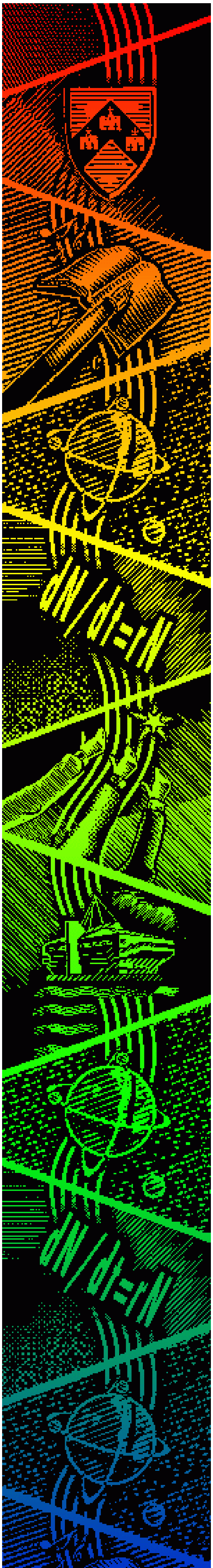

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A Two Factor Black-Karasinski Sovereign Credit Default Swap Pricing Model

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A TWO FACTOR BLACK-KARASINSKI SOVEREIGN CREDIT DEFAULT SWAP PRICING MODEL (forthcoming in the Icfai Journal of Derivatives Markets Vol.IV, n.04, October 2007; all copyrights rest with the Icfai University press)

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Abstract

This paper presents, estimates and tests a reduced form sovereign credit default swap (CDS) pricing model where the default intensity is driven by two latent Black-Karasinski-type processes. CDS pricing requires finite difference numerical solutions, but parameter estimation is

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still feasible. Evidence from a sample of sovereign CDS rates shows the good empirical performance of the model and that a second stochastic factor driving the default intensity is statistically significant. Surprisingly the evidence fails to support the view that the risk associated with the dynamics of the default intensity is priced. For all countries the bulk of variations of the default intensity are explained by just one factor. As a by-product, a viable methodology for maximum likelihood estimation of pricing models with two latent factors is provided despite the fact that the pricing requires numerical solutions through finite difference methods.

Key words: sovereign CDS pricing, reduced-form credit risk model, Black-Karasinski, implicit finite difference method, maximum likelihood estimation.

JEL classification: G13.

1 Introduction

After the major development of the sovereign CDS market in recent years, the pricing of sovereign CDS's has become a topical issue. To such issue the recent academic literature is also dedicating much attention, partly thanks to the wide availability of CDS rates. The particular theme of this paper is to explore the empirical performance and estimability of a reduced form credit risk pricing model which does not exhibit closed form solutions for pricing purposes. The model assumes that the default intensity is driven by two latent uncorrelated Black-Karasinski-type stochastic factors. Not requiring pricing closed form solutions provides much modelling flexibility and represents a departure from most of the empirical literature that tests continuous-time pricing models. Importantly although the CDS pricing models will only be solved numerically through finite difference methods, maximum likelihood parameter estimation remains feasible. The main contribution of the paper can be listed as follows.

First the Black-Karasinski (BK) reduced form model first proposed by Pan and Singleton (2005) is generalised by introducing a second latent stochastic factor driving the default intensity. The addition of a second latent factor also makes the estimation more burdensome, especially because CDS rates must be "inverted" on any observation date to infer the values of the latent factors. The empirical evidence shows the presence of the second stochastic factor is statistically and economically significant. Two factors generally enable the pricing model to explain well more than 90% of variation in observed CDS rates, with the exception of Thailand. Overall the model fits the observed CDS rates well.

Second, surprisingly the empirical evidence fails to support the view that the risk associated with the dynamics of the default intensity is priced. This is the case for all countries in the sample and it implies that the real world process of the risk-adjusted default intensities does not seem to differ significantly from the risk-neutral process of the risk-adjusted default intensities. For each country the bulk of the variations of the default intensity is explained by one factor.

Third the paper provides a viable method to estimate continuous time pricing model driven by two latent factors, which furthers the boundaries of the estimable pricing models.

The paper is organised as follows. The next section briefly reviews the most relevant literature on CDS pricing. Then the CDS pricing model is presented. Thereafter the model is estimated and tested using a sample of sovereign CDS rates. The conclusions follow.

2 Literature

The credit risk pricing literature based on reduced form models has concentrated on affine and quadratic models, which are well known from the interest rate term structure literature. According to affine or quadratic models discount bonds and risk-neutral survival probability for a given obligor are a exponential affine or an exponential quadratic functions of latent factors that follow affine processes and that drive the instantaneous default intensity (e.g. Duffie and Singleton (1999) or Chen, Filipovic and Poor (2004)). In other words in affine and quadratic models the survival probility is known in quasi-closed form and this makes these models amenable to econometric estimation. For example Duffee (1999), Driessen (2005) and Bakshi-Madan-Zhang (2006) show how to estimate affine reduced form credit risk models using Extended Kalman Filters. Longstaff et al. (2005) calibrate a CIR-type reduced form affine model to corporate bond and CDS data. Zhang (2003) employs again a CIR-type model to study the sovereign CDS rates of Argentina. Recently Chen-Cheng-Fabozzi-Liu (2006) use the Extended Kalman Filter to estimate a two factor quadratic model using corporate CDS rates. Overall reduced form credit risk models have mainly been confined to affine and quadratic models for the sake of tractability and ease of estimation.

Recent important exceptions to the choice of affine and quadratic models to model credit risk are Berndt-Douglas-Ferguson-Duffie-Schrantz (2004) for corporate credit risk and Pan and Singleton (2005) for sovereign credit risk. The first of these papers assumes that the default intensity follows a Black-Karasinski-type process and finds significant variability of credit risk premia in the corporate CDS market. Pan and Singleton (2005) test reduced form CDS pricing models under alternative specifications of the default intensity process. They test the Cox-Ingersoll-Ross (1985) process, the (1991) process and the Ahn-Gao (1999) process. They conclude that the Black-Karasinski (BK) process seems the most appropriate. Importantly, the BK model does not have closed form solutions for survival probabilities and defaultable bonds and yet its parameters can be estimated through maximum likelihood. The computational that estimation involves is still affordable. This paper extends the analysis of Pan and Singleton as two Black-Karasinski-type latent factors drive the default intensity of sovereigns, which improves the empirical fit of model predictions to the observed CDS rates.

3 The CDS pricing model

As suggested in Pan-Singleton (2005), a second stochastic factor may be needed to better capture the joint dynamics of sovereign CDS rates. Moreover the assumption of constant interest rates may not always be satisfactory. Hence this section proposes a two-factor Black-Karasinski-type reduced form CDS pricing model under stochastic interest rates. The model is later estimated. We assume that in the risk-neutral measure Q the default intensity is

$$\lambda^Q = e^x + e^z \tag{1}$$

where x and z are latent factors that in the risk-neutral measure follow the uncorrelated diffusion processes

$$dx = (b_x - a_x x)\, dt + s_x dw_x^Q \tag{2}$$

$$dz = (b_z - a_z z)\, dt + s_z dw_z^Q \tag{3}$$

whereas in the real measure x and z follow the uncorrelated processes

$$dx = (b_x + s_x b_x^* - (a_x + s_x a_x^*)\, x)\, dt + s_x dw_x \tag{4}$$

$$dz = (b_z + s_z b_z^* - (a_z + s_z a_z^*)\, z)\, dt + s_z dw_z. \tag{5}$$

$a_z, b_z, a_z^*, b_z^*, s_z, a_x, b_x, a_x^*, b_x^*, s_x$ are all constant. The default-free instantaneous short interest rate in the risk-neutral measure follows the "Vasicek" process

$$dr = (b - ar)\, dt + sdw_r^Q \tag{6}$$

with $dw \cdot dw_x = \rho_x dt$ and with $dw \cdot dw_x = \rho_z dt$. Thus the default intensity can be correlated with the default-free interest rate r.

We denote the value of a defaultable zero coupon bond with unit face value as $D(t,T)$ or more simply as D. t stands for the current date and T for the bond maturity date. $Z(t,T)$ denotes the value of a default-free zero coupon bond with the same maturity and face value as $D(t,T)$. For now we make the "recovery of Treasury" assumption, so that upon default the bond recovery value is $Z(t,T) \cdot \pi$, with $0 \leq \pi \leq 1$. Then the absence of arbitrage opportunities implies that D satisfies the following equation

$$\begin{aligned}
&\frac{\partial D}{\partial t} + \frac{\partial^2 D}{\partial z \partial r}\rho_z s s_z + \frac{\partial^2 D}{\partial x \partial r}\rho_x s s_x + \frac{\partial^2 D}{\partial r^2}s^2 + \frac{\partial D}{\partial r}(b - ar) - (e^x + e^z + r) D \\
&+ \frac{\partial^2 D}{\partial x^2}s_x^2 + \frac{\partial D}{\partial x}(b_x - a_x x) + \frac{\partial^2 D}{\partial z^2}s_z^2 + \frac{\partial D}{\partial z}(b_z - a_z z) = 0
\end{aligned} \tag{7}$$

subject to

$$\lim_{x\to\infty} D(t,T) \to Z(t,T)\cdot\pi \tag{8}$$

$$\lim_{x\to-\infty} D(t,T) \to Z(t,T) \tag{9}$$

$$\lim_{z\to\infty} D(t,T) \to Z(t,T)\cdot\pi \tag{10}$$

$$\lim_{z\to-\infty} D(t,T) \to Z(t,T) \tag{11}$$

$$D(T,T) = 1. \tag{12}$$

The first and the third conditions state the as $x \to \infty$ and $z \to \infty$ immediate default becomes certain and the value $D(t,T)$ of the defaultable bond approaches the recovery value $Z(t,T)\cdot\pi$ according to the "recovery of Treasury" assumption. Instead the second and the fourth conditions state that as $x \to -\infty$ and $z \to -\infty$ default is so remote that the value $D(t,T)$ of the defaultable bond approaches the value $Z(t,T)$ of the corresponding default-free bond. The last condition tells us that the bond has unit face value. The solution to equation 7 and to its conditions is

$$D(t,T) = Z(t,T)\cdot\left(\pi + (1-\pi)\cdot P^{T}(t,T)\right) \tag{13}$$

where $Z(t,T)$ is given by the Vasicek (1977) formula

$$
\begin{aligned}
Z(t,T) &= e^{A(t,T)-rB(t,T)} \\
B(t,T) &= \frac{1-e^{-a(T-t)}}{a} \\
A(t,T) &= \left(\frac{s^2}{2a^2}-\frac{b}{a}\right)[T-t-B(t,T)]-\frac{s^2B(t,T)^2}{4a}
\end{aligned} \tag{14}
$$

and where $P^T(t,T)$ is the survival probability over the period $]t,T]$ in a world that is forward risk neutral with respect to $Z(t,T)$. It can be shown that

$$
P^T(t,T) = P_x^T(t,T)\cdot P_z^T(t,T) \tag{15}
$$

where $P_x^T(t,T)$ can be found numerically through the implicit finite difference method by solving

$$
\frac{\partial P_x}{\partial t}+\frac{\partial^2 P_x}{\partial x^2}\frac{1}{2}s_x^2+\frac{\partial P_x}{\partial x}\left(b_x-a_x x-B(t,T)\rho_x s_x s\right)-e^x P_x=0 \tag{16}
$$

subject to the conditions $P_x(T,T)=1$, $\lim_{x\to-\infty}\frac{\partial^2 P_x}{\partial x^2}\to 0$, $\lim_{x\to\infty}\frac{\partial^2 P_x}{\partial x^2}\to 0$. Similarly and $P_z^T(t,T)$ can be found by solving

$$
\frac{\partial P_z}{\partial t}+\frac{\partial^2 P_z}{\partial z^2}\frac{1}{2}s_z^2+\frac{\partial P_z}{\partial z}\left(b_z-a_z z-B(t,T)\rho_z s_z s\right)-e^z P_z=0 \tag{17}
$$

subject to the conditions $P_z(T,T)=1$, $\lim_{x\to-\infty}\frac{\partial^2 P_z}{\partial z^2}\to 0$, $\lim_{x\to\infty}\frac{\partial^2 P_z}{\partial z^2}\to 0$. Here we have used the simpler notation P_x and P_z in place of $P_x^T(t,T)$ and

$P_z^T(t,T)$ when preferable. The boundary conditions state that P_x tends to become linear in x when $x \to \pm\infty$, because of the dominant effect of the drift of x. As x tends to infinity the drift term $(b - ax)\,dt$ dominates over the diffusion sdw^Q. The same is valid for P_z as $z \to \pm\infty$. We notice that what above is still valid with minor adjustments even when b is chosen to be a deterministic function of time to be calibrated to the default-free yield curve as shown in Hull and White (1990). Although the above solutions give the value $D(t,T)$ of a defaultable zero coupon bond, they can be used also to value defaultable coupon bonds and credit default swaps.

3.1 Quasi recovery of face value assumption and CDS valuation

So far we have maintained the tractable "recovery of Treasury" assumption for illustrative purposes. Now we introduce a more accurate assumption about the bond recovery value upon default, an assumption that is as tractable as the "recovery of Treasury" assumption and that approximates the more accurate "recovery of face value" assumption. We call this new recovery assumption "quasi recovery of face value" (QRF). If today's date is t and T is the bond maturity date, the period $]t,T]$ is the bond residual life. We set m dates during $[t,T]$ such that $t \leq T_1 < T_2 < .. < T_m = T$ and such that $(T_k - T_{k-1})$ is constant for $k = 2, 3, ..m$. Denote with $R(t, T_{k-1}, T_k)$ the value at time $t \leq T_1$ of a claim that pays 1 at time T_k if default occurs in the time interval $]T_{k-1}, T_k]$. It follows that

$$R\left(t,T_{k-1},T_k\right)=Z\left(t,T_k\right)\cdot E_t^k\left(\frac{1_{\tau>T_{k-1}}\cdot Z\left(T_{k-1},T_k\right)}{Z\left(T_{k-1},T_k\right)}\right)-D\left(t,T_k\right) \tag{18}$$

where $E_t^k\left(..\right)$ denotes time t conditional expectation in the $Z\left(t,T_k\right)$ forward risk neutral measure, where τ is the default time, where $1_{\tau>T_{k-1}}$ is the indicator function of the survival event $\tau>T_{k-1}$ and where $Z\left(t,T_k\right)\cdot E_t^k\left(\frac{1_{\tau>T_{k-1}}\cdot Z\left(T_{k-1},T_k\right)}{Z\left(T_{k-1},T_k\right)}\right)$ is the present value of a defaultable claim that pays off $Z\left(T_{k-1},T_k\right)$ at T_k. Then notice that

$$E_t^k\left(\frac{1_{\tau>T_{k-1}}\cdot Z\left(T_{k-1},T_k\right)}{Z\left(T_{k-1},T_k\right)}\right)=E_t^k\left(1_{\tau>T_{k-1}}\right)=P^k\left(t,T_{k-1}\right) \tag{19}$$

and $P^k\left(t,T_{k-1}\right)$ is the survival probability up to time T_{k-1} in the $Z\left(t,T_k\right)$ forward risk neutral measure. It follows that we can write

$$R\left(t,T_{k-1},T_k\right)=Z\left(t,T_k\right)\left(P^k\left(t,T_{k-1}\right)-P^k\left(t,T_k\right)\right). \tag{20}$$

The expression $P^k\left(t,T_{k-1}\right)-P^k\left(t,T_k\right)$ denotes the probability calculated at time t in the $Z\left(t,T_k\right)$ forward risk neutral measure that default will occur in the time interval $]T_{k-1},T_k]$. We can now determine the present value of what bond holders expect to recover upon default. At time t such present value is equal to the value of a claim that pays π at T_k if default time τ falls during the interval $]T_{k-1},T_k]$ for $k=1,2,..m$, and it is equal to

$$\pi \sum_{k=1}^{m} R\left(t, T_{k-1}, T_k\right). \tag{21}$$

We can readily compute this expression since we have closed form solutions for $Z(t,T_k)$ and $P^k(t,T_k)$ from above. Thus this QRF assumption is as tractable as the "recovery of Treasury" assumption. Moreover as the bond residual life $]t,T]$ is partitioned in a greater number m of sub-intervals, the bond recovery value approaches the recovery value we obtain under the proper "recovery of face" assumption, which is commonly regarded as the most realistic and least tractable recovery assumption. According to the "recovery of face" assumption π is received at the exact time of default, whereas the QRF assumption posits that the recovery value is received soon after default, which seems a good and tractable approximation. Then the value of a defaultable fixed coupon bond with face value of 1 and which promises to pay coupons at times T_k for $k = 1, 2, ..m$ equal to $c(T_k - T_{k-1})$ is

$$\sum_{k=1}^{m} c\left(T_k - T_{k-1}\right) D\left(t, T_k\right) + D\left(t, T_m\right) + \pi \sum_{k=1}^{m} R\left(t, T_{k-1}, T_k\right). \tag{22}$$

Then we can readily derive the following closed form solution for CDS rates at time t as

$$C_t = \frac{(1-\pi) \cdot \sum_{k=1}^{m} R\left(t, T_{k-1}, T_k\right)}{\sum_{k=1}^{m} \left(T_k - T_{k-1}\right) \cdot D\left(t, T_k\right)}. \tag{23}$$

In stating this formula for a CDS spread we retain all previous assumptions, in particular the assumption about the bond recovery value to be received at times T_k. Although not necessary, for simplicity we also assume, without much loss in accuracy, that the CDS fee payment dates are T_k, so that each fee payment amounts to $C_t\left(T_k - T_{k-1}\right)$ for a CDS initiated at time t. Again this seems a good approximation to how CDS fees are periodically paid. The above formula for CDS rates is still tractable enough to allows us to estimate the model parameters. In particular the above solution for $D\left(t,T\right)$ involves separation of variables, which lends much tractability to the computations.

[FIGURE 1]

Figure1: CDS rates predicted by the model for maturities of 1 year, 3 years and 7 years (this graph uses the parameter values estimated for Russia).

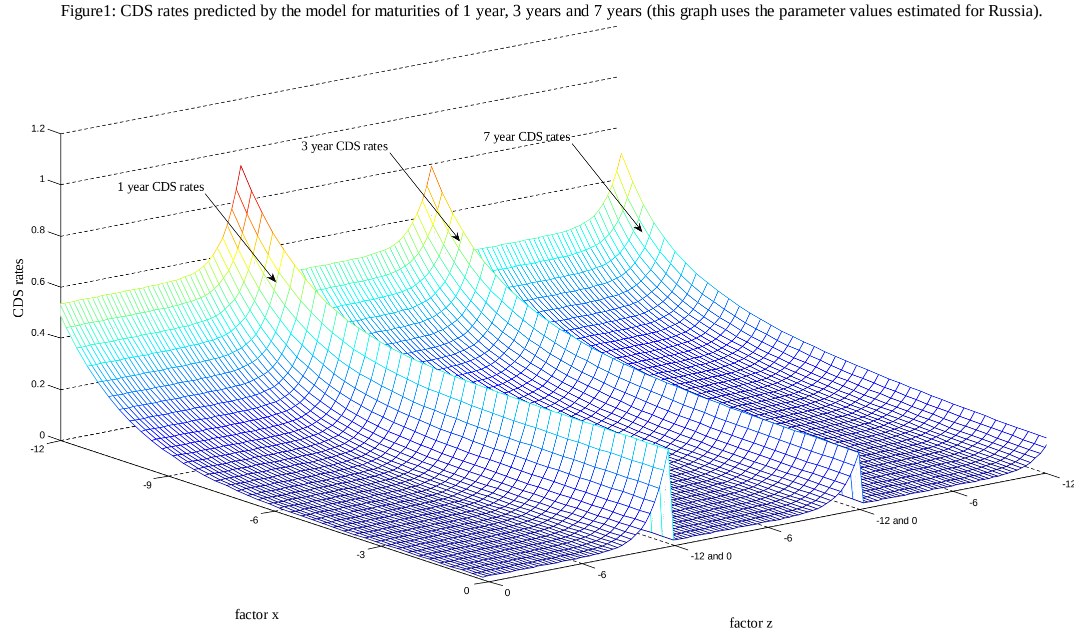

Figure 1 merges three figures in order to show how the model predicted CDS rates vary with x and z for three different CDS maturities. The foremost figure refers to one year CDS rates, the intermediate figure to three year CDS rates and the last figure to seven year CDS rates. Notice how CDS rates of all maturities are monotonic in z and x, which makes it possible to infer z and x

if only we observed the one year and three year CDS rates without error. This feature is key in the estimation to follow.

3.2 The likelihood function

The above assumptions allow us to "invert" the observed CDS rates in order to infer the latent factors x and z on any date. Appendix 2 describes how this "inversion" is accomplished. "Inversion" is perhaps the main technical obstacle to estimating the model of this paper, but it is "inversion " that makes maximum likelihood estimation feasible. We denote the set of M dates on which we observe CDS rates in the market as t_i with $i = 0, 1, 2.., M$. x_i and z_i denote the values of x and z at time t_i. We denote with $l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right)$ the conditional density of (z_i, x_i) given (z_{i-1}, x_{i-1}). In the real measure $l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right)$ is a bivariate Gaussian density such that

$$l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right) = \frac{1}{2\pi\sqrt{v_x \cdot v_z \cdot}} \cdot e^{-\frac{\frac{(x_i - m_{x,i})^2}{v_x} + \frac{(z_i - m_{z,i})^2}{v_z}}{2}} \tag{24}$$

with

$$\begin{aligned}
m_{x,i} &= (b_x + s_x b_x^*) + (x_{i-1} - (b_x + s_x b_x^*))\, e^{-(a_x + s_x a_x^*)(t_i - t_{i-1})} \\
v_x &= \frac{s_x^2}{2\left(a_x + s_x a_x^*\right)}\left(1 - e^{-2(a_x + s_x a_x^*)(t_i - t_{i-1})}\right) \\
m_{z,i} &= (b_z + s_z b_z^*) + (z_{i-1} - (b_z + s_z b_z^*))\, e^{-(a_z + s_z a_z^*)(t_i - t_{i-1})} \\
v_z &= \frac{s_z^2}{2\left(a_z + s_z a_z^*\right)}\left(1 - e^{-2(a_z + s_z a_z^*)(t_i - t_{i-1})}\right).
\end{aligned}$$

To infer x_i and z_i and estimate the model parameters, for every country we assume that the CDS rates for the two shortest maturity are observed without error, while rates for longer maturities are assumed to be observed with errors. This entails that information implied by short term CDS rates is used to predict long term CDS rates on any date. Then the model is assessed according to how well it predicts observed CDS rates. Pan and Singleton assumed that the 5 year CDS rates are observed without error and found that in their sample one principal component could explain long term CDS rates, but not the 1 year CDS rates. Since 1 year CDS contracts are very liquid, here we assume that 1 year CDS rates are observed without error. We also assume that the CDS rates for the second shortest maturity, which for the countries in the sample is either the three year or the five year maturity, are observed without error. Then the model is asked to predict CDS rates for the longer maturities. To clarify the point, we consider the case of Russia.

Let $O_{i,n}$ denote the observed CDS rate on date t_i for the maturity equal to n years. In the case of Russia we observe $O_{i,1}$ and $O_{i,3}$ without error, while $O_{i,7}$ is observed with errors. Such errors are a white noise series uncorrelated with the factors x and z. At any time the errors are normally distributed with mean 0 and variance σ_7^2. Observing $O_{i,1}$ and $O_{i,3}$ without error enables us to infer z_i, x_i for all dates t_i. In the real measure the conditional density of the vector $(z_i, x_i, O_{i,7})$ is

$$l\left(z_i, x_i, O_{i,7}\right) = l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right) \cdot l\left(O_{i,7} \mid z_i, x_i\right) \tag{25}$$

where $l\left(O_{i,7} \mid z_i, x_i\right)$ is the conditional density of $O_{i,7}$ given (z_i, x_i). Let $(C_{i,1}, C_{i,3}, C_{i,7})$ denotes the CDS rates at time t_i for maturities of 1 year, 3 years and 7 years as predicted by the model. $(C_{i,1}, C_{i,3}, C_{i,7})$ are found numerically through an implicit finite difference as described in Appendix 1. Notice that we impose $C_{i,1} = O_{i,1}$ and $C_{i,3} = O_{i,3}$. Then $l\left(O_{i,7} \mid z_i, x_i\right)$ is a uni-variate normal density with mean of $C_{i,7}$ and variance of σ_7^2, such that

$$l\left(O_{i,7} \mid z_i, x_i\right) = \frac{1}{\sigma_7\sqrt{2\pi}} e^{-\frac{\left(O_{i,7}-C_{i,7}\right)^2}{2\sigma_7^2}}. \tag{26}$$

It follows that the conditional density of the observed CDS rates $(O_{i,1}, O_{i,3}, O_{i,7})$ is

$$l\left(O_{i,1}, O_{i,3}, O_{i,7}\right) = |J_i| \cdot l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right) \cdot l\left(O_{i,7} \mid z_i, x_i\right) \tag{27}$$

where $|J_i|$ is the absolute value of the Jacobian determinant for time t_i, such that

$$|J_i| = \begin{vmatrix} \frac{\partial z_i}{\partial O_{i,1}} & \frac{\partial x_i}{\partial O_{i,1}} \\ \frac{\partial z_i}{\partial O_{i,3}} & \frac{\partial x_i}{\partial O_{i,3}} \end{vmatrix} = \left(\frac{\partial O_{i,1}}{\partial z_i}\right)^{-1} \cdot \frac{\partial x_i}{\partial O_{i,3}} - \frac{\partial x_i}{\partial O_{i,1}} \cdot \left(\frac{\partial O_{i,3}}{\partial z_i}\right)^{-1}$$

since $\frac{\partial z_i}{\partial O_{i,1}} = \left(\frac{\partial O_{i,1}}{\partial z_i}\right)^{-1}$, $\frac{\partial z_i}{\partial O_{i,3}} = \left(\frac{\partial O_{i,3}}{\partial z_i}\right)^{-1}$. Notice that $C_{i,1}$ and $C_{i,3}$ are monotonic in x_i and z_i. Then the logarithm of the joint likelihood function of $(O_{i,1}, O_{i,3}, O_{i,7})$ for $i = 1, .., M$ is

$$L = \sum_{t=1}^{M} \ln |J_i| + \ln l\left(z_i, x_i \mid z_{i-1}, x_{i-1}\right) + \ln l\left(O_{i,10} \mid z_i, x_i\right). \qquad (28)$$

Maximising L gives the parameter estimates for the model presented below. A similar procedure is repeated for all countries in the sample.

4 CDS model estimation and tests

This section presents estimates and tests of the above two factor CDS pricing model using daily CDS rate observations collected from Datastream for various countries, namely Russia, Philippines, Malaysia, South Africa, Argentina, Brazil, Mexico, Romania, Turkey, Thailand. Table 1 displays descriptive statistics for the observed CDS rates expressed in basis points.

[TABLE 1 about here]

TABLE I

	Turkey	Romania	Argentina	Thailand	Mexico	South Africa	Peru	Russia	Malaysia	Brazil	Philippines
Period	1/6/04 to 1/2/07	21/3/05 to 24/10/06	10/6/05 to 24/10/06	10/9/03 to 24/10/06	10/9/03 to 24/10/06	15/6/04 to 24/10/06	1/1/04 to 24/10/06	213/05 to 24/10/06	1/9/03 to 24/10/06	1/3/04 to 24/10/06	21/3/05 to 24/10/06
Observations	714	417	358	815	995	616	734	417	822	692	417
maturity	1 year	1 year	1 year	1 year	1 year	3 years	1 year	1 year	3 years	1 year	3 years
mean	84	17	84	14	64	45	104	36	25	124	193
stdev	55	5	33	3	64	16	89	10	8	127	79
max	361	37	222	23	269	101	387	65	54	589	351
min	23	6	41	5	10	17	20	19	12	19	83
median	70	16	78	13	34	42	74	34	22	81	162
maturity	3 years	5 years	3 years	3 years	3 years	5 years	3 years	3 years	5 years	3 years	5 years
mean	178	45	204	27	101	64	184	57	34	252	283
stdev	89	9	50	5	69	22	100	20	10	173	96
max	512	75	359	43	262	131	453	121	70	795	451
min	68	24	118	17	28	33	53	30	20	54	141
median	160	43	198	26	73	59	154	52	30	226	257
maturity	5 years	7 years	5 years	5 years	5 years	7 years	10 years	7 years	10 years	10 years	10 years
mean	247	55	372	37	127	78	312	88	49	424	376
stdev	95	9	51	5	71	25	116	29	12	188	104
max	615	84	505	53	259	151	691	180	92	990	566
min	115	34	275	25	45	43	108	56	28	139	219
median	228	53	367	37	103	72	278	78	46	419	344
maturity	10 years	10 years	10 years	10 years	10 years						
mean	319	65	429	51	165						
stdev	95	10	53	7	76						
max	677	97	545	67	381						
min	180	45	337	32	75						
median	299	63	423	52	141						

The CDS contracts considered in this paper are settled in Euros, so that we need to discount cash flows using the Euro term structure of interest rates, which we approximate to be flat and constant at 5% will little loss in accuracy. Also Pan and Singleton (2005) assume a flat and constant term structure. Unreported simulations confirmed that this assumption does not entail any significant sacrifice of accuracy. The recovery rate is exogenously set equal to $\pi = 0.25$, in keeping with market practice. Table 2 summarises the estimation results for the two factor Black-Karasinski CDS pricing model.

[TABLE 2 about here]

TABLE II

	Turkey		Romania		Argentina		Thailand		Mexico		South Africa	
s_x	2.81	0.00	2.10	0.00	3.23	0.00	2.59	0.01	2.58	0.00	1.91	0.03
a_x	0.07	0.00	0.06	0.00	0.19	0.01	0.06	0.01	0.13	0.01	-0.12	0.01
b_x	-0.09	0.02	-0.31	0.01	-0.75	0.05	-0.45	0.06	-0.39	0.03	0.98	0.09
s_z	2.10	0.00	2.99	0.00	3.56	0.49	3.45	0.00	4.54	0.02	2.85	0.00
a_z	0.31	0.02	0.04	0.00	0.26	0.42	0.24	0.05	0.13	0.01	0.23	0.07
b_z	-2.44	0.04	-2.72	0.01	-8.21	2.54	-4.70	0.18	-4.33	0.07	-5.15	0.19
stdev err 1	0.0017	0.00	0.0003	0.00	0.0019	0.00	0.0004	0.00	0.0009	0.00		
stdev err 2	0.0023	0.00	0.0004	0.00	0.0022	0.00	0.0007	0.00	0.0022	0.00	0.0006	0.00
a_x^*	0.00	0.62	0.08	1.85	-0.12	1.52	-0.04	3.47	-0.04	0.75	-0.04	2.79
b_x^*	-	6.24	-1.28	20.01	0.77	13.77	-0.00	39.61	-0.00	8.24	-0.00	28.08
a_z^*	-	2.14	-3.17	3.56	1.07	4.87	-0.00	3.32	0.00	0.74	0.00	2.52
b_z^*	0.00	7.13	-17.25	20.14	-0.54	12.47	3.24	20.66	0.74	5.62	0.00	7.51
L	16,849		13,635		7,897		24,660		25,226		12,377	
avg L	23.60		32.70		22.06		30.26		25.35		20.09	
R^2-5yr	0.99	R^2-7yr	0.95	R^2-7yr	0.91	R^2-5yr	0.76	R^2-5yr	0.99			
R^2-10yr	0.95	R^2-10yr	0.9	R^2-10yr	0.89	R^2-10yr	0.36	R^2-10yr	0.95	R^2-7yr	0.98	

	Peru		Russia		Malaysia		Brazil		Philippines	
s_x	2.97	0.00	2.70	0.00	1.93	0.00	2.89	0.00	2.48	0.00
a_x	0.01	0.00	0.04	0.00	-0.03	0.00	0.01	0.01	0.06	0.00
b_x	0.36	0.00	-0.37	0.04	0.06	0.02	0.39	0.04	0.05	0.03
s_z	5.91	0.00	1.65	0.00	1.67	0.05	5.20	0.33	1.10	0.15
a_z	0.08	0.09	-0.33	0.19	-0.72	0.14	0.07	0.40	-0.83	0.14
b_z	-13.52	0.29	-1.29	0.35	-0.44	0.25	-16.00	2.10	-0.36	0.39
stdev err 1										
stdev err 2	0.0049	0.00	0.0008	-	0.0005	0.00	0.0049	0.00	0.0021	0.00
a_x^*	0.00	0.73	-0.23	1.57	0.54	1.81	0.00	0.44	0.04	0.94
b_x^*	-0.21	6.93	-3.62	16.10	4.51	17.86	-0.01	3.89	-0.33	8.30
a_z^*	-0.77	1.46	0.00	17.55	-0.00	10.53	-0.00	1.68	-0.00	13.94
b_z^*	0.00	5.15	-0.00	48.40	0.00	32.28	0.00	3.74	0.00	20.91
L	12,611		9,051		17,559		11,379		6,307	
avg L	17.18		21.70		21.36		16.44		15.12	
R^2-10yr	0.85	R^2-7yr	0.95	R^2-10yr	0.89	R^2-10yr	0.96	R^2-10yr	0.99	

In Table 2 the pricing model was solved on a grid of size: 40x40x10. This means that the finite difference scheme employs 40 steps both in the dimension of the x factor and in the dimension of the z factor, as well as 10 time steps per year. We notice that the whole estimation hinges on numerical solutions to find the model predicted CDS rates, but the results are not particularly sensitive as we vary the number of grid points. $stdev\_err1$ denotes the estimate of the standard error of the daily difference between model predicted CDS rate and observed CDS rate for the second longest CDS maturity for any country. $stdev\_err2$ denotes the estimate of the standard error of the daily difference between model predicted CDS rate and observed CDS rate for the longest CDS maturity for any country. R^2 provides a measure of how well the model predicts the long term observed CDS rates. For example R^2-$5yr$ denotes the R^2 for the five year CDS rates, R^2-$10yr$ denotes the R^2 for the ten year CDS rates. L is the value of the log-likelihood function that is maximised in estimation. $avg\ L$ is given by L divided by the number of daily observations in the sample.

Overall Table 2 reveals the good pricing performance of the CDS pricing model. With the exception of Thailand, the model explains from 89% to 99% of the observed changes in long term CDS rates, as measured by the R^2 coefficients. As could be expected, the longest maturity rates are predicted with less accuracy than the rates for the second longest maturity. The parameter values that characterise the risk-neutral processes of the two latent factors are generally are significant. In other words the inclusion of the second latent factor appears statistically significant.

As in Pan and Singleton (2005) for some countries, such as Russia and Malaysia, the latent factors x and z are mean-averting in the risk-neutral measure, but this is not the case in general. Perhaps the most striking result in Table 2 is that for no country any of the parameters that determine risk premia are significant, although the parameters b_x^* and b_z^* tend to have the negative sign implied by risk-aversion. In other words, the evidence based on the use of the two factor Black-Karasinski model does not support the existence of risk premia. This means that the dynamics of default intensities implied by the observed CDS time series does not seem significantly different in the real measure and in the risk-neutral measure.

This result is confirmed by likelihood ratio tests that assess the restrictions $a_x^* = b_x^* = a_z^* = b_z^* = 0$. For example, for Turkey the maximised restricted log-likelihood function has value $16,848$, which is almost the same as $16,849$ and which implies that the likelihood ratio statistic is certainly not significantly different from 0 according to the X^2 distribution with 4 degrees of freedom.

4.1 The representative case of Argentina

We discuss the case of Argentina as it is representative of how the CDS model fits the observed CDS rates in the sample of countries we consider. By construction the model perfectly matches the time series of rates for the two shortest CDS maturities (one year and three years for Argentina) and quite accurately predicts the CDS rates for the two longest maturities (five years and ten years for Argentina). The R^2 estimates imply that the model can explain about 90% of the variation in observed CDS rates for the five year and ten year CDS maturities. The longest maturity CDS rates are predicted less accurately. The prediction standard error for five year rates is 19 basis points while the prediction standard error for the ten year rates is 22 basis points.

Although unreported, estimation provides time series of the latent factors x and z. The average of x across observation dates is -9.32 and that of z is

-2.61. Since $e^{-9.32} = 0.0001$ we can approximately estimate that x contributes to the instantaneous default intensity λ^Q for 1 basis point on average, while z contributes to the instantaneous default intensity for 734 basis points on average, since $e^{-2.61} = 0.0734$. Thus one factor, namely z, explains most of the level of the default intensity implied by the model and by the observed CDS rates. Moreover, since the minimum and maximum values for x are -10.81 and -7.62 respectively and since $e^{-10.81} = 0.00002$ and $e^{-7.62} = 0.00049$, x contributes to the variations of the instantaneous default intensity λ^Q for a maximum of about 5 basis points during the sample period. Instead, since the minimum and maximum values for z are -3.13 and -1.68 respectively and since $e^{-3.13} = 0.0437$ and $e^{-1.68} = 0.1864$, x contributes to the variations of the instantaneous default intensity λ^Q for a maximum of about 1400 basis points during the sample period. Thus one factor, namely x, explains most of the variation of the default intensity and hence the variation of the model predicted CDS rates during the sample period. Although unreported, a similar pattern repeats itself for all the countries in the sample.

For Argentina $a_x = 0.19$ and $a_z = 0.26$, thus both x and z follow mean reverting processes. This is the case especially for the first factor, which is significantly different from 0. The estimated model on average predicts upward sloping term structures of credit spreads. This is due to the fact that the estimated long term mean of x is $b_x = -0.75$, well above the mean level of x which is -9.32. Both x and z are very volatile as $s_x = 3.23$ and $s_z = 3.56$. Also for the other countries in the sample the risk premia parameters are also not significantly different from 0 and the up-ward sloping term structures of predicted CDS rates can be explained in a similar way.

Figure 2 graphically displays how well the CDS pricing model fits the observed CDS rates for Argentina. The bold lines denote the model predicted time series of CDS rates. By construction the one-year and three-year CDS rates are (almost) perfectly reproduced by the model, as the latent factors are inferred from the one year and three year maturities. The five year and ten year maturities are predicted quite accurately. The two factor Black-Karasinski CDS pricing model provides a good fit to observed CDS rates.

[FIGURE 2 about here]

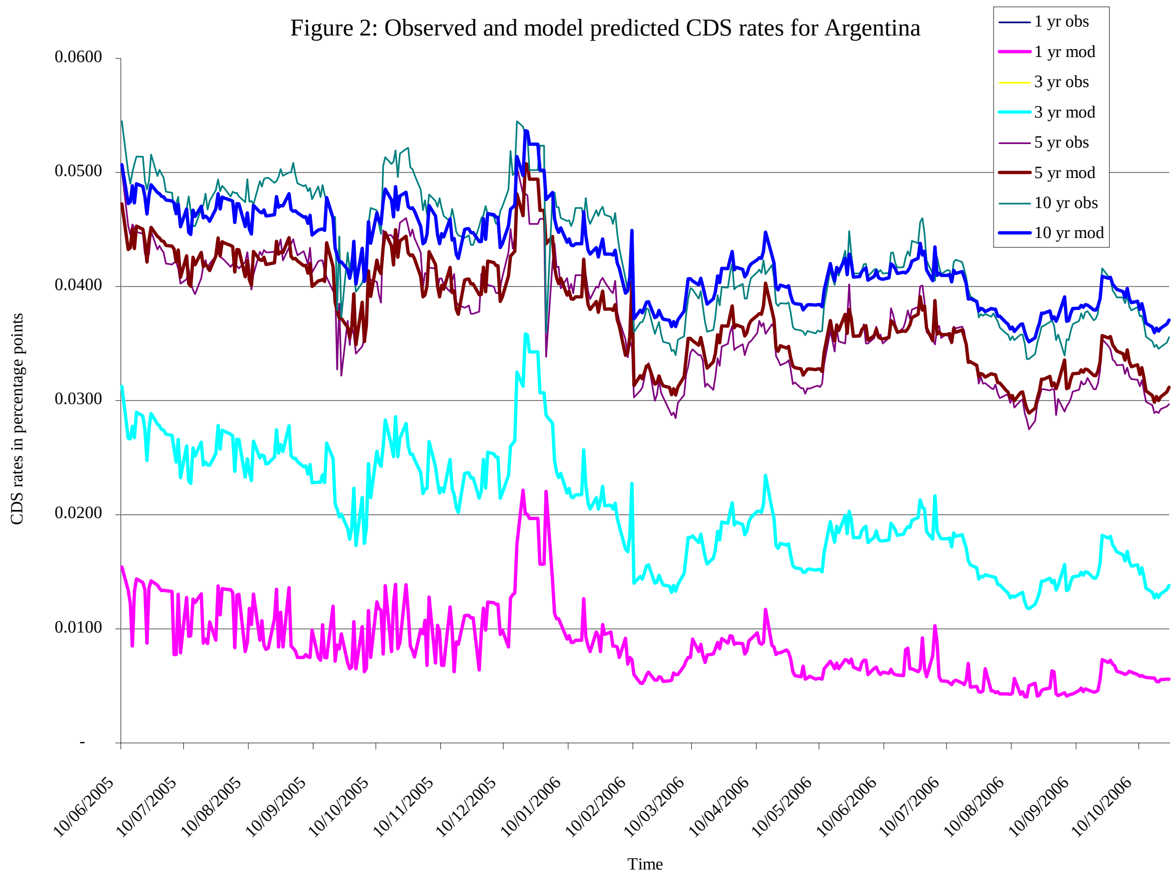

Figure 2: Observed and model predicted CDS rates for Argentina

5 Conclusions

This paper has presented and tested a two-factor reduced form sovereign CDS pricing model. The model extends the one factor Black-Karasinski (BK) model of Pan and Singleton (2005) to envisage a second latent factor driving the instantaneous default intensity of any sovereign. Although the pricing model can only be solved numerically through finite differences, and despite the presence of two latent factors, we can still estimate the parameters through maximum likelihood. The model has been estimated and tested on a sample of credit default swap rates for Brazil, Russia, South Africa, Malaysia, Philippines, Turkey, Mexico, Argentina, Romania, Peru and Thailand.

The introduction of a second stochastic factor driving the default intensity improves the fit of the model to the observed CDS rates. The presence of the second stochastic factor is statistically significant. In most cases two latent factors enable the pricing model to explain well more than 90% of the variation in observed CDS rates.

The evidence based on the two factor model fails to support the view that the risk associated with uncertainty about the evolution of the default intensity of each sovereign is priced. In other words the real world process of the default intensities does not seem to differ significantly from the risk-neutral process. Moreover for all countries the variations of the default intensity are mainly driven by one factor.

Overall these empirical results highlight how envisaging a second latent factor driving the default intensity is a worthy extensions to reduced form sovereign CDS pricing models. As a by-product, the paper has also provided a viable method for maximum likelihood estimation of pricing models with two latent factors for cases in which pricing models can only be solved numerically through finite difference methods. Such method furthers the boundaries of the estimable pricing models.

A The finite difference scheme

This Appendix shows how equation (16) and the respective conditions are solved through the implicit finite difference method. Here it is convenient to discretise $P_x(t,T)$ and re-express it as a function of x and t as follows. We use the discretisation $x = -12 + h \cdot \delta x,\ T - t = k \cdot \delta\tau,\ u_h^k \backsimeq P_x(-12 + h \cdot \delta x, k \cdot \delta\tau)$ with $h = 0, 1, ..., h^*$. δx and $\delta\tau$ are fixed "steps". The grid upper boundary in the x-dimension is $x_{\max} = -12 + h^* \cdot \delta x = 0$ and the grid lower boundary is $x_{\min} = -12 + 0 \cdot \delta x = -12$. The grid parameters used in estimation are $\delta\tau = 1/10$, $\delta x = 12/40$, $h^* = 40$, $T - t = 10$. This is a coarse grid but it speeds up the optimisation of the likelihood function with respect to the parameters and entails little sacrifice in accuracy when compared with finer grids. Equation 16 is approximated as

$$
\begin{aligned}
u_h^k = {} & \delta\tau \left(\frac{1}{\delta\tau} + r + e^{h\delta x} + \left(\frac{s}{\delta x} \right)^2 \right) u_h^{k+1} \\
& - \delta\tau \left(\frac{1}{2} \left(\frac{s}{\delta x} \right)^2 + \frac{(b - ah\delta x)}{2\delta x} \right) u_{h+1}^{k+1} - \delta\tau \left(\frac{1}{2} \left(\frac{s}{\delta x} \right)^2 - \frac{(b - ah\delta x)}{2\delta x} \right) u_{h-1}^{k+1}
\end{aligned}
\tag{29}
$$

subject to condition $u_0^{k+1} = 2u_1^{k+1} - u_2^{k+1}$, to condition $u_h^0 = 1$, and to condition $u_{h^*}^{k+1} = 2u_{h^*-1}^{k+1} - u_{h^*-2}^{k+1}$. At every time step Gaussian elimination is used to implement the time-stepping from any u_h^k to any u_h^{k+1} and to retain the stability of the solution. Gaussian elimination is used since successive under-relaxation or over-relaxation could not converge thus compromising the solution.

B Inversion of the model to infer the latent factors

This appendix describes how the latent factors are inferred from observed CDS rates. For the sake of clarity here we assume the case of Russia, where $O_{i,1}$ and $O_{i,3}$, which are the CDS rates for the two shortest maturities (1 year and three years), are observed without error on any date t_i. $O_{i,7}$ is the observed seven year CDS rate and it is observed with error. Then we can infer z_i and x_i on any date t_i. This can be done more accurately as the grid that calculates $u_h^k \backsimeq P_x\left(-12 + h \cdot \delta x, k \cdot \delta\tau\right)$ is finer, i.e. as δx is smaller. It is important to notice that $P_x\left(-12 + h \cdot \delta x, k \cdot \delta\tau\right)$ and the cross section of model predicted CDS rates need only be computed once for every parameter set in order to obtain the cross sections of the model predicted CDS rates C_1, C_3 and C_7 for maturities of 1, 3 and 7 years. C_1, C_3 and C_7 are matrixes of CDS rates. For example C_1 is a matrix in two dimensions: x and z. The elements of C_1 are $C_1(h,j)$ for $h = 0, 1, .., h^*$ and $j = 0, 1, .., j^*$, where $z = -12 + j \cdot \delta z$. To find $C_{i,1}$, $C_{i,3}$, $C_{i,7}$, z_i and x_i for any date t_i we use the following interpolation. For every date t_i first we find the values of h and j such that

$$
\begin{aligned}
C_1(h,j) &\leq O_{i,1} \leq C_1(h+1, j+1) \\
C_3(h,j) &\leq O_{i,3} \leq C_3(h+1, j+1).
\end{aligned}
$$

Then we find the interpolation weights for time t_i as

$$\begin{bmatrix} wh(i) \\ wj(i) \end{bmatrix} = \begin{bmatrix} C_1(h+1,j) - C_1(h,j) & C_1(h,j+1) - C_1(h,j)) \\ C_3(h+1,j) - C_3(h,j) & C_3(h,j+1) - C_3(h,j) \end{bmatrix}^{-1} \cdot \begin{bmatrix} O_{i,1} - C_1(h,j) \\ O_{i,3} - C_3(h,j) \end{bmatrix}. \tag{30}$$

Finally we employ the interpolation weights to compute

$$x_i = wh(i) \cdot (-12 + (h+1) \cdot \delta x) + (1 - wh(i)) \cdot (-12 + h \cdot \delta x) \tag{31}$$

$$z_i = wj(i) \cdot (-12 + (j+1) \cdot \delta z) + (1 - wj(i)) \cdot (-12 + j \cdot \delta z) \tag{32}$$

$$C_{i,1} = C_1(i,j)(1 - wh(i))(1 - wj(i)) + C_1(i+1,j)wh(i)(1 - wj(i)) \tag{33}$$

$$+C_1(i,j+1)(1 - wh(i))wj(i) + C_1(i+1,j+1)wh(i)wj(i)$$

$$C_{i,3} = C_3(i,j)(1 - wh(i))(1 - wj(i)) + C_3(i+1,j)wh(i)(1 - wj(i)) \tag{34}$$

$$+C_3(i,j+1)(1 - wh(i))wj(i) + C_3(i+1,j+1)wh(i)wj(i)$$

$$C_{i,7} = C_7(i,j)(1 - wh(i))(1 - wj(i)) + C_7(i+1,j)wh(i)(1 - wj(i)) \tag{35}$$

$$+C_7(i,j+1)(1 - wh(i))wj(i) + C_7(i+1,j+1)wh(i)wj(i).$$

This interpolation is repeated for every date t_i.

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