strong if $|\partial RC_h/\partial x_k| > 1$. For positive externalities preferences are well defined for each $a_1 > 0$.

The market and Nash equilibrium must involve x_h, x_k, p solving

$$\begin{aligned} X/2 &= p^{1/(\beta-1)}(A + a_1 X/2)^{\alpha/(1-\beta)} \\ x_h &= x_k = X/2 \end{aligned}$$

yielding

$$p = (X/2)^{\beta-1}(A + a_1 X/2)^{\alpha}$$

and in this equilibrium

$$\frac{dp}{dX} = (X/2)^{\beta-1}(A + a_1 X/2)^{\alpha}(\frac{\beta-1}{X} + \frac{a_1\alpha}{2A + a_1 X}).$$

At the market/Nash equilibrium the slope of the reaction curve is

$$\frac{dRC_h}{dx_k} = \frac{\alpha a_1 X}{2(1-\beta)(A + a_1 X/2)}.$$

For particular parameter values Table 1 shows the directions of change of the equilibrium price, of individual utility and the eigenvalues of the tatonnement/Cournot adjustment system linearised about the equilibrium.

Table1: Weak & Strong Strategic Complementarities

$A = 10., a_1 = 4., \alpha = 0.75, \beta = 0.5, \delta = 0.5, \delta_p = 1.0$

X	4.	8.	12.	20.
dx_h/dx_k	0.67	0.92	1.06	1.20
p	6.18	5.76	5.75	5.95
dp	-0.26	-0.03	0.014	0.03
du_h	> 0	> 0	> 0	> 0
λ_1	-0.83	-0.96	-1.03	-1.10
λ_2	-0.73*	-0.64	-0.59	-0.55
λ_3	-0.59*	-2.18	-3.56	-6.07
D	-0.45	2.38	8.84	30.46

* real part of complex conjugate root

Table 1 shows the slope of the reaction curve at the equilibrium point, the equilibrium price, the changes in this and in the equilibrium utility level when the aggregate endowment of x changes (dp and du_h respectively). In addition the eigenvalues at the equilibrium point are shown together with the function D defined in Section 4. With strategic complementarity (Table 1) as the aggregate endowment increases so does the slope of the reaction curves and, with weak complementarity, the equilibrium price falls. However when the interactions are strong the price increases with the endowment. In Table 1 the effect looks small but for different parameter values we can magnify the perverse price changes. For example when $A = 5.0$, $a_1 = 2.0, \alpha := 1.5, \beta = 0.75$ the prices and comparative statics of equilibrium prices are

X	4.	8.	12.	20.
p	22.70	33.14	44.78	70.29
dp	2.36	2.79	3.02	3.34

Nevertheless equilibrium utility still rises with the endowment. The eigenvalues show that with weak or strong complementarity there is local stability.

The stability properties are sensitive to the speed of price adjustment. If we set $\delta_p = 0$ the eigenvalues become

Table2: Weak & Strong Complementarities, $\delta_p = 0$

$A = 10., a_1 = 4., \alpha = 0.75, \beta = 0.5, \delta = 0.5, \delta_p = 0$

X	4.	8.	12.	20.
λ_1	-0.83	-0.96	0.03	0.10
λ_2	-0.17	-0.04	-1.03	-1.10
λ_3	0.0	0.0	0.0	0.0
D	0.03	0.001	0.0008	0.01

The main effect in this example is that the lack of price adjustment makes the strong complementarity case locally unstable. This result holds also when $\delta_p \neq 0$ but it is small as Table 3 reveals.

Table 3: Slow Price Adjustment

$A = 10., a_1 = 4., \alpha = 0.75, \beta = 0.5, \delta = 0.9, \delta_p = 0.01$

X	4.	8.	12.	20.
λ_1	-1.5	-1.73	-1.86	-1.98
λ_2	-0.04	-0.05*	0.006*	0.056*
λ_3	-0.27	-0.05*	0.006*	0.056*
D	0.05	-0.1	-0.15	-0.23

*real part of complex conjugate root

Since the price adjustment is not fast enough to overcome the quantity adjustment, with strong complementarities the equilibrium is still unstable.

9 Conclusions

In this paper we assess the effect of price flexibility on market behaviour and equilibrium of an exchange economy with strategic complementarities. Individuals have heterogeneous endowments but identical preferences. We show that, despite the presence of strategic complementarities, a unique Nash-market equilibrium exists in the system with equal consumptions of the externality inducing good. If the strategic interactions are strong, the unique equilibrium displays anomalous behaviour. By contrast with previous literature which held prices constant, we show that price flexibility may make this equilibrium stable even if it is characterised by strong strategic interactions. The stability of anomalous behaviour relies on the relative speed of the adjustment process in quantities and prices. It occurs when prices move faster than quantities.

Finally, we show that if the externality is detrimental corrections are required to improve efficiency. But with beneficial externalities the unique equilibrium is Pareto optimal despite the externality. Price flexibility may actually eliminate instability and inefficiency in markets.

A Appendix

A.1 Proof of Proposition 1

Consider $H = 3$ and suppose there is an asymmetric equilibrium in which $x_1 > x_2, x_1 \geq x_3$, that is individual 1 has the highest consumption of x which is strictly greater than the consumption of x for at least one other individual. Then

$$x_1 + x_2 \geq x_2 + x_3, x_1 + x_3 > x_2 + x_3$$

Since the reaction curve RC_X is increasing in z_k we have

$$\begin{aligned} x_1 &= RC_X(p, \frac{x_2 + x_3}{2}) \leq RC_X(p, \frac{x_1 + x_2}{2}) = x_3 \\ x_1 &= RC_X(p, \frac{x_2 + x_3}{2}) < RC_X(p, \frac{x_1 + x_3}{2}) = x_2 \end{aligned}$$

Taking the mean of these two equations

$$x_1 < \frac{x_2 + x_3}{2}$$

which is a contradiction. The same argument applies for any number of individuals H. This means any equilibrium must be on the 45° line. Corner solutions are ruled out because they exhibit excess demand or excess supply. Only the equilibrium with equal consumption of the good by each individual is selected.■

A.2 Proof of Proposition 2

The Jacobian (2) has $H-1$ repeated eigenvalues of $-\delta(1+a/(H-1))$ and a pair of roots

$$\lambda_{1,2} = \frac{(a-1)\delta}{2} + \frac{H\delta_p c}{2} \pm \frac{\{(a-1)\delta + H\delta_p c]^2 + 4Hc\delta\delta_p\}^{0.5}}{2}. \tag{9}$$

Under weak or strong complementarities at an interior Nash equilibrium, $a > 0$ and we know $c < 0$. The repeated roots $-\delta(1+a/(H-1))$ are then all real and negative.

If

$$D = [(a-1)\delta + H\delta_p c]^2 + 4Hc\delta\delta_p < 0$$

the conjugate roots have real part equal to

$$\frac{(a-1)\delta}{2} + \frac{H\delta_p c}{2}$$

With weak complementarity the real part of these roots is always negative. With strong complementarity, it is negative if $-H\delta_p c > (a-1)\delta$.

On the other hand if $D > 0$ the second and third roots are real and equal to (10):

$$\lambda_{1,2} = \frac{(a-1)\delta}{2} + \frac{H\delta_p c}{2} \pm \frac{\{(a-1)\delta + H\delta_p c]^2 + 4Hc\delta\delta_p\}^{0.5}}{2} \tag{10}$$

Since $c < 0$, $[D]^{0.5} < [(a-1)\delta + H\delta_p c]$ and so $\lambda_1 < [(a-1)\delta + H\delta_p c]$. If $[(a-1)\delta + H\delta_p c] < 0$, then $\lambda_1 < 0$. Since $\lambda_2 < \lambda_1$, if $\lambda_1 < 0$, this implies also $\lambda_2 < 0$.

Hence with strong complementarity whatever the sign of D, the real part of the conjugate roots is negative if $[(a-1)\delta + H\delta_p c] < 0$. On the other hand if $[(a-1)\delta + H\delta_p c] > 0$, the result is ambiguous: both the roots may be positive.■

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