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ABSTRACT

This paper is concerned with estimating preference functionals for choice under risk from the choice behaviour of individuals. We note that there is heterogeneity in behaviour between individuals and within individuals. By 'heterogeneity between individuals' we mean that people are different, in terms of both their preference functionals and their parameters for these functionals. By 'heterogeneity within individuals' we mean that behaviour may be different even by the same individual for the same choice problem. We propose methods of taking into account all forms of heterogeneity, concentrating particularly on using a Mixture Model to capture the heterogeneity of preference functionals.

Keywords: expected utility theory, maximum simulated likelihood, mixture models, rank dependent expected utility theory, heterogeneity.

JEL codes: C15, C29, C51, C87, C91, D81

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1. Introduction

As is clear from Starmer (2000), the past five decades have witnessed intensive theoretical and empirical research into finding a good descriptive theory of behaviour under risk. Since the general acceptance of the criticisms of Expected Utility made by Allais (for example, in Allais 1953) and others, theorists have been active in developing new theories to explain the deficiencies of Expected Utility theory. Hey (1997) provides a list[2] of the major theories at that time: Allais' 1952 theory, Anticipated Utility theory, Cumulative Prospect theory, Disappointment theory, Disappointment Aversion theory, Implicit Expected (or linear) Utility theory, Implicit Rank Linear Utility theory, Implicit Weighed Utility theory, Lottery Dependent EU theory, Machina's Generalised EU theory, Perspective theory, Prospect theory, Prospective Reference theory, Quadratic Utility theory, Rank Dependent Expected (or Linear) Utility theory, Regret theory, SSB theory, Weighted EU theory, Yaari's Dual theory. All these theories were motivated by the inability of Expected Utility theory to explain all observed behaviour. This burst of theoretical activity took place in the last thirty of years or so of the 20th century. Since then, activity has been concentrated more on discovering which of these theories are empirically most plausible and robust – see, for example, Hey and Orme (1994). This period of empirical work revealed clearly that there is considerable heterogeneity of behaviour both between individuals and within individuals. By 'heterogeneity between individuals' we mean that people are different, not only in terms of which type of preference functional that they have, but also in terms of their parameters for these functionals. By 'heterogeneity within individuals' we mean that behaviour may be different even

[2] Full references can be found in Hey (1997).

for the same choice problem. Econometric investigation has to take these heterogeneities into account.

Some of the empirical literature adopted the strategy of trying to find the best preference functional individual by individual; see, for example, Hey and Orme (1994) and Gonzales and Wu (1999). Another part of the literature attempted to find the best preference functional across a group of individuals, by, in some way, pooling or aggregating the data; see, for example, Harless and Camerer (1994). In fitting data subject by subject, the problem of heterogeneity *within* subjects becomes immediately apparent in two different ways. First, when confronted with the same decision problem on different occasions, people respond differently. Second, and perhaps more importantly, it was soon realised that *none* of the long list of preference functionals listed above fitted any (non-trivial) data exactly. Economists responded in their usual fashion – by declaring that individuals were *noisy* in their behaviour, or that they made errors of some kind when taking decisions. At this point, interest centred on ways of describing such noise and incorporating it into the econometric investigation. A number of solutions were proposed: the constant-probability-of-making-a-mistake model of Harless and Camerer (1994), the Fechner-error model adopted by Hey and Orme (1994), and the random-preference model of Loomes and Sugden (1998). In the first of these, subjects in experiments are thought of as implementing their choices with a constant error; in the second, subjects were perceived as measuring the value of each option with some error; in the third, subjects were thought of as not having precisely defined preferences, but preferences drawn randomly from some probability distribution. The tremble model, analysed in Moffatt and Peters (2001), can be considered like the constant-probability model but perhaps appended to one of the other two types. A useful discussion of the relative merits of these different models can be found in Ballinger and Wilcox (1997), which concludes that the constant-probability model on its own is dominated by the other two approaches. Further results can be found in Buschena and Zilberman (2000).

Those economists who followed the measurement error story soon realised that the error might not be homoscedastic and could well depend on the nature of the choice problem (see, for example, Hey 1995). Indeed, Blavatskyy (2007) argues that, with the appropriate heteroscedastic error specification, Expected Utility theory can explain the data at least as well as any of the generalisations (after allowing for degrees of freedom). Not all would go as far as this, but the incorporation of some kind of error story has led to the demise of many of the theories noted in the list above. Two remain pre-eminent: Expected Utility theory – henceforth EU, and Rank Dependent Expected Utility theory (Quiggin 1982) – henceforth RDEU. Machina (1994) comments that the Rank Dependent model is "the most natural and useful modification of the classical expected utility formula". In certain contexts, for example the Cumulative Prospect theory of Tversky and Kahneman (1992), the theory is enriched with a context-dependent reference point. Nevertheless, the consensus seems to be that EU and RDEU remain the leading contenders for the description of behaviour under risk.

As we have already remarked, some of the investigations of the appropriate preference functional have taken each individual separately and have carried out econometric work individual by individual. There are problems here with degrees of freedom and with possible over-fitting. Other investigations have proceeded with pooled data – from a set of individuals. The problem with this latter approach, even though it saves on degrees of freedom, is that individuals are clearly different. They are different, not only in terms of which type of preference functional that they have, but also in terms of their parameters for these functionals. The latter can be taken care of by assuming a *distribution* of the relevant parameters over the individuals concerned and in estimating the parameters of this distribution (of the underlying economic parameters). This *heterogeneity* may depend on observable and observed (demographic) characteristics of the individuals or it may just be unobserved heterogeneity. In either case, estimating the parameters of the distribution saves on degrees of freedom compared with estimating the underlying economic parameters for each individual. Moreover, the resulting estimates may be preferred if they are going to be used for predicting the behaviour of the same, or a similar, group of individuals. Some economists are now taking into account such heterogeneity. The dangers of not so doing are well illustrated by Wilcox (2006), who shows that serious distortions in the econometric results may well be the consequence. Similarly, the paper by Myung *et al* (2000) shows clearly the problems with fitting a single agent model to a heterogeneous population.

Taking into account the fact that different individuals may have different preference functionals is more difficult. In this paper we adopt a solution: that of using a *Mixture Model* – see McLachlan and Peel (2000). We emphasise that we are by no means the first to use such a solution in such a context – a very useful reference is Harrison and Rutstrom (2007), which includes a discussion of the previous use of mixture models in economics[3].

We restrict attention to EU and RDEU, and we proceed by assuming that a proportion *(1-p)* of the population from which the sample is drawn have EU preference functionals, and the remaining proportion have RDEU preference functionals. The parameter *p* is known as the *mixing proportion* and it is estimated along with the other parameters of the model. Obviously the method can be extended to more than two functionals, but the purpose of this paper is to illustrate the power of the approach. Moreover, within each model we shall assume heterogeneity of parameters. Thus we take into account both types of heterogeneity *between* individuals, without sacrificing degrees of freedom, and without getting distorted results. Finally, to take into account heterogeneity *within* subjects we shall incorporate both a Fechner-type error and a tremble.

We illustrate the approach with data from an experiment reported in Hey (2001). The next section describes the experiment. Section 3 details the specification of EU and RDEU, while

[3] We note that, while this paper and that of Harrison and Rutstrom (2007), are similar in many respects, there are differences, in particular that we include unobserved heterogeneity of parameter values across individuals. They, however, include demographic effects, which we do not.

Section 4 discusses econometric detail, including the application of the Mixture Model (with unobserved heterogeneity) in this context. Section 5 discusses the results and Section 6 concludes.

2. The experiment and the data

The data used in this study, previously analysed by Hey (2001) and more recently by Moffatt (2005), was obtained from 53 subjects, drawn from the student population of the University of York. Each subject faced a set of 100 pairwise-choice problems between two different lotteries, repeated on five different days over a two-week period, so that the total number of problems faced by each subject is 500. The ordering of the problems changed between days and also between subjects. The probabilities defining the 100 problems are listed in Table A1 of the Appendix. All 100 problems involved three of the four outcomes £0, £50, £100 and £150. The random lottery incentive system was applied: at the end of the final session, one of the subject's 500 chosen lotteries was selected at random and played for real. For each subject and for each pairwise-choice problem we know the lottery chosen by the subject. The resulting matrix, of size 500 by 53, is our data.

3. The preference functionals under consideration[4]

We denote the four outcomes in the experiment by x_i $(i = 1, 2, 3, 4)$[5]. In both the EU formulation and the RDEU formulation there is a utility function and we denote the corresponding utility values by u_i $(i = 1, 2, 3, 4)$. We normalise[6] so that $u_1 = 0$ and $u_4 = 1$. Each choice problem involves two lotteries – the **p**-lottery and the **q**-lottery. We denote the probabilities of the four outcomes in these two lotteries in pairwise-choice problem t ($t = 1, \ldots, 500$) by $p_{1t}, p_{2t}, p_{3t}, p_{4t}$ and $q_{1t}, q_{2t}, q_{3t}, q_{4t}$ respectively.

[4] A glossary of notation can be found in Table A2.
[5] Respectively £0, £50, £100 and £150.
[6] The utility function in both specifications is unique only up to a linear transformation.

The EU specification envisages subjects evaluating the expected utilities $EU(\mathbf{p_t})$ and $EU(\mathbf{q_t})$ of the two lotteries in pairwise-choice problem t as in equation (1).

$$\begin{aligned} EU(\mathbf{p_t}) &= p_{2t}u_2 + p_{3t}u_3 + p_{4t} \\ EU(\mathbf{q_t}) &= q_{2t}u_2 + q_{3t}u_3 + q_{4t} \end{aligned} \tag{1}$$

In the absence of error, the EU specification envisages the subject choosing $\mathbf{p_t}$ ($\mathbf{q_t}$) if and only if

$$d_{2t}u_2 + d_{3t}u_3 + d_{4t} > (<) 0 \tag{2}$$

where $d_{jt} = p_{jt} - q_{jt}$ $(i = 2, 3, 4)$.

To incorporate the fact that subjects are noisy in their choice behaviour, we add a (Fechnerian) stochastic term to (2), implying that the subject chooses $\mathbf{p_t}$ ($\mathbf{q_t}$) if and only if

$$d_{2t}u_2 + d_{3t}u_3 + d_{4t} + \varepsilon_t > (<) 0 \tag{3}$$

where we assume that each ε_t is independently and normally distributed with mean 0 and standard deviation σ. The magnitude of σ indicates the noisiness in the choices – the larger the value of this parameter, the greater the noise. We estimate σ along with the other parameters. Later we add a tremble, and at the end of section 4 we note how our estimation procedure would need to be modified for the random preferences story of randomness in behaviour.

The RDEU specification looks similar to that of EU but the subjects are envisaged as transforming the objective probabilities in a specific way. As a consequence, under RDEU subjects evaluate the rank dependent expected utilities $RDEU(\mathbf{p_t})$ and $RDEU(\mathbf{q_t})$ of the two lotteries as in equation (4).

$$\begin{aligned} RDEU(\mathbf{p_t}) &= P_{2t}u_2 + P_{3t}u_3 + P_{4t} \\ RDEU(\mathbf{q_t}) &= Q_{2t}u_2 + Q_{3t}u_3 + Q_{4t} \end{aligned} \tag{4}$$

where the *P's* and *Q's* are not the correct probabilities though they are derived from the correct probabilities in the manner specified in (5):

$$\begin{aligned} P_{2t} &= w(p_{2t} + p_{3t} + p_{4t}) - w(p_{3t} + p_{4t}) & Q_{2t} &= w(q_{2t} + q_{3t} + q_{4t}) - w(q_{3t} + q_{4t}) \\ P_{3t} &= w(p_{3t} + p_{4t}) - w(p_{4t}) & Q_{3t} &= w(q_{3t} + q_{4t}) - w(q_{4t}) \\ P_{4t} &= w(p_{4t}) & Q_{4t} &= w(q_{4t}) \end{aligned} \tag{5}$$

Here the function *w(.)* is a *probability weighting function* which is monotonically non-decreasing everywhere in the range [0,1] and for which $w(0) = 0$ and $w(1) = 1$. Note that if $w(p) = p$ everywhere, RDEU reduces to EU.

In the absence of error, the RDEU specification envisages the subject choosing $\mathbf{p_t}$ ($\mathbf{q_t}$) if and only if

$$D_{2t}u_2 + D_{3t}u_3 + D_{4t} > (<) 0 \tag{6}$$

where $D_{jt} = P_{it} - Q_{jt}$ $(i = 2, 3, 4)$. Once again, to incorporate the fact that subjects are noisy in their choice behaviour, we add a (Fechnerian) stochastic term to equation (6), implying that the subject chooses $\mathbf{p_t}$ ($\mathbf{q_t}$) if and only if

$$D_{2t}u_2 + D_{3t}u_3 + D_{4t} + \varepsilon_t > (<) 0 \tag{7}$$

To proceed to estimation, we now need to parameterise both the utility function and the weighting function. For the former, there are two possibilities: (1) we could estimate u_2 and u_3; or (2) we could adopt a particular functional form and estimate the parameter(s) of that function. As we want to have a parsimonious specification, and as we want to introduce unobserved heterogeneity, we follow the second route. The most obvious contenders for the functional form are Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA). In each of these there is one parameter. Given our normalisation, these forms can be written as in equations (8) and (9).

$$\begin{aligned} \text{CARA: } u(x) &= \frac{1-\exp(-rx)}{1-\exp(-150r)} \qquad && r \in (-\infty, 0) \cup (0, \infty) \\ &= x/150 && r = 0 \end{aligned} \tag{8}$$

$$\text{CRRA: } u(x) = (x/150)^r \tag{9}$$

We will assume later that the parameter r in both formulations is distributed over the population (from which our subjects were recruited) and we will estimate the parameters of that distribution. For the CARA function, the parameter r can take any value between $-\infty$ and $+\infty$, and r is positive for risk averters, 0 for risk-neutral agents (for whom the functions become linear), and negative for risk-loving agents. For the CRRA function, the parameter r has to be positive[7], and r is less than 1 for risk-averter agents, equal to 1 for risk-neutral agents, and greater than 1 for risk-loving agents.

For the weighting function we follow a similar route. The most parsimonious functions used in the literature are the Quiggin function and the Power function. These can be written as in equation (10), where we call γ the *weighting-function* parameter.

$$\begin{aligned} &\text{Quiggin: } w(p) = \frac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{1/\gamma}} && \gamma > \gamma^* \\ &\text{Power: } \quad w(p) = p^{\gamma} && \gamma > 0 \end{aligned} \tag{10}$$

For those subjects who act in accordance with RDEU we will assume that the risk-aversion parameter, r, and the weighting-function parameter, γ, are jointly distributed over the population from which our sample is drawn and we will estimate the parameters of that distribution. For the Quiggin function, γ must be greater than γ^* (otherwise *w(.)* is not monotonic)[8], while for the Power function, γ must be positive. For both functions, RDEU reduces to EU when $\gamma = 1$. When $\gamma \neq 1$, the Power function is either completely above or completely below the 45^0-line. In contrast, the Quiggin function does cross the 45^0-line, either with an S-shape or an inverted S-shape. This is often seen as an advantage of the Quiggin function.

In what follows, we estimate all four combinations: CARA with Quiggin, CARA with Power, CRRA with Quiggin and CRRA with Power, so we can test for the robustness of our results[9].

4. The econometric specification

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[7] We note that in some formulations the parameter r can be negative (in which case a different functional form is required). Wakker (2006) has an extended discussion of this case. However, we exclude this for a number of reasons, not least that the utility of zero is minus infinity, which makes it impossible to apply our normalisation and renders meaningless the interpretation of σ as a measure of the noisiness of the subjects' responses.

[8] $\gamma^* = 0.279095$.

[9] Here we use just four possible combinations. There are clearly other possibilities – as is discussed in Stott (2006).

Let us use the binary indicator $y_t = 1$ (-1) to indicate that the subject chose $\mathbf{p_t}$ ($\mathbf{q_t}$) on problem t. We start with the EU specification. From the choice rule given by (3), we obtain the likelihood contribution for a single subject's choice in problem t:

$$\mathrm{P}(y_t \mid r, \sigma) = \Phi\left[y_t \left(d_{2t} u_2 + d_{3t} u_3 + d_{4t} \right) / \sigma \right] \quad y_t \in \{1, -1\} \tag{11}$$

where $\Phi(.)$ is the unit normal cumulative distribution function. Note that this depends on the risk aversion parameter r and the standard deviation of the error term σ. We now introduce a *tremble*. By this we mean that the individual implements the choice indicated by equation (3) with probability $(1-\omega)$, and chooses at random between the two lotteries with probability ω. The parameter ω is called the "tremble probability". Introducing this parameter into (11), the likelihood contribution becomes:

$$\mathrm{P}(y_t \mid r, \sigma, \omega) = (1-\omega)\Phi[y_t (d_{2t} u_2 + d_{3t} u_3 + d_{4t}) / \sigma] + \omega / 2 \quad y_t \in \{1, -1\} \tag{12}$$

Following the same route for the RDEU specification, we obtain the likelihood contribution:

$$\mathrm{P}(y_t \mid r, \gamma, \sigma, \omega) = (1-\omega)\Phi[y_t (D_{2t} u_2 + D_{3t} u_3 + D_{4t}) / \sigma] + \omega / 2 \quad y_t \in \{1, -1\} \tag{13}$$

Note that the rank-dependent parameter γ now enters the likelihood in (13), through the D variables defined in (5) and (6).

We now assume that, for the population of EU individuals, the parameter r (ln(r) in the case of the CRRA specification) is distributed normally over the population with mean θ and variance δ^2, and we denote this normal density function as $f(r;\theta, \delta)$. For the population of RDEU individuals, we assume that the parameters r and γ have a joint distribution, such that the two quantities r (ln(r) in the case of the CRRA specification) and ln(γ - γ_{min}) have a bivariate normal distribution, where γ_{min} is 0 in the Power case and γ^* in the Quiggin case (see (10)). The parameters of this bivariate normal are specified for CARA in (14) and for CRRA in (15):

$$\begin{pmatrix} r \\ \ln(\gamma - \gamma_{\min}) \end{pmatrix} \sim N\left[\begin{pmatrix} \Theta \\ M \end{pmatrix}, \begin{pmatrix} \Delta^2 & \rho\Delta S \\ \rho\Delta S & S^2 \end{pmatrix} \right] \tag{14}$$

$$\begin{pmatrix} \ln(r) \\ \ln(\gamma - \gamma_{\min}) \end{pmatrix} \sim N\left[\begin{pmatrix} \Theta \\ M \end{pmatrix}, \begin{pmatrix} \Delta^2 & \rho\Delta S \\ \rho\Delta S & S^2 \end{pmatrix}\right] \tag{15}$$

The joint density function of r and γ will be denoted as $g(r,\gamma;\Theta, \Delta, M, S, \rho)$. Note that this function is not actually a bivariate normal density, since it is not the case that both arguments are normally distributed; either one or both arguments are log-normally distributed. Note that this formulation is assuming that the distribution of the risk-aversion parameter for the EU subjects may be different from that for the RDEU subjects.

Finally we assume that a proportion p of the population is RDEU and a proportion $(1-p)$ is EU. Hence, the contribution to the likelihood for any given subject is as given in (16).

$$\begin{aligned} &L(\theta,\delta,\Theta,\Delta,M,S,\rho,\sigma,\omega,p) = \\ &(1-p)\int_{-\infty}^{\infty}\left\{\prod_{t=1}^{500}\left[(1-\omega)\Phi\left[y_t\left(d_{2t}u_2 + d_{3t}u_3 + d_{4t}\right)/\sigma\right] + \omega/2\right]\right\} f\left(r;\theta,\delta\right)dr \\ &+p\int_{-\infty}^{\infty}\int_{\gamma_{\min}}^{\infty}\left\{\prod_{t=1}^{500}\left[(1-\omega)\Phi\left[y_t\left(D_{2t}u_2 + D_{3t}u_3 + D_{4t}\right)/\sigma\right] + \omega/2\right]\right\} g\left(r,\gamma;\Theta,\Delta,M,S,\rho\right)d\gamma dr \end{aligned} \tag{16}$$

The overall log-likelihood for all 53 subjects is just the sum of the log of L given by (16) over all 53 subjects. Estimation proceeds by maximum simulated likelihood[10] (see Gourieroux and Monfort, 2002, for the general principles) because of the computational problems with the double integral in the likelihood function. We estimate the parameters θ, δ, Θ, Δ, M, S, ρ, ω, σ, and p. The program (written in GAUSS) is available on request. We carry out estimation for all four combinations of the utility function and the weighting function.

A final note before proceeding is in order. It would be possible to modify our formulation to take into account the random preference model of Loomes and Sugden (1998) – where subjects are envisaged as taking a fresh drawing from his or her set of preferences on every problem – by re-writing (16) as in (17).

[10] Integration over the distribution of r in the EU model and over the joint distribution of r and γ in the RDEU model is performed by simulation. In particular we use 100 draws for each subject based on Halton sequences (Train 2003).

$$L\left(\theta,\delta,\Theta,\Delta,M,S,\rho,\sigma,\omega,p\right)=$$

$$\int_{-\infty}^{\infty}\prod_{t=1}^{500}(1-p_t)\left[(1-\omega)\Phi\left[y_t\left(d_{2t}u_2+d_{3t}u_3+d_{4t}\right)/\sigma\right]+\omega/2\right]f\left(r;\theta,\delta\right)dr \tag{17}$$

$$+\int_{-\infty}^{\infty}\int_{\gamma_{\min}}^{\infty}\prod_{t=1}^{500}p_t\left[(1-\omega)\Phi\left[y_t\left(D_{2t}u_2+D_{3t}u_3+D_{4t}\right)/\sigma\right]+\omega/2\right]g\left(r,\gamma;\Theta,\Delta,M,S,\rho\right)d\gamma dr$$

Note the difference: in formulation (16) it is as if the subject's preferences are a random drawing from the set of all preferences, but these preferences then remain fixed for the duration of the experiment; while in (17) there is a fresh drawing on every choice problem.

5. Results

The results for the CRRA/Quiggin specification are reported in Table 1. Those for the other three specifications are reported in Appendix Tables A3, A4 and A5 and Appendix Figure A1. We conclude from these tables that the CRRA/Quiggin specification fits best, and so we will concentrate the discussion that follows on this specification. Our justification for this can be found in the following Table 2, which reports the maximised log-likelihoods for each of the specifications. The column headed 'EU only' ('RDEU only') shows the maximised log-likelihoods when it is assumed that all the subjects are EU (RDEU); and that headed 'Mixture Model' shows the maximised log-likelihoods when our Mixture Model, as specified by equation (16), is fitted to the data. Whether we assume that all the subjects are EU or all are RDEU, the CRRA specification clearly emerges as the better utility function. If we assume that all subjects are RDEU, then Quiggin is marginally better when combined with the CARA and marginally worse when combined with CRRA. However, the Vuong (1989) tests reported in Table 3, while showing that the CRRA/Quiggin specification is superior to the other specifications, do not show it to be significantly better than the other specifications.

However, and crucially for the purposes of this paper, the log-likelihoods in the table above show clearly that, for all specifications, the Mixture Model fits significantly better (at very small significance levels[11]) than either of the two preference functionals individually.

This is one of the crucial points of this paper, and we expand on it here – in relation to the CRRA/Quiggin specification. Table 1 shows that the Mixture Model fits the data significantly better than either of the two preference functionals individually. Hence, it follows that assuming that, in the population from which our subjects were drawn, agents are either all EU or all RDEU gives a distorted view of the truth. The mixing proportion, *p*, is estimated to be slightly above 0.8 – suggesting that 20% of the population are EU and 80% are RDEU. Figures 1 and 2 show the log-likelihood as a function of the mixing proportion and the peak is well-defined. A 95% confidence interval for *p* is (0.692, 0.913). We can use our results to calculate the posterior probabilities of each subject being either EU or RDEU, conditional on their 500 choices. Using Bayes rule we have the following posterior probabilities:

$$P(subject\ is\ EU \mid y_1 \cdots y_{500}) = \frac{(1-p)\int_{-\infty}^{\infty}\left\{\prod_{t=1}^{500}\left[(1-\omega)\Phi\left[y_t\left(d_{2t}u_2+d_{3t}u_3+d_{4t}\right)/\sigma\right]+\omega/2\right]\right\}f\left(r;\theta,\delta\right)dr}{L}$$

$$P(subject\ is\ RDEU \mid y_1 \cdots y_{500}) = \frac{p\int_{-\infty}^{\infty}\int_{\gamma_{\min}}^{\infty}\left\{\prod_{t=1}^{500}\left[(1-\omega)\Phi\left[y_t\left(D_{2t}u_2+D_{3t}u_3+D_{4t}\right)/\sigma\right]+\omega/2\right]\right\}g\left(r,\gamma;\Theta,\Delta,M,S,\rho\right)d\gamma dr}{L} \quad (18)$$

where *L* is as given in equation (16). The resulting histogram of the posterior probabilities is shown in Figure 3. Apart from one apparently very confused subject, that partition is close to perfect.

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[11] The log-likelihood test-statistics for the Mixture Model *v* EU are 988, 874, 961 and 694 (for CRRA/Quiggin, CRRA/Power, CARA/Quiggin and CARA/Power respectively; the critical value at 1% is 18.475 (7 degrees of freedom). The log-likelihood test-statistics for the Mixture Model *v* RDEU are 287, 143, 399 and 139 (for CRRA/Quiggin, CRRA/Power, CARA/Quiggin and CARA/Power respectively; the critical value at 1% is 13.277 (4 degrees of freedom).

In the Mixture Model (with the CRRA/Quiggin specification), the estimated parameters of the distribution of the *log* of the risk-aversion parameter show a mean and standard deviation of 0.76438 and 0.32431 for the EU subjects and a mean and standard deviation of -0.95425 and 0.53947 for the RDEU subjects. This implies a mean and standard deviation of the risk-aversion parameter, r, for the EU subjects of 0.491 and 0.027. It also implies a 95% confidence interval for the EU subjects for ln(r) given by (-1.400, -0.129) - implying a 95% confidence interval for r given by (0.247, 0.839). To interpret these figures it may be useful to note that, for a subject with a CRRA parameter of 0.247 (0.839) his or her certainty equivalent for a 50-50 gamble between £0 and £150 is £9.06 (£68.17). An equivalent calculation for the RDEU subjects shows a 95% confidence interval for r given by (0.134, 1.109) – with corresponding certainty equivalents for a 50-50 gamble between £0 and £150 given by (£0.85, £80.29). A small fraction of the RDEU subjects are risk-loving.

Again within the Mixture Model and the CRRA/Quiggin specification, our results show that the distribution of $\ln(\gamma - \gamma^*)$ has an estimated mean and standard deviation of -0.55465 and 0.24031 respectively. This implies that approximately 95% of the values of γ in the population lie between 0.637 and 1.199. The implied weighting functions at these two 'extremes' and the weighting function at the mean are plotted in Figure 4. It is interesting to note that this range of the possible weighting functions includes the (unique) function estimated by Tversky and Kahneman (1992). It can also be seen from Figure 1 that the RDEU estimates also include some subjects whose weighting function is close to that of the EU subjects (for whom $w(p) = p$). The proportion of the population who are therefore *strictly* RDEU is therefore somewhat less than the 80% implied by the estimate of the weighting parameter It is interesting to note that the correlation ρ between $\ln(r)$ and $\ln(\gamma - \gamma^*)$ is estimated to be 0.33793 (and is significantly different from zero). This implies that in general, the more risk-loving is a subject, the higher the value of the weighting parameter.

Finally we should comment on the within-subject errors. The estimates of σ (the standard deviation of the Fechnerian error) are 0.07438 and 0.03398 for the EU and RDEU subjects respectively. These have meaning with respect to the normalisation of the utility function – constrained to lie between 0 and 1 for the outcomes in the experiment. So, for example, a 50-50 gamble between £0 and £150, which has an expected utility of 0.5, is valuated by subjects to have an expected utility with mean 0.5 and standard deviation 0.07438 (0.03398) by the EU (RDEU) subjects. So the EU [RDEU] subjects evaluate it with 95% probability in the range (0.354, 0.646) [(0.433, 0.567)]. This error appears in line with previous estimates. The tremble probability is estimated to be 0.01139 – indicating a tremble of just over 1%.

As we have already noted, the CRRA/Quiggin specification appears superior to the others. Appendix Tables A3, A4 and A5 rather obviously show variations in the estimates obtained, particularly in the estimates of the mixing parameter. But all specifications show clearly that the Mixture Model fits the data better than either of the two models (assuming subjects are either all EU or all RDEU). To demonstrate this result is one of the main purposes of this paper.

6. Conclusions

This paper started from the observation that there is considerable heterogeneity in the behaviour of subjects in experiments[12]. This heterogeneity is both within subjects and between subjects. If we want to use the data to estimate the underlying preference functionals of subjects, we need to take this heterogeneity into account. Heterogeneity within subjects can be incorporated by appending some kind of error story into our analysis. This is already a common feature of empirical work in this area. As for heterogeneity between subjects, if we wish to save on degrees of freedom by pooling our data in some way, we cannot ignore this heterogeneity. This heterogeneity can be taken into account by modelling the parameters as being distributed within the population from

[12] The same is true in data obtained from other sources.

which our subjects are drawn. This kind of heterogeneity has already been considered in the literature (see Botti *et* al, 2007). Heterogeneity of preference functionals across individuals is more difficult to take into account, and this we do by using a Mixture Model (like and Harrison and Rutstrom, 2007) - we assume that different agents in the population have different functionals and we estimate the proportion of each type. This is the main contribution of the paper. We show that such a Mixture Model adds significantly to the explanatory power of our estimates. We thus present a method of using the data to take into account all forms of heterogeneity.

We have applied the Mixture Model to the problem of estimating preference functionals from a sample of 53 subjects for each of whom we have 500 observations. Our results show that it is misleading to assume a representative agent model – not all agents are EU and not all agents are RDEU – there is a mixture in the population. Moreover, there is significant heterogeneity in both the risk-aversion parameter and the weighting function parameter. And, of course, there is considerably heterogeneity of behaviour within subjects. Our estimations take all these forms of heterogeneity into account[13].

Acknowledgements

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[13] Clearly there is scope for further investigations. For example, one of the parameters that we have assumed the same for all members of the population, ω, might also be distributed over the population.

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Table 1: Estimates for CRRA Quiggin Specification

Parameter estimates. Maximum simulated likelihood. CRRA specification. Quiggin weighting function. (53 individuals, 500 observations each, standard errors in parentheses)				
			mixture model	
	EU only	*RDEU only*	*EU-type*	*RDEU-type*
θ (EU) Θ (RDEU)	-1.15599 (0.07217)	-0.95338 (0.02332)	-0.76438 (0.09492)	-0.95425 (0.01996)
δ (EU) Δ (RDEU)	0.55348 (0.03040)	0.54652 (0.01755)	0.32431 (0.06369)	0.53947 (0.01500)
M	-	-0.49629 (0.02043)	-	-0.55465 (0.03041)
S	-	0.24676 (0.01965)	-	0.24031 (0.01820)
ρ	-	0.25563 (0.10011)	-	0.33793 (0.08296)
ω	0.01375 (0.00173)	0.01534 (0.00184)	0.01139 (0.00157)	
σ	0.04823 (0.00089)	0.04134 (0.00089)	0.07438 (0.00297)	0.03398 (0.00081)
p	-	-	0.80266 (0.05623)	
Log-likelihood	-7210.27887	-6860.18750	-6716.49907	

Table 2: The Maximised Log-Likelihoods for the Different Specifications

	EU only	RDEU only	Mixture Model
CRRA/Quiggin	-7210.27887	-6860.18750	-6716.49907
CRRA/Power	-7210.27887	-6845.15422	-6773.37108
CARA/Quiggin	-7766.48390	-7485.00936	-7285.69205
CARA/Power	-7766.48390	-7488.98040	-7419.30892

Table 3: Vuong Tests Between the Various Specifications

Vuong Tests *H0: model1 and model2 are equally close to the true model* *H1: model1 is closer to the true model than model2*		
model1/model2	Voung_statistic	*p*-value
cara_quiggin/cara_power	0.16708	0.43365
crra quiggin /crra power	0.03012	0.48799
crra_quiggin/cara_quiggin	0.28884	0.38635
crra_quiggin/cara_power	0.32326	0.37325
crra power/cara quiggin	0.19832	0.42140
crra power/cara power	0.60596	0.27227
The Vuong-statistic is distributed N(0,1) under the null hypothesis – One-sided test		

Figure 1: Log-likelihood function (CRRA Quiggin)

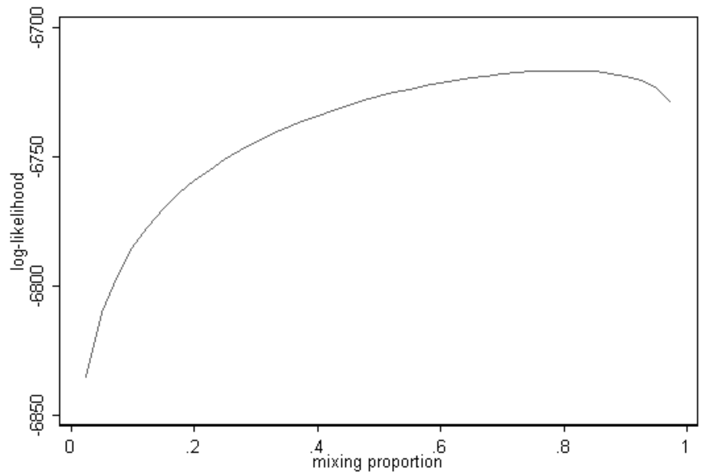


Figure 2: Log-likelihood function (CRRA Quiggin)

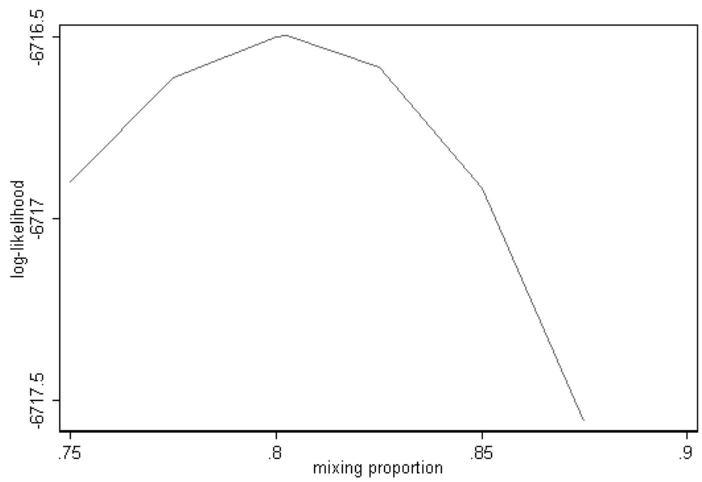


Figure 3: Posterior Probabilities (CRRA Quiggin)

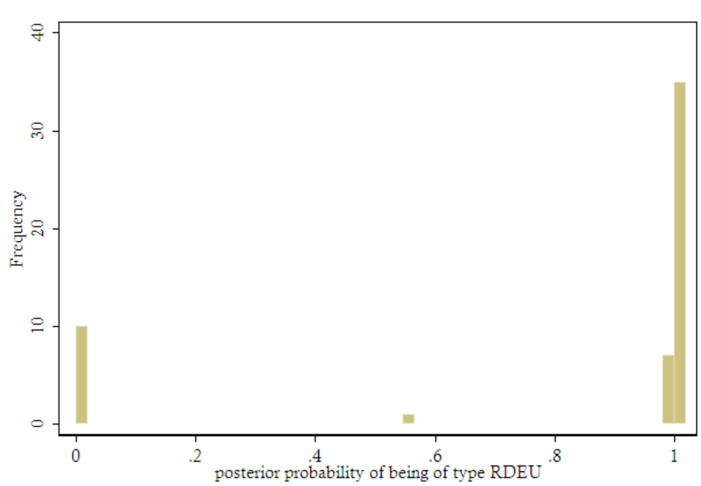
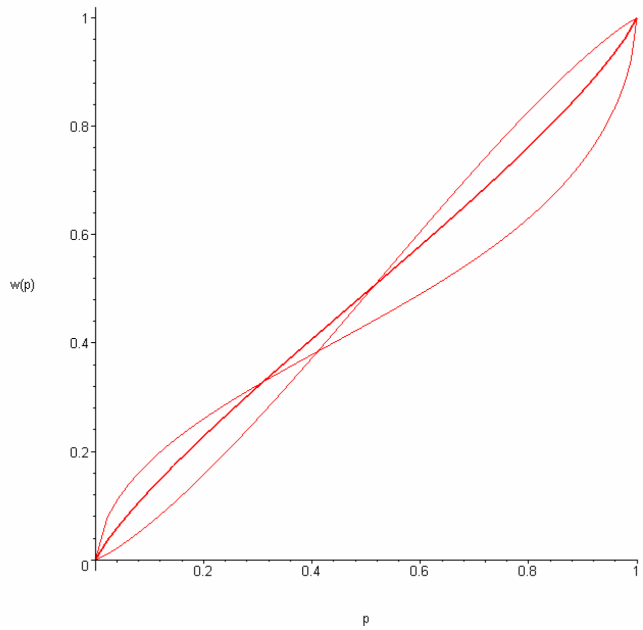


Figure 4: the mean and 95% bounds for the weighting function: Mixture Model CRRA/Quiggin Specification



Appendix

Table A1: The 100 choice problems[14]

t	q_1	q_2	q_3	q_4	p_1	p_2	p_3	p_4
1	.000	.000	.875	.125	.000	.125	.000	.875
2	.000	.000	.875	.125	.000	.125	.000	.875
3	.000	.000	.875	.125	.000	.125	.500	.375
4	.000	.000	.875	.125	.000	.375	.000	.625
5	.000	.000	.875	.125	.000	.375	.125	.500
6	.000	.000	.875	.125	.000	.375	.250	.375
7	.000	.000	.875	.125	.000	.625	.000	.375
8	.000	.125	.500	.375	.000	.375	.000	.625
9	.000	.125	.500	.375	.000	.375	.125	.500
10	.000	.125	.875	.000	.000	.375	.000	.625
11	.000	.125	.875	.000	.000	.375	.125	.500
12	.000	.125	.875	.000	.000	.375	.250	.375
13	.000	.125	.875	.000	.000	.375	.500	.125
14	.000	.125	.875	.000	.000	.625	.000	.375
15	.000	.125	.875	.000	.000	.875	.000	.125
16	.000	.250	.750	.000	.000	.375	.000	.625
17	.000	.250	.750	.000	.000	.375	.125	.500
18	.000	.250	.750	.000	.000	.375	.250	.375
19	.000	.250	.750	.000	.000	.375	.500	.125
20	.000	.250	.750	.000	.000	.375	.500	.125
21	.000	.250	.750	.000	.000	.625	.000	.375
22	.000	.250	.750	.000	.000	.875	.000	.125
23	.000	.375	.500	.125	.000	.625	.000	.375
24	.000	.125	.875	.000	.000	.250	.750	.000
25	.000	.375	.125	.500	.000	.375	.250	.375
26	.000	.000	.500	.500	.125	.000	.250	.625
27	.000	.000	.500	.500	.125	.000	.250	.625
28	.000	.000	.875	.125	.125	.000	.250	.625
29	.000	.000	.875	.125	.125	.000	.625	.250
30	.000	.000	.875	.125	.375	.000	.375	.250
31	.000	.000	.875	.125	.500	.000	.000	.500
32	.000	.000	.875	.125	.750	.000	.000	.250
33	.000	.000	1.000	.000	.125	.000	.250	.625
34	.000	.000	1.000	.000	.125	.000	.625	.250
35	.000	.000	1.000	.000	.375	.000	.375	.250
36	.000	.000	1.000	.000	.500	.000	.000	.500
37	.000	.000	1.000	.000	.750	.000	.000	.250
38	.000	.000	1.000	.000	.750	.000	.000	.250
39	.000	.000	1.000	.000	.750	.000	.125	.125
40	.125	.000	.625	.250	.500	.000	.000	.500
41	.250	.000	.750	.000	.375	.000	.375	.250
42	.250	.000	.750	.000	.500	.000	.000	.500
43	.250	.000	.750	.000	.750	.000	.000	.250
44	.250	.000	.750	.000	.750	.000	.125	.125
45	.375	.000	.375	.250	.500	.000	.000	.500
46	.375	.000	.625	.000	.500	.000	.000	.500
47	.375	.000	.625	.000	.750	.000	.000	.250
48	.375	.000	.625	.000	.750	.000	.125	.125
49	.250	.000	.750	.000	.375	.000	.625	.000

[14] Note that the questions were presented to the subjects in a random sequence with left and right randomly interchanged.

50	.750	.000	.000	.250	.750	.000	.125	.125
51	.000	.750	.000	.250	.250	.375	.000	.375
52	.000	.750	.000	.250	.375	.125	.000	.500
53	.000	.750	.000	.250	.625	.000	.000	.375
54	.000	.875	.000	.125	.250	.375	.000	.375
55	.000	.875	.000	.125	.375	.125	.000	.500
56	.000	.875	.000	.125	.500	.250	.000	.250
57	.000	.875	.000	.125	.625	.000	.000	.375
58	.000	.875	.000	.125	.625	.125	.000	.250
59	.125	.750	.000	.125	.250	.375	.000	.375
60	.125	.750	.000	.125	.375	.125	.000	.500
61	.125	.750	.000	.125	.500	.250	.000	.250
62	.125	.750	.000	.125	.625	.000	.000	.375
63	.125	.750	.000	.125	.625	.125	.000	.250
64	.125	.875	.000	.000	.250	.375	.000	.375
65	.125	.875	.000	.000	.375	.125	.000	.500
66	.125	.875	.000	.000	.500	.250	.000	.250
67	.125	.875	.000	.000	.625	.000	.000	.375
68	.125	.875	.000	.000	.625	.125	.000	.250
69	.125	.875	.000	.000	.750	.125	.000	.125
70	.125	.875	.000	.000	.875	.000	.000	.125
71	.125	.875	.000	.000	.875	.000	.000	.125
72	.250	.375	.000	.375	.375	.125	.000	.500
73	.500	.250	.000	.250	.625	.000	.000	.375
74	.500	.250	.000	.250	.625	.000	.000	.375
75	.000	.750	.000	.250	.125	.750	.000	.125
76	.000	.750	.250	.000	.125	.000	.875	.000
77	.000	.750	.250	.000	.125	.375	.500	.000
78	.000	.750	.250	.000	.375	.125	.500	.000
79	.000	.750	.250	.000	.375	.250	.375	.000
80	.000	.750	.250	.000	.500	.000	.500	.000
81	.000	.750	.250	.000	.500	.125	.375	.000
82	.000	1.000	.000	.000	.125	.000	.875	.000
83	.000	1.000	.000	.000	.125	.375	.500	.000
84	.000	1.000	.000	.000	.250	.625	.125	.000
85	.000	1.000	.000	.000	.375	.125	.500	.000
86	.000	1.000	.000	.000	.375	.250	.375	.000
87	.000	1.000	.000	.000	.500	.000	.500	.000
88	.000	1.000	.000	.000	.500	.000	.500	.000
89	.000	1.000	.000	.000	.500	.125	.375	.000
90	.000	1.000	.000	.000	.750	.125	.125	.000
91	.250	.625	.125	.000	.375	.125	.500	.000
92	.250	.625	.125	.000	.375	.250	.375	.000
93	.250	.625	.125	.000	.500	.000	.500	.000
94	.250	.625	.125	.000	.500	.125	.375	.000
95	.375	.250	.375	.000	.500	.000	.500	.000
96	.375	.250	.375	.000	.500	.000	.500	.000
97	.375	.625	.000	.000	.500	.000	.500	.000
98	.375	.625	.000	.000	.500	.125	.375	.000
99	.375	.625	.000	.000	.750	.125	.125	.000
100	.375	.125	.500	.000	.500	.125	.375	.000

Table A2: glossary of notation

variables	
x_i	i'th outcome
u_i	utility of i'th outcome = $u(x_i)$
$\mathbf{p_t}$	P-lottery on problem t
$\mathbf{q_t}$	Q- lottery on problem t
p_{it} (q_{it})	probability of outcome i in P- (Q-) lottery on problem number t
P_{it} (Q_{it})	modified probability of outcome i in P- (Q-) lottery number on problem t (see (5))
d_{it}	p_{it} - q_{it}
D_{it}	P_{it} - Q_{it}
y_t	decision on lottery t (1: P-lottery; -1: Q-lottery)
ε_t	measurement error on problem number t
r	risk-aversion parameter
γ	weighting-function parameter
functions	
EU(.)	Expected Utility function (see (1))
RDEU(.)	Rank Dependent Expected Utility function (see (4))
u(.)	utility function (see (8) and (9))
w(.)	weighting function (see (10))
$\Phi(.)$	unit normal cumulative density function
f(.)	probability density function of risk-aversion parameter (EU agents)
g(.)	joint probability density function of risk-aversion and (log of) weighting-function parameters (RDEU agents)
L(.)	likelihood function (see (16))
parameters	
θ	mean of the (marginal) distribution of the risk-aversion parameter (EU agents)
Θ	mean of the (marginal) distribution of the risk-aversion parameter (RDEU agents)
δ	standard deviation of the (marginal) distribution of the risk-aversion parameter (EU agents)
Δ	standard deviation of the (marginal) distribution of the risk-aversion parameter (RDEU agents)
M	mean of the (marginal) distribution of the (transformation of the) weighting-function parameter (RDEU agents)
S	standard deviation of the (marginal) distribution of the (transformation of the) weighting-function parameter (RDEU agents)
ρ	correlation between risk-aversion parameter and the (transformation of the) weighting-function parameter (RDEU agents)
ω	probability of a tremble
σ	standard deviation of the Fechnerian error
p	the mixing parameter (proportion of RDEU agents in population)

Table A3: Estimates for CRRA Power Specification

Parameter estimates. Maximum simulated likelihood. CRRA specification. Power weighting function. (53 individuals, 500 observations each, standard errors in parentheses)				
			mixture model	
	EU only	*RDEU only*	*EU-type*	*RDEU-type*
θ (EU) Θ (RDEU)	-1.15599 (0.07217)	-1.06849 (0.03109)	-0.94253 (0.07489)	-1.04301 (0.05346)
δ (EU) Δ (RDEU)	0.55348 (0.03040)	0.72066 (0.02313)	0.46515 (0.03238)	0.80237 (0.04294)
M	-	-0.02251 (0.02224)	-	-0.30098 (0.03833)
S	-	0.34156 (0.01314)	-	0.87514 (0.02717)
ρ	-	-0.21909 (0.05699)	-	-0.02117 (0.02247)
ω	0.01375 (0.00173)	0.01235 (0.00333)	0.01288 (0.00168)	
σ	0.04823 (0.00089)	0.04309 (0.00089)	0.05522 (0.00178)	0.03049 (0.00125)
p	-	-	0.58638 (0.07542)	
Log-likelihood	-7210.27887	-6845.15422	-6773.37108	

Table A4: Estimates for CARA Quiggin Specification

Parameter estimates. Maximum simulated likelihood. CARA specification. Quiggin weighting function. (53 individuals, 500 observations each, standard errors in parentheses)				
			mixture model	
	EU only	*RDEU only*	*EU-type*	*RDEU-type*
θ (EU) Θ (RDEU)	0.02440 (0.00146)	0.02548 (0.00108)	0.03618 (0.00499)	0.01693 (0.00034)
δ (EU) Δ (RDEU)	0.01905 (0.00142)	0.01915 (0.00088)	0.02180 (0.02769)	0.01492 (0.00039)
M	-	-0.46542 (0.02492)	-	-0.10515 (0.01987)
S	-	0.31122 (0.02058)	-	0.59426 (0.01988)
ρ	-	-0.55347 (0.05475)	-	0.10194 (0.03750)
ω	0.01272 (0.00483)	0.01172 (0.00177)	0.00724 (0.00146)	
σ	0.06142 (0.00110)	0.05609 (0.00109)	0.09635 (0.00326)	0.04778 (0.00099)
p	-	-	0.75248 (0.05998)	
Log-likelihood	-7766.48390	-7485.00936	-7285.69205	

Table A5: Estimates for CARA Power Specification

Parameter estimates. Maximum simulated likelihood. CARA specification. Power weighting function. (53 individuals, 500 observations each, standard errors in parentheses)				
			mixture model	
	EU only	*RDEU only*	*EU-type*	*RDEU-type*
θ (EU) Θ (RDEU)	0.02440 (0.00146)	0.02998 (0.00092)	0.01873 (0.00240)	0.03760 (0.00114)
δ (EU) Δ (RDEU)	0.01905 (0.00142)	0.02347 (0.00088)	0.00853 (0.00141)	0.02406 (0.00094)
M	-	-0.17251 (0.03479)	-	-0.29090 (0.04924)
S	-	0.36198 (0.01542)	-	0.50443 (0.02153)
ρ	-	-0.25988 (0.08560)	-	-0.05176 (0.08508)
ω	0.01272 (0.00483)	0.01033 (0.00007)	0.00873 (0.00146)	
σ	0.06142 (0.00110)	0.05183 (0.00122)	0.07657 (0.00268)	0.04257 (0.00170)
p	-	-	0.69953 (0.07494)	
Log-likelihood	-7766.48390	-7488.98040	-7419.30892	

Figure A1: The log-likelihoods and the posterior probabilities for the CARA and CRRA/Power Specifications