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Extended-Gaussian Term Structure Models and Credit Risk Applications

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EXTENDED-GAUSSIAN TERM STRUCTURE MODELS AND CREDIT RISK APPLICATIONS

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Abstract

This paper presents three factor "Extended Gaussian" term structure models (EGM) to price default-free and defaultable bonds. To price default-free bonds EGM assume that the instantaneous interest rate is a possibly non-linear but monotonic function of three latent factors that follow correlated Gaussian processes. The bond pricing equation can be solved conveniently through separation of variables and finite difference methods. The merits of EGM are hetero-schedastic yields, unrestricted correlation between factors and the absence of the admissibility restrictions that affect canonical affine models. Unlike quadratic term structure models, EGM are amenable to maximum likelihood estimation, since observed yields are sufficient statistics to infer the latent factors. Empirical evidence from US Treasury yields shows that EGM fit observed yields quite well and are estimable. EGM are of even greater interest to price fixed and floating rate defaultable bonds. A reduced form, a credit rating based and a structural credit risk valuation model are presented: these credit risk models are EGM and their common merit is that bond pricing remains tractable through separation of variables even if interest rate risk and credit risk are arbitrarily correlated.

Key words: bond pricing, Gaussian term structure models, Vasicek model, separation of variables, finite difference method, reduced form credit risk model, credit ratings model, structural model.

JEL classification: G13.

1 Introduction

This paper presents three factor Extended Gaussian term structure models (EGM). In EGM the instantaneous interest rate is driven by latent factors that follow correlated Gaussian processes, but the instantaneous interest rate is

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non-linear and monotonic in one or more of the latent factors. Although this non-linearity requires numerical solutions for bond prices, the computations remain tractable and parameter estimation remains feasible. EGM whereby the instantaneous interest rate is linear in all three factors coincide with the Gaussian models proposed in Langetieg (1980), Babbs and Nowman (1999) or Dai-Singletgon (2002). We concentrate on a class of EGM whereby the instantaneous interest rate is linear in two of the latent factors and non-linear in the third latent factor. We apply such class of EGM to the pricing of default-free and defaultable bonds.

To price default-free bonds the focus is on two specific EGM: the GBK model whereby one latent factor follows a Black-Karasinski (1991) process and the GB model whereby one latent factor follows the Black (1995) truncated Gaussian process. Empirical evidence based on US Treasury yields shows that EGM are estimable despite the numerical solutions they involve and the GB model fits observed default-free yields better than the GBK model.

EGM are of particular interest to price fixed and floating rate defaultable bonds and the paper presents three EGM for credit risk pricing: a reduced form, a credit rating based and a structural credit risk valuation model. These credit risk models have the common merit that bond pricing remains tractable even if interest rate risk and credit risk are arbitrarily correlated. In particular the solution to the bond pricing equation involves convenient separation of variables. The credit rating based model can re-produce stylised facts observed in the credit markets, such as rating momentum and stochastic credit spreads even in the absence of rating transitions.

The paper is organised as follows. First the most relevant literature is reviewed. Then three-factor EGM for pricing default-free bonds are characterised and estimated using US Treasury yields. Then three credit risk EGM are presented. The conclusions follow.

1.1 Literature

The focus of this paper on EGM is motivated by the limitations of affine and quadratic term structure models. Affine term structure models are affected by admissibility restrictions on the market price of risk specification and on the correlation between factors, as shown in Dai-Singleton (2000, 2002). In particular Dai and Singleton (2002) found that these restrictions explain why general affine models perform worse than Gaussian models in explaining stylized facts that challenge the expectations hypothesis. In fact these restrictions affect neither Gaussian nor quadratic models, but Gaussian models cannot rule out negative nominal interest rates and cannot generate hetero-schedastic yields, whereas quadratic model can rule out negative interest rates and can generate hetero-schedastic yields. On these counts quadratic models seem preferable to Gaussian models and general affine models. Ahn-Dittmat-Gallant (2002) report the very good empirical performance of quadratic models. Unfortunately quadratic models have a major drawback in that observed bond yields are not sufficient statistics to infer the latent factors. Filtering method are then needed, but this is not entirely satisfactory and computationally demanding.

Against this backdrop this paper presents "extended Gaussian" models (EGM) to price default-free and defaultable bonds. EGM are similar to the Gaussian models of Langetieg (1980) or Babbs-Nowman (1999), but for the fact that the instantaneous interest rate is non-linear in one or more of the latent factors. EGM retain tractability through bond price formulae that still involve separation of variables. Moreover EGM have some advantages over both affine and quadratic term structure models. Unlike Gaussian models, EGM can generate hetero-schedastic yields. Unlike quadratic models, yields can be "inverted" to infer the latent factors, which makes maximum likelihood estimation feasible. EGM are not affected by admissibility restrictions on the specification of market prices of risk and on the correlation between factors. In particular negative factor correlation is admissible.

The paper also presents applications of EGM to price fixed and floating rate defaultable bonds. This is probably the most fruitful application of EGM. A reduced form, a credit rating based and a structural credit risk valuation model are presented. The credit rating based model is close in spirit to Jarrow-Lando-Turnbull (1997) and the structural model to Zhou (2001). All these EGM for credit risk pricing have the common merit that bond pricing remains tractable, even if interest rate risk and credit risk are arbitrarily correlated.

2 Three factor Extended Gaussian term structure models

This section presents three factor Extended Gaussian term structure models. The name "Extended Gaussian" is due to the fact that the instantaneous interest rate r is a possibly non-linear function of one or more latent factors that follow correlated Gaussian processes. A merit of EGM is that the bond pricing partial differential equation (PDE) can be solved through separation of variables even if the latent factors driving the instantaneous interest rate r are correlated and even if r is non-linear in some of the factors. Separation of variables provides much tractability to EGM models. Define a vector process X such that

$$X = (x, x_1, x_2)' \tag{1}$$

and such that in the risk-neutral world

$$dX = K \cdot (\Theta - X) \cdot dt + \Sigma \cdot dW^Q \tag{2}$$

where

$$\begin{aligned} K &= diag\,(k, k_1, k_2) \\ \Theta &= [\theta, \theta_1, \theta_2]' \\ \Sigma &= diag\,(\sigma, \sigma_1, \sigma_2)\,. \end{aligned}$$

W^Q is a vector of correlated Wiener processes in the risk-neutral world such that

$$dW^Q = \left[dw^Q, dw_1^Q, dw_2^Q\right]' \tag{3}$$

and

$$dw^Q dw_1^Q = \rho_1, dw^Q dw_2^Q = \rho_2, dw_1^Q dw_2^Q = \rho_{1,2}. \tag{4}$$

Later we impose the identification conditions $\theta_1 = \theta_2 = 0$. In the real probability measure X follows the process

$$dX = K^* \cdot (\Theta^* - X) \cdot dt + \Sigma \cdot dW \tag{5}$$

where

$$K^* = diag\,(k^*, k_1^*, k_2^*)$$
$$\Theta^* = [\theta^*, 0, 0]'$$

and W is a vector of correlated Wiener processes in the real world such that

$$dwdw_1 = \rho_1, dwdw_2 = \rho_2, dw_1 dw_2 = \rho_{1,2}. \tag{6}$$

Next we assume that the instantaneous interest rate is

$$r = c + f\,(x) + x_1 + x_2 \tag{7}$$

where c is a non-negative constant and $f\,(x)$ is a continuous and monotonic increasing function of x. Later we compare the empirical performance of two alternative specifications of $f\,(x)$, which are

$$f\,(x) = e^x \tag{8}$$
$$f\,(x) = \max(x, 0)^q \tag{9}$$

where q is a positive constant. When $f\,(x) = e^x$ we call the model the Gaussian-Black-Karasinski (GBK) model after the Black-Karasinski (1991) model. When $f\,(x) = \max(x, 0)^q$ we call the model the Gaussian-Black (GB) model, after the Black (1995) model. We notice that the EGM of this section is the same as the three-factor Gaussian model in Babbs and Nowman (1999) if $f\,(x) = x$. We also notice that, as in Langetieg (1980), K and K^* need not be diagonal, although we so assume for simplicity.

Let $Z\,(t, T)$ or more simply Z denote the value of a default-free zero coupon bond with residual life equal to the time interval $]t, T]$. Under the above assumptions the absence of arbitrage opportunities implies that

$$\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial x_1\partial x_2}\rho_{1,2}\sigma_1\sigma_2+\frac{\partial^2 Z}{\partial x\partial x_1}\rho_1\sigma\sigma_1+\frac{\partial^2 Z}{\partial x\partial x_2}\rho_2\sigma\sigma_2 \tag{10}$$
$$+\frac{\partial^2 Z}{\partial x_1^2}\frac{\sigma_1^2}{2}+\frac{\partial^2 Z}{\partial x_2^2}\frac{\sigma_2^2}{2}+\frac{\partial^2 Z}{\partial x^2}\frac{\sigma^2}{2}$$
$$+\frac{\partial Z}{\partial x_1}k_1\left(\theta_1-x_1\right)+\frac{\partial Z}{\partial x_2}k_2\left(\theta_2-x_2\right)+\frac{\partial Z}{\partial x}k\left(\theta-x\right)-\left(c+f\left(x\right)+x_1+x_2\right)Z=0$$

subject to $Z\left(T,T\right)=1$. The discount bond price solution involves separation of variables and is

$$Z\left(t,T\right)=e^{A(t,T)-x_1B(t,T)-x_2C(t,T)}\cdot H\left(x,t\right) \tag{11}$$

with

$$B\left(t,T\right)=\frac{1-e^{-k_1(T-t)}}{k_1} \tag{12}$$
$$C\left(t,T\right)=\frac{1-e^{-k_2(T-t)}}{k_2} \tag{13}$$

and $A\left(t,T\right)$ is the solution to

$$\frac{\partial A\left(t,T\right)}{\partial t}+B\left(t,T\right)C\left(t,T\right)\rho_{1,2}\sigma_1\sigma_2+\frac{B\left(t,T\right)^2\sigma_1^2}{2}+\frac{C\left(t,T\right)^2\sigma_2^2}{2}-c=0 \tag{14}$$

subject to $A\left(T,T\right)=0$. This ODE can be quickly solved numerically or in closed form. $H\left(x,t\right)$ or more simply H is the solution to

$$\frac{\partial H}{\partial t}+\frac{\partial^2 H}{\partial x^2}\frac{\sigma^2}{2}+\frac{\partial H}{\partial x}\left(k\left(\theta-x\right)-B\left(t,T\right)\rho_1\sigma_1\sigma-C\left(t,T\right)\rho_2\sigma_2\sigma\right)-f\left(x\right)H=0 \tag{15}$$

subject to $H\left(x,T\right)=1$. This last PDE can be quickly solved through an implicit finite difference scheme, which gives accurate and fast numerical solutions. In spite of these numerical solutions for $H\left(x,t\right)$, we can still estimate the model parameters through maximum likelihood as shown below. Such estimation is of interest as past literature has hardly ever estimated a pricing model that is solved numerically through finite difference methods. The separation of variables in the solution for $Z\left(t,T\right)$ entails that the finite difference grid is just two-dimensional grid (one dimension for time t and the other for x) rather than four-dimensional (one dimension for time t, one for x, one for x_1 and one for x_2). A two dimensional grid is much quicker to compute bond prices and to optimise the log-likelihood function of observed yields.

The merits of this EGM are that bond yields are hetero-schedastic, the factors can be freely correlated and the market prices of risk can be freely specified without any admissibility problems. In other words EGM do not suffer from the drawbacks of general affine term structure models. Negative yields cannot be ruled out, but observed bond yields are sufficient statistics to infer the latent factors X, unlike in the case of quadratic term structure models.

2.1 Maximum likelihood estimation

We can estimate the parameters of Extended Gaussian models using maximum likelihood if we assume that some bond prices are observed without error. The observation dates are t_i for $i = 1, 2, .., n$. We collect US Treasury discount bond yields from Datastream. We use the 3-month, 2-year, 5-year and 10-year yields observed monthly from March 1997 to March 2007. For every yield we have 120 monthly observations, thus $t_i - t_{i-1} = \frac{1}{12}$ is the time interval, measured in years, between one observation and the next. t_1 denotes March 1997 and t_{120} denotes March 2007. $o_{i,0.25}$ is the 3-month yield, $o_{i,2}$ is the 2-year yield, $o_{i,5}$ is the 5-year yield and $o_{i,10}$ is the 10-year yield at time t_i. The 10-year yield $o_{i,10}$ is assumed to be observed with error, while the other yields $(o_{i,0.25}, o_{i,2}, o_{i,5})$ are assumed to be observed without error and are used to infer the values of the latent factors $X_i = (x_i, x_{1,i}, x_{2,i})'$ on any observation date t_i. The Appendix further explains how this is accomplished, provided $f(x)$ is monotonic in x. Denote with O_i the vector of observed yields at time t_i such that $O_i = (o_{i,0.25}, o_{i,2}, o_{i,5}, o_{i,10})'$. The transition density of $X_i = (x_i, x_{1,i}, x_{2,i})'$ is conditionally and unconditionally Gaussian. The conditional density of the vector $(X_i, o_{i,10})$ can be written as

$$l(X_i, o_{i,10}) = l(X_i \mid X_{i-1}) \cdot l(o_{i,10} \mid X_i) \tag{16}$$

where $l(X_i \mid X_{i-1})$ denotes the conditional density of X_i given X_{i-1} and where $l(o_{i,10} \mid X_i)$ is the conditional density of $o_{i,10}$ given X_i. $l(X_i \mid X_{i-1})$ is a trivariate Gaussian density, such that

$$l(X_i \mid X_{i-1}) = (2\pi)^{-3/2} |\Omega|^{-1/2} \cdot e^{-\frac{1}{2}(X-M)'\Omega^{-1}(X-M)} \tag{17}$$

with

$$M = \begin{pmatrix} \theta_1^* + (x_{1,i-1} - \theta_1^*) e^{-k_1/12} \\ \theta_2^* + (x_{2,i-1} - \theta_2^*) e^{-k_2/12} \\ \theta_3^* + (x_{3,i-1} - \theta_3^*) e^{-k_3/12} \end{pmatrix}$$

$$\Omega = \begin{bmatrix} \frac{\sigma_1^2}{2k_1}\left(1 - e^{-2k_1/12}\right) & \frac{\rho_{1,2}\sigma_2\sigma_1}{k_2+k_1}\left(1 - e^{-(k_2+k_1)/12}\right) & \frac{\rho_{1,3}\sigma_3\sigma_1}{k_3+k_1}\left(1 - e^{-(k_3+k_1)/12}\right) \\ \frac{\rho_{2,1}\sigma_2\sigma_1}{k_2+k_1}\left(1 - e^{-(k_2+k_1)/12}\right) & \frac{\sigma_2^2}{2k_2}\left(1 - e^{-2k_2/12}\right) & \frac{\rho_{2,3}\sigma_3\sigma_2}{k_3+k_2}\left(1 - e^{-(k_3+k_2)/12}\right) \\ \frac{\rho_{3,1}\sigma_3\sigma_1}{k_3+k_1}\left(1 - e^{-(k_3+k_1)/12}\right) & \frac{\rho_{3,2}\sigma_3\sigma_2}{k_3+k_2}\left(1 - e^{-(k_3+k_2)/12}\right) & \frac{\sigma_3^2}{2k_3}\left(1 - e^{-2k_3/12}\right) \end{bmatrix}.$$

We denote with $Z(t_i, t_i + 10)$ the price of a 10-year discount bond at time t_i as predicted by the Extended Gaussian model. $Z(t_i, t_i + 10)$ is computed as explained above. The observation error affecting the ten-year yield $O_{i,10}$ is assumed to be a white noise time series uncorrelated with the factors X_i. It follows that $l(O_{i,10} \mid X_i)$ is a univariate normal density with mean $-\frac{Z(t_i,t_i+10)}{10}$, which is the model predicted 10-year yield, and variance σ_ε^2, such that

$$l(O_{i,10} \mid X_i) = \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} e^{-\frac{\left(O_{i,10} - \left(-\frac{Z(t_i,t_i+10)}{10}\right)\right)^2}{2\sigma_\varepsilon^2}}. \tag{18}$$

σ_ε^2 is the variance of the error with which the 10-year yield is observed. It follows that the conditional density of the observed yields is

$$l\left(O_{i,0.25}, O_{i,2}, O_{i,5}, O_{i,10} \mid X_{i-1}\right) = abs\left(J_i\right) \cdot l\left(X_i \mid X_{i-1}\right) \cdot l\left(O_{i,10} \mid X_i\right) \quad (19)$$

where $abs\left(J_i\right)$ is the absolute value of the Jacobian determinant for time t_i, such that

$$J_i = \begin{vmatrix} \frac{\partial x_i}{\partial O_{1,i}} & \frac{\partial x_{1,i}}{\partial O_{1,i}} & \frac{\partial x_{2,i}}{\partial O_{1,i}} \\ \frac{\partial x_i}{\partial O_{2,i}} & \frac{\partial x_{1,i}}{\partial O_{2,i}} & \frac{\partial x_{2,i}}{\partial O_{2,i}} \\ \frac{\partial x_i}{\partial O_{5,i}} & \frac{\partial x_{1,i}}{\partial O_{5,i}} & \frac{\partial x_{2,i}}{\partial O_{5,i}} \end{vmatrix}. \quad (20)$$

We have expressed J_i in this way since $O_{1,i} = -\frac{\ln Z(t_i,t_i+1)}{1}$, $O_{2,i} - \frac{\ln Z(t_i,t_i+2)}{2}$, $O_{5,i} = -\frac{\ln Z(t_i,t_i+5)}{5}$. The joint log-likelihood function for the observed yields is

$$L = \textstyle\sum_{t=1}^{120} \ln abs\left(J_i\right) + \ln l\left(X_i \mid X_{i-1}\right) + \ln l\left(O_{10,i} \mid X_i\right). \quad (21)$$

Maximising L gives the parameter estimates that follow.

2.2 Empirical results

Table 1 presents descriptive statistics for our sample of observed yields.

[Table 1]

Table 1	Mean	Std dev
$O_{i,0.25}$	4.15%	2.01%
$O_{i,2}$	4.68%	1.69%
$O_{i,5}$	5.27%	1.22%
$O_{i,10}$	5.77%	0.94%

Table 2 displays the estimation results for the GB and GBK models. In brackets the standard deviations of the parameter estimates are reported. These standard deviations are calculated with the BHHH estimator. The row labelled L reports the values of the maximised log-likelihood function. The first column refers to the GB model. The estimate of c is not significantly different from 0. As we set $c = 0$ the log-likelihood function hardly decreases. If we impose the restriction $q = 1$ in the GB model, the value of the likelihood L just decreases from $2,069.75$ to 2069.41. The corresponding likelihood ratio test statistic for this parameter restriction gives $2\left(2,069.75 - 2069.41\right) = 0.68$ with entails a p-value of 0.41 according to the X^2 distribution with one degree of freedom. Thus the data does not seem to reject the $q = 1$ restriction. The second column of Table 3 shows the estimation results for the GBK model. The GB model, either with or without the restriction $q = 1$, fits the observed yields better than the GBK model: the standard deviation of the 10-year yield observation error is $\sigma_\varepsilon = 0.0037$ for the GB model and $\sigma_\varepsilon = 0.0055$ for the GBK model. Here we limit the comparison of the models to their in-sample goodness of fit. Anyway this section makes the main point that EGM are estimable despite the numerical solutions they involve.

[Table 2]

Table 2	GB	GBK
σ	0.48 (0.11)	1.59 (0.56)
k	37.07 (7.32)	33.76 (15.33)
k^*	0.72 (3.53)	0.88 (0.99)
$k\theta$	11.61 (5.19)	-75.89 (25.11)
$k^*\theta^*$	0.24 (1.01)	-1.65 (2.59)
c	0.0002 (0.22)	0.0000 (3.45)
σ_1	0.06 (0.04)	0.01 (0.002)
k_1	1.36 (0.87)	-0.001 (0.002)
σ_2	0.03 (0.02)	0.034 (0.016)
k_2	0.03 (0.04)	1.04 (0.37)
ρ_1	-0.73 (NA)	-0.72 (NA)
σ_ε	0.0037 (0.0007)	0.0055 (0.0012)
ρ_2	0.01 (NA)	-0.72 (NA)
$\rho_{1,2}$	-0.50 (NA)	-0.10 (NA)
k_1^*	0.21 (1.26)	0.03 (0.81)
k_2^*	-0.02 (0.24)	0.15 (0.47)
q	1.31 (0.37)	
L	2,069.75	2,055.66

3 Extended Gaussian reduced form credit risk model

The above EGM price default-free bonds, but they can also be immediately applied to price defaultable bonds if only we re-interpret the variables. To do so we can re-define the default-free instantaneous interest rate as $r = x_1 + x_2$. In fact even when pricing defaultable bonds it seems important to allow two factors to drive the default-free yield curve. This is also what Bakshi-Madan-Zhang (2006) suggested. Then the default intensity in the risk-neutral world can be chosen to be $\lambda = f(x)$. If $f(x) = e^x$ or $f(x) = \max(x,0)^q$ as before, the default intensity is non-negative. A virtue of such a reduced form credit risk model is that the default intensity λ and the instantaneous interest rate r are freely correlated, in particular they can be negatively correlated while λ is never negative. Moreover λ is monotonic in x. This makes parameter estimation and calibration much simpler than in quadratic reduced form credit risk models, which assume $f(x) = x^2$. Estimation of quadratic models typically requires Kalman Filters since the default intensity is not monotonic in x, hence we cannot simply "invert" the observed bond yields to infer the latent value of x.

Under these assumptions and if bond holders recover nothing as the bond defaults, the value $D(t,T)$ of a defaultable zero coupon bond with residual life equal to the interval $]t,T]$ is

$$D(t,T) = Z(t,T) \cdot P^T(t,T) \tag{22}$$

where $P^T(t,T)$ is the survival probability over $]t,T]$ calculated in the forward risk-neutral measure induced by the default-free bond $Z(t,T)$. The value of $Z(t,T)$ now becomes

$$Z(t,T) = e^{A(t,T)-x_1 B(t,T)-x_2 C(t,T)}.$$

It can be shown that $P^T(t,T)$ is equal to $H(x,t)$ given above. Thus to price defaultable bonds we just need to re-interpret the variables in the GB or GBK models above.

3.1 Quasi recovery of face value assumption (QRF)

To price bonds and credit default swaps (CDS's) we can more realistically introduce a tractable and accurate assumption about a bond recovery value upon default. Following Realdon (2007) we make the "quasi recovery of face value" (QRF) assumption, which approximates the "recovery of face value" assumption. Let π denote the bond recovery value expressed as a fraction of the bond face value and let the bond face value be 1. According to the "recovery of face" assumption π is received at the exact time of default, rather than later. According to the QRF assumption π is received shortly after default as follows. If today's date is t and T is the bond maturity date, the period $]t,T]$ is the bond residual life. Then we set m dates during $[t,T]$ such that $t \leq T_1 < T_2 < .. < T_m = T$ and such that $(T_k - T_{k-1})$ is constant for $k = 2,3,..m$. According to the QRF assumption bond holders recover π at time T_k if default occurs in the time interval $]T_{k-1},T_k]$.

If we denote with $R(t,T_{k-1},T_k)$ the value at time $t \leq T_{k-1}$ of a claim that pays 1 at time T_k if default occurs in the time interval $]T_{k-1},T_k]$, we can conclude that

$$R(t,T_{k-1},T_k) = Z(t,T_k) \cdot E_t^k\left(\frac{1_{\tau>T_{k-1}} \cdot Z(T_{k-1},T_k)}{Z(T_{k-1},T_k)}\right) - D(t,T_k) \tag{23}$$

where $E_t^k(..)$ denotes time t conditional expectation in the $Z(t,T_k)$ forward risk neutral measure, where τ is the default time, where $1_{\tau>T_{k-1}}$ is the indicator function of the survival event $\tau > T_{k-1}$. We notice that

$$E_t^k\left(\frac{1_{\tau>T_{k-1}} \cdot Z(T_{k-1},T_k)}{Z(T_{k-1},T_k)}\right) = E_t^k\left(1_{\tau>T_{k-1}}\right) = P^k(t,T_{k-1}) \tag{24}$$

is the survival probability up to time T_{k-1} in the $Z(t,T_k)$ forward risk neutral measure. $Z(t,T_k) \cdot E_t^k\left(\frac{1_{\tau>T_{k-1}} \cdot Z(T_{k-1},T_k)}{Z(T_{k-1},T_k)}\right)$ is the present value of a defaultable claim that pays off 1 at T_k provided that $\tau > T_{k-1}$. Substituting for $D(t,T_k)$ it follows that

$$R(t,T_{k-1},T_k) = Z(t,T_k)\left(P^k(t,T_{k-1}) - P^k(t,T_k)\right). \tag{25}$$

The term $P^k(t,T_{k-1}) - P^k(t,T_k)$ denotes the probability calculated at time t in the $Z(t,T_k)$ forward risk neutral measure that default will occur in the time interval $]T_{k-1},T_k]$. We can readily compute $R(t,T_{k-1},T_k)$ since we we can readily compute $Z(t,T_k)$ and $P^k(t,T_k)$ as shown above. We can now determine the present value of what bond holders expect to recover upon default under the QRF assumption. At time t such present value is equal to the value of a claim that pays π at T_k if default time τ falls during any interval $]T_{k-1},T_k]$ for $k = 1,2,..m$, and it is equal to

$$\pi \textstyle\sum_{k=1}^{m} R(t,T_{k-1},T_k). \tag{26}$$

Moreover as m rises, the period $[t,T]$ is partitioned in a greater number of sub-intervals and the bond recovery value under the QRF assumption approaches the recovery value we obtain under the proper "recovery of face" assumption. The recovery of face value assumption is commonly regarded as the most realistic and least tractable recovery assumption (see Schonbucher (2003)), while the QRF assumption can approximate the "recovery of face" assumption and is more tractable.

Then under the QRF assumption the value of a defaultable fixed coupon bond with face value of 1 and with promised coupons equal to $c(T_i - T_{i-1})$ at times T_i for $i = 1,2,..n$ is

$$C(t) = \sum_{i=1}^{n} c(T_i - T_{i-1}) D(t,T_i) + D(t,T_n) + \pi \sum_{k=1}^{m} R(t,T_{k-1},T_k). \tag{27}$$

For completeness we also notice that the QRF assumption immediately provides the following formula for CDS spreads

$$s_{cds} = \frac{(1-\pi)\cdot \sum_{k=1}^{m} R(t,T_{k-1},T_k)}{\sum_{k=1}^{m} (T_k - T_{k-1})\cdot D(t,T_k)}. \tag{28}$$

Although not necessary, for simplicity in this formula we assume that also the CDS fee payment dates are T_k and that each protection fee payment amounts to $s_{cds}(T_k - T_{k-1})$.

3.2 Valuation of defaultable floating rate bonds

The above setting and the QRF assumption also imply convenient closed form solutions to price defaultable floating rate bonds. Consider such a bond with face value of 1 and promising to pay coupons at times T_i for $k = 1,2,..n$ equal to

$$L_{i-1}(T_i - T_{i-1}) \tag{29}$$

where L_{i-1} is the Libor rate for the period $[T_{i-1}, T_i]$. Let t be today's date, T_1 be the next coupon payment date and $T_n = T$ be the bond maturity date. For simplicity we compute the floating rate bond value $C'(t)$ at time t net of the value of the coupon payment due at time T_1. Such value corresponds to the bond "clean price". We also assume that default entails the entire loss of all future coupon payments. Then, at $t \leq T_1$ the value $C'(t)$ is

$$C'(t) = \sum_{i=1}^{n} F(t, T_{i-1}, T_i) D(t, T_i) + D(t, T_n) + \pi \sum_{k=1}^{m} R(t, T_{k-1}, T_k)$$

where $F(t, T_{i-1}, T_i) = \left(\frac{Z(t,T_{i-1})}{Z(t,T_i)} - 1\right)$ denotes the default-free forward rate at time t for the period $[T_{i-1}, T_i]$. We notice that $F(T_{i-1}, T_{i-1}, T_i) = L_{i-1}(T_i - T_{i-1})$. The first term in the equation for $C'(t)$ is the present value of the defaultable floating rate coupon payments. To clarify this notice that the present value at time t of a default-free floating coupon is $Z(t, T_{i-1}) - Z(t, T_i) = Z(t, T_i) \cdot F(t, T_{i-1}, T_i)$. Then

$$Z(T_{i-1}, T_i) \cdot F(T_{i-1}, T_{i-1}, T_i) \cdot P^i(T_{i-1}, T_i)$$

is the time T_{i-1} value of a defaultable floating rate coupon that is set at time T_{i-1} and paid at time T_i provided no default occurs until T_i. As above $P^i(T_{i-1}, T_i)$ is the survival probability in the $Z(T_{i-1}, T_i)$ forward neutral measure. Thus the time t value of a defaultable floating coupon is

$$\begin{aligned} (Z(t, T_{i-1}) - Z(t, T_i)) \cdot P^i(t, T_i) &= Z(t, T_i) \cdot P^i(t, T_i) \cdot F(t, T_{i-1}, T_i) \quad (30) \\ &= D(t, T_i) \cdot F(t, T_{i-1}, T_i). \end{aligned}$$

Again we notice that the value $C'(t)$ of the floating rate bond can be quickly computed once we have computed $Z(t, T_i)$ and $P^i(t, T_i)$ as above.

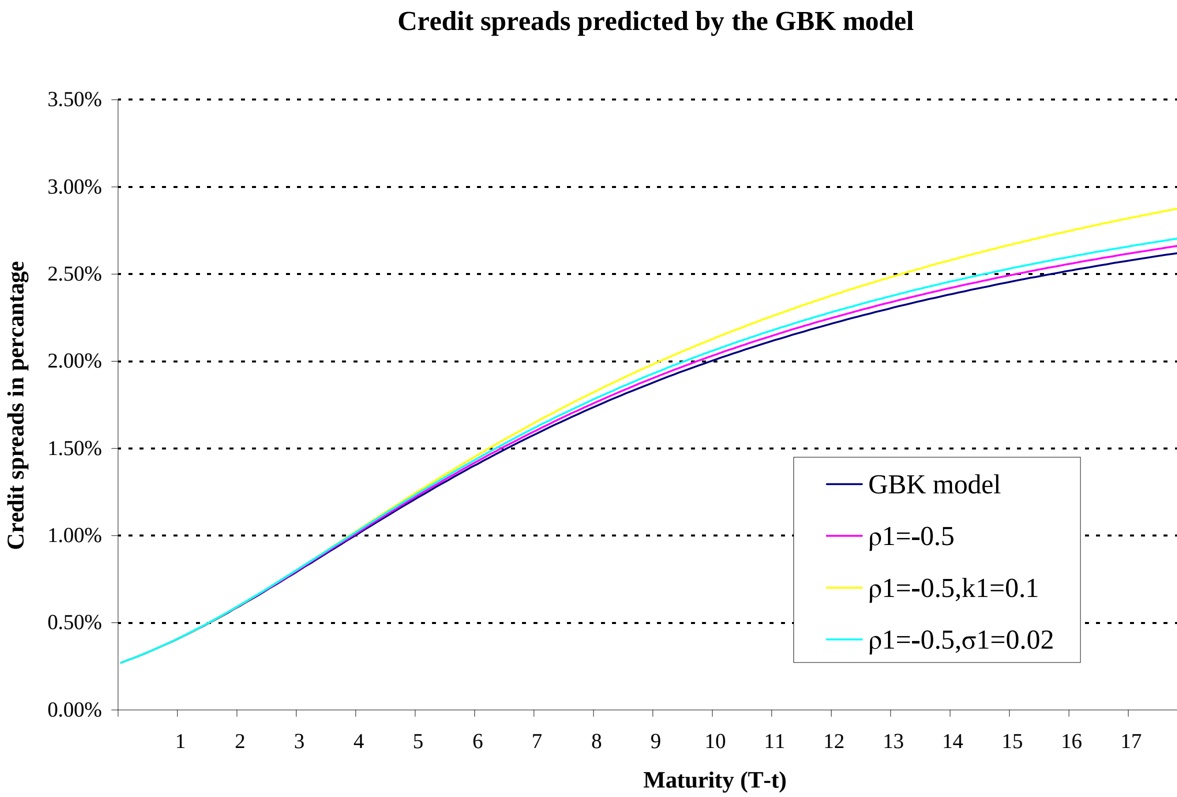
3.3 Simulations

Without much loss in generality, in Figure 1 we concentrate on term structures of credit spreads on zero coupon bonds predicted by the EG reduced form credit risk model of this section. In Figure 1 we assume for simplicity that $r = x_1$, $\lambda = e^x$ and $\pi = 0$, so that credit spreads are given by the formula $-\ln \frac{P^T(t,T)}{T-t}$. In the line headed "GBK model", the parameters are assumed to be $\lambda = e^x = 0.0025$, $\sigma = 1$, $k = 0.2$, $\sigma_1 = 0.01$, $k_1 = 1$, $\rho_1 = 0$, $\pi = 0$. The other lines assume the same parameters as in the line called "GBK model", but for the different parameter values shown in the respective line headings. Figure 1 shows how credit spreads are affected by the correlation ρ_1 and by the parameters σ_1 and k_1 that drive the default-free yield curve when $r = x_1$. To interpret these results we notice that credit spreads rise with the term

$$\left(k(\theta - x) - \frac{1 - e^{-k_1(T-t)}}{k_1} \rho_1 \sigma_1 \sigma\right) \quad (31)$$

that appears in front of the first derivative in the PDE satisfied by $P^T(t,T)$. We can think of 31 as the drift of x in the forward risk-neutral measure induced by $Z(t,T)$ when $r = x_1$. We recall that when $\rho_1 = 0$ the survival probability $P^T(t,T)$ in the forward risk-neutral measure induced by $Z(t,T)$ becomes equal to the more familiar survival probability in the risk-neutral world. Then as the correlation parameter ρ_1 rises and $k_1 > 0$, the expression in 31 decreases and so do credit spreads. Hence credit spreads decrease with the degree of instantaneous correlation between the default free instantaneous interest rate r and the default intensity. This is shown by comparing the lines "$\rho_1 = -0.5$" and "GBK model". Credit spreads also depend on the parameters σ_1 and k_1 that determine the variance of the instantaneous interest rate r. The sensitivity of credit spreads to σ_1 and k_1 depends on the sign of the instantaneous correlation ρ_1. When $\rho_1 < 0$ ($\rho_1 > 0$) credit spreads increase (decrease) in σ_1. This emerges by comparing the line headed "$\rho_1 = -0.5$", which assumes $\sigma_1 = 0.01$, with the line headed "$\rho_1 = -0.5, \sigma_1 = 0.02$". When $\rho_1 < 0$ ($\rho_1 > 0$) credit spreads increase (decrease) as the mean reversion speed k_1 decreases. This is shown by comparing the line headed "$\rho_1 = 0. - 5$", which assumes $k_1 = 1$, with the line headed "$\rho_1 = -0.5, k_1 = 0.1$". We can summarise these results by stating that, if the default intensity is positively correlated with the instantaneous interest rate r, then credit spreads increase with the conditional and unconditional variance of r. The opposite is true when the default intensity is negatively correlated with r: credit spreads decrease with the conditional and unconditional variance of r.

[Figure 1 here]



As expected, as the time to maturity $(T-t)$ increases, credit spreads become more sensitive to changes in the parameters ρ_1, σ_1 and k_1. Thus when pricing bonds with relatively short maturities up to 5 years or so, there may often be a case for ignoring the correlation between the instantaneous interest rate and the default intensity. The loss in accuracy may be tolerable in the light of the simplification of the pricing model. All these considerations are valid for the GBK model applied to credit risk, which assumes $\lambda = e^x$, but they are also valid for GB model applied to credit risk, which assumes $\lambda = \max(x, 0)^q$.

4 Extended Gaussian credit rating model

This section presents an Extended Gaussian model for credit risk pricing that makes use of credit rating information. As before we assume that $r = x_1 + x_2$. Thus as before the value of a default-free zero coupon bond is $Z(t,T) = e^{A(t,T)-x_1B(t,T)-x_2C(t,T)}$. The time t value of a defaultable zero coupon bond with maturity T is now denoted as $D_i(t,T)$ or more simply D_i. i now denotes the current rating class of the bond. Assume n rating classes, so that $i =$

$1, 2, .., n$. $i = n$ is the default rating and $i = 1$ is the highest rating, say AAA. The firm cannot leave the default rating, i.e. default is an "absorbing" state. x can now be interpreted as the latent process driving the risk-neutral probability of a change in the credit rating as described below. x still follows the Gaussian process in the risk-neutral world defined above. The risk-neutral probability of a change in the bond credit rating from i to $i+1$ during the infinitesimal period dt is

$$\max\left(x - \alpha_i, 0\right)^q hdt \tag{32}$$

while the probability of a change in the bond credit rating from i to $i-1$ during dt is

$$\max\left(\alpha_i - x, 0\right)^q hdt. \tag{33}$$

h is a constant. q is equal either to 1 or 3 as explained below. $\alpha_1, \alpha_2, .., \alpha_n$ are parameters associated with the rating classes $i = 1, 2, .., n$ and are such that $\alpha_1 < \alpha_2 < .. < \alpha_n$. The parameters α_i can be determined by calibrating the model to market prices or by econometric estimation when this is feasible. Notice that the lower (higher) x is, the more likely it is that the rating will "improve" ("worsen"). The expected change in bond value due to a rating change during $]t, t + dt]$ is $E_t\left(D_{i\pm1} - D_i\right) dt$ with

$$E_t\left(D_{i\pm1} - D_i\right) = \left(D_{i+1} - D_i\right)\max\left(x - \alpha_i, 0\right)^q h + \left(D_{i-1} - D_i\right)\max\left(\alpha_i - x, 0\right)^q h \tag{34}$$

where $E_t(..)$ is the expectation in the risk-neutral measure conditional on information at t. This expectation assumes that during $]t, t + dt]$ the rating can only increase by one notch or decrease by one notch. We set $i = 1$ to correspond to the AAA rating and $\alpha_1 = 0$. Then, since a AAA rated bond cannot be upgraded, and since we do not preclude x from turning negative, it is fitting to impose that

$$E_t\left(D_2 - D_1\right) = \left(D_2 - D_1\right)\max\left(x, 0\right)^q hdt. \tag{35}$$

We assume that upon default the bond is worthless, i.e. $D_n(t, T) = 0$, only to relax this assumption below. The absence of arbitrage opportunities implies that, for $i = 1, 2, .., n$, $D_i(t, T)$ or more simply D_i must satisfy the PDE

$$\frac{\partial D_i}{\partial t} + \frac{\partial^2 D_i}{\partial x_1 \partial x_2}\rho_{1,2}\sigma_1\sigma_2 + \frac{\partial^2 D_i}{\partial x \partial x_1}\rho_1\sigma\sigma_1 + \frac{\partial^2 D_i}{\partial x \partial x_2}\rho_2\sigma\sigma_2 + \frac{\partial^2 D_i}{\partial x_1^2}\frac{\sigma_1^2}{2} + \frac{\partial^2 D_i}{\partial x_2^2}\frac{\sigma_2^2}{2} + \frac{\partial^2 D_i}{\partial x^2}\frac{\sigma^2}{2} \tag{36}$$

$$+ \frac{\partial D_i}{\partial x_1}k_1\left(\theta_1 - x_1\right) + \frac{\partial D_i}{\partial x_2}k_2\left(\theta_2 - x_2\right) + \frac{\partial D_i}{\partial x}k\left(\theta - x\right)$$
$$- \left(\max\left(\alpha_i - x, 0\right)^q h + \max\left(x - \alpha_i, 0\right)^q h + x_1 + x_2\right) D_i + E_t\left(D_{i\pm1} - D_i\right) = 0 \tag{37}$$

subject to the terminal condition $D_i(t, T) = 1$ where T is the bond maturity date. The solution to this system of pricing equation is such that

$$D_i(t, T) = P_i^T(t, T) \cdot e^{A(t,T) - x_1 B(t,T) - x_2 C(t,T)} \tag{38}$$

where $A(t,T)$ and $B(t,T)$ are given above and $P_i^T(t,T)$ can be computed numerically. $P_i^T(t,T)$ is the survival probability over $]t,T]$ calculated in the forward risk-neutral measure induced by the default-free bond $Z(t,T)$ given that the current bond rating is i. $P_i^T(t,T)$ or more simply P_i^T solves

$$\begin{aligned}&\frac{\partial P_i^T}{\partial t}+\frac{\partial^2 P_i^T}{\partial x^2}\frac{1}{2}\sigma^2+\frac{\partial P_i^T}{\partial x}\left(k(\theta-x)-B(t,T)\rho_1\sigma_1\sigma-C(t,T)\rho_2\sigma_2\sigma\right)\\&+\left(P_{i-1}^T-P_i^T\right)\max(\alpha_i-x,0)^q h+\left(P_{i+1}^T-P_i^T\right)\max(x-\alpha_i,0)^q h=0\end{aligned}\tag{39}$$

subject to $P_i^T(T,T)=1$, $P_n^T(t,T)=0$, $\lim_{x\to-\infty}P_i^T(t,T)\to 0$, $\lim_{x\to\infty}P_i^T(t,T)\to 1$ for $i=2,..,n-1$ and

$$\begin{aligned}&\frac{\partial P_1^T}{\partial t}+\frac{\partial^2 P_1^T}{\partial x^2}\frac{1}{2}\sigma^2+\frac{\partial P_1^T}{\partial x}\left(k(\theta-x)-B(t,T)\rho_1\sigma_1\sigma-C(t,T)\rho_2\sigma_2\sigma\right)\\&\left(P_2^T-P_1^T\right)\max(x,0)^q h=0.\end{aligned}\tag{40}$$

The PDE's satisfied by P_i^T can be quickly solved numerically through a system of $n-1$ implicit finite difference grids. We notice that it is possible that bonds with lower rating have lower credit spreads than bonds with higher rating, and this seems to happen in the market. The extent to which this is possible is mitigated if we make rating migration probabilities more sensitive to the distance $(x-\alpha_i)$ for very rating class. We can do this by setting $q=3$ rather than $q=1$. We notice that, as for example x rises from 0 to 1, the intensity and hence the probability of a rating downgrade from $i=1$ to $i=2$ rises, while the intensity and hence the probability of a rating upgrade from $i=2$ to $i=1$ decreases.

An alternative specification or rating transition intensities is such that

$$E_t\left(D_{i\pm1}-D_i\right)=\left(D_{i+1}-D_i\right)\max\left(e^x-\alpha_i,0\right)h+\left(D_{i-1}-D_i\right)\max\left(\alpha_i-e^x,0\right)h\tag{41}$$

for and

$$\begin{aligned}&\frac{\partial P_i^T}{\partial t}+\frac{\partial^2 P_i^T}{\partial x^2}\frac{1}{2}\sigma^2+\frac{\partial P_i^T}{\partial x}\left(k(\theta-x)-B(t,T)\rho_1\sigma_1\sigma-C(t,T)\rho_2\sigma_2\sigma\right)\\&+\left(P_{i-1}^T-P_i^T\right)\max(\alpha_i-e^x,0)h+\left(P_{i+1}^T-P_i^T\right)\max(e^x-\alpha_i,0)h=0\end{aligned}\tag{42}$$

for $i=1,..,n-1$. Notice that as $\alpha_1=0$, $\max(\alpha_1-e^x,0)=0$ for all values of x.

We notice that the formulae shown above for pricing fixed and floating coupon bonds and credit default swap under the convenient QRF assumption are all still applicable also in the current setting once we have computed P_i^T for $i=1,..,n-1$.

This EG credit rating model recalls that of Jarrow-Lando-Turnbull (1997) and that of Lando (2000). As in Lando (2000) or Consigli et al. (2007) credit spreads can change even in the absence of a rating transition. Notice that in this model rating migration probabilities are endogenous rather than exogenous. A merit of the model is that it remains tractable even when r and x are correlated, i.e. even if interest rate risk is correlated with credit risk. The model seems relatively parsimonious, since it does not require a specific spread process for every rating class and specific correlation parameters of each spread process with r as in Consigli et al. (2007). The model can reproduce rating momentum: negative (positive) rating changes are likely to be followed by further negative (positive) rating changes.

The number n of rating classes in the model may be less than the actual number of rating classes of a rating agency. For example Moody's ratings may be aggregated into a coarser rating scale with only four rating classes, e.g. A, B, C and D. This would simplify the model and it would only make partial use of rating information, which seems acceptable. After all the literature has proposed a number of reduced form models that make no use at all of rating information.

This EG credit rating model can be calibrated to or estimated from the prices of bonds and credit derivatives. As is typical of ratings models, calibration or estimation should make simultaneous use of instruments that belong to all rating classes. Notice that the parameter values $\alpha_1, \alpha_2, .., \alpha_n, k, \theta, \sigma$ determine the prices of all calibration instruments, be they bonds or credit default swaps. The value x of the latent factor is specific to a single obligor, at least under the assumption that the values of all calibration instruments issued by the same obligor are driven by x. Maximum likelihood estimation is hampered by the fact that the intensities that drive rating transitions are not constant over time. Yet a useful approximation to the likelihood function of the observed credit spreads is possible. In fact the transition density of x is simply Gaussian and the transition intensity may be regarded as approximately constant during one single time period of one day.

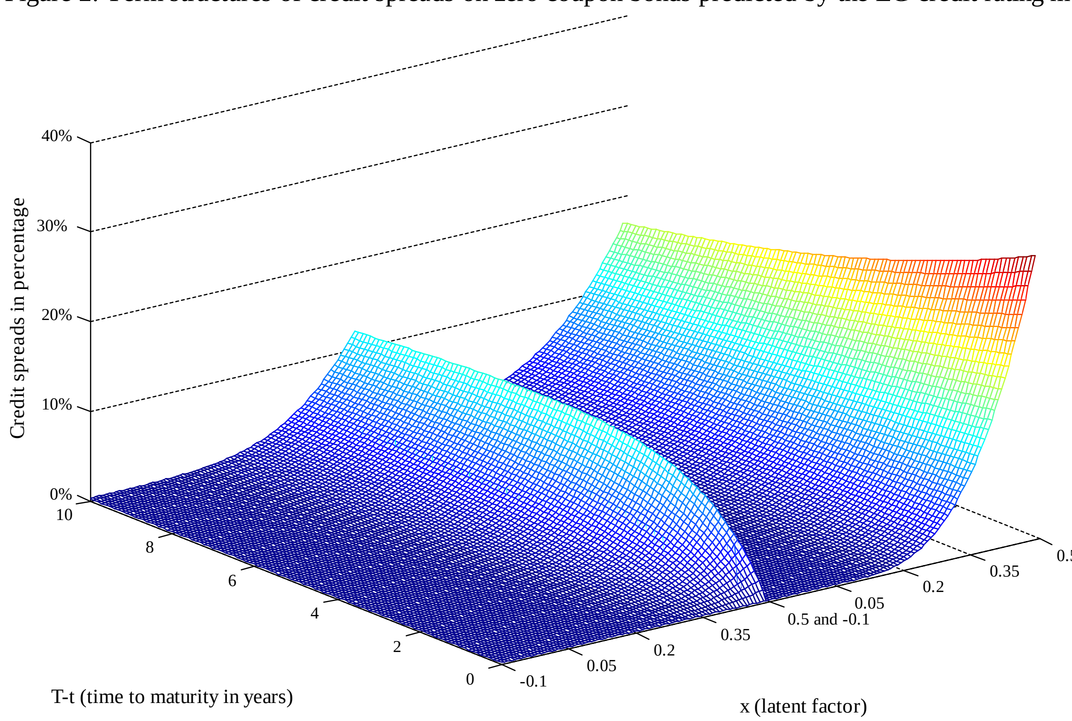
5 Simulations

Figure 2 displays the term structures of credit spreads on a zero coupon bond for maturities up to 10 years. Given that we assume the bond to have zero recovery value in case of default, credit spreads are given by the formula $-\ln \frac{P_i^T(t,T)}{T-t}$. Figure 2 only assumes two possible rating classes $i = 1, 2$ before default. Default is denoted as $i = 3$. Figure 2 assumes: $q = 3, a_1 = 0, a_2 = 0.2, k = 0.1, \theta = 0.05, \sigma = 0.05, \sigma_1 = 0.01, k_1 = 0.5, \sigma_2 = 0.01, k_2 = 0.5, \rho_1 = -0.5, \rho_1 = 0.1, \pi = 0$.

Figure 2 merges two figures. The figure in the forefront displays term structures of credit spreads when the bond rating is $i = 1$ for values of x in the range $[-0.1, 0.5]$. The figure in the background displays term structures of credit spreads when the bond rating is $i = 2$ for values of x in the range $[-0.1, 0.5]$. We recall that the bond can only default if its current rating is $i = 2$. If the bond is rated $i = 1$, it must be downgraded to $i = 2$ before experiencing default. Then for a given value of x and a given parameter set, credit spreads for a bond rated $i = 1$ are always lower than for a bond rated $i = 2$. Yet Figure 2 shows how a bond rated $i = 1$ may have higher credit spreads than a bond rated $i = 2$. From the Figure we see that this is possible if say $x = 0.4$ for the bond rated $i = 1$ and $x = 0.05$ for the bond rated $i = 2$. Of course this would be an unlikely case of extremely "sticky" ratings, i.e. of ratings not tracking the level of credit risk indicated by the value of x.

We also notice that credit spreads on the bond rated $i = 1$ are upward sloping, while the credit spreads on the bond rated $i = 2$ can be downward sloping. This result recalls similar predictions from structural models: low (high) grade bonds with downward (upward) sloping term structures of credit spreads. Unlike structural models, this model predicts significantly high short term spreads when either x is very high or when the rating is low ($i = 2$). In fact, when x is high enough a downgrade from $i = 1$ to $i = 2$ soon followed by default are possible even in the short term. Overall the model seems of practical interest even as we assume fewer rating classes than the rating agencies.

Figure 2: Term structures of credit spreads on zero coupon bonds predicted by the EG credit rating model.



6 Extended Gaussian structural credit risk model

This final section presents a structural credit risk model that can be regarded as an Extended Gaussian model. We again assume that $r = x_1 + x_2$, so that the value of a default-free zero coupon bond is again $Z(t,T) = e^{A(t,T)-x_1 B(t,T)-x_2 C(t,T)}$. Following Cathcart and El-Jahel (1998, 2003), now we also assume that $x = \ln S$ and that S is a latent factor that is *not* the value of the firm's assets. When x hits the lower barrier $\ln K$ default occurs. K is not observable either. In the risk-neutral world x now follows the process

$$dx = \left(\mu - \lambda_s \sigma - \frac{1}{2}\sigma^2\right) \cdot dt + \sigma \cdot dw^Q + d\Pi \tag{43}$$

where μ, λ_s, σ are constant. $\mu - \frac{1}{2}\sigma^2$ is the drift of x in the real world, λ_s is the market price of risk due to the randomness of S. $d\Pi$ is the differential of a jump process, which we assume to be the same under the real and the risk-neutral measures. $d\Pi$ is such that

$$d\Pi = \begin{cases} j \backsim n(a,b) \text{ with risk-neutral probability } \lambda dt; \\ 0 \text{ with risk-neutral probability } (1 - \lambda dt). \end{cases} \tag{44}$$

j is a random variable distributed according to $n(a,b)$. $n(a,b)$ is the normal density with mean a and standard deviation b. We may want to impose that $a \leq 0$. λ is a constant. For clarity, we assume that upon default the bond recovers nothing. In this setting the value of a defaultable zero coupon bond is

$$D(t,T) = e^{A(t,T)-x_1 B(t,T)-x_2 C(t,T)} \cdot P^T(t,T) \tag{45}$$

where $P^T(t,T)$ or more simply P is the survival probability over the period $]t,T]$ in the forward risk-neutral measure induced by the default-free bond $Z(t,T)$. P satisfies

$$\begin{aligned} &\frac{\partial P}{\partial t} + \frac{\partial^2 P}{\partial x^2}\frac{1}{2}\sigma^2 + \frac{\partial P}{\partial x}\left(\mu - \lambda_s\sigma - \frac{1}{2}\sigma^2 - B(t,T)\rho_1\sigma_1\sigma - C(t,T)\rho_2\sigma_2\sigma\right) + \\ &- \lambda P + \lambda \textstyle\int_{-\infty}^{\infty} P(x+j) \cdot n(a,b) \cdot dj = 0 \end{aligned} \tag{46}$$

subject to $P^T(T,T) = 1$, $\lim_{x\to\infty} P^T(t,T) \to 1$ and $P^T(t,T) = 0$ when $x \leq \ln K$. $P(x+j)$ denotes P just after a jump from x to $x+j$. Again this equation can be quickly solved numerically through finite difference methods. The formulae presented above for pricing defaultable fixed and floating coupon bonds and credit default swaps under the convenient QRF assumption are all still applicable also in the current setting.

7 Conclusions

This paper has presented the family of three factor "Extended Gaussian" term structure models (EGM). In EGM the instantaneous interest rate is a function of multiple correlated latent Vasicek-type processes. Unlike in the Gaussian models of Langetieg (1980) or Babbs and Nowman (1999), the instantaneous interest rate may be a non-linear but monotonic function of one or more of the three latent factors. Yet the bond pricing equation can be solved through separation of variables, which provides much tractability despite the need for finite difference numerical solutions. Maximum likelihood estimation of model parameters is feasible even though bond prices are computed numerically.

Using US Treasury yields, two specific EGM are estimated and tested: the "Gaussian-Black " (GB) model, whereby one of the three factors follows a truncated Gaussian process as in Black (1995), and the "Gaussian-Black-Karasinski" (GBK) model, whereby one of the three latent factors follows a Black-Karasinski (1991) process. The merits of the GB and GBK models are similar to those of quadratic models: bond yields are hetero-schedastic and no admissibility problems affect the specification of the correlation between the latent factors or of market risk-premia. In particular correlation between factors is unrestricted. Unlike quadratic models, the GB and GBK models cannot rule out a negative instantaneous interest rate, which is unlikely anyway, and observed yields are sufficient statistics to infer the latent factors. The latter is a major advantage for calibration and estimation purposes and one that is missing in quadratic models. The empirical evidence from US Treasury yields shows the good pricing performance of the GB and GBK models. The GB model performs better than the GBK model.

The paper has also presented Extended Gaussian models to value fixed and floating rate defaultable bonds and credit default swaps. A reduced form model, a credit rating based model and a structural model of credit risk have been presented. The common feature of these three models is that the default-free yield curve is driven by two Gaussian latent factors, while a third Gaussian factor drives the default probability. The common merit of these EGM is that defaultable bond pricing remains tractable. "Separation of variables" remains possible even if interest rate risk and credit risk are correlated and such correlation is unrestricted. Maximum likelihood estimation also remains feasible due to the Gaussian process of the latent factors. The credit rating based model can reproduce stylised facts observed in the credit markets.

7.1 Appendix: inferring latent factors from observed yields

This Appendix shows how, in estimating the three factor GB and GBK models, the latent factors $x_{1,i}$,$x_{2,i}$ and x_i are inferred from the observed yields for any date t_i. For example, in the GBK model on any date t_i the instantaneous interest rate is $r_i = x_{1,i} + x_{2,i} + e^{x_i}$, where $x_{1,i}$, $x_{2,i}$ and x_i are latent factors. We assume that we observe the three discount bond yields $O_{i,0.25}$, $O_{i,2}$ and $O_{i,5}$ without error on any date t_i and infer $x_{1,i}$,$x_{2,i}$ and x_i by using the following equations

$$
\begin{aligned}
x_{1,i} &= \frac{C(t_i,t_i+2)\left(0.25O_{i,0.25}+A(t_i,t_i+0.25)+\ln P(x_i,0.25)\right)}{C(t_i,t_i+2)B(t_i,t_i+0.25)-B(t_i,t_i+2)C(t_i,t_i+0.25)} \\
&- \frac{C(t_i,t_i+0.25)\left(2O_{i,2}+A(t_i,t_i+2)+\ln P(x_i,2)\right)}{C(t_i,t_i+2)B(t_i,t_i+0.25)-B(t_i,t_i+2)C(t_i,t_i+0.25)} \\
x_{2,i} &= \frac{2O_{i,2}+A(t_i,t_i+2)-x_{1,i}B(t_i,t_i+2)+\ln P(x_i,2)}{C(t_i,t_i+2)} \\
O_{i,5} &= -\frac{A(t_i,t_i+5)-x_{1,i}B(t_i,t_i+5)-x_{2,i}C(t_i,t_i+5)+\ln P(x_i,5)}{5}.
\end{aligned}
$$

As $\ln P(x_i,T)$ is monotonic in x_i, we can numerically find x_i^*, which is the value of x_i that solves the last of these equations and then use the first two equations to find $x_{1,i}^*$ as a function of x_i^* and $x_{2,i}^*$ as a function of x_i^*. In other words, as $\ln P(x_i,T)$ is monotonic in x_i and at least three yields ($O_{i,0.25}$, $O_{i,2}$ and $O_{i,5}$ in our case) are observed without error, we can infer the values of the latent factors, $x_{1,i}^*, x_{2,i}^*$ and x_i^*, at any time t_i, which permits us to employ maximum likelihood estimation.

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