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## Discussion Papers in Economics

## No. 2006/12

Equity Valuation Under Stochastic Interest Rates
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## STOCHASTIC INTEREST RATES

## 15/5/06


#### Abstract

This paper presents an equity valuation model that employs riskneutral valuation under stochastic interest rates along the lines of Ohlson and Feltham (1999). Closed form valuation formulae for equities are presented in a discrete time setting whereby the short term interest rate is modelled by a quadratic term structure model. Earnings are driven by mean reverting return on equity (ROE). The term structure of interest rates, and in particular the variance of the future short rates, is found to be a primary determinant of equity value that has been largely overlooked by the previous equity valuation literature. Equity value decreases in the correlation between the short interest rate and ROE and can be very sensitive to such correlation when the ROE process is very persistent. This suggets that equity value decreases in the degree of pro-cyclicality of the firm's profitability.


Key words: equity valuation, residual income valuation, stochastic in-
terest rates, quadratic term structure model in discrete time, mean revert-
ing return on equity.
JEL classification: G12; G13; M41.

## 1 Introduction

This paper presents new equity valuation formulae that accommodate stochastic interest rates. The focus on stochastic interest rates is due to the fact that the past literature has largely overlooked the modelling of stochastic interest rates in equity valuation, apart from the two important exceptions of Ohlson and Feltham (1999) and Gode and Ohlson (2004). Yet in principle equities are perpetuities and as such they should be very sensitive to the dynamics of the term structure of interest rates. This is confirmed by the well documented reactions of equity markets to changes or to perceived changes in monetary policy or in inflation expectations. Hence this paper explores the impact on equity valuation of stochastic interest rates and of the correlation between future profitability and interest rates.

New equity pricing formulae are presented that are based on risk-neutral valuation along the lines of Ohlson and Feltham (1999). The model is in discrete time and arbitrage-free. Earnings are driven by return on equity (ROE) and ROE follows a mean reverting process in keeping with empirical evidence provided by Fama and French (2000) and Penman and Nissim (2001) among others.

The model shows how the variance of the future short interest rate and the
correlation between the short interest rate and earnings are primary determinants of equity value. Equity value decreases in the correlation between interest rates and ROE and can be very sensitive to such correlation especially when the short rate and ROE are highly persistent. This result implies that equity value decreases in the degree of pro-cyclicality of the firm's profitability, at least under the assumption that peaks of the business cycle are associated with higher short interest rate levels and with higher profitability of cyclical stocks.

The paper is organised as follows. The next section reviews the relevant literature. Section 3 introduces the equity valuation framework of Ohlson and Feltham (1999). Section 4 presents the equity valuation formulae under stochastic interest rates. Section 5 explores the predictions of the formulae. Then the conclusions follow.

## 2 Literature

This paper intends to contribute to the equity valuation literature, which has developed significantly over the past decade. Notable developments are the linear information models proposed in Ohlson (1995), Feltham and Ohlson (1995 and 1996) and Myers (1999). Unfortunately the literature on equity valuation has paid little attention to the relation between equity valuation and the term structure of interest rates and this is the issue addressed in this paper. This paper is close in spirit to Feltham and Ohlson (1999), who also address this issue. Feltham and Ohlson (1999) show that the dividend valuation relation is
equivalent to the accounting valuation relation even when risk-neutral valuation is employed and interest rates are stochastic. They also show how risk-neutral valuation still implies the irrelevancy of the dividend policy for equity valuation. Their equity valuation model is more realistic and accurate than existing models, which most often discount all future cash flows at the same risk-adjusted discount rate. Feltham and Ohlson's model seems a worthwhile improvement given that equities are perpetuities and thus their value is potentially very sensitive to changes in discount rates. But Feltham and Ohlson stop short of specifying a dynamic term structure model to be embedded in the equity valuation model. Hence the contribution of this paper is in taking this further step. A realistic default free term structure model is embedded in the equity valuation model.

The embedded term structure model is a quadratic one in discrete time. A discrete time term structure model is employed because the firm's earnings process is in discrete time, in keeping with the equity valuation literature and with the periodic disclosure of earnings. The specific quadratic model we employ is the discrete time quadratic model of Realdon (2006), who provides the discrete time version of the general continuous time quadratic term structure models of Ahn-Dittmar-Gallant (2002). The choice of a quadratic model is motivated by the recent empirical success of such models as reported by Ahn-Dittmar-Gallant (2002) and Lieppold and Wu (2003) in reproducing empirical regularities of the observed term structures of interest rates. Ahn-Dittmar-Gallant show how quadratic models offer advantages over affine term structure models.

## 3 Equity risk-neutral valuation and accounting value relation

The equity valuation model later presented builds on Feltham and Ohlson's (1999) risk-neutral valuation and accounting value relation. Hence here we introduce risk-neutral valuation of equity along the lines of Feltham and Ohlson (1999). Let us denote the market value of equity at time $t$ with $V_{t}$, the book value of equity at $t$ with $B o_{t}$, the constant required return from investing in equity with $\pi$. Let $n$ denote a number of time periods of equal length. For simplicity let $\Delta=1$ year denote the length of one period. The earnings produced over the period $] t+n-1, t+n]$ are disclosed at time $t+n$ and are denoted with $e a_{t+n}$. Then the accounting value relation used in residual income equity valuation is

$$
\begin{equation*}
V_{t}=B o_{t}+E_{t}\left[\sum_{n=1}^{\infty} \frac{e a_{t+n}-\pi B o_{t+n-1}}{(1+\pi)^{n}}\right] \tag{1}
\end{equation*}
$$

where $E_{t}[.$.$] is the time t$ conditional expectation operator in the real probability measure. Thus equity fair value is equal to equity book value plus the present value of abnormal earnings discounted at the constant opportunity cost of capital. This formula and its variants are the war-horse of the residual income valuation literature and have the merit to anchor equity valuation to the observable book value of equity. Moreover the formula implies that equity value does not depend on the dividend policy, even though it typically gives the same estimate of value as discounting expected dividends. A drawback of the formula is that the risk-adjusted discount rate $\pi$ is assumed constant over time and
needs estimating. Feltham and Ohlson (1999) showed that the accounting value relation is still valid even if we employ risk-neutral valuation, in which case the fair value of equity becomes

$$
\begin{equation*}
V_{t}=B o_{t}+E_{t}^{*}\left[\sum_{n=1}^{\infty} e^{\sum_{i=1}^{n} r_{t+i-1}}\left(e a_{t+n}-r_{t+n-1} \cdot B o_{t+n-1}\right)\right] \tag{2}
\end{equation*}
$$

where $E_{t}^{*}[.$.$] denotes conditional expectation in the risk-neutral probability mea-$ sure rather than in the real measure. $r_{t}$ is the one period default-free interest rate at time $t$ for the period $] t, t+1]$. Formula 2 allows the short interest rate $r_{t}$ to change over time, which is realistic and a major improvement over valuations that discount all cash flows using the same discount rate. Formula 2 preserves the important benefits of formula 1. As explained in Feltham and Ohlson (1999), the risk-neutral version of the accounting value relation preserves the advantage that equity value $V_{t}$ is not affected by changes in the dividend policy, i.e. it preserves dividend irrelevance. Moreover formula 2 gives the same equity value as discounting risk-neutral expected dividends, even under the assumption of stochastic interest rates.

In formula 2 the problem of estimating the risk-adjusted discount rate $\pi$ is substituted by the new problem of estimating the risk-neutral dynamics of earnings $e a_{t}$. Moreover the risk-neutral dynamics of the default free short interest rate $r_{t}$ are also needed in order to compute the risk-neutral expectation. Thus the equity valuation model that now follows is a tractable special case of
formula 2 that assumes a realistic specification of the processes of earnings and interest rates.

## 4 Equity valuation model

This section presents a discrete time arbitrage-free model for equity risk-neutral valuation using accounting value relation 2 . The model assumes that return on equity (ROE) follows an auto-regressive mean-reverting process correlated with stochastic interest rates. The ROE process is consistent with linear information models appeared in the equity valuation literature such as in Ohlson (1995), Feltham and Ohlson (1995), Myers (1999). Also the empirical literature provides support for the assumption that ROE typically follows a mean reverting process. A text-book explanation along these lines is given in chapter 7 of Palepu, Healy and Bernard (2003). But the major innovation of the model is the realistic modelling of interest rates in keeping with advances in the term structure literature. This seems topical since equities are perpetuities in nature and hence can be expected to be very sensitive to the dynamics of interest rates.

### 4.1 Assumptions

Let $R_{t+n+1}$ denote the continuously compounded return on the book value of equity (ROE) during the period $] t+n, t+n+1]$. It follows that

$$
\begin{equation*}
e a_{t+n+1}=B o_{t+n} \cdot\left(e^{R_{t+n+1}}-1\right) \tag{3}
\end{equation*}
$$

where again $e a_{t+n+1}$ denotes the earnings and $B o_{t+n}$ denotes the book value of equity. As a matter of convenience we assume that earnings $e a_{t+n+1}$ are entirely received by the firm at time $t+n+1$. Notice that, if we assume $\Delta=1$ year, as we observe $e a_{t+1}$ from yearly disclosure, we can immediately infer the corresponding continuously compounded ROE since

$$
\begin{equation*}
R_{t+1}=\ln \left(\frac{e a_{t+1}}{B o_{t}}+1\right) . \tag{4}
\end{equation*}
$$

This is an important advantage in model estimation, which we cannot retain if we assume that time periods are of length $\Delta<1$ year; in the latter case we cannot immediately infer $R_{t+n+1}$, but $R_{t+n+1}$ could still be implied by the current stock price at time $t+n+1$ much in the same way as volatility is implied by option prices. Of course this is only possible if we can express stock prices as functions of current ROE $R_{t}$. This can be done using the valuation model proposed below. Alternatively $R_{t}$ could be implied by earnings forecasts, if again we can express earnings forecasts as functions of $R_{t}$. Again this can be done since, as shown below, we can express expected future ROE and future earnings as functions of current ROE $R_{t}$.

We now assume that the risk-neutral process for ROE is such that

$$
\begin{equation*}
R_{t+1}=R_{t}(1-\theta)+\theta m+v \rho \cdot \xi_{t+1}+v \sqrt{1-\rho^{2}} \varsigma_{t+1} \tag{5}
\end{equation*}
$$

where $\theta, m$ and $v$ are scalar constants, $\varsigma_{t+1} \backsim N(0,1), \xi_{t+1} \backsim N(0,1)$ and $N(0,1)$ denotes the normal density with mean 0 and variance 1. $\xi_{t+1}$ and $\varsigma_{t+1}$
are independent. $\rho$ is the conditional correlation between ROE $R_{t+1}$ and $x_{t+1}$, where $x$ drives the the short term interest rate as we are going to see below. In other words $\rho$ is the correlation between $x_{t}-x_{t-1}$ and $R_{t}-R_{t-1}$.

We assume that interest rates are stochastic. $P_{n, t}$ is the value of a discount bond at time $t$ with maturity date at time $t+n . P_{n, t}$ is determined by a quadratic discrete time default free bond pricing model in discrete time along the lines of Realdon (2006). This choice is inspired by the recent successes of quadratic term structure models in reproducing the dynamics of the term structure of interest rates. The empirical performance of quadratic models seems superior to that of linear models. Moreover a quadratic model has the advantage of permitting us to obtain closed form equity valuation formulae even if $r_{t}$ and $R_{t}$ are correlated, and such correlation can have a material impact on equity valuation as we are going to see. Instead, if for example we assumed that $r_{t}$ were driven by the discrete time equivalent of the popular Cox-Ingersoll-Ross (1985) model as in Campbell-Lo-McKinlay (chapter 11, 1997), correlation between $r_{t}$ and $R_{t}$ would spoil model tractability and the $r_{t}$ process would not even be well defined since it could turn negative.

As in all quadratic models, $r_{t}$ is driven by the latent factor $x_{t}$. We can summarise the set of assumptions concerning interest rates as

$$
\begin{align*}
& r_{t}=\alpha+\beta x_{t}+\Psi x_{t}^{2}  \tag{6}\\
& x_{t+1}=(1-\phi) x_{t}+\phi \mu+\Sigma \xi_{t+1}  \tag{7}\\
& \xi_{t+1} \backsim N(0,1)  \tag{8}\\
& P_{n, t}=e^{A_{n}^{*}+B_{n}^{*} x_{t}+C_{n}^{*} x_{t}^{2}} \tag{9}
\end{align*}
$$

where $\alpha, \beta, \Psi, \phi, \mu, \Sigma$ are scalar constants and $A_{n}^{*}, B_{n}^{*}, C_{n}^{*}$ are scalar functions of $n$, where $n$ is the number of time periods until the bond maturity date. The random variable $\xi_{t+1}$ is again distributed according to the normal density $N(0,1)$. The above auto-regressive process for $x_{t}$ is the risk-neutral process rather than the real process.

### 4.2 Closed form solution

First we can solve for the discount bond price in closed form. In fact risk-neutral valuation and the above assumptions imply that

$$
\begin{align*}
& P_{n, t}=E_{t}^{*}\left[e^{-r_{t}} \cdot P_{n-1, t+1}\right]  \tag{10}\\
& e^{A_{n}^{*}+B_{n}^{*} x_{t}+C_{n}^{*} x_{t}^{2}}=E_{t}^{*}\left[e^{-\left(\alpha+\beta x_{t}+\Psi x_{t}^{2}\right)} \cdot e^{A_{n-1}^{*}+B_{n-1}^{*} x_{t+1}+C_{n-1}^{*} x_{t+1}^{2}}\right]
\end{align*}
$$

subject to the terminal conditions

$$
\begin{equation*}
A_{0}^{*}=B_{0}^{*}=C_{0}^{*}=0 \tag{11}
\end{equation*}
$$

Realdon (2006) shows that the solution to this recursive equation implies that

$$
\begin{align*}
A_{n}^{*} & =-\alpha+A_{n-1}+\phi \mu B_{n-1}+C_{n-1} \phi^{2} \mu^{2}  \tag{12}\\
& -\ln \Sigma-\frac{1}{2} \ln \left(\frac{1}{\Sigma^{2}}-2 C_{n-1}\right)+\frac{\Sigma^{2}\left(B_{n-1}+2 C_{n-1} \phi \mu\right)^{2}}{\left(2-4 C_{n-1} \Sigma^{2}\right)} \\
B_{n}^{*} & =-\beta+(1-\phi) B_{n-1}+2(1-\phi) \phi \mu C_{n-1}  \tag{13}\\
& +\frac{2 C_{n-1}(1-\phi) \Sigma^{2}\left(B_{n-1}+2 C_{n-1} \phi \mu\right)}{\left(1-2 C_{n-1} \Sigma^{2}\right)} \\
C_{n}^{*} & =-\Psi+(1-\phi)^{2} C_{n-1}+\frac{\Sigma^{2}\left(2 C_{n-1}(1-\phi)\right)^{2}}{\left(2-4 C_{n-1} \Sigma^{2}\right)} . \tag{14}
\end{align*}
$$

Then the solution for the discount bond yield for maturity $n$ is

$$
\begin{equation*}
y_{n, t}=\frac{-\ln P_{n, t}}{n}=\frac{-A_{n}^{*}-B_{n}^{*} x_{t}-C_{n}^{*} x_{t}^{2}}{n} . \tag{15}
\end{equation*}
$$

Notice also that

$$
\begin{align*}
& A_{1}^{*}=-\alpha, B_{1}^{*}=-\beta, C_{1}^{*}=-\Psi  \tag{16}\\
& y_{1, t}=-\ln P_{1, t}=\alpha+\beta x_{t}+\Psi x_{t}^{2}=r_{t} \tag{17}
\end{align*}
$$

This is a one factor quadratic term structure model. The model can be extended to multiple correlated factors as shown in Realdon (2006), but we abstract from
this complication. The model can easily accommodate time dependent parameters and can be calibrated to the yield curve observed at time $t$. Notice that $x_{t}$ cannot be observed, but it can be treated as a parameter in the calibration to the cross section yields at a given time $t$.

In the light of these results and of the previous assumptions, we can solve the accounting value relation of equation 2. To this end define the residual earnings generated over the accounting period $] t+n, t+n+1]$ as

$$
\begin{equation*}
\left(e^{R_{t+n+1}} P_{1, t+n}-1\right) \cdot B o_{t+n} . \tag{18}
\end{equation*}
$$

Then we can re-write equation 2 as

$$
\begin{equation*}
V_{t}=B o_{t}+E_{t}^{*}\left[\sum_{n=0}^{\infty} P_{n+1, t} \cdot\left(e^{R_{t+n+1}} P_{1, t+n}-1\right) \cdot B o_{t+n}\right] \tag{19}
\end{equation*}
$$

The solution to this equation depends on the process we assume for equity book value $B o_{t+n}$. We assume that the future book value of equity does not change from its current value $B o_{t}$, i.e.

$$
\begin{equation*}
B o_{t+n}=B o_{t} \tag{20}
\end{equation*}
$$

for all $n>0$. For equity valuation purposes this assumption is not as restrictive as it may first appear and it can be justified as follows. $B o_{t+n}$ can be assumed constant if:

- all positive earnings after $t$ are either immediately distributed as they are
reported or are invested in zero-net-present-value investments in money market securities;
- all negative earnings (losses) are either covered by immediate funds injections of equity holders as the losses are reported or covered by reducing zero-net-present-value investments in money market securities or covered by zero-net-present-value issuance of new debt;
- all dividends are financed by either reducing zero-net-present-value investments in money market securities or by zero-net-present-value issuance of new debt.

These arguments imply the irrelevance to equity valuation of the dividend policy and of changes in the book value of equity that are due to positive or negative earnings, since such changes are offset by zero-net-present-value transactions that keep the book value of equity constant over time. In other words assumption 20 is valid for valuation purposes even when $B o_{t+n} \neq B o_{t}$. Similar arguments highlighting the irrelevance of the dividend policy are offered also in Ohlson and Feltham (1999) for the general case of equation 2 in the presence of stochastic interest rates.

Assumption 20 is particularly tractable, but it can to some extent be relaxed without losing closed form solutions for equity value. For example the book value of equity $B o_{t+n}$ may become a deterministic function of time. Again it does not matter if future equity book values turn out different from the values $B o_{t+n}$ used in the valuation, provided zero-net-present value transactions could to be put in place to adjust equity book value accordingly. Another tractable
alternative is to assume that all earnings are re-invested in the firm to earn the same return as other assets and all losses are covered by liquidating assets that earn the same return as other assets. The equity valuation formulae that follow can be easily adapted also to this case.

Hereafter we proceed by retaining assumption 20. We denote with $V_{1, t}$ the present value at $t$ of residual earnings to be paid at $t+1$. Residual earnings are defined in equation 18. Then risk-neutral valuation tells us that $V_{1, t}$ is

$$
\begin{align*}
V_{1, t} & =E_{t}^{*}\left[e^{-r_{t}} \cdot V_{0, t+1}\right]  \tag{21}\\
& =E_{t}^{*}\left[e^{-r_{t}} \cdot B o_{t} \cdot\left(e^{R_{t+1}-r_{t}}-1\right)\right]  \tag{22}\\
& =B o_{t} \cdot P_{1, t} \cdot P_{1, t} \cdot E_{t}^{*}\left(e^{R_{t+1}}\right)-B o_{t} \cdot P_{1, t} \tag{23}
\end{align*}
$$

where $E_{t}^{*}[.$.$] denotes conditional expectation in the risk-neutral probability mea-$ sure and where

$$
\begin{align*}
& V_{0, t+1}=B o_{t} \cdot\left(e^{R_{t+1}-r_{t}}-1\right)  \tag{24}\\
& E_{t}^{*}\left(e^{R_{t+1}}\right)=E_{t}^{*}\left[e^{(1-\theta) R_{t}+\theta m+v \xi_{t+1}}\right]=e^{D_{1}+F_{1} R_{t}}  \tag{25}\\
& D_{1}=\left(\theta m+\frac{v^{2}}{2}\right), F_{1}=(1-\theta)  \tag{26}\\
& D_{0}=0, F_{0}=1 \tag{27}
\end{align*}
$$

More generally risk-neutral valuation implies the recursive equation

$$
\begin{equation*}
V_{n, t}=E_{t}^{*}\left[e^{-r_{t}} \cdot V_{n-1, t+1}\right] \tag{28}
\end{equation*}
$$

The solution to this recursive equation can be shown to be

$$
\begin{equation*}
V_{n, t}=B o_{t} \cdot e^{A_{n}+B_{n} x_{t}+C_{n} x_{t}^{2}+D_{n}+F_{n} R_{t}}-B o_{t} \cdot P_{n, t} \tag{29}
\end{equation*}
$$

Substituting this solution into equation 28 entails that

$$
\begin{equation*}
e^{A_{n}+B_{n} x_{t}+C_{n} x_{t}^{2}+D_{n}+F_{n} R_{t}}=E_{t}^{*}\left[e^{-r_{t}} \cdot e^{A_{n-1}+B_{n-1} x_{t+1}+C_{n-1} x_{t+1}^{2}+D_{n-1}+F_{n-1} R_{t+1}}\right] \tag{30}
\end{equation*}
$$

subject to terminal conditions 27 and $A_{0}=B_{0}=C_{0}=0$. Now we switch to matrix notation and define

$$
\begin{align*}
& e^{\mathbf{A}_{n}+\mathbf{B}_{n}^{\prime} \mathbf{x}_{t}+\mathbf{x}_{t}^{\prime} \mathbf{C}_{n} \mathbf{x}_{t}}=e^{A_{n}+B_{n} x_{t}+C_{n} x_{t}^{2}+D_{n}+F_{n} R_{t}}  \tag{31}\\
& \mathbf{r}_{t}=\alpha+\boldsymbol{\beta}^{\prime} \mathbf{x}_{t}+\mathbf{x}_{t}^{\prime} \boldsymbol{\Psi} \mathbf{x}_{t}  \tag{32}\\
& \mathbf{x}_{t+1}=(\mathbf{I}-\boldsymbol{\phi}) \mathbf{x}_{t}+\boldsymbol{\phi} \boldsymbol{\mu}+\boldsymbol{\Sigma} \boldsymbol{\xi}_{t+1}  \tag{33}\\
& \boldsymbol{\xi}_{t+1}^{\prime}=\left[\xi_{t+1}, \varsigma_{t+1}\right] \tag{34}
\end{align*}
$$

where $\mathbf{I}$ is the $2 \times 2$ identity matrix and

$$
\begin{align*}
& \mathbf{A}_{n}=A_{n}+D_{n}  \tag{35}\\
& \mathbf{B}_{n}^{\prime}=\left[B_{n}, F_{n}\right]  \tag{36}\\
& \mathbf{C}_{n}=\left[\begin{array}{ll}
C_{n} & 0 \\
0 & 0
\end{array}\right]  \tag{37}\\
& \mathbf{x}_{t}^{\prime}=\left[x_{t}, R_{t}\right]  \tag{38}\\
& \boldsymbol{\beta}^{\prime}=[\beta, 0]  \tag{39}\\
& \boldsymbol{\Psi}=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]  \tag{40}\\
& \boldsymbol{\phi}=\left[\begin{array}{ll}
\phi & 0 \\
0 & \theta
\end{array}\right]  \tag{41}\\
& \boldsymbol{\mu}^{\prime}=[\mu, m]  \tag{42}\\
& \boldsymbol{\Sigma}=\left[\begin{array}{ll}
\Sigma \\
v \rho & v \sqrt{1-\rho^{2}}
\end{array}\right]  \tag{43}\\
& \boldsymbol{\xi}_{t+1}^{\prime}=\left[\begin{array}{ll}
\left.\xi_{t+1}, \varsigma_{t+1}\right] .
\end{array}\right. \tag{44}
\end{align*}
$$

The terminal conditions can now be re-expressed as

$$
\begin{align*}
& \mathbf{A}_{1}=2 A_{1}^{*}+D_{1}  \tag{45}\\
& \mathbf{B}_{1}^{\prime}=\left[2 B_{1}^{*}, F_{1}\right]  \tag{46}\\
& \mathbf{C}_{1}=\left[\begin{array}{cc}
2 C_{1}^{*} & 0 \\
0 & 0
\end{array}\right] . \tag{47}
\end{align*}
$$

From Realdon (2006) we know that the solutions for $\mathbf{A}_{n}, \mathbf{B}_{n}$ and $\mathbf{C}_{n}$ are

$$
\begin{align*}
\mathbf{A}_{n} & =-\alpha+\mathbf{A}_{n-1}+\mathbf{B}_{n-1}^{\prime} \boldsymbol{\phi} \boldsymbol{\mu}+(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \boldsymbol{\phi} \boldsymbol{\mu}+\ln \frac{|\gamma|}{a b s|\boldsymbol{\Sigma}|}  \tag{48}\\
& +\frac{1}{2} \sum_{i=1}^{N}\binom{\mathbf{B}_{n-1}^{\prime} \gamma_{i} \mathbf{B}_{n-1}^{\prime} \gamma_{i}+\mathbf{B}_{n-1}^{\prime} \gamma_{i} 2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i}}{+2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i} \mathbf{B}_{n-1}^{\prime} \gamma_{i}+2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i} 2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i}}
\end{align*}
$$

$$
\begin{aligned}
& \mathbf{B}_{n}^{\prime}=-\boldsymbol{\beta}^{\prime}+\mathbf{B}_{n-1}^{\prime}(1-\boldsymbol{\phi})+2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1}(I-\boldsymbol{\phi}) \\
& \quad+\sum_{i=1}^{N}\binom{\mathbf{B}_{n-1}^{\prime} \gamma_{i}\left(\mathbf{C}_{n-1} \gamma_{i}\right)^{\prime}(I-\boldsymbol{\phi})+\mathbf{B}_{n-1}^{\prime} \gamma_{i} \gamma_{i}^{\prime} \mathbf{C}_{n-1}(I-\boldsymbol{\phi})}{+2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i} \gamma_{i}^{\prime} \mathbf{C}_{n-1}(I-\boldsymbol{\phi})+2(\boldsymbol{\phi} \boldsymbol{\mu})^{\prime} \mathbf{C}_{n-1} \gamma_{i} \gamma_{i}^{\prime} \mathbf{C}_{n-1}(I-\boldsymbol{\phi})} \\
& \mathbf{C}_{n}=-\boldsymbol{\Psi}+(I-\boldsymbol{\phi})^{\prime} \mathbf{C}_{n-1}(I-\boldsymbol{\phi})+2 \sum_{i=1}^{N}(I-\boldsymbol{\phi})^{\prime} \mathbf{C}_{n-1} \gamma_{i} \gamma_{i}^{\prime} \mathbf{C}_{n-1}^{\prime}(I-\boldsymbol{\phi})
\end{aligned}
$$

with $\gamma_{i}$ being the $i$-th column of the $N \times N$ matrix $\gamma=\left(\left(\boldsymbol{\Sigma} \boldsymbol{\Sigma}^{\prime}\right)^{-1}-2 \mathbf{C}_{n-1}\right)^{-1 / 2}$.
$|\gamma|$ is the determinant of $\boldsymbol{\Sigma} . a b s|\boldsymbol{\Sigma}|$ is the absolute value of the determinant of $\boldsymbol{\Sigma}$. Then we can state the value of any claim to future earnings as

$$
\begin{equation*}
V_{n, t}=B o_{t} \cdot\left(e^{\mathbf{A}_{n}+\mathbf{B}_{n}^{\prime} \mathbf{x}_{t}+\mathbf{x}_{t}^{\prime} \mathbf{C}_{n} \mathbf{x}_{t}}-P_{n, t}\right) . \tag{51}
\end{equation*}
$$

At this point we have a closed form solution for equity value $V_{t}$. In fact equity value is

$$
\begin{equation*}
V_{t}=B o_{t}+\lim _{N \rightarrow \infty} \sum_{n=1}^{N} V_{n, t} \tag{52}
\end{equation*}
$$

$N$ is the time horizon over which future residual earnings are supposed to differ from 0 . This model can easily accommodate time-dependent parameters. Time dependent parameters are relevant if the quadratic term structure model is calibrated to observed interest rates or if earnings forecasts introduce time dependent parameters in the ROE process.

A theoretical drawback of the model is that the stock price can become negative, but for profitable firms the actual chance of that happening usually seems negligible under realistic parameters, as is also implied by the simulations illustrated in the following.

### 4.3 Independent interest rates and earnings

If earnings are independent of interest rates, i.e. if $\rho=0$, the equity valuation formula simplifies considerably. In fact the formula for any claim $V_{n, t}$ on future earnings becomes

$$
\begin{equation*}
V_{n, t}=B o_{t} \cdot\left(P_{n, t}^{\prime} \cdot e^{D_{n}^{\prime}+F_{n}^{\prime} R_{t}}-P_{n, t}\right) \tag{53}
\end{equation*}
$$

where $P_{n, t}^{\prime}=e^{A_{n}^{\prime}+B_{n}^{\prime} x_{t}+C_{n}^{\prime} x_{t}^{2}} \cdot A_{n}^{\prime}, B_{n}^{\prime}, C_{n}^{\prime}$ obey the same recursion equations as $A_{n}^{*}, B_{n}^{*}, C_{n}^{*}$ but for the different terminal conditions $A_{1}^{\prime}=-2 \alpha, B_{1}^{\prime}=-2 \beta$, $C_{1}^{\prime}=-2 \Psi \cdot e^{D_{n}^{\prime}+F_{n}^{\prime} R_{t}}$ must satisfy the recursion

$$
\begin{equation*}
e^{D_{n}^{\prime}+F_{n}^{\prime} R_{t}}=E_{t}^{*}\left[e^{D_{n-1}^{\prime}+F_{n-1}^{\prime} R_{t+1}}\right]=e^{D_{n-1}^{\prime}+F_{n-1}^{\prime}\left((1-\theta) R_{t}+\theta m\right)+\frac{\left(F_{n-1}^{\prime} v\right)^{2}}{2}} \tag{54}
\end{equation*}
$$

Again $E_{t}^{*}(.$.$) denotes conditional time t$ expectation in the risk-neutral measure.
The solution to this last equation is

$$
\begin{equation*}
D_{n}^{\prime}+F_{n}^{\prime} R_{t}=D_{n-1}^{\prime}+F_{n-1}^{\prime}\left((1-\theta) R_{t}+\theta m\right)+\frac{\left(F_{n-1}^{\prime} v\right)^{2}}{2} \tag{55}
\end{equation*}
$$

which implies the recursion

$$
\begin{aligned}
& D_{n}^{\prime}=D_{n-1}^{\prime}+F_{n-1}^{\prime} \theta m+\frac{\left(F_{n-1}^{\prime} v\right)^{2}}{2} \\
& F_{n}^{\prime}=F_{n-1}^{\prime}(1-\theta)
\end{aligned}
$$

subject to the terminal conditions

$$
\begin{align*}
D_{1}^{\prime} & =\left(\theta m+\frac{v^{2}}{2}\right)  \tag{56}\\
F_{1}^{\prime} & =(1-\theta) \tag{57}
\end{align*}
$$

### 4.4 Re-interpreting the model

The model could be re-interpreted under the assumption that $R_{t}$ is the return on net operating assets (RNOA) rather than the return on equity ROE. Then $B o_{t}$ would be the book value of net operating assets (NOA) and $V_{t}$ the market value of NOA. Again $B o_{t}$ could be assumed constant for valuation purposes, or to change deterministically over time, or to change because of retained earnings growing the operating assets and of losses eroding the operating assets. Assuming that the book value of NOA is constant could again be justified by assuming that losses are covered by zero-net-present-value issuance of new debt, profits are invested in zero-net-present-value investments in money market securities, dividends are financed by either reducing zero-net-present-value investments in money market securities or by zero-net-present-value issuance of new debt. Of course to value equity in this case the market value of currently outstanding net financial obligations (NFO) would have to be estimated and subtracted from the estimated market value of NOA.

## 5 Model predictions

The above equity valuation model provides theoretical insights into the determinants of equity value. To clarify the exposition we proceed incrementally in illustrating the model predictions. First we assume that the interest rate term structure is flat and constant and assess the effect of the ROE process on equity value. Then we allow interest rates to be stochastic, but uncorrelated with ROE. Then correlation between ROE and interest rates is introduced and its possible material impact on equity value is highlighted.

We assume a base case scenario whereby the interest rate term structure is flat and constant: $r_{t}=x_{t}=0.07$ for all times $t$ and the parameter values are $\Delta=1, B o=1, \alpha=0, \beta=1, \Psi=0, \phi=0, \mu=0, \Sigma=0, \theta=0.4, m=0.07$, $v=0.04, \rho=0$. We assume that residual earnings become negligible in the long term by setting the long term average ROE at $m=r_{t}=0.07$. Figure 1 displays equity value $V_{t}$ in the base case scenario as a function of ROE $R_{t}$ and of the valuation horizon $N$. Of course equity value rises in the current value of continuously compounded ROE $R_{t}$. Figure 1 shows how equity value is sensitive to the valuation horizon when $N$ is relatively short. Instead, for valuation horizons of 20 years or more equity value settles to a constant level. This is due to the fact that we assume, as is common, that abnormal earnings disappear in the long term. Then $V_{t} / B o_{t}$ differs from 1 because of the expected abnormal profits or losses in the first part of the valuation horizon until such expected profits or losses vanish in the long term.

Equity value $V_{t}$ critically depends on the parameters defining the ROE
process. Equity value is sensitive to the ROE long term mean $m$. When assuming that $m \backsim \mu$, where $\mu$ is the long term mean short interest rate level, we assume that long term residual earnings become negligible so that shorter valuation horizons are required for accurate estimates of equity value.

Equity value rises in the volatility of $\operatorname{ROE} v$. The reason is that equity value is driven by expected (under the risk-neutral measure) future earnings, which are convex in $R_{t+n}$. In fact

$$
\begin{equation*}
E_{t}^{*}\left(e^{R_{t+n}}-1\right)=e^{E_{t}^{*}\left(R_{t+n}\right)+\frac{1}{2} \operatorname{Var}_{t}\left(R_{t+n}\right)}-1 \tag{58}
\end{equation*}
$$

where $\operatorname{Var}_{t}\left(R_{t+n}\right)$ is the variance of $R_{t+n}$ conditional on time $t$ information and

$$
\begin{equation*}
\operatorname{Var}_{t}\left(R_{t+n}\right)=v^{2} \sum_{i=1}^{n}(1-\theta)^{2(i-1)} . \tag{59}
\end{equation*}
$$

The last two equations imply that expected future earnings rise in $\operatorname{Var}_{t}\left(R_{t+n}\right)$ and that $\operatorname{Var}_{t}\left(R_{t+n}\right)$ rises in the volatility of ROE $v$.

As for the mean reversion parameter $\theta$, if $\theta<1$ the $R_{t}$ process is stationary. If $\theta=1$ the $R_{t}$ process is similar to a white noise process with mean $m . \theta$ affects equity value by driving both the time $t$ conditional variance of future ROE $\operatorname{Var}_{t}\left(R_{t+n}\right)$ as well as the time $t$ conditional expected value $E_{t}^{*}\left(R_{t+n}\right)$ since

$$
\begin{equation*}
E_{t}^{*}\left(R_{t+n}\right)=R_{t}(1-\theta)^{n}+\sum_{i=1}^{n} \theta m(1-\theta)^{n-i} \tag{60}
\end{equation*}
$$

But as $n \rightarrow \infty$ and $\theta<1$, it follows that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \operatorname{Var}_{t}\left(R_{t+n}\right)=\frac{v^{2}}{1-(1-\theta)^{2}}  \tag{61}\\
& \lim _{n \rightarrow \infty} E_{t}^{*}\left(R_{t+n}\right)=m  \tag{62}\\
& \lim _{n \rightarrow \infty} E_{t}^{*}\left(e^{R_{t+n}}-1\right)=e^{E_{t}^{*}\left(R_{t+n}\right)+\frac{1}{2} \operatorname{Var}_{t}\left(R_{t+n}\right)}-1 \rightarrow e^{m+\frac{1}{2} \frac{v^{2}}{1-(1-\theta)^{2}}}-1 . \tag{63}
\end{align*}
$$

The last equation highlights that long term expected earnings decrease in $\theta$, but the impact of $\theta$ on equity value $V_{t}$ is less obvious. The effect of $\theta$ on shorter term earnings and on equity value depends on the relative level of $R_{t}$ with respect to the ROE long term mean level $m$. When $R_{t}>m\left(R_{t}<m\right)$, $V_{t}$ decreases (increases) in $\theta$, since greater mean reversion speed would reduce (raise) expected future abnormal earnings. Generally equity value is sensitive to $\theta$.
[Figure 1 about here]
Figure 2 displays equity value in a scenario that is the same as the base case scenario of Figure 1, but for the fact that interest rates are stochastic and uncorrelated with ROE. The parameter values underlying Figure 2 are $\Delta=1$, $B o=1, \alpha=0, \beta=1, \Psi=0, r_{t}=0.07, \phi=0.3, \mu=0.07, \Sigma=0.01, \theta=0.4$, $m=0.07, v=0.04, \rho=0$. In particular these parameter values imply that the short interest rate follows an $\mathrm{AR}(1)$ process, that can be thought of as a discrete time version of the continuous time Vasicek process, in that

$$
\begin{equation*}
x_{t+1}=r_{t+1}=(1-\phi) r_{t}+\phi \mu+\Sigma \xi_{t+1} \text { with } \xi_{t+1} \sim N(0,1) \tag{64}
\end{equation*}
$$

This process is here assumed for the sake of expositional clarity, even though it implies that $r_{t}$ can turn negative. When $\Psi \neq 0, r_{t}$ can be easily constrained to be non-negative, which is one of the advantages of quadratic term structure models as the one specified above.

Notice that, as in the base case scenario, $r_{t}=0.07=m$, but this time $r_{t}$ is stochastic. Comparing Figure 1 and 2 shows that equity value is sensitive to the variance of the future short interest rate: equity value rises with the variance of the future short interest rate. This entails that equity value decreases in the mean reversion speed $\phi$ and rises in the interest rate volatility $\Sigma$. Really this is no surprise, since we know that bond yields decrease (and discount factors increase) with the variance of the future short interest rate: a phenomenon often referred to as the "volatility kicker". Of course equity value decreases in the interest rate mean reversion level $\mu$.
[Figure 2 about here]
Figure 3 shows equity value in the same scenario as in Figure 2, but for the fact that the short interest rate and ROE are correlated: the correlation is -0.5 rather than 0 . The parameter values underlying Figure 3 are: $\Delta=1, B o=1$, $\alpha=0, \beta=1, \Psi=0, r_{t}=0.07, \phi=0.3, \mu=0.07, \Sigma=0.01, \theta=0.4, m=0.07$, $v=0.04, \rho=-0.5$. Figure 3 implies hardly any change in equity value when compared with Figure 2.
[Figure 3 about here]
Figure 4 shows the difference between $V_{t}$ under the assumptions of $\rho=-0.5$ and of $\rho=0.5$. Equity value decreases in the correlation by up to about $1.5 \%$
of equity book value.
[Figure 4 about here]
This inverse relationship between equity value and correlation between ROE and the short interest rate may seem puzzling, since expected future residual earnings are convex in $R_{t}$ and $r_{t}$, in fact

$$
\begin{equation*}
E_{t}^{*}\left(e^{R_{t+n}-r_{t+n}}-1\right)=e^{E_{t}^{*}\left(R_{t+n}-r_{t+n}\right)+\frac{1}{2} \operatorname{Var}_{t}\left(R_{t+n}-r_{t+n}\right)}-1 . \tag{65}
\end{equation*}
$$

As the correlation between $R_{t}$ and $r_{t}$ decreases, the variance of residual earnings $\operatorname{Var}_{t}\left(R_{t+n}-r_{t+n}\right)$ decreases and hence we could expect expected residual earnings and equity value to decrease too. But the negative relation between $V_{t}$ and $\rho$ is due to another effect, which is of prevailing magnitude on equity present value. As correlation decreases, future earnings are expected to be discounted at a higher (lower) rate precisely if they are lower (higher). In other words, the higher the correlation, the more high earnings tend to be discounted at high rates and the more low earnings tend to be discounted at low rates. Thus equity value must decrease in correlation since equity value is given by expected discounted earnings.

As this explanation suggests, the magnitude of the correlation effect on equity value rises materially with the variance of future ROE and of the future short interest rate. In other words $\frac{\partial V_{t}}{\partial \rho}<0$ and $\frac{\partial V_{t}}{\partial \rho}$ markedly decreases as the volatility of ROE $v$ and of interest rates $\Sigma$ rise or as the mean reversion
parameters $\theta$ and $\phi$ decrease. For example Figure 5 is a plot of

$$
\begin{equation*}
V_{t}(\rho=-0.5, \theta=0.1)-V_{t}(\rho=0.5, \theta=0.1) \tag{66}
\end{equation*}
$$

where $V_{t}(\rho=-0.5, \theta=0.1)$ is equity present value under the assumption that $\rho=-0.5, \theta=0.1$ and all other parameter values are as in Figure 2. $V_{t}(\rho=0.5, \theta=0.1)$ has a similar interpretation. Thus Figure 5 assumes a more persistent ROE process than Figure 4: $\theta=0.1$ rather than 0.4 . In this case the difference in equity value is no longer of the order of about $2 \%$ of equity book value, but of $8 \%$ as correlation changes.

The actual correlation between ROE and the short rate would typically depend on the pro-cyclicality of the firm's economic performance. For procyclical equities, at the through (peak) of the cycle, earnings are low (high) and interest rates are low (high), implying a certain degree of positive correlation between $r_{t}$ and $R_{t}$. We can conclude that the less pro-cyclical an equity is, the more valuable it should be. This conclusion had already been reached by past literature, but through another explanation. The explanation was that procyclical stocks have higher betas and higher cost of capital, implying a lower equity present value for a given stream of expected future earnings.
[Figure 5 about here]
As the persistency of ROE process magnifies the importance of correlation between $r_{t}$ and $R_{t}$ for equity valuation, so does the persistency of the short rate process. But the persistency of the short rate seems of secondary importance
in this respect as shown by Figure 6. Figure 6 plots

$$
\begin{equation*}
V_{t}(\rho=-0.5, \theta=0.4, \phi=0.1)-V_{t}(\rho=0.5, \theta=0.4, \phi=0.1) \tag{67}
\end{equation*}
$$

where $V_{t}(\rho=-0.5, \theta=0.4, \phi=0.1)$ is equity value when $\rho=-0.5, \theta=0.4$, $\phi=0.1$ and all other parameter values are as in Figure 2.
[Figure 6 about here]
We conclude that, not only the shape of the term structure of interest rates is a primary determinant of equity value, but also the variance of the future short rate is. Moreover equity value can be also very sensitive the correlation between the short rate and ROE, especially if the ROE process is very persistent. As a consequence, assuming a flat and constant term structure is a simplification that can lead to important inaccuracies.

A note is due since in the above simulations have we retained the assumption that $\mu=m$, the long term level of ROE is the same as the long term level of the short interest rate. This assumption corresponds to the intuition that abnormal profitability, either positive or negative, cannot persist forever. Moreover the $\mu=m$ assumption entails that equity valuation is not particularly sensitive to the forecast horizon: typically the expected residual earnings for the next 20 to 30 years suffice to provide an accurate estimate of equity value. If we assumed $\mu \neq m$ longer horizons would be required.

### 5.1 More general assumptions about ROE

The above equity valuation model can be generalised in a number of ways. For example multiple auto-regressive stochastic factors could drive the short interest rate $r_{t}$ and the return on equity $R_{t}$. For example Myers (1999) explains how, in the context of linear factor models, earnings could be driven by several man reverting auto-regressive $\operatorname{AR}(1)$ processes. The same can be assumed for ROE in the above model. For example ROE could be driven by macroeconomic variables like the output-gap, to capture the phase of the business cycle, and by industry specific measures of economic activity. We could even envisage that common macroeconomic variables drive both the short term interest rate and ROE. Finally the above model could be extended by assuming that ROE follows an $\mathrm{AR}(2), \mathrm{AR}(3)$ or other auto-regressive mean reverting process.

## 6 Conclusions

This paper presents an equity pricing model that employs risk-neutral valuation under stochastic interest rates along the lines of Ohlson and Feltham (1999). Closed form valuation formulae for equity pricing are presented in a discrete time setting. Earnings are driven by mean reverting return on equity (ROE).

The shape of the term structure of interest rates, the variance of the short term interest rate and the correlation between interest rates and earnings are primary determinants of equity value, which the past literature on equity valuation has largely overlooked. Equity value decreases in the correlation between
interest rates and ROE and can be very sensitive to such correlation when the ROE process is highly persistent. This result implies that equity value decreases in the degree of pro-cyclicality of the firm's profitability.

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