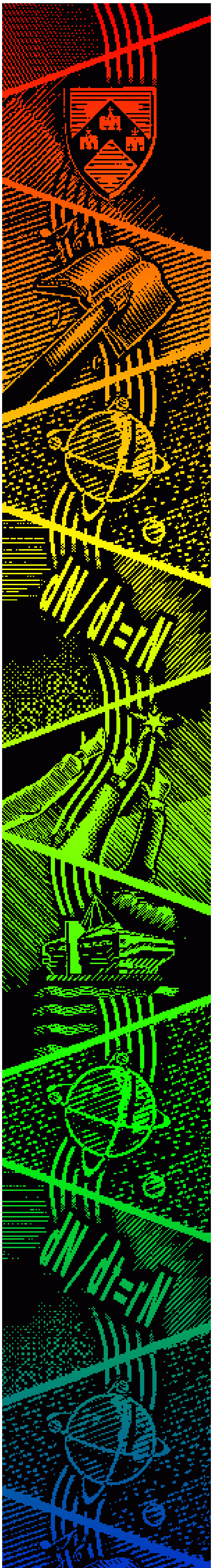

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Term Structure Dynamics in a Monetary Economy with Learning

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Abstract

This paper investigates an equilibrium model of the term structure of nominal interest rates on default-free, zero coupon bonds. In a pure exchange economy with incomplete information, a representative agent is unable to observe the expected growth rates of both exogenous real output and money supply and, therefore, engages in dynamic Bayesian inference. The dependence of term premia on beliefs allows the model to introduce a GARCH property, which interacts with the volatility of the macro variables. In particular, the volatility of excess returns is inversely related to noise in the macro variables, implying that erratic monetary policy may reduce fluctuations in interest rates.

JEL Classification: G12, E43, D83.

Keywords: Term structure of interest rates; Monetary equilibrium model; Uncertainty in parameters; Learning.

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1 Introduction

A key framework of analysis for the term structure of interest rates on default-free, zero coupon bonds is based on endogenous, time-varying risk premia determined by rational expectations models. Lucas (1978) led researchers to consider equilibrium models of the term structure with a representative agent.[1] A model of the Lucas-type exchange economy is elegant and tractable from an analytical point of view, but they have not performed well in empirical tests. For example, Backus, Gregory, and Zin (1989) show that a monetary version of the Mehra-Prescott's (1985) model cannot account for either the sign or the magnitude and the variability of term premia in holding period returns on US multi-period bonds. Additionally, data produced by simulation from the model invariably accept the null hypothesis of no ARCH effect in the term premium. Another explanation of the term structure dynamics highlights over- or underreaction of expected future long rates to changes in short rates (see Froot (1989), Campbell and Shiller (1991), and Hardouvelis (1994)). Using survey data, Froot (1989) found evidence that when forming their expectations of future long rates, market participants place more weight on the contemporaneous short rate than is warranted, thus producing over reaction of the yield spread.

This paper focuses on an idea that contains elements of the previous two explanations. In this regard, I introduce incomplete information in an equilibrium approach in continuous time, i.e. learning as a source of time-variations in the term premium and the under- and overreaction of economic fundamentals. In an economy with incomplete information, a representative investor cannot observe expected growth rates of economic fundamentals such as real output and money supply that depend on the current regime of the economy. Because these drifts are key parameters of the fundamental processes characterizing the economy, market participants will optimally estimate them. Due to potential estimation errors, investors may appear to under- or overreact to switches

[1]Another strand of equilibrium term structure models is developed by Cox, Ingersoll, and Ross (1985a, 1985b), (henceforth CIR) and Duffie and Kan (1996) in which the yields of zero-coupon bonds are an affine function of a set of state variables. The closed-form solution for yields is found only when the representative agent has a log utility function. In contrast, the Lucas-type of exchange economy derives the closed form solution under more general classes of utility functions such as the power and the exponential utility functions.

of the current economic state although the market's expectations are perfectly rational. Empirical evidence by Tzavalis and Wickens (1997) suggests the effects of combining the under- and overreaction hypothesis with time-varying risk premia are important in developing a term structure model. Introducing parameter uncertainty into a forecasting model of future interest rates, Ederington and Huang (1995) show that the model is able to generate an interest rate series that is more consistent with bond return forecasting of a yield spread than a model without parameter uncertainty. In addition, a reason for information to be incomplete is found in empirical success of regime-switching models, in which an unobservable regime is a key driving force of economic dynamics.[2]

In the incomplete information economy the model obtains closed form expressions for equilibrium prices of the commodity and nominal bonds and the process of investor's beliefs. Bond prices are expressed as the weighted average of the state dependent bond prices, and the weights are given by the investor's beliefs on the unobservable states of the mean growth rates. Discrete changes in the macro variable volatility induce jumps in bond prices, while bond prices move continuously and gradually with learning in the unobservable state. Beliefs and random shocks to economic fundamentals also play an important role in generating time-variation in the conditional volatility of excess holding period returns. Conditional volatility of the excess returns shows volatility clustering due to mean-reversion in the process of beliefs. Furthermore, the model implies that the conditional volatility of the excess returns is inversely related with the volatility parameters in the process of the economic fundamentals. As a result, a large volatility of the excess returns occurs even with small fluctuations in the economic fundamentals, which is consistent with an empirical fact (see Schwert (1989)). This volatility feature can be explained in the following way. When the volatility of the economic fundamentals is low, the investor obtains precious information on the states, thus their beliefs tend to be updated frequently. These frequent changes in beliefs repeatedly shift the excess holding period returns, thus resulting in a large volatility. This point addresses the effect of monetary policy conduct on term structure volatility. That is, erratic monetary policy, i.e., higher variance of money supply shock, does not necessarily lead to greater volatility of either interest

[2]For examples of regime-switching interest rate models, see Hamilton (1988), Sola and Driffill (1994), Gray (1996), Bekaert, Hodrick, and Marshall (2001), Ang and Bekaert (2002a), Bansal and Zhou (2002), Evans (2003), and Guidolin and Timmermann (2007).

rates or term premia.

Most of the existing term structure literature consists of models of real bond prices and relies on an exogenous process for inflation and the Fisher equation to obtain nominal bond returns. However, the Fisher equation itself may not hold when inflation is stochastic.[3] A few papers model continuous time nominal bond prices in equilibrium. For instance, Bakshi and Chen (1996) have developed a monetary model that addresses the endogenous and simultaneous determination of the price level and of the term structure of interest rates, both real and nominal. Money in the utility function (MIUF) formulations are used as a method to incorporate money. They show that the term structure of real versus nominal interest rates may have completely different properties, which contrasts with the Fisher framework. Buraschi and Jiltsov (2005) incorporate an endogenously determined monetary policy and derive closed form solutions for the nominal term structure of interest rates. Their empirical analysis finds that the inflation risk premium plays an important role in accounting for rejections of the expectations hypothesis.[4]

The incorporation of incomplete information into asset pricing theory has been introduced in the term structure literature. Dothan and Feldman (1986) are the first to examine the effects of learning on real bond prices. They demonstrate that, in contrast to a complete information economy, high volatility of interest rates is not necessarily implied by a highly volatile investment opportunity set. Stulz (1986) models the effects of uncertainty on monetary policy on the nominal short rate, finding that this type of uncertainty leads to a higher volatility of the nominal short rate. Those papers assume similar learning processes characterized by continuous-time Kalman filtering in which the unobservable drift of a state variable follows a Gaussian diffusion and is characterized by a large number (a continuum) of states. In such filtering problems, the conditional variance of the investor's estimates evolves deterministically and converges to a constant. Instead, David (1997) develops a CIR-type model in which the growth opportunities of firms can switch between two

[3] Evidence against the Fisher equation is found in Evans (1998) and Buraschi and Jiltsov (2005). Applying the CIR type term structure model in a monetary economy, Lioui and Poncet (2004) found that inflation uncertainty results in non-neutrality of money.

[4] There are several monetary models in discrete time to explain term structrure dynamics. Ang and Piazzesi (2003) relate term structure dynamics using factor models to macro variables. Ellingsen and Söderström (2001), Rudebusch and Wu (2004), and Gallmeyer, Hollifield and Zin (2005) incorporate monetary policy rules into term structure models.

distinct values and investors have to infer the unobservable current growth rate. In this filtering process, the estimation error exhibits stochastic variation over time.[5] With the filtering process applied in David (1997), Veronesi and Yared (1999) introduce a partial equilibrium model of the term structure under the assumption that real consumption and the commodity price follow a joint log-normal process with unobservable discrete drift rates.[6]

My research contributes to the term structure literature in several ways. First, among the models with incomplete information using the filtering process that generates stochastic estimation errors, my model is the first to endogenously derive the commodity price level, nominal bond prices, and the process of the investor's beliefs. A monetary model of Bakshi and Chen (1996), in which economic fundamentals follow a joint stochastic process with constant parameters, can be recovered as a special case. In Veronesi and Yared (1999), money plays no role. As matter of fact, money is found to affect the term structure in the US market, robust to monetary policy regimes for both pre- and post-1979 samples of data.[7] For example, Ireland (2004) presents empirical evidence supporting the inclusion of money growth in the interest rate rule for monetary policy. His estimation is the result of a New Keynesian model showing that money growth helps predict the target nominal short rate. Additionally, Hur (2005) shows negative relationships between the past history of money growth rate (M1) and yields of various maturities, confirming the presence of the short-run liquidity effect.

Second, my model incorporates stochastic time-variation in instantaneous growth rate volatilities and correlations between fundamentals, thus producing time variation in precautionary savings and in asset pricing functions. Moreover, the speed of learning is allowed to change over time due to the changing quality of the information content of signals. None of the previous papers on incomplete information incorporates time-variation in volatility of economic fundamentals, although it is widely

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[5] In a Lucas-type economy, Veronesi (2000) demonstrates that the equity premium tends to increase if the agent receives more precise signals on the unobserved growth rate of dividends.

[6] Extending homogeneous belief setting, Basak (2005) studies a pure-exchange economy where investors have heterogeneous beliefs about the expected growth rates of economic fundamentals.

[7] The Federal Reserve announced a change in operating procedures from the federal funds rate to nonborrowed reserves as the operating target in October 1979.

known that the conditional volatility of financial and macroeconomic variables is time-varying.[8]

The remainder of the paper is organized as follows. Section 2 presents a model where a representative agent uses money for transaction purposes. Section 3 introduces unobservable time-varying mean growth rates of economic fundamentals. In this incomplete information economy, the dynamic process of investors' beliefs is characterized, and commodity and nominal discount bonds are priced. Section 4 analyzes the properties of the term premium, the conditional volatility of excess holding period returns on nominal bonds, and the nominal short rate. Section 5 concludes the paper.

2 The model

This section presents a continuous-time variation of Lucas' (1978) pure-exchange economy in which a single representative agent has a preference over consumption of a non-storable good and wishes to hold fiat money for transaction purposes. The following assets are traded: one risky nominal stock in unit exogenous supply and n maturities of nominal default-free discount bonds in zero net exogenous supply. The risky stock is a claim to an exogenous flow of dividends paid in cash. The nominal bonds pay one dollar at maturity in cash, which involve inflation risks. The investor selects consumption and the fraction of wealth to be invested in the stock and in nominal bonds in order to maximize a time-separable, power utility function. In equilibrium, the representative investor consumes dividends paid by the risky asset and invests their whole wealth in the stock. This way, equilibrium nominal bond prices and commodity price are endogenously determined.

2.1 The stochastic processes of economic fundamentals

Assumption 1 (Exogenous stochastic processes): There are two fundamental variables in the economy, real output $y(t)$ and money supply $M^s(t)$. They follow the stochastic processes:

$$dy(t) = \theta(t)y(t)dt + \sigma_y(t)y(t)dZ_y(t), \text{ with } y(0) = y_0 > 0, \tag{1}$$

[8]See Schwert (1989) for empirical evidence for time-varying volatility of a variety of economic variables and Bollerslev, Chou, and Kroner (1992) for a survey of ARCH models. Lee (1995) found that based on a monetary equilibrium model the volatility of industrial production and money supply affects the term structure of interest rates.

$$dM^s(t) = \mu(t)M^s(t)dt + \sigma_M(t)M^s(t)dZ_M(t), \text{ with } M^s(0) = M_0^s > 0, \tag{2}$$

where $\{Z_y(t),\ Z_M(t)\}$ are two-dimensional Brownian motions under a complete probability space $(\Omega,\ P,\ \mathcal{F})$ with instantaneous correlation $\rho(t)$ ($|\rho(t)| \leq 1$). $\theta(t)$, $\mu(t)$, $\sigma_y(t)$, $\sigma_M(t)$, and $\rho(t)$ are exogenously specified bounded Markovian processes. I assume that all necessary and sufficient conditions hold so that all the stochastic differential equations have a unique strong solution $\{(y(z),\ M^s(z)),\ \mathcal{F}(z)\}_{0\leq z\leq t}$, where $\{\mathcal{F}(z)\}_{0\leq z\leq t}$ is a filtration of $\mathcal{F}$.[9] This economy has several distinctive features. For instance, the money supply management by the monetary authority is taken as exogenous. Monetary policy, measured by money supply, is related to the real output representing the real side of the economy. This is because the real output growth rate and the money growth rate are correlated due to non-zero correlation of their means and volatilities. Bakshi and Chen (1996) consider the similar processes of real output and money supply with complete information.

2.2 The optimization problem

Based on the stochastic processes of economic fundamentals (1) and (2), an investor solves an optimization problem described as follows.

Assumption 2 (Preferences): There exists a representative investor who maximizes expected utility with preferences given by:

$$E\left[\int_0^\infty e^{-\phi t} U\left(c(t), m(t)\right) |\mathcal{F}(0)\right] dt, \tag{3}$$

where ϕ is the subjective discount rate, $c(t)$ is the amount of time-t consumption. $m(t) \equiv \left(\frac{M^d(t)}{P(t)}\right)$ is the real money demand, where $M^d(t)$ is time-t nominal money demand and $P(t)$ is the price of the consumption good. Furthermore, the time-t utility function is twice continuously differentiable, increasing and concave in both real money demand and consumption, such that $U_c > 0$, $U_m > 0$, $U_{cc} < 0$, $U_{mm} < 0$, $U_{cc}U_{mm} - (U_{cm})^2 > 0$, with the subscripts on U denoting partial derivatives.[10]

[9] The necessary and sufficient conditions hold if the coefficients satisfy Lipschitz and growth conditions (see Liptser and Shiryaev 2000, p. 134-135).

[10] Sidrauski (1967) originally establishes a monetary model that puts real balances in the utility function. Feenstra (1986) shows that this way of incorporating money in the utility function is equivalent to assuming that money holdings create a transaction service.

Let $S(t)$ be the time-t price of the stock, and let $B(t,\tau_j)$ the time-t price of a τ_j-maturity nominal bond. Let $\alpha(t)$ be the number of equity shares in the investor's portfolio, and $\eta_j(t)$ the number of units held of the τ_j-maturity nominal bond. The investor's optimization problem is to select $\{c, M^d, \alpha, \eta_1, ., \eta_n\}$ in order to maximize the expected utility (3) subject to the budget constraint

$$dW(t) = (dS(t) + P(t)y(t)dt)\alpha(t) + \sum_{j=1}^{n} dB(t,\tau_j)\eta_j(t) - P(t)c(t)dt - dM^d(t), \tag{4}$$

where $W(t)$ is the time-t wealth in nominal terms. Because aggregate output is perishable and exogenous, in equilibrium the rational investor must consume the entire real output (dividends) at each time: $c(t) = y(t)$. In equilibrium, money supply $M^s(t)$ and money demand $M^d(t)$ must be equalized at each point in time: $M(t) \equiv M^s(t) = M^d(t)$. To finance this consumption/money-holding plan, the equilibrium portfolio policy then becomes $\alpha(t) = 1$, and $\eta_j(t) = 0$ for $j = 1, ..., n$.

2.3 Commodity and bond prices in equilibrium

In a continuous-time pure exchange economy, Bakshi and Chen (1996) show that the value of bonds depends only on the exogenous output and money processes subject to transversality conditions. In my economy, one transversality condition ensures the existence of an interior optimum, when deriving equilibrium commodity and bond prices:

$$e^{-\phi T} E\left[\frac{U_c(y(T), m(T))}{U_c(y(t), m(t))}\frac{1}{P(T)}|\mathcal{F}(t)\right] \overset{T\to\infty}{\to} 0. \tag{5}$$

If this transversality condition does not hold, investors are expected to obtain positive discounted (real) values from holding cash at any future time without consuming at a current time. Hence it is possible that they do not consume at a current time, and thus interior solutions may not be achieved.

Subjected to the transversality condition, the valuation of assets can be performed without facing the difficulty of a nonlinear Bellman equation. Applying Bakshi and Chen's (1996) pricing equation to the pure-exchange monetary economy, the time-t equilibrium commodity price level satisfies

$$\frac{1}{P(t)} = E\left[\int_t^{\infty} e^{-\phi(z-t)}\frac{U_m(y(z), m(z))}{U_c(y(t), m(t))}\frac{1}{P(z)}dz|\mathcal{F}(t)\right]. \tag{6}$$

The intuition behind this condition is that the real value of one dollar of cash $\frac{1}{P(t)}$ is, in equilibrium, equal to the total discounted value of future marginal benefits resulting from saving one unit of cash. Notice that U_m/U_c represents the marginal rate of substitution between consumption and real cash holding, and it can be viewed as the marginal benefit of holding one additional unit of cash. Thus, in equilibrium, the marginal cost of holding a cash, or the nominal interest rate, must equal the marginal benefit U_m/U_c. In equilibrium, the time-t prices of the nominal discount bond $B(t,\tau)$ are:

$$B(t,\tau) = e^{-\phi\tau} E\left[\frac{U_c(y(t+\tau), m(t+\tau))}{U_c(y(t), m(t))} \frac{P(t)}{P(t+\tau)} |\mathcal{F}(t)\right]. \tag{7}$$

The pricing formula for the nominal bond implies that its price is the conditional expectation of the discounted value of one unit of cash paid at maturity, relative to the current discounted value of one unit of cash.

Next, a specific form of the utility function is presented in order to obtain more explicit representations for commodity and bond prices. For this purpose, assume that the representative agent has nonseparable Cobb-Douglas preference in which real cash balances enter directly. This money-in-the-utility-function formulation (MIUF) captures a more general role of liquidity service than a cash-in-advance (CIA) constraint. In addition to offering transaction motives for holding money, the MIUF captures precautionary and store-of-value demands for money.

Assumption 3 (Utility function): The utility function is given by

$$U(c(t), m(t)) = \frac{1}{1-\gamma}[c(t)^{\alpha} m(t)^{1-\alpha}]^{1-\gamma}, \tag{8}$$

where $0 < \alpha < 1$, $0 < \gamma$.[11]

The parameter γ characterizes the coefficient of relative risk aversion over the composite good $c(t)^{\alpha} m(t)^{1-\alpha}$ while $\frac{1}{\gamma}$ represents the elasticity of intertemporal substitution. The parameter α measures the weight of consumption in the utility function. When $\alpha = 1$, money holdings do not provide any transaction services. Notice that for $\gamma = 1$, the Cobb-Douglas form becomes a separable

[11] These restrictions on the parameters guarantee increasing and concave features of the utility function.

logarithmic utility:

$$U(c(t), m(t)) = \alpha \ln c(t) + (1-\alpha) \ln m(t). \tag{9}$$

This separable utility implies that the real balances do not affect the marginal utility of consumption and that consumption does not affect the marginal utility of real balances.

3 Learning under incomplete information

Assume that the investor operates under incomplete information about the mean growth rates of real output and money supply. In this setting the investor's choice problem can be solved in two-stages. In the first stage, the unobserved mean growth rates are estimated. The optimal mean square estimate is an expectation, conditional on all past realizations of real output and money supply. In the second stage, employing the estimated mean growth rates, the investor solves their portfolio problem.[12]

3.1 The Dynamics of Investors' Beliefs

In this subsection, the dynamics of investors' beliefs are derived. The following results are closely related to those in Veronesi and Yared (1999), in which signals stem from dividends and the commodity price level.

Let $s(t)$ be an index function taking values on the set $\{1, 2, ..., I\}$ so that $\theta(t) = \theta_1$ and $\mu(t) = \mu_1$ when $s(t) = 1$, $\theta(t) = \theta_2$ and $\mu(t) = \mu_2$ when $s(t) = 2$,..., $\theta(t) = \theta_I$ and $\mu(t) = \mu_I$ when $s(t) = I$.

Assumption 4 (Mean growth rates): The investor cannot observe the expected growth rates, $\theta(t)$ and $\mu(t)$. The investor only knows that $\theta(t)$ can take any of I possible values $-\infty < \theta_1, \theta_2, ..., \theta_I < \infty$ and $\mu(t)$ can be any of I possible values $-\infty < \mu_1, \mu_2, ..., \mu_I < \infty$. $\theta(t)$ and $\mu(t)$ are assumed to follow a joint Markov chain process whose transition probabilities within an infinitesimal period Δ are given by

$$\Pr(s(t+\Delta) = i^{'} | s(t) = i) = \left\{ \begin{array}{c} 1 + \lambda_{ii}^{s} \Delta, \text{ for } i = i^{'} \\ \lambda_{ii^{'}}^{s} \Delta, \text{ for } i \neq i^{'} \end{array} \right\}. \tag{10}$$

[12] This two-stage optimization procedure is proven viable by Dothan and Feldman (1986).

Let Λ^s be the infinitesimal matrix of the Markov chain process so that $[\Lambda^s]_{ii'} = \lambda^s_{ii'}$. Notice that $\lambda^s_{ii} = -\sum_{i' \neq i} \lambda^s_{ii'}$ for $i = 1, ..., I$. For $i = i^{'}$, $1 + \lambda^s_{ii'}\Delta$ gives the probability of state i' at time $t + \Delta$ given state i at t. And, for $i \neq i^{'}$, $\lambda^s_{ii'}\Delta$ gives the probability of state i' at time $t + \Delta$ given state i at t.

Assumption 5 (Growth rate volatility): The investor knows that the diffusion parameters $\sigma_y(t)$, $\sigma_M(t)$, and $\rho(t)$ can take any values such that $0 < \sigma_{y1}, \sigma_{y2}, ..., \sigma_{yJ} < \infty$, $0 < \sigma_{M1}, \sigma_{M2}, ..., \sigma_{MJ} < \infty$, $-1 \leq \rho_1, \rho_2, ..., \rho_J \leq 1$. J denotes the total number of the values of $\sigma_y(t)$, $\sigma_M(t)$, and $\rho(t)$. More important, the investor can observe the true current values of these parameters. The instantaneous volatility and correlation coefficients $\sigma_y(t)$, $\sigma_M(t)$, and $\rho(t)$ are assumed to follow a joint Markov chain process whose transition probabilities within an infinitesimal period Δ are given by

$$\Pr(v(t+\Delta) = j^{'} | v(t) = j) = \left\{ \begin{array}{c} 1 + \lambda^v_{jj}\Delta, \text{ for } j = j^{'} \\ \lambda^v_{jj'}\Delta, \text{ for } j \neq j^{'} \end{array} \right\}, \tag{11}$$

where $v(t)$ is an index function taking values in the set $\{1, 2, ..., J\}$ with $\sigma_y(t) = \sigma_{y1}$, $\sigma_M(t) = \sigma_{M1}$, $\rho(t) = \rho_1$ when $v(t) = 1$, $\sigma_y(t) = \sigma_{y2}$, $\sigma_M(t) = \sigma_{M2}$, $\rho(t) = \rho_2$ when $v(t) = 2$, and so on. Call $[\Lambda^v]_{jj'} \equiv \lambda^v_{jj'}$ the infinitesimal matrix of the Markov chain process. Notice that $\lambda^v_{jj} = -\sum_{j' \neq j} \lambda^v_{jj'}$ for $j = 1, ..., J$.

From Assumptions 4 and 5, the investor's information set consists of $\mathcal{F}^{'}(t) = \{y(z), M^s(z), \sigma_y(z), \sigma_M(z), \rho(z)\}_{0 \leq z \leq t}$ and $\mathcal{F}^{'}(t) \subset \mathcal{F}(t)$ so that processes $\{\theta(t),\ \mu(t),\ Z_y(t),\ Z_M(t)\}$ are adapted to $\mathcal{F}(t)$, but not to $\mathcal{F}^{'}(t)$ due to the unobservability of the drift rates. Although the assumption of observability of the volatility and its correlation is for analytical tractability only, Merton (1980) claims that under an Itô process such as (1) and (2), the volatility of the process can be estimated far more accurately than the expected value. In addition, judging from the success of modeling volatility dynamics by means of GARCH processes shown in Bollerslev, et al. (1992), the volatility may be accurately estimated.

Assumption 6 (Independence): The joint Markov processes of $\theta(t)$ and $\mu(t)$, and of $\sigma_y(t)$, $\sigma_M(t)$ and $\rho(t)$ are independent. This assumption implies that information on second moments gives no direct information on the means.

Assumption 7 (Filtering rule): The investor uses Bayes' Law to update her beliefs regarding the current state of the fundamentals' growth rates in the economy.

Under Assumptions 4-7, Proposition 1 presents the stochastic process followed by the investor's beliefs.

Proposition 1:

(1) *The investor's information set,* $F^{'}(t) = \{y(z), M^s(z), \sigma_y(z), \sigma_M(z), \rho(z)\}_{0\leq z\leq t}$ *is equivalent to* $\{y(z), M^s(z)\}_{0\leq z\leq t}$. *In other words, the investor correctly estimates the current volatility parameters using data for the fundamentals.*

(2) *Let* $\pi_i(t)$ *be the representative investor's posterior probability that the state of expected growth rates is* $s(t) = i$ *at time* t*, conditional on their information* $F^{'}(t)$*:*

$$\pi_i(t) = \Pr(s(t) = i|\mathcal{F}^{'}(t)). \tag{12}$$

Then, $\pi_i(t)$ *is the solution of a* I*-dimensional system of stochastic differential equations,*

$$\begin{aligned} d\pi_i(t) &= \left(\sum_{l=1}^{I} \pi_l(t)\lambda_{li}\right) dt \\ &+\frac{\pi_i(t)}{\sqrt{1-\rho(t)^2}}\left[\frac{\theta_i - m_y(t)}{\sigma_y(t)} + \frac{\sqrt{|\rho(t)|}\,(\mu_i - m_M(t))}{\sqrt{\sigma_y(t)\sigma_M(t)}}\right] d\widetilde{Z}_1(t) \\ &+\frac{\pi_i(t)}{\sqrt{1-\rho(t)^2}}\left[\frac{\sqrt{|\rho(t)|}\,(\theta_i - m_y(t))}{\sqrt{\sigma_y(t)\sigma_M(t)}} + \frac{\mu_i - m_M(t)}{\sigma_M(t)}\right] d\widetilde{Z}_2(t) \quad i = 1, ..., I, \end{aligned} \tag{13}$$

where $m_y(t) = \sum_{i=1}^{I} \theta_i\pi_i(t)$, $m_M(t) = \sum_{i=1}^{I} \mu_i\pi_i(t)$. $\left\{\widetilde{Z}_1(t), \widetilde{Z}_2(t)\right\}$ *are independent Brownian motions adapted to* $\mathcal{F}^{'}(t)$*, in other words, they are observable to investors.*

According to equation (13), the investor's beliefs follow a two-factor Itô process with mean-reverting drift terms and stochastic volatility terms. There are two important remarks concerning the stochastic processes of investor's beliefs. According to an argument in Veronesi (2000), information quality is measured as the inverse of the instantaneous volatility term. When information about the real output and money supply is of good quality, the noise terms in (1) and (2) are apt to be small. Therefore, the investor quickly learns the state of the economy. Conversely, learning is slow if instantaneous volatilities are large. The volatility coefficients change over time in my model.

Consequently, the speed of learning also changes over time. Next, the investor's attitudes toward risk do not affect beliefs because the investor's choice problem is conducted in two separate stages.

A simple example intuitively shows the properties of the belief process. Suppose there are two possible states and there is no instantaneous correlation between the real output and money growth rates, $\rho(t)=0$. Let $\pi(t)=\Pr(s(t)=1|\mathcal{F}'(t))$. Equation (13) can then be written as

$$\begin{aligned} d\pi(t) &= (\lambda_{12}^s+\lambda_{21}^s)\left[\left(\frac{\lambda_{21}^s}{\lambda_{12}^s+\lambda_{21}^s}\right)-\pi(t)\right]dt \\ &\quad +\pi(t)(1-\pi(t))\left[\left(\frac{\theta_1-\theta_2}{\sigma_y(t)}\right)d\widetilde{Z}_1(t)+\left(\frac{\mu_1-\mu_2}{\sigma_M(t)}\right)d\widetilde{Z}_2(t)\right]. \end{aligned} \tag{14}$$

This updating process has two parts. The first part shows that the process is mean-reverting. In the long run, the belief converges to $\frac{\lambda_{21}^s}{\lambda_{12}^s+\lambda_{21}^s}$, which is the unconditional probability of the first state. The second part is the sum of the product of the information weights ($\pi(t)(1-\pi(t))\left(\frac{\theta_1-\theta_2}{\sigma_y(t)}\right)$ and $\pi(t)(1-\pi(t))\left(\frac{\mu_1-\mu_2}{\sigma_M(t)}\right)$) and new information on the economic fundamentals ($d\widetilde{Z}_1(t)$ and $d\widetilde{Z}_2(t)$). The information weights are at their peak when the investor is more uncertain about current economic states, i.e. π is close to 0.5. If this is the case, the investor is more likely to revise their beliefs after receiving new information on fundamentals. On the other hand, when the investor is confident on the identity of the current regime ($\pi\approx 0$ or 1), they will not significantly change their beliefs. For the same reason, the investor tends to change their beliefs slowly in response to new information when signals are very noisy, i.e. $\sigma_y(t)$ and $\sigma_M(t)$ are large. The more each state is significantly different from other states ($|\theta_1-\theta_2|$ or $|\mu_1-\mu_2|$ is larger), the better are the investors' chances to correctly figure out the current state.

Under optimally filtered beliefs, the investor computes the expected growth rates at time t as:

$$m_y(t)\equiv E(\theta|\mathcal{F}'(t))=\sum_{i=1}^{I}\theta_i\pi_i(t),\ m_M(t)\equiv E(\mu|\mathcal{F}'(t))=\sum_{i=1}^{I}\mu_i\pi_i(t). \tag{15}$$

Therefore, to the investor's perception the real output process and the money supply process become

$$dy(t)=m_y(t)y(t)dt+\sigma_y(t)y(t)d\widetilde{Z}_1(t)+\sqrt{|\rho(t)|\,\sigma_y(t)\sigma_M(t)}y(t)d\widetilde{Z}_2(t), \tag{16}$$

$$dM^s(t)=m_M(t)M^s(t)dt+\sqrt{|\rho(t)|\,\sigma_y(t)\sigma_M(t)}M^s(t)d\widetilde{Z}_1(t)+\sigma_M(t)M^s(t)d\widetilde{Z}_2(t). \tag{17}$$

All coefficients and the two Brownian motions are adapted to the investor's information set $\mathcal{F}'(t)$.

3.2 The Equilibrium prices of the commodity and of the nominal bonds

Under Assumptions 1-7, it follows that:

Proposition 2: *The equilibrium price of the commodity is*

$$P(t) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{M(t)}{y(t)}\right)\left(\sum_{i=1}^{I}\sum_{j=1}^{J} A_{ij}\pi_i(t)1(v(t)=j)\right)^{-1}, \tag{18}$$

where $1(.)$ *is an indicator function, and the* $A_{ij}s$ *are positive constants satisfying the following system of equations:*

$$\left(-\lambda_{ii}^s - \lambda_{jj}^v + \phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 + \mu_i - \sigma_{M_j}^2 + (1-\gamma)\rho_j\sigma_{yj}\sigma_{M_j}\right)A_{ij} +$$

$$-\sum_{l\neq i}\lambda_{il}^s A_{lj} - \sum_{n\neq j}\lambda_{jn}^v A_{in} = 1. \tag{19}$$

From price formula (18), the commodity price is affected by the current levels of real output, showing that an increase in real output reduces the commodity price: When output increases and money supply is fixed, one unit of money can purchase more commodities, thus leading to an increase in the value of money and a decline in the good price. Additionally, (18) states that the money supply is directly proportional to the commodity price. This can be explained by the fact that liquidity in the economy increases with money supply so that demand for the commodity increases. As a result, the commodity becomes more expensive. By (18), the investor's beliefs also affect the good price, which is higher when the belief gives more weight on a lower value of A_{ij}.

The time-t velocity of money $Vel(t)$ can be derived from (18):

$$Vel(t) = \left(\frac{\alpha}{1-\alpha}\right)\left(\sum_{i=1}^{I}\sum_{j=1}^{J} A_{ij}\pi_i(t)1(v(t)=j)\right)^{-1}. \tag{20}$$

The velocity formula (20) reveals that the beliefs and the growth rate volatilities affect the velocity of money. $\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{A_{ij}}\right)$ can also be interpreted as the state-dependent velocity of money. To see further implications for the velocity, I impose an assumption that the means and volatilities of the growth rates do not change frequently and there is no correlation between them, that is, $\lambda_{il}^s \approx 0$, for $i, l = 1, ..., I$, and $\lambda_{jn}^v \approx 0$, and $\rho_j = 0$ for $j, n = 1, ..., J$. Under this assumption, the velocity of money has a closed-form expression:

$$Vel(t) = \left(\frac{\alpha}{1-\alpha}\right) \sum_{i=1}^{I} \sum_{j=1}^{J} \left(\phi - (1-\gamma)\theta_i + \mu_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 - \sigma_{M_j}^2\right) \pi_i(t) 1(v(t) = j). \tag{21}$$

This velocity equation offers some implications. First, the velocity of money is positively related to the mean growth rate of money. Suppose that the consumer expects the mean money growth rate to increase: Due to a hedging effect, the current demand for money falls, which means the velocity rises. The beliefs on the mean money growth rate also affects the velocity. From the equation, the velocity increases when investors think the mean growth rate is higher. Second, an increase in the volatility of the money growth rate reduces the velocity, which is consistent with a finding in Friedman (1983). He shows that increases in the volatility of money growth increases the level of money stock relative to nominal income due to precautionary purposes. As a result, the velocity decreases.[13]. Lastly, the effects of the mean and the volatility of the output growth rate depend on the investor's attitude towards risk, γ. If the hedging motive dominates ($\gamma > 1$), the effect of the mean and the volatility of output growth rate are similar to money growth rate. However, if the substitution motive dominates ($\gamma < 1$), the effect has the opposite sign. When $\gamma = 1$, the velocity is unrelated to the parameters in the real output process since the utility function has a separable form in output and real balances.

Proposition 3: *Under Assumptions 1-7, the nominal price for a τ-maturity zero-coupon bond is given by*

$$B(t,\tau) = \sum_{i=1}^{I} \sum_{j=1}^{J} B_{ij}(\tau) \pi_i(t) 1(v(t) = j), \tag{22}$$

where the positive constants $B_{ij}(\tau)$ are given by

$$B_{ij}(\tau) = E\left[e^{-\phi\tau} \frac{U_c(y(t+\tau), m(t+\tau))}{U_c(y(t), m(t))} \frac{P(t)}{P(t+\tau)} | s(t) = i, v(t) = j\right]. \tag{23}$$

The parameter values are subject to the following condition:

$$\phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 + \mu_i - \sigma_{M_j}^2 + (1-\gamma)\rho_j \sigma_{yj} \sigma_{M_j} > 0, \text{ for } i = 1, .., I \text{ and } j = 1, .., J. \tag{24}$$

[13]Applying a multivariate GARCH-M model of money growth and velocity, Serletis and Shahmoradi (2006) found that the variability of money growth (M2) contains information in predicting the velocity over the pre- and post-October 1979 periods. An increase in the current volatility of M2 growth will decrease the velocity in the following two (month) periods.

$\{B_{ij}(\tau)\}_{i=1,..,I,\ j=1,..,J}$ represent the investor's expectation on the relative discounted value of one unit of cash paid at maturity, conditional on the state (value) of the means, volatilities and correlation of growth rates. I call $B_{ij}(\tau)$ the state-dependent price of nominal bonds. Condition (24) rules out negative state-dependent prices as well as negative A_{ij}. These state-dependent prices depend on maturity, but not on time. A large $B_{ij}(\tau)$ implies that the investor is willing to pay a high price for one unit of cash at maturity, relative to a unit of cash in state i, j. From equation (22), the prices of nominal bonds are shown as weighted averages of the state-dependent prices, $B_{ij}(\tau)$, and the weights are given by the beliefs. Time-dependence in prices arises from the dynamics of investor's beliefs and the dynamics of the instantaneous volatility and the correlation coefficients. In turn, these coefficients also affect the evolution of prices through the investor's belief, as can been seen in the belief equation (13).

Proposition 4:

The instantaneous nominal interest rate is given by

$$R(t) = \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} \pi_i(t) 1(v(t) = j), \tag{25}$$

where

$$h_{ij} = \phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 + \mu_i - \sigma_{M_j}^2 + (1-\gamma)\rho_j \sigma_{yj} \sigma_{M_j} > 0 \text{ for any } i \text{ and } j. \tag{26}$$

According to equation (25), the time-variation in the nominal short rate arises from changes in the investor's beliefs and the covariances of the growth rates of real output and money. Equation (26) implies that the state-dependent short rate is a function of parameters in the utility function and the stochastic process of real output and money. The short rate is increasing with the discount rate and the mean growth rate of money and decreasing with the volatility of the money growth rate. Also, if investors are sufficiently risk-averse, namely, $\gamma > 1$, the short rate increases with the mean growth rate of real output and decreases with the volatility of the output growth rate. These comparative static properties are supported by the following intuition. When the subjective discount rate is higher, the consumer favors less future consumption or less future real balances, thus they demand more bonds. Moreover, a more risk-averse investor prefers to diversify their consumption and money over time and also prefers a small amount of uncertainty regarding the economy. Hence, when the mean output and money growth rates fall or the volatility of the output and money growth rates rise, the current demand for the (risk-free) bond increases, that is, its return decreases. These effects are strengthened by increasing investor's risk aversion. (25) also shows that the state-dependent short rate and the mean growth rate of real output (consumption) are perfectly positively correlated when $\gamma > 1$. In other words, the state-dependent short rate is always higher at business cycle peaks than at troughs. If the investor has logarithmic utility, the nominal rate is independent of the real output growth rate.

4 Properties of the term premium, the conditional volatility and the short rate

In this section I explore the implications for the ex ante term premium of nominal bonds, for the conditional volatility of excess holding period returns on nominal bonds, and for the nominal short rate.

4.1 Term premium for nominal bonds

Applying Itô's formula to the bond price (22) and the short rate (25), I derive the conditional mean of the instantaneous excess returns on nominal bonds as:

$$\frac{1}{dt}E\left[\frac{dB(t,\tau)}{B(t,\tau)} - R(t)dt|\mathcal{F}^{'}(t)\right] = \frac{1}{B(t,\tau)}\left[\sum_{i=1}^{I}\sum_{j=1}^{J} B_{ij}(\tau)(h_{ij} - R(t))1(v(t) = j)\pi_i(t)\right.$$
$$\left. + \sum_{i=1}^{I}\sum_{j,k}\left(B_{ij}(\tau)1(v(t) = j) - B_{ik}(\tau)1(v(t-) = k)\right)\pi_i(t)\right], \tag{27}$$

where $v(t-)$ denotes limits from the left. This ex ante term premium contains the continuous-time and discontinuous-time parts that arise from the investor's beliefs and jumps in the growth volatilities, respectively. Assume that the mean and volatility of the growth rates take two possible values, so that $I = 2$ and $J = 2$. With this assumption, the continuous-time part has the following expression when $v(t) = 1$:

$$\frac{(h_{11}-h_{21})(B_{11}(\tau)-B_{21}(\tau))\pi_1(t)\pi_2(t)}{B(t,\tau)}$$

$$+\left[B_{11}(\tau)-B_{12}(\tau)1(v(t-)=2)\right]\pi_1(t)+\left[B_{21}(\tau)-B_{22}(\tau)1(v(t-)=2)\right]\pi_2(t). \tag{28}$$

From this expression the sign of the continuous part of the term premium is determined by the sign of $(h_{11}-h_{21})(B_{11}(\tau)-B_{21}(\tau))$. Hence, for the term premium to be positive, the differences between the state-dependent short rates and bond prices are required to have opposite signs. Additionally, the absolute value of the term premium increases as these differences widen. Furthermore, assume that the mean and the volatility of the growth rates do not change frequently so that $\lambda^s_{12}, \lambda^s_{21} \approx 0$ and $\lambda^v_{12}, \lambda^v_{21} \approx 0$. With this assumption, the continuous part of the term premium always becomes negative because $(h_{1j}-h_{2j})(B_{1j}(\tau)-B_{2j}(\tau)) = (h_{1j}-h_{2j})(e^{-h_{1j}\tau}-e^{-h_{2j}\tau})$ is negative. By taking account of the effect of the discontinuous part, the term premium cannot be positive unless the state-dependent bond prices, B_{ij} are sufficiently different across the states of the growth volatilities, j. In addition, because jumps in the growth volatility rarely occur, the effect of the discontinuous-time part on the term premium is small. As a result, it is likely the term premium will be negative for any time and any maturity when the state does not change frequently and the number of states is two.

4.2 The conditional volatility of excess nominal bond returns

Itô's formula implies that the conditional volatility of the excess returns is of the form

$$
\begin{aligned}
&\frac{d}{dt}Var\left(\frac{dB(t,\tau)}{B(t,\tau)}-R(t)dt|\mathcal{F}'(t)\right)\\
&=\frac{1}{[B(t,\tau)]^2}\left\{\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\pi_i(t)}{\sqrt{1-\rho_j^2}}\left(\frac{\theta_i-m_y(t)}{\sigma_{yj}}+\frac{\sqrt{|\rho_j|}\,(\mu_i-m_M(t))}{\sqrt{\sigma_{yj}\sigma_{Mj}}}\right)B_{ij}(\tau)1(v(t)=j)\right]^2\right.\\
&+\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\pi_i(t)}{\sqrt{1-\rho_j^2}}\left(\frac{\sqrt{|\rho_j|}\,(\theta_i-m_y(t))}{\sqrt{\sigma_{yj}\sigma_{Mj}}}+\frac{\mu_i-m_M(t)}{\sigma_{Mj}}\right)B_{ij}(\tau)1(v(t)=j)\right]^2\\
&\left.+\sum_{j\neq k}\left[\sum_{i=1}^{I}\left(B_{ij}(\tau)-B_{ik}(\tau)\right)\pi_i(t)1(v(t)=j)\right]^2\lambda_{kj}^v\right\}. \qquad (29)
\end{aligned}
$$

Equation (29) gives several implications. First, the model captures GARCH effects, or volatility clustering, and identifies the sources of GARCH in the investor's beliefs that exhibit mean reversion. Time-variations in the conditional volatility also stem from random shocks (conditional volatility in the growth rates). The first two terms on the right hand side of the conditional volatility result from the unobservability of the mean growth rates. On the other hand, the last term arises from (observable) time-varying volatility of growth rates.[14] Second, unlike the ex ante term premium, the conditional volatility contains no jump. Finally, the conditional volatility of the excess returns and the volatility parameters of the economic fundamentals are inversely related. Consequently, the volatility of the excess returns can be large although macro variable volatility is small, which has been noticed by Schwert (1989). This fact suggests that interest rates can be stable even in a volatile period of monetary policy. In order to confirm the inverse relationship between noise parameters in economic fundamentals and the conditional volatility of bond returns, Panel A of Figure 1 plots the conditional volatility (29) for the calibrated parameter reported in Table 1. This figure offers insightful observations. First, it highlights the inverse relationship between the conditional volatility of term premium and macroeconomic growth. Second, the figure shows that

[14] If mean growth rates are observable, $\theta_i - m_y(t) = \mu_i - m_M(t) = 0$ for all i and the first two terms vanish. If volatility is constant, $\lambda_{kj}^v = 0$ for $k \neq j$ and the last term vanishes.

the conditional volatility of bond returns is largest at the point of maximum uncertainty, $\pi(t) = 0.5$. In other words, the more uncertain on economic conditions the investor is, the more volatile the bond market becomes. Let us next look at the impact of risk aversion on the volatility of term premium. Panel B of Figure 1 plots the conditional volatility in relation to investor's belief given the value of the coefficient of relative risk aversion, γ. In this figure, given an investor's beliefs about the mean growth rates of real output and money the conditional volatility monotonically increases with γ, implying that the investor's risk attitude and beliefs independently affect volatility. This result should not be surprising due to the two-stage optimization process (the estimation of unobservable mean economic growths and the maximization of utility).

4.3 The nominal short rate

Next, applying Itô's formula to the short rate (25) gives the process of the short rate:

$$dR(t) = \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} d\pi_i(t) 1(v(t) = j) + \sum_{i=1}^{I} \sum_{j,k} \left(h_{ij} 1(v(t) = j) - h_{ik} 1(v(t-) = k)\right) \pi_i(t), \tag{30}$$

where $v(t-)$ denotes limits from the left.

The stochastic process of the short rate is given by the sum of two terms. The first term arises from learning effects, resulting in smooth changes in the short rates. This term can be represented as a Markov switching process, which is somewhat similar to the regime switching model of Ang and Bekaert (2002b). While Ang and Bekaert (2002b) exogenously specify the regimes in the level of short rate, the short rate regimes in the incomplete information model are derived by the regimes in the mean growth rates in real output and money supply. The belief process (13) demonstrates that the instantaneous volatility of beliefs becomes larger as the investor becomes more uncertain about the economic states. If the uncertainty is small when the mean real output growth is high (this is possible if the mean and the volatility of the real output growth are negatively correlated), then the conditional volatility of the short rates is negatively correlated with the mean growth rate or business cycle. This negative correlation has been empirically observed by Den Haan and Spear (1998), who explain this short rate property introducing financial frictions quantified using the spread between borrowing and lending rate into their equilibrium model. On the other hand, I have shown that the investor's beliefs may represent an alternative key to understand this feature of the conditional short rate volatility over the business cycles.

The second term is driven by jumps in the volatility of growth rates in which the jump size is also stochastic. According to Das (2002), jump processes offer a good empirical description of the short rate behavior, which cannot be explained by i.i.d. Gaussian models. He claims the jumps are caused by surprises in information releases, such as economic news announcements. However, in the incomplete information model, jumps in short rates are endogenously determined by discrete shifts in economic fundamentals. In sum, the stochastic processes of the short rate in incomplete information (30) can be recognized to belong to the class of jump models mixed with Markov switching processes.

5 Conclusion

In this paper I have developed a dynamic equilibrium model of the term structure of nominal interest rates for a monetary economy with incomplete information. In this economy, investors cannot observe the mean growth rates of fundamentals. Therefore, they infer the current state of the drift coefficients by Bayes' rule. Closed form solutions for bond prices and a commodity price have then been derived. The dynamic properties of prices depend on investors' beliefs on the current state and random shocks to fundamentals. An interesting feature in the model is that a GARCH effect in excess bond returns results from time-variations in beliefs. In addition, calibrating the model for US output and money supply data shows that the conditional volatility of a bond return is maximum when investors are most uncertain about economic fundamentals. The model also implies that noisy monetary policy can decrease the volatility in bond markets.

The incomplete information model in this paper offers superior tractability for pricing several asset classes. First, the model can be used to infer structural parameters from real interest rates. Evans (1998) discusses several stylized facts regarding indexed bonds in the UK market. It would be useful to check whether the model can replicate these empirical findings, some of which differ from those of nominal rates. For instance, the UK real term structure is, on average, downward sloping and the real long-term rates appear to be highly stable. Furthermore, the analysis in this article can be applied to price stocks. For instance, the relationship between stock returns and inflation and between stock returns and bond returns could be investigated. There are some intriguing empirical findings on these issues. For instance, Fama (1981) has found a negative relation between stock returns and inflation. Shiller (1982) has stressed the failure of bond prices to move with stock prices.

Appendix

1. Proof of Proposition 1

Theorem 7.17 in Liptser and Shiryayev (2000, p. 286) presents explicit equations of optimal filtering when an unobservable process takes discrete values. David (1997), Veronesi (2000), and Veronesi and Yared (1999) apply the theorem in case of constant diffusion terms. I extend the case of constant diffusion terms to stochastic diffusion terms, described in Assumption 5 (Growth rate volatility). To apply the theorem on optimal filtering in Liptser and Shiryayev (2000) I need to show that the stochastic diffusion terms are adapted to $\mathcal{F}'(t) = \{y(z), M^s(z)\}_{0\leq z\leq t}$, that is, the current diffusion terms are observable to investors. Indeed, the quadratic variation of the process (1) and (2) gives correct current diffusion terms. Hence, Theorem 9.1 in Liptser and Shiryayev (2000, p. 355) can be applied in the stochastic diffusion case, obtaining the required differential equation (13) in Proposition 1.

2. Proof of Proposition 2

From the equilibrium condition for the commodity price (6) and investors' utility function (8),

$$\frac{1}{P(t)} = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{y(t)}{M(t)}\right)\sum_{i=1}^{I}\sum_{j=1}^{J}\pi_i(t)1(v(t)=j) \times E\left[\int_t^\infty e^{-\phi(z-t)}\right. .$$

$$\left. \times \left(\frac{y(z)}{y(t)}\right)^{\alpha(1-\gamma)}\left(\frac{M(z)}{M(t)}\right)^{-\alpha-\gamma(1-\alpha)}\left(\frac{P(z)}{P(t)}\right)^{(1-\alpha)(1-\gamma)} dz|s(t)=i, v(t)=j\right]. \tag{A.1}$$

Next, guess the functional form of the commodity price in the following way,

$$\frac{1}{P(t)} = By(t)^a M(t)^b \left(\sum_{i=1}^{I}\sum_{j=1}^{J} A_{ij}\pi_i(t)1(v(t)=j)\right)^c, \tag{A.2}$$

where a, b, c, B and A_{ij} are indeterminate coefficients. Substituting (A.1) into (A.2) and comparing both sides of the equation yields the coefficients as follows:

$$a=1,\, b=-1,\, c=1,\, B=\frac{1-\alpha}{\alpha},\, A_{ij} = E\left[\int_t^\infty \frac{f(z)}{f(t)}dz|s(t)=i, v(t)=j\right], \tag{A.3}$$

where $f(t) \equiv e^{-\phi t}y(t)^{1-\gamma}M(t)^{-1}\left(\sum_{l=1}^{I} A_{lj}\pi_l(t)\right)^{(1-\alpha)(1-\gamma)}$.

A system of equations for A_{ij} (19) will be derived in the following manner. First, let $V_{ij}(t)$ ($\equiv A_{ij}$) be

$$V_{ij}(t) = E\left[\int_t^\infty \frac{f(z)}{f(t)}dz|s(t)=i, v(t)=j\right]$$

$$= E\left[\int_t^{t+\Delta} \frac{f(z)}{f(t)}dz|s(t)=i, v(t)=j\right] + E\left[\int_{t+\Delta}^\infty \frac{f(z)}{f(t)}dz|s(t)=i, v(t)=j\right], \quad \text{(A.4)}$$

and Δ denotes an infinitesimal interval, the limit of which will be taken as $\Delta \to 0$ below. The discontinuous vector processes $s(t)$ and $v(t)$ are assumed to be right-continuous. Thus, the probability that a shift from one state to the other occurs in $[t, t+\Delta]$ is of order $o(\Delta)$. This means that as $\Delta \to 0$, its probability of changing states goes to zero.

Next, I will evaluate the two conditional expectations in (A.4). For this, I will obtain the stochastic process of $f(t)$, given $s(t)=i, v(t)=j$. As explained above, $s(t)=i$ and $v(t)=j$ do not change during the infinitesimal period Δ. Thus, $\Pr(s(z)=i)=1$, and $1(v(z)=j)=1$ at any z in $[t, t+\Delta]$. With this result, the Bayes formula presented in Liptser and Shiryayev (2000, p. 314) implies $\pi_l(t)=1$ for $l=i$ and $\pi_l(t)=0$ for $l \neq i$ during the small period of time. Hence, (in the small period, given $s(t)=i, v(t)=j$) the processes of the three random variables, $\pi_l(t)$, $y(t)$ and $M(t)$ are

$$d\pi_l(t) = 0, \text{ for any } l, \quad \text{(A.5)}$$

$$dy(t) = \theta_i y(t)dt + \sigma_{11j}y(t)\widetilde{Z}_1(t) + \sigma_{12j}y(t)\widetilde{Z}_2(t), \quad \text{(A.6)}$$

$$dM(t) = \mu_i M(t)dt + \sigma_{21_j}M(t)d\widetilde{Z}_1(t) + \sigma_{22_j}M(t)d\widetilde{Z}_2(t), \quad \text{(A.7)}$$

where $\sigma_{11j} = \sigma_{yj}$, $\sigma_{12j} = \sigma_{21j} = \sqrt{|\rho_j|\,\sigma_{yj}\sigma_{Mj}}$, $\sigma_{22j} = \sigma_{Mj}$. $\left\{\widetilde{Z}_1(t), \widetilde{Z}_2(t)\right\}$ are independent Brownian motions adapted to $\mathcal{F}'(t)$. Using (A.5)-(A.7) and applying Itô's formula to $\log f$ give

$$d\log f = \left[-k_{ij} - \frac{\{-\gamma(1-\gamma)\sigma_{11j}+\sigma_{21_j}\}^2}{2} - \frac{\{-\gamma(1-\gamma)\sigma_{21j}+\sigma_{22_j}\}^2}{2}\right]dt$$

$$+[-\gamma(1-\gamma)\sigma_{11j}+\sigma_{21_j}]d\widetilde{Z}_1(t) - [-\gamma(1-\gamma)\sigma_{21j}+\sigma_{22_j}]d\widetilde{Z}_2(t), \quad \text{(A.8)}$$

where

$$k_{ij} = \phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 + \mu_i - \sigma_{M_j}^2 + (1-\gamma)\rho_j\sigma_{yj}\sigma_{M_j} \tag{A.9}$$

Taking the integral of the both sides of (A.8) from t to $t+\Delta$ gives

$$\begin{aligned}\frac{f(t+\Delta)}{f(t)} = \exp\Bigg\{ & \left[-k_{ij} - \frac{\{-\gamma(1-\gamma)\sigma_{11j} + \sigma_{21_j}\}^2}{2} - \frac{\{-\gamma(1-\gamma)\sigma_{21j} + \sigma_{22_j}\}^2}{2}\right]\Delta + \\ & + \left[-\gamma(1-\gamma)\sigma_{11j} + \sigma_{21_j}\right]\left[\widetilde{Z}_1(t+\Delta) - \widetilde{Z}_1(t)\right] \\ & - [-\gamma(1-\gamma)\sigma_{21j} + \sigma_{22_j}]\left[\widetilde{Z}_2(t+\Delta) - \widetilde{Z}_2(t)\right]\Bigg\}\end{aligned} \tag{A.10}$$

By applying Fubini's theorem and using (A.10), the first conditional expectation in (A.4) can be written as

$$\begin{aligned} E\left[\int_t^{t+\Delta}\frac{f(z)}{f(t)}dz | s(t)=i, v(t)=j\right] &= \int_0^{\Delta} E\left[\frac{f(t+h)}{f(t)} | s(t)=i, v(t)=j\right] dh \\ &= \frac{1-\exp(-k_{ij}\Delta)}{k_{ij}}. \end{aligned} \tag{A.11}$$

In a similar manner, the second conditional expectation in (A.4) can be rewritten as:

$$\begin{aligned} E\left[\int_{t+\Delta}^{\infty}\frac{f(z)}{f(t)}dz | s(t)=i, v(t)=j\right] = \exp(-k_{ij}\Delta)\Big[& (1+\lambda_{ii}^s\Delta + \lambda_{jj}^v\Delta)V_{ij}(t+\Delta) \\ & + \Delta\sum_{l\neq i}\lambda_{il}^s V_{lj}(t+\Delta) + \Delta\sum_{n\neq j}\lambda_{jn}^v V_{in}(t+\Delta)\Big] \end{aligned} \tag{A.12}$$

Substituting (A.11) and (A.12) into (A.4) and applying Taylor's theorem ($V_{ln}(t+\Delta) = V_{ln}(t) + \Delta V_{ln}'(t) + o(\Delta)$ where V' denotes the first derivative of V with respect of time t.) give

$$\begin{aligned} V_{ij}(t) = & \frac{1-\exp(-k_{ij}\Delta)}{k_{ij}} + \exp(-k_{ij}\Delta)\Big\{(1+\lambda_{ii}^s\Delta + \lambda_{jj}^v\Delta)V_{ij}(t) \\ & + \Delta V_{ln}'(t) + \Delta\sum_{l\neq i}\lambda_{il}^s V_{lj}(t) + \Delta\sum_{n\neq j}\lambda_{jn}^v V_{in}(t) + o(\Delta)\Big\}. \end{aligned} \tag{A.13}$$

Taking the limit $\Delta \to 0$ on (A.13) obtains the following differential equations

$$V_{ij}'(t) = (-\lambda_{ii}^s - \lambda_{jj}^v + k_{ij})V_{ij}(t) - \sum_{l\neq i}\lambda_{il}^s V_{lj}(t) - \sum_{n\neq j}\lambda_{jn}^v V_{in}(t) - 1, \text{ for } i=1,...,I \text{ and } j=1,...,J. \tag{A.14}$$

Because V does not directly depend on t (Remind $V_{ij}(t) = A_{ij}$ just depends on states i and j), $V'_{ij}(t) = 0$. As a result, (A.14) can be rewritten as

$$\left(-\lambda^s_{ii} - \lambda^v_{jj} + \phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma^2_{yj} + \mu_i - \sigma^2_{M_j} + (1-\gamma)\rho_j\sigma_{yj}\sigma_{M_j}\right) A_{ij}$$

$$- \sum_{l\neq i} \lambda^s_{il} A_{lj} - \sum_{n\neq j} \lambda^v_{jn} A_{in} = 1. \quad \text{(A.15)}$$

where $i = 1, ..., I$ and $j = 1, ..., J$. By solving the system of linear equations (A.15), A_{ij} can be obtained.

3. Proof of Proposition 3

From the equilibrium condition for bond prices (7), investors' preference (8), and commodity price (18), bond prices are given by

$$B(t,\tau) = e^{-\phi\tau} E\left[\frac{U_c(y(t+\tau), m(t+\tau)}{U_c(y(t), m(t))}\frac{P(t)}{P(t+\tau)}|\mathcal{F}'(t)\right] = \sum_{i=1}^{I}\sum_{j=1}^{J} B_{ij}(\tau)\pi_i(t)1(v(t)=j) \quad \text{(A.16)}$$

where

$$B_{ij}(\tau) \equiv E\left[\frac{U_c(y(t+\tau), m(t+\tau)}{U_c(y(t), m(t))}\frac{P(t)}{P(t+\tau)}|\mathcal{F}'(t)\right] \quad \text{(A.17)}$$

$B_{ij}(\tau)$ can be interpreted as the state-dependent price of nominal bonds.

Let $g(t) \equiv e^{-\phi t} y(t)^{1-\gamma} M(t)^{-1}\left[\sum_{l=1}^{I}\sum_{n=1}^{J} A_{ln}\pi_l(t)1(v(t)=n)\right]^{1+(1-\alpha)(1-\gamma)}$. (A.17) can then be written as

$$B_{ij}(\tau) = E\left[\frac{g(t+\Delta)}{g(t)}\frac{g(t+\tau)}{g(t+\Delta)}|s(t)=i, v(t)=j\right], \quad \text{(A.18)}$$

where Δ is an infinitesimal period of time. As in Proof of Proposition 2, $s(t) = i, v(t) = j$ do not change during the infinitesimal period Δ so that in $[t, t+\Delta]$ the processes of $\pi_l(t)$, $y(t)$ and $M(t)$ are given in (A.5)-(A.7). Then, applying Itô's formula to $g(t)$, its stochastic process can be derived:

$$\frac{dg}{g} = -h_{ij}dt - \gamma\sigma_{11j}d\widetilde{Z}_1(t) - \gamma\sigma_{12j}d\widetilde{Z}_2(t), \quad \text{(A.19)}$$

where $h_{ij} = \phi - (1-\gamma)\theta_i + \frac{\gamma(1-\gamma)}{2}\sigma_{yj}^2 + \mu_i - \sigma_{Mj}^2 + (1-\gamma)\rho_j\sigma_{Yj}\sigma_{Mj}$, $\sigma_{11j} = \sigma_{yj}$, $\sigma_{12j} = \sqrt{|\rho_j|\,\sigma_{yj}\sigma_{Mj}}$. $\left\{\widetilde{Z}_1(t), \widetilde{Z}_2(t)\right\}$ are independent Brownian motions adapted to $\mathcal{F}'(t)$. Integrating the differential equation (A.19) from t to $t+\Delta$ yields

$$\begin{aligned}\frac{g(t+\Delta)}{g(t)} &= \exp\left\{\int_t^{t+\Delta}[-h_{ij}ds - \gamma\sigma_{11j}d\widetilde{Z}_1(s) - \gamma\sigma_{12j}d\widetilde{Z}_2(s)]ds\right\} \\ &= \exp\left\{[-h_{ij} - \frac{(\gamma\sigma_{11j})^2}{2} - \frac{(\gamma\sigma_{12j})^2}{2}]\Delta - \gamma\sigma_{11j}[\widetilde{Z}_1(t+\Delta) - \widetilde{Z}_1(t)]\right. \\ &\quad \left. -\gamma\sigma_{12j}[\widetilde{Z}_2(t+\Delta) - \widetilde{Z}_2(t)]\right\}. \qquad \text{(A.20)}\end{aligned}$$

Taking the conditional expectation on (A.20) gives

$$E\left[\frac{g(t+\Delta)}{g(t)}|s(t)=i, v(t)=j\right] = \exp(-h_{ij}\Delta). \qquad \text{(A.21)}$$

Using the fact that $\frac{g(t+\Delta)}{g(t)}$ and $\frac{g(t+\tau)}{g(t+\Delta)}$ are independent and applying Taylor's theorem,

$$\begin{aligned}B_{ij}(\tau) &= E\left[\frac{g(t+\Delta)}{g(t)}\frac{g(t+\tau)}{g(t+\Delta)}|s(t)=i, v(t)=j\right] \\ &= E\left[\frac{g(t+\Delta)}{g(t)}|s(t)=i, v(t)=j\right] E\left[\frac{g(t+\tau)}{g(t+\Delta)}|s(t)=i, v(t)=j\right] \\ &= \exp(-h_{ij}\Delta)\left[B_{ij}(\tau) - \Delta B'_{ij}(\tau) + \Delta\sum_{l=1}^{I}\lambda_{il}^s B_{lj}(\tau) + \Delta\sum_{n=1}^{J}\lambda_{jn}^v B_{in}(\tau)\right] + o(\Delta), \qquad \text{(A.22)}\end{aligned}$$

where $B'_{ij}(\tau)$ denotes the first derivative of $B_{ij}(\tau)$ with respect of the term to maturity τ. Taking the limit on (A.22) as $\Delta \longrightarrow \infty$ and rearranging,

$$B'_{ij}(\tau) = (\lambda_{ii}^s + \lambda_{jj}^v - h_{ij})B_{ij}(\tau) + \sum_{l\neq i}\lambda_{il}^s B_{lj}(\tau) + \sum_{n\neq j}\lambda_{jn}^v B_{in}(\tau) \text{ for } i=1,...,I \text{ and } j=1,...,J. \qquad \text{(A.23)}$$

(A.23) represents the system of first-order differential equations for $B_{ij}(\tau)$ with constant coefficients. Writing (A.23) in vector form,

$$B'(\tau) = \Phi B(\tau), \qquad \text{(A.24)}$$

where $B(\tau)$ is a $(I \cdot J)$ vector and Φ is a $(I \cdot J)\times(I \cdot J)$ constant matrix. Note that the initial condition for the differential equations is given by $B_{ij}(0) = 1$ for any i and j. Hence, applying the solution method of the system of first-order differential equations with constant coefficients gives the solutions: $B(\tau) = \Pi \begin{pmatrix} \exp(\omega_1 \tau) \\ . \\ \exp(\omega_{I \cdot J} \tau) \end{pmatrix}$, where ω_n for $n = 1, ..., I \cdot J$ and Π are the eigenvalues and the normalized eigenvectors of Φ, respectively.

4. Proof of Proposition 4

Applying L'Hôspital's Rule,

$$
\begin{aligned}
R(t,\tau) &= -\lim_{\tau \to 0} \frac{\ln(B(t,\tau))}{\tau} \\
&= -\frac{\sum\limits_{i=1}^{I}\sum\limits_{j=1}^{J}\left[(\lambda_{ii}^{s} + \lambda_{jj}^{v} - h_{ij}) + \sum\limits_{l \neq i} \lambda_{il}^{s} + \sum\limits_{n \neq j} \lambda_{jn}^{v}\right] \pi_i(t) 1(v(t) = j)}{\sum\limits_{i=1}^{I}\sum\limits_{j=1}^{J} \pi_i(t) 1(v(t) = j)} \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} h_{ij} \pi_i(t) 1(v(t) = j), \qquad \text{(A.25)}
\end{aligned}
$$

To derive the last equation in (A.25), I use the following facts: $\lambda_{ii}^{s} + \sum\limits_{l \neq i} \lambda_{il}^{s} = 0$, for any i and $\lambda_{jj}^{v} + \sum\limits_{n \neq j} \lambda_{jn}^{v} = 0$, for any j and $\sum\limits_{i=1}^{I}\sum\limits_{j=1}^{J} \pi_i(t) 1(v(t) = j) = 1$.

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Table 1: Parameter values employed for illustrating volatility graphs:

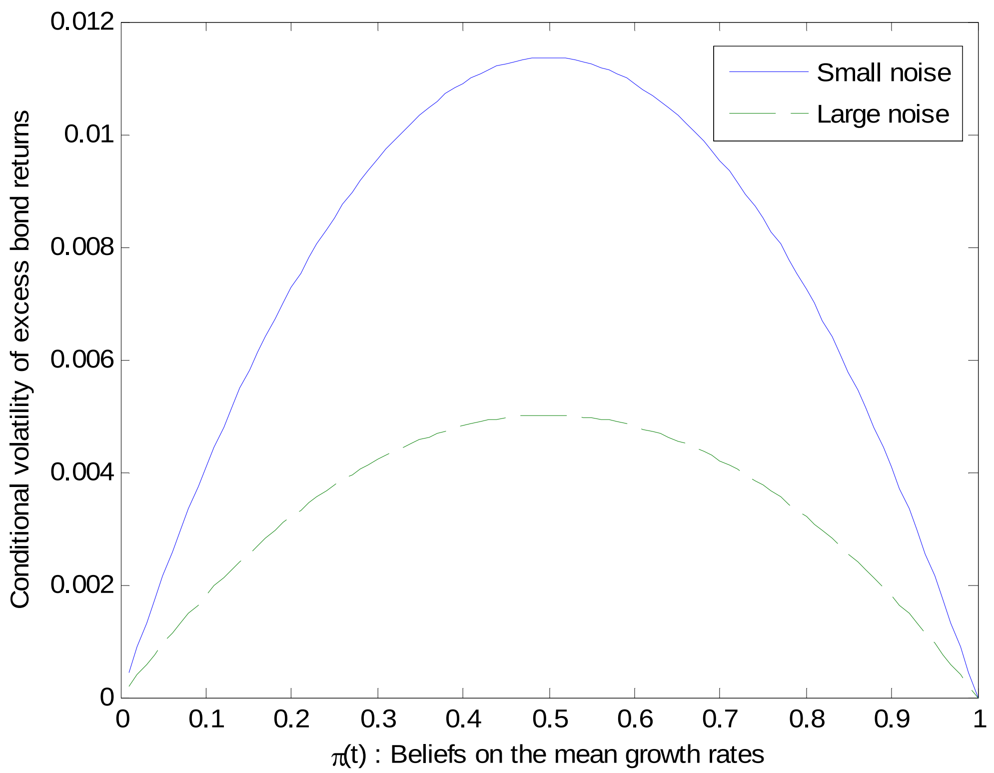
The table summarizes the selection of parameter values in the model used for developing Figure 1. Transition probabilities are on a monthly basis while the other parameters are on a yearly basis. The parameter values in the fundamental process are estimated by a two-state regime switching model by Hamilton (1989), and the parameter values in the utility function are chosen such that the model-implied mean short rate is roughly matched to its actual sample mean. For this calibration I employ monthly US data for industrial production as real output, M2 as money supply, and a yield on the 3-month Treasury bill as the short rate. All of the series are obtained from the official release of the Federal Reserve Board of Governors. The sample spans from January 1960 through December 2005.

Parameters	Values
α : Weight of consumption vs. real balances	0.57
γ : Coefficient of relative risk aversion	0.6
$e^{-\phi}$: Subjective discount rate	0.99
θ_L, μ_L : First (Low) state of the mean growth rate	3.2%, 2.9%
θ_H, μ_H : Second (High) state of the mean growth rate	4.4%, 8.0%
σ_{yL}, σ_{ML}, ρ_L : First (Low) state of covariances	1.5%, 0.4%, -0.09
σ_{yH}, σ_{MH}, ρ_H : Second (High) state of covariances	3.2%, 1.0%, 0.10
λ^s_{LH}, λ^s_{HL} : Transition probabilities of mean growth rates	6.7%, 3.2%
λ^v_{LH}, λ^v_{HL} : Transition probabilities of covariances	2.7%, 4.7%

Figure 1: Conditional volatility of excess bond returns and investor's beliefs:

This figure illustrates the conditional volatility of excess (two-year maturity) bond returns (Equation (29)) as a function of an investor's beliefs on the current economic state, implied by the parameters in Table 1. The term of jump effects on the volatility is ignored. The two plots depend on the conditional covariance (noise parameter) of real output and money growths, $\sigma_{y,}$ $\sigma_{M,}$ and ρ (Panel A), and on the coefficient of relative risk aversion, γ (Panel B). For Panel B, the covariances of growth rates of fundamentals are set as high.

Panel A: Relationship between bond volatility and noise in the fundamentals



Panel B: Relationship between bond volatility and risk aversion