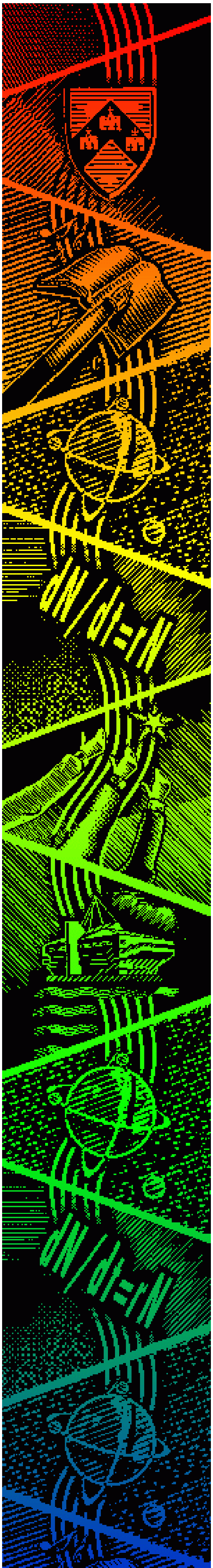

THE UNIVERSITY of York

Discussion Papers in Economics

No. 2006/01

Quadratic Term Structure Models in Discrete Time

by

Marco Realdon

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

QUADRATIC TERM STRUCTURE MODELS IN DISCRETE TIME

Marco Realdon*

9/1/06

Abstract

This paper extends the results on quadratic term structure models in continuos time to the discrete time setting. The continuos time setting can be seen as a special case of the discrete time one. Closed form solutions for zero coupon bonds are provided even in the presence of multiple correlated underlying factors. The model, which can also be used for pricing credit risk in a reduced form fashion, is useful when the factors are or depend on periodically released macroeconomic data or corporate financial reports. Model estimation does not require a restrictive choice of the market price of risk.

Key words: quadratic term structure model, discrete time, bond valuation, recursive solution, bond option.

JEL classification: G12; G13.

*Department of Economics, University of York, Alcuin College, University Rd, YO10 5DD, UK; tel: +44/(0)1904/433750; email: mr15@york.ac.uk.

1 Introduction

This paper presents a general class of quadratic term structure models (hereafter QTSM) in discrete time rather than in continuous time. The latter can be seen as a special case of the former as the discrete time steps converge to zero. QTSM in discrete time retain the main advantages of QTSM in continuous time. The short rate can be constrained to be non negative. A closed form solution for zero coupon bonds is available in a one factor setting and a simple integration gives also the price of a European bond option.

But the discrete time setting also offers advantages over the continuous time one. Full closed form solutions are available even in the presence of multiple correlated factors. This is the case even if the model parameters change over time. The discrete time setting provides much flexibility in specifying the market price of risk while the factors transition density remains Gaussian. This is an advantage in estimation as already noted by Dai-Le-Singleton (2005). The discrete time setting is more suitable to study models whose factors are or depend on macroeconomic time series or corporate accounting data, both of which are released periodically rather than continuously.

2 Literature

The literature most directly relevant to this paper is that on term structure models in discrete time and that on QTSM. Noteworthy discrete time models are that of Sun (1992), who proposes a discrete time version of the Cox-Ingersoll-Ross model, that of Ang and Piazzesi (2003), who propose a Gaussian model driven by macroeconomic factors, and more recently that of Dai-Le-Singleton (2005), who study the discrete time counterparts of the continuous time term structure models of Duffie-Kan (1996) and Dai-Singleton (2000). Dai-Le-Singleton show that in discrete time, unlike in continuos time, affine models permit a flexible specification of the market price of risk while still providing tractable likelihood functions for the observed yields. The latter being of interest for the econometric testing of the models. Also the present paper considers the discrete time setting, but it analyses quadratic rather than affine term structure models.

QTSM are been studied in continuos time, but not yet in discrete time, at least to the best of our knowledge. The first QTSM of Beaglehole and Tenney (1991) and Constantinides (1992) have been recently extended in Lieppold and Wu (2001, 2002), who price various contingent claims in the quadratic set up, in Ahn-Dittmar-Gallant (2002), who provide the maximally flexible QTSM, and in Chen-Filipovic-Poor (2005), who highlight the applicability of QTSM to credit risk pricing.

This paper extends some of these results about continuos time QTSM to the discrete time setting. Then the continuos time setting can be seen as a special case of the discrete time one.

3 Single factor discrete time QTSM

This section presents the basic one factor QTSM in discrete time. We define with $P_{n,t}$ the price of a zero coupon bond at time t and with n time periods to maturity. Each time period is of length $\Delta = 1$, thus the bond expires at time $t + n$. We define with r_t the risk free interest rate at time t for the maturity equal to one period. Equivalently r_t can be viewed as the one period yield of the risk free zero coupon bond $P_{1,t}$, i.e. $r_t = -\ln P_{1,t}$.

The we invoke the basic risk-neutral valuation equation

$$\begin{aligned} P_{n,t} &= E_t\left[e^{-\Sigma_{i=t}^{t+n-1} r_i}\right] && (1) \\ &= E_t\left[e^{-r_t} \cdot P_{n-1,t+1}\right] && (2) \end{aligned}$$

where $E_t[..]$ denotes conditional expectation at time t under the risk-neutral measure. Using risk-neutral valuation as opposed to the stochastic discount factor pricing formulation has the advantages that indeed $r_t = -\ln P_{1,t}$ and that, as it will be apparent later, it is simpler to specify the market risk premium through appropriate choice of the Radon-Nykodim derivative.

We state the following equation, which summarises the assumptions underlying the pricing model of this section, and then we explain the assumptions in turn:

$$r_t = \alpha + \beta x_t + \Psi x_t^2 \tag{3}$$

$$x_{t+1} = x_t (1 - \phi) + \phi\mu + \xi_{t+1} \tag{4}$$

$$\xi_t \backsim N\left(0, \sigma^2\right) \tag{5}$$

$$P_{n,t} = e^{A_n + B_n x_t + C_n x_t^2} \tag{6}$$

where α, β, Ψ, σ^2 are constant and A_n, B_n and C_n only depend on n. The first equation states that the one period interest rate r_t is a quadratic function of the underlying factor x_t. The second equation states that the factor x follows a Gaussian autoregressive process. The third equation defines the noise term ξ_{t+1} driving x_{t+1} as normally distributed with mean of 0 and variance σ^2. The fourth equation states our conjectured solution for $P_{n,t}$, which we are now going to verify.

In fact we can restate the pricing equation as

$$P_{n,t} = E_t\left[e^{-\alpha - \beta x_t - \Psi x_t^2} e^{C_{n-1} x_{t+1}^2 + B_{n-1} x_{t+1} + A_{n-1}}\right]. \tag{7}$$

which we can also re-write as

$$\begin{aligned} A_n + B_n x_t + C_n x_t^2 \quad = \quad & -\alpha - \beta x_t - \Psi x_t^2 + A_{n-1} + x_t (1 - \phi) B_{n-1} \\ & + \phi\mu B_{n-1} + C_{n-1} \left(x_t (1 - \phi) + \phi\mu\right)^2 \\ & + \ln E_t \left[e^{(B_{n-1} + 2C_{n-1} x_t (1-\phi) + 2C_{n-1}\phi\mu)\xi_{t+1} + C_{n-1}\xi_{t+1}^2}\right] \end{aligned}$$

where

$$\ln E_t\left[e^{(B_{n-1}+2C_{n-1}x_t(1-\phi)+2C_{n-1}\phi\mu)\xi_{t+1}+C_{n-1}\xi_{t+1}^2}\right] \tag{8}$$
$$= -\ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2}-2C_{n-1}\right) + \frac{\sigma^2\left(B_{n-1}+2C_{n-1}x_t\left(1-\phi\right)+2C_{n-1}\phi\mu\right)^2}{\left(2-4C_{n-1}\sigma^2\right)}.$$

In our setting $\frac{1}{\sigma^2}-2C_{n-1}\geq 0$, so the logarithm in the last equation is always well defined. Appendix 1 shows how equation 8 is derived and also shows that the above equation imply that

$$\begin{aligned}
A_n &= -\alpha + A_{n-1} + \phi\mu B_{n-1} + C_{n-1}\phi^2\mu^2 && (9)\\
&\quad -\ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2}-2C_{n-1}\right) + \frac{\sigma^2\left(B_{n-1}+2C_{n-1}\phi\mu\right)^2}{\left(2-4C_{n-1}\sigma^2\right)}\\
B_n &= -\beta + \left(1-\phi\right)B_{n-1} + 2\left(1-\phi\right)\phi\mu C_{n-1} && (10)\\
&\quad + \frac{2C_{n-1}\left(1-\phi\right)\sigma^2\left(B_{n-1}+2C_{n-1}\phi\mu\right)}{\left(1-2C_{n-1}\sigma^2\right)}\\
C_n &= -\Psi + \left(1-\phi\right)^2 C_{n-1} + \frac{\sigma^2\left(2C_{n-1}\left(1-\phi\right)\right)^2}{\left(2-4C_{n-1}\sigma^2\right)}. && (11)
\end{aligned}$$

These recursive difference equations are subject to the initial conditions $A_0 = B_0 = C_0 = 0$. We notice that technically these equations provide a closed form solution for zero coupon bonds. At this point we can verify that at time t the one-period yield $y_{1,t}$ is

$$\begin{aligned} y_{1,t} &= -\ln P_{1,t} = -A_1 - B_1 x_t - C_1 x_t^2 \\ &= \alpha + \beta x_t + \Psi x_t^2 = r_t \end{aligned} \tag{12}$$

since $A_1 = -\alpha$, $B_1 = -\beta$ and $C_1 = -\Psi$.

Following Ahn-Dittmar-Gallant (2002) we note that, even in this discrete time setting, $r_t \geq 0$ as long as $\alpha \geq \frac{\beta^2}{4\Psi}$ and $\Psi > 0$. In fact $\frac{\partial r_t}{\partial x_t} = \beta + 2\Psi x_t = 0$, giving $x_t = -\frac{\beta}{2\Psi}$ and the corresponding lower bound $r_t^* = \alpha - \beta\frac{\beta}{2\Psi} + \Psi\frac{\beta^2}{4\Psi^2} = \alpha - \frac{\beta^2}{4\Psi}$. This implies that $\alpha \geq \frac{\beta^2}{4\Psi}$ if the lower bound is $r_t^* \geq 0$. As in Ahn-Dittmar-Gallant, we simply assume that this condition is valid. Thus, whereas the discrete time version of Cox-Ingersoll-Ross type affine models, as in Sun (1992), poses the problem of possible negative values of r, the present discrete time QTSM does not pose such a problem.

It is worth highlighting that ξ_{t+1} needs to have a Gaussian distribution in order for the above results to hold.

4 Multiple factors

Now we extend the previous single factor analysis to a setting of multiple factors. We redefine x_t, β, μ, ξ_{t+1}, B_n as $N \times 1$ vectors, and Ψ, ϕ, C_n, Σ, as $N \times N$ matrixes. r_t, A_n and α are still scalars. Then we reformulate the model assumptions as

$$\xi_{t+1} \backsim N(0, I) \tag{13}$$

$$r_t = \alpha + \beta' x_t + x_t' \Psi x_t \tag{14}$$

$$x_{t+1} = (I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1} \tag{15}$$

$$P_{n,t} = e^{A_n + B_n' x_t + x_t' C_n x_t} \tag{16}$$

where I is the $N \times N$ identity matrix. The assumptions implies that $Var(x_{t+1} - x_t) = \Sigma\Sigma'$. Without loss in generality we assume that Ψ and C_n are symmetric, which are conditions for the econometric identification of Ψ and C_n as Ahn-Dittmar-Gallant pointed out for the continuos time case.

We can derive closed form solutions for A_n, B_n and C_n also in this multi-factor setting. To how, first we restate the pricing equation for $P_{n,t}$ as

$$P_{n,t} = E_t \left[e^{-\alpha - \beta' x_t - x_t' \Psi x_t} e^{A_{n-1} + B_{n-1}' x_{t+1} + x_{t+1}' C_{n-1} x_{t+1}} \right] \tag{17}$$

Noting that

$$\begin{aligned}
x_{t+1}' C_{n-1} x_{t+1} &= \left((I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1} \right)' C_{n-1} \left((I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1} \right) \\
&= x_t' (I - \phi)' C_{n-1} (I - \phi) x_t + (\phi\mu)' C_{n-1} \phi\mu + \left(\Sigma\xi_{t+1} \right)' C_{n-1} \Sigma\xi_{t+1} \\
&\quad + 2 x_t' (I - \phi)' C_{n-1} \phi\mu + 2 x_t' (I - \phi)' C_{n-1} \Sigma\xi_{t+1} + 2 (\phi\mu)' C_{n-1} \Sigma\xi_{t+1}
\end{aligned}$$

we define

$$
\begin{aligned}
Q &= x_t'(I-\phi)' C_{n-1}(I-\phi)x_t + (\phi\mu)' C_{n-1}\phi\mu + 2x_t'(I-\phi)' C_{n-1}\phi\mu \\
F' &= 2x_t'(I-\phi)' C_{n-1} + 2(\phi\mu)' C_{n-1}.
\end{aligned}
$$

Then we can rewrite equation 17 as

$$
\begin{aligned}
A_n + B_n' x_t + x_t' C_n x_t &= -\alpha - \beta' x_t - x_t'\Psi x_t + A_{n-1} + B_{n-1}'((1-\phi)x_t + \phi\mu) \\
&\quad + ((1-\phi)x_t + \phi\mu)' C_{n-1}((1-\phi)x_t + \phi\mu) + Q \\
&\quad + \ln E_t\left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1} + \xi_{t+1}'\Sigma' C_{n-1}\Sigma\xi_{t+1}}\right]
\end{aligned} \tag{18}
$$

where

$$
\ln E_t\left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1} + \xi_{t+1}'\Sigma' C_{n-1}\Sigma\xi_{t+1}}\right] = \ln\frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left((B_{n-1}+F)'\gamma_i\right)^2 \tag{19}
$$

with γ_i being the i-th column of the $N \times N$ matrix $\gamma = \left((\Sigma\Sigma')^{-1} - 2C_{n-1}\right)^{-1/2}$. Appendix 2 derives equation 19 and shows that equations 18 and 19 imply three recursive equations for A_n, B_n and C_n, which are

$$A_n = -\alpha + A_{n-1} + B'_{n-1}\phi\mu + (\phi\mu)' C_{n-1}\phi\mu + \ln \frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N} \left(\begin{array}{c} B'_{n-1}\gamma_i B'_{n-1}\gamma_i + B'_{n-1}\gamma_i 2 (\phi\mu)' C_{n-1}\gamma_i \\ +2 (\phi\mu)' C_{n-1}\gamma_i B'_{n-1}\gamma_i + 2 (\phi\mu)' C_{n-1}\gamma_i 2 (\phi\mu)' C_{n-1}\gamma_i \end{array} \right) \quad (20)$$

$$B'_n = -\beta' + B'_{n-1}(1-\phi) + 2(\phi\mu)' C_{n-1}(I-\phi) + \sum_{i=1}^{N} \left(\begin{array}{c} B'_{n-1}\gamma_i (C_{n-1}\gamma_i)' (I-\phi) + B'_{n-1}\gamma_i\gamma'_i C_{n-1}(I-\phi) \\ +2(\phi\mu)' C_{n-1}\gamma_i\gamma'_i C_{n-1}(I-\phi) + 2(\phi\mu)' C_{n-1}\gamma_i\gamma'_i C_{n-1}(I-\phi) \end{array} \right) \quad (21)$$

$$C_n = -\Psi + (I-\phi)' C_{n-1}(I-\phi) + 2\sum_{i=1}^{N}(I-\phi)' C_{n-1}\gamma_i\gamma'_i C'_{n-1}(I-\phi) \quad (22)$$

subject to the terminal conditions $A_n = 0$, $B'_n = \mathbf{1}\cdot\mathbf{0}$ and $C_n = \mathbf{0}$, where $\mathbf{0}$ is an $N \times N$ matrix of zeros and $\mathbf{1}$ is a $1 \times N$ vector of ones. Of course these equations are a generalisation of the corresponding ones derived above in the single factor setting. We notice that technically these equations are again closed form solutions and imply a closed form solution for zero coupon bonds even in this multifactor setting. On the other hand in continuous time closed form solutions are not known for QTSM in the presence of multiple factors. Rather in continuos time a system of ODE's need to be solved numerically. Moreover the above closed form solutions are ideal to accommodate parameters whose values change deterministically from one time period to the next.

We also notice that in this multifactor setting $r_t \geq 0$ as long as $\alpha \geq \frac{1}{4}\beta'\Psi^{-1'}\beta$ and $\Psi > 0$. In fact $\frac{\partial r_t}{\partial x_t} = \beta + 2\Psi x_t = 0$, gives $x_t = -\frac{1}{2}\Psi^{-1}\beta$ and the corresponding lower bound $r_t^* = \alpha - \frac{1}{4}\beta'\Psi^{-1}\beta$. This lower bound for r_t implies that, if $\alpha \geq \frac{1}{4}\beta'\Psi^{-1}\beta$, then the lower bound is $r_t^* \geq 0$.

4.1 Convergence to the continuos time counterpart

We can consider the above model as the discrete time counterpart of the QTSM in continuos time such those in Ahn-Dittmar-Gallant (2002) or Lieppold Wu (2002). In continuos time the state vector x follows the stochastic differential equation $dx = k(\mu - x)\,dt + \sigma dz$, where k and σ are $N \times N$ square matrixes of constants, μ and x are $N \times 1$ column vectors and dz is an $N \times 1$ column vector of differentials of independent Wiener processes. But the above discrete time autoregressive Markov process can be re-expressed as $x_{t+\Delta} - x_t = \phi(\mu - x_t) + \Sigma\xi_{t+1}$, where Δ is the length of the time step. Above we set $\Delta = 1$. But if $\Delta \to 0$, $x_{t+\Delta} - x_t$ converges to dx if only we set $\phi = k\Delta$ and $\Sigma\Sigma' = \sigma\sigma'\Delta$. This is why we can think of the continuos time QTSM as special cases of the above discrete time model as $\Delta \to 0$.

4.2 Conditions for parameter identification

If the state variables x are not observable, we need to add some restrictions to the above QTSM in order to be able to uniquely identify the model parameters. As already shown by Ahn-Dittmar-Gallant (2002) in a continuos time setting, also in the present discrete time setting we need to impose that:

- Ψ be symmetric; we normalise Ψ by requiring that its diagonal be made up 1's;

- $\mu \geq 0$, $\alpha \geq 0$, $\beta = 0$ in order for μ to be identifiable;

- Σ be diagonal (triangular) and ϕ be triangular (diagonal).

This restrictions are explained in Appendix 3. In other words the conditions for the econometric identification of the discrete time model are similar to the corresponding conditions in continuos time.

4.3 Credit risk

The above term structure model can be reinterpreted as a reduced form model of credit risk. For example r can be reinterpreted as a risk-neutral default intensity, $P_{n,t}$ as the survival probability between t and $t + \Delta n$, $P_{n,t} - P_{n+1,t}$ as the probability of default in the time period $]t + n, t + n + 1]$, and so on.

5 Physical process

The above multifactor model was built while assuming that, under the risk neutral measure, which we denote as Q, the process of the state variables was $x_{t+1} = (I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1}$. Now we specify the process for x under the physical measure, which is of interest for econometric estimation and risk management. To do so we need to specify a market price. As highlighted by Dai-Le-Singleton (2005), the discrete time setting allows very flexible specifications of the market price of risk while still retaining tractable transition densities for the time series of the underlying factors or of the observed yields.

To switch to the physical measure, which we denote with P, we assume that the Radon-Nykodim derivative is

$$\frac{dP}{dQ} = e^{\frac{1}{2}\left(2\xi'_{t+1} f(x_t) - f(x_t)' f(x_t)\right)} \tag{23}$$

where $f(x_t)$ is an $N \times 1$ vector or functions of x_t that do not depend on ξ_{t+1}. Then the probability density of ξ_{t+1} under the physical measure, which we denote with $P\left(\xi_{t+1}\right)$, is

$$\begin{aligned} P\left(\xi_{t+1}\right) &= Q\left(\xi_{t+1}\right) \cdot \frac{dP}{dQ} = \frac{1}{\sqrt{(2\pi)^N}} e^{-\frac{1}{2}\xi'_{t+1}\xi_{t+1}} \cdot e^{\frac{1}{2}\left(2\xi'_{t+1} f(x_t) - f(x_t)' f(x_t)\right)} \\ &= \frac{1}{\sqrt{(2\pi)^N}} e^{-\frac{1}{2}\sum_{i=1}^{N}\left(\xi_{t+1,i} - f(x_t)_i\right)^2}. \end{aligned} \tag{24}$$

It follows that under the physical measure the process of x becomes

$$x_{t+1} = x_t (1-\phi) + \phi\mu + \Sigma f(x_t) + \xi_{t+1} \tag{25}$$

$$= x_t (1-\phi) + \phi\mu + \Sigma f(x_t) + \Sigma\xi_{t+1} \tag{26}$$

where again $\xi_{t+1} \backsim N\left(0, \sigma^2\right)$. Here the point to note is that $f(x_t)$ is a constant at time t, so the choice of the function $f(x_t)$ can be very wide. On the other hand x_{t+1} will still be have a Gaussian distribution, irrespective of $f(x_t)$. This fact guarantees the tractably of econometric testing, while allowing the researcher to chose $f(x_t)$ from a wide range of candidates, provided that the choice is consistent with the absence of arbitrage.

$f(x_t)$ will determine the risk-premia demanded by the market as revealed by the level of excess expected return on the bond worth $P_{n,t}$ over and above the risk free one period yield r_t. To see this we can calculate the one period excess expected log return as

$$\ln \frac{E_t^P\left[P_{n-1,t+1}\right]}{P_{n,t}} - r_t = \ln E_t^P\left[P_{n-1,t+1}\right] - \left(A_n + B_n' x_t + x_t' C_n x_t\right) - r_t$$

where $E_t^P[..]$ denotes conditional expectation with respect to the physical measure. Invoking again equation 19 we obtain

$$\begin{aligned} E_t^P\left[P_{n-1,t+1}\right] &= A_{n-1} + B_{n-1}'\left((1-\phi)x_t + \phi\mu + \Sigma f(x_t)\right) + Q_P \\ &\quad + \left((1-\phi)x_t + \phi\mu + \Sigma f(x_t)\right)' C_{n-1}\left((1-\phi)x_t + \phi\mu + \Sigma f(x_t)\right) \\ &\quad + \ln \frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(B_{n-1}+F_P\right)'\gamma_i\right)^2 \end{aligned} \tag{27}$$

with

$$
\begin{aligned}
Q_P &= x_t' (I-\phi)' C_{n-1} (I-\phi) x_t + (\phi\mu + \Sigma f(x_t))' C_{n-1} (\phi\mu + \Sigma f(x_t)) \\
&\quad +2x_t' (I-\phi)' C_{n-1} (\phi\mu + \Sigma f(x_t))
\end{aligned} \tag{28}
$$

$$
F_P' = 2x_t' (I-\phi)' C_{n-1} + 2(\phi\mu + \Sigma f(x_t))' C_{n-1}. \tag{29}
$$

Although unrepoerted, we can also find closed form expressions for the expected value and variance of future bond yields under the physical measure. These tractable expressions for expected bond returns, expected future yields and variance of future yields under the physical measure are of interest for the econometric testing of the model. For example, they can be used to provide moment conditions to be used in GMM estimation.

6 Bond options

In this section we provide a semi closed form solution also for bond options in discrete time and in a single factor setting. We denote with $O_{n,t}$ the price of a European call option at time t that expires at time $t+n$. The call gives the right to buy a zero coupon bond which expires at time $t+m$ and whose value at t is denoted as $P_{m,t}$. At the option expiry the bond is worth $P_{m-n,t+n} = e^{A_* + B_* x_{t+n} + C_* x_{t+n}^2}$, where A_*, B_* and C_* can be found as shown above in the single factor setting. Invoking again the risk neutral valuation argument we can write

$$O_{n,t} = E_t\left[e^{-\Sigma_{i=t}^{t+n-1} r_i} \max\left(e^{A_* + B_* x_{t+n} + C_* x_{t+n}^2} - K, 0\right)\right]. \quad (30)$$

We notice that the the option expires at the money when

$$e^{A_* + B_* x_{t+n} + C_* x_{t+n}^2} = \ln K \quad (31)$$

which implies that the call will be exercise as long as the following two conditions are simultaneously met

$$\frac{-B_* - \sqrt{(B_*)^2 - 4C_*(A^* - \ln K)}}{2C_*} = x_{t+1}^* \leq x_{t+1} \quad (32)$$

$$x_{t+1} \leq x_{t+1}^{**} = \frac{-B_* + \sqrt{(B_*)^2 - 4C_*(A_* - \ln K)}}{2C_*}. \quad (33)$$

To determine the option value $O_{n,t}$ we proceed as follows. We denote with $O_{t,n}\left(x_{t+1}^{**}\right)$ the value of the contingent claim option that pays off the bond at time $t+1$ if and only if $x_{t+1} = x_{t+1}^{**}$, in which case the bond is worth $e^{A_* + B_* x_{t+1}^{**} + C_* x_{t+1}^{**2}} = H$. Then we can write the pricing equation for the one period option $O_{t,1}\left(x_{t+1}^{**}\right)$ as

$$O_{t,1}\left(x_{t+1}^{**}\right) = e^{-\alpha - \beta x_t - \Psi x_t^2} \cdot E_t\left[1_{x_{t+1} = x_{t+1}^{**}} \cdot e^{A_* + B_* x_{t+1} + C_* x_{t+1}^2}\right] \quad (34)$$

with

$$E_t\left[1_{x_{t+1}=x_{t+1}^{**}}\cdot e^{A_*+B_*x_{t+1}+C_*x_{t+1}^2}\right] = e^{A_*+B_*x_t(1-\phi)+B_*\phi\mu+C_*(x_t(1-\phi)+\phi\mu)^2}\cdot \tag{35}$$
$$\cdot e^{(B_*+2(\phi\mu+x_t(1-\phi))C_*)\xi^{**}+C_*(\xi^{**})^2}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2}$$

where

$$\begin{aligned}\xi^{**} &= x_{t+1}^{**}-x_t(1-\phi)-\phi\mu \\ &= \frac{-B_*+\sqrt{(B_*)^2-4C_*(A_*-\ln H)}}{2C_*}-x_t(1-\phi)-\phi\mu.\end{aligned} \tag{36}$$

Then we assume that $O_{t,1}\left(x_{t+1}^{**}\right)=e^{A_1^o+B_1^ox_t+C_1^ox_t^2}$ so that we can write

$$e^{A_1^o+B_1^ox_t+C_1^ox_t^2} = e^{-\alpha-\beta x_t-\Psi x_t^2}\cdot e^{A_*+B_*x_t(1-\phi)+B_*\phi\mu+C_*(x_t(1-\phi)+\phi\mu)^2}\cdot \tag{37}$$
$$\cdot e^{(B_*+2(\phi\mu+x_t(1-\phi))C_*)\xi^{**}+C_*\xi^{**2}}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2}.$$

A_1^o, B_1^o and C_1^o only depend on n, the time to the option expiry. The superscript o highlights that these functions refer to the bond option value. Solving the last equation gives

$$A_1^o = A_1 = -\alpha + \ln \frac{1}{\sigma\sqrt{2\pi}} - \frac{\left(x_{t+1}^{**} - \phi\mu\right)^2}{2\sigma^2} + A_* + B_* x_{t+1}^{**} + C_* x_{t+1}^{**2} \quad (38)$$

$$B_1^o = -\beta + \left(x_{t+1}^{**} - \phi\mu\right) \frac{(1-\phi)}{\sigma^2} \quad (39)$$

$$C_1^o = -\Psi - \frac{(1-\phi)^2}{2\sigma^2} \quad (40)$$

Now we highlight the dependence of x_{t+1}^{**} on H explicitly by writing $x_{t+1}^{**}(H)$. Then in order to find $O_{t,n}\left(x_{t+1}^{**}(H)\right)$, we notice that, given A_1^o, B_1^o and C_1^o, we can find A_n^o, B_n^o, C_n^o for $n > 1$ as we found A_n, B_n, C_n in the single factor bond valuation setting. Similarly we can find $O_{t,n}\left(x_{t+1}^{*}(H)\right)$. Then integrating $O_{t,n}\left(x_{t+1}^{**}(H)\right)$ and $O_{t,n}\left(x_{t+1}^{*}(H)\right)$ over H we can find the solution for the present value of the option since

$$\begin{aligned} O_{t,n} = & \int_K^\infty \left[O_{t,n}\left(x_{t+1}^{**}(H)\right) + O_{t,n}\left(x_{t+1}^{*}(H)\right)\right] dH \\ & -K_1 \int_K^\infty \left[C_{t,n}\left(x_{t+1}^{**}(H)\right) + C_{t,n}\left(x_{t+1}^{*}(H)\right)\right] dH \end{aligned} \quad (41)$$

where $C_{t,n}\left(x_{t+1}^{**}(H)\right)$ is a claim that pays 1 if $x_{t+1} = x_{t+1}^{**}$ such that bond is worth H. It can be shown that

$$C_{t,1}\left(x_{t+1}^{**}\right) = e^{-\alpha-\beta x_t - \Psi x_t^2} E_t\left[1_{x_{t+1}=x_{t+1}^{**}}\right] \quad (42)$$

with

$$E_t\left[1_{x_{t+1}=x^{**}_{t+1}}\right]=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\xi^{**}}{\sigma}\right)^2} \tag{43}$$

giving

$$A^c_1+B^c_1x_t+C^c_1x^2_t=-\alpha-\beta x_t-\Psi x^2_t+\ln\frac{1}{\sigma\sqrt{2\pi}}-\frac{1}{2}\left(\frac{x^{**}_{t+1}-x_t\left(1-\phi\right)-\phi\mu}{\sigma}\right)^2 \tag{44}$$

which implies that

$$\begin{aligned} A^c_1 &= -\alpha+\ln\frac{1}{\sigma\sqrt{2\pi}}-\frac{\left(x^{**}_{t+1}-\phi\mu\right)^2}{2\sigma^2} && (45)\\ B^c_1 &= B^o_1 && (46)\\ C^c_1 &= C^o_1. && (47)\end{aligned}$$

7 Conclusion

This paper has studies quadratic term structure models in discrete time, providing closed form solutions for zero coupon bonds even in the presence of multiple correlated factors. Closed forms are available also for state prices and the valuation of zero coupon bond options requires one simple numerical integration. The discrete time setting provides much flexibility in specifying the market price of risk while the factors transition density remains Gaussian, which is an advantage in estimation. This point had already been noted by Dai-Le-Singleton

(2005) in the context of affine term structure models, but the same is true for quadratic term structure models. This paper has shown that the same is true for quadratic term structure models.

Overall quadratic models in discrete time retain the main advantages as quadratic models in continuos time, but also offer some additional advantages. An obvious step for future research is the empirical testing of the model here presented. Moreover the discrete time setting paves the way to relating the unobserved factors driving the yield curve to discretely observed macroeconomic variables.

A Appendixes

A.1 The one factor model

This Appendix considers the setting with one single factor. We can derive equation 8 as follows. We define $u \sim N\left(0, \sigma^2\right)$, and a and b arbitrary constants. Then we want to evaluate the expectation

$$E\left[e^{au+bu^2}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int e^{\left(-\frac{u^2}{2\sigma^2}+au+bu^2\right)} du. \tag{48}$$

But if we put $\gamma = \sqrt{\left(\frac{1}{\sigma^2} - 2b\right)^{-1}}$, we can write

$$\begin{aligned} -\left(\frac{1}{2\sigma^2} - b\right)u^2 + au &= -\frac{u^2}{2\gamma^2} + au \\ &= -\frac{u^2}{2\gamma^2} + au - \frac{\gamma^2 a^2}{2} + \frac{\gamma^2 a^2}{2} \\ &= -\frac{1}{2}\left(\frac{u}{\gamma} - \gamma a\right)^2 + \frac{\gamma^2 a^2}{2}. \end{aligned} \tag{49}$$

It follows that

$$\begin{aligned} \frac{1}{\sigma\sqrt{2\pi}} \int e^{\left(-\frac{u^2}{2\sigma^2} + au + bu^2\right)} du &= \frac{\gamma}{\gamma} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{\gamma^2 a^2}{2}} \int e^{-\frac{\left(u - \gamma^2 a\right)^2}{2\gamma^2}} du \\ &= \frac{\gamma}{\sigma} e^{\frac{\gamma^2 a^2}{2}} \\ &= \frac{1}{\sigma\sqrt{\left(\frac{1}{\sigma^2} - 2b\right)}} e^{\frac{\sigma^2 a^2}{2\left(1 - 2b\sigma^2\right)}}. \end{aligned} \tag{50}$$

and thus

$$\ln E\left[e^{au + bu^2}\right] = -\ln\sigma - \frac{1}{2}\ln\left(\frac{1}{\sigma^2} - 2b\right) + \frac{\sigma^2 a^2}{\left(2 - 4b\sigma^2\right)}. \tag{51}$$

Now if we substitute $u = \xi_{t+1}$, $a = B_{n-1} + 2C_{n-1}x_t\left(1 - \phi\right) + 2C_{n-1}\phi\mu$ and $b = C_{n-1}$ into this equation, we get equation 8 in the text.

A.2 The multi-factor model

This Appendix considers the setting with multiple factors. Equation **??** is derived as follows. We define $w = \Sigma\xi_{t+1}$ and notice that $w \sim N\left(\mathbf{0}, \Sigma\Sigma'\right)$. Then we set $a = B_{n-1} + F$ and notice that

$$E_t\left[e^{(B_{n-1}+F)'\Sigma\xi_{t+1}+\xi_{t+1}'\Sigma' C_{n-1}\Sigma\xi_{t+1}}\right] \tag{52}$$

$$= \frac{1}{\sqrt{(2\pi)^N}}\int e^{-\frac{1}{2}\xi_{t+1}'\xi_{t+1}+a'w+w'C_{n-1}w}d\xi_{t+1} \tag{53}$$

$$= \frac{1}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}}\int e^{-\frac{1}{2}w'\left(\Sigma\Sigma'\right)^{-1}w+a'w+w'C_{n-1}w}dw$$

$$= E\left[e^{a'w+w'C_{n-1}w}\right]. \tag{54}$$

where we have made the substitutions $\xi_{t+1} = \Sigma^{-1}w$ and $d\xi_{t+1} = abs\left|\Sigma^{-1}\right|dw$, where $abs\left|\Sigma^{-1}\right|$ denotes the absolute value of the determinant of Σ^{-1}.

Then $\left(\left(\Sigma\Sigma'\right)^{-1} - 2C_{n-1}\right)$ is positive semidefinite and symmetric. This is the case since $\Sigma\Sigma'$ is symmetric and so is $\left(\Sigma\Sigma'\right)^{-1}$. Then C_{n-1} can be also be assumed symmetric and negative definite for our purposes without loss in generality. It follows that $\gamma = \left(\left(\Sigma\Sigma'\right)^{-1} - 2C_{n-1}\right)^{-1/2}$ exists and is symmetric. Then we can write the following

$$\begin{aligned} -\frac{1}{2}w'\left(\Sigma\Sigma'\right)^{-1}w + a'w + w'C_{n-1}w &= -\frac{1}{2}w'\left(\left(\Sigma\Sigma'\right)^{-1} - 2C_{n-1}\right)w + a'w \quad (55)\\ &= -\frac{1}{2}w'\gamma^{-2}w + a'w \\ &= -\frac{1}{2}\left(\gamma^{-1}w\right)'\gamma^{-1}w + a'w \\ &= -\frac{1}{2}v'v + a'\gamma v \end{aligned}$$

where $v = \gamma^{-1}w$. Hence, if γ is of full rank, it follows that the differential dw is such that

$$dw = abs\left|\gamma\right| dv = \left|\gamma\right| dv \tag{56}$$

where $abs\left|\gamma\right|$ is the absolute value of $\left|\gamma\right|$ and $abs\left|\gamma\right| = \left|\gamma\right|$ since γ is non-negative definite.

At this point we can write

$$\begin{aligned}
\frac{1}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \int e^{-\frac{1}{2}w'\left(\Sigma\Sigma'\right)^{-1}w + a'w + w'C_{n-1}w} du &= \frac{1}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \int e^{-\frac{1}{2}v'v + a'\gamma v} \left|\gamma\right| dv \\
&= \frac{\left|\gamma\right|}{\sqrt{(2\pi)^N abs\left|\Sigma\right|}} \Pi_{i=1}^{N} \int e^{-\frac{v_i^2}{2} + a'\gamma_i v_i} dv_i \\
&= \frac{\left|\gamma\right|}{abs\left|\Sigma\right|} \Pi_{i=1}^{N} e^{\frac{(a'\gamma_i)^2}{2}}
\end{aligned} \tag{57}$$

where γ_i denotes the i-th column of γ and substituting for $a = B_{n-1} + F$ into the last line we get equation 19. We notice that the last line makes use of the fact that

$$\frac{1}{\sqrt{2\pi}} \int e^{\left(-\frac{u^2}{2} + au\right)} du = \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}} \int e^{-\frac{(u-a)^2}{2}} du = e^{\frac{a^2}{2}}. \tag{58}$$

Then we can find the recursive solutions for A_n, B_n and C_n in the multifactor setting. Equations 18 and 19 imply

$$
\begin{aligned}
& A_n + B_n' x_t + x_t' C_n x_t && (59) \\
= \; & -\alpha - \beta' x_t - x_t' \Psi x_t + A_{n-1} + B_{n-1}' \left((1-\phi) x_t + \phi\mu \right) \\
& + x_t' (I-\phi)' C_{n-1} (I-\phi) x_t + (\phi\mu)' C_{n-1} \phi\mu + 2 x_t' (I-\phi)' C_{n-1} \phi\mu \\
& \ln \frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2} \sum_{i=1}^{N} \left(\left(B_{n-1} + \left(2 x_t' (I-\phi)' C_{n-1} + 2 (\phi\mu)' C_{n-1} \right)' \right)' \gamma_i \right)^2 && (60)
\end{aligned}
$$

Then, we also invoke the matching principle to separate the variables, we find that equation 59 implies the following system of difference equations

$$
\begin{aligned}
A_n &= -\alpha + A_{n-1} + B_{n-1}' \phi\mu + (\phi\mu)' C_{n-1} \phi\mu + \ln \frac{|\gamma|}{abs\,|\Sigma|} \\
& + \frac{1}{2} \sum_{i=1}^{N} \left(\begin{array}{c} B_{n-1}' \gamma_i B_{n-1}' \gamma_i + B_{n-1}' \gamma_i 2 (\phi\mu)' C_{n-1} \gamma_i \\ +2 (\phi\mu)' C_{n-1} \gamma_i B_{n-1}' \gamma_i + 2 (\phi\mu)' C_{n-1} \gamma_i 2 (\phi\mu)' C_{n-1} \gamma_i \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
B_n' x_t &= -\beta' x_t + B_{n-1}' (1-\phi) x_t + 2 (\phi\mu)' C_{n-1} (I-\phi) x_t \\
& + \sum_{i=1}^{N} \left(\begin{array}{c} B_{n-1}' \gamma_i (C_{n-1} \gamma_i)' (I-\phi) x_t + B_{n-1}' \gamma_i \gamma_i' C_{n-1} (I-\phi) x_t \\ +2 (\phi\mu)' C_{n-1} \gamma_i \gamma_i' C_{n-1} (I-\phi) x_t + 2 (\phi\mu)' C_{n-1} \gamma_i \gamma_i' C_{n-1} (I-\phi) x_t \end{array} \right)
\end{aligned}
$$

$$
x_t' C_n x_t = -x_t' \Psi x_t + x_t' (I-\phi)' C_{n-1} (I-\phi) x_t + 2 \sum_{i=1}^{N} x_t' (I-\phi)' C_{n-1} \gamma_i \gamma_i' C_{n-1}' (I-\phi) x_t
$$

The equations for B'_n and for C_n in the text follow immediately.

B Conditions for econometric identification of parameters

This Appendix discusses the conditions for the econometric identification of the model parameters. We focus on the general setting with multiple factor. We consider linear invariant transformations of x, since only linear transformations will retain the Gaussian distribution given that x has Gaussian distribution. We denote the generic invariant transformation as $x = \Omega y + \Theta$, where Θ and y are $N \times 1$ vectors and Ω is an $N \times N$ matrix. Ω^{-1} is assumed to exist. Then, since we assumed that $r_t = \alpha + \beta' x_t + x_t' \Psi x_t$ and $x_{t+1} = (I - \phi) x_t + \phi\mu + \Sigma\xi_{t+1}$, we can re-express these assumptions as

$$r_t = \alpha + \beta'\Theta + \Theta'\Psi\Theta + \beta'\Omega y_t + y_t'\Omega'\Psi\Theta + \Theta'\Psi\Omega y_t + y_t'\Omega'\Psi\Omega y_t \quad (61)$$

$$y_{t+1} - y_t = \Omega^{-1}\phi (\mu - \Theta - \Omega y_t) + \Omega^{-1}\Sigma\xi_{t+1} \quad (62)$$

Then we notice that only if $\beta = 0$ also $\Theta = 0$ in order for the transformation to be invariant. And only if $\Theta = 0$ can μ be uniquely identified in estimation.

Then, since Ψ is symmetric, $\beta = 0$ and $\Theta = 0$, we can re-express r_t and $y_{t+1} - y_t$ under the invariant transformation as

$$r_t = \alpha + y_t' \Omega' \Psi \Omega y_t \tag{63}$$

$$y_{t+1} - y_t = \Omega^{-1} \phi \Omega \left(\Omega^{-1} \mu - y_t \right) + \Omega^{-1} \Sigma \xi_{t+1}. \tag{64}$$

Then, in order for the transformation to be invariant and for Ω to be constrained to be equal to the identity matrix I, either Σ is diagonal and ϕ triangular or Σ is diagonal and ϕ triangular.

References

[1] Ahn D., Dittmar R., Gallant R., 2002, "Quadratic term structure models: theory and evidence", The Review of financial studies 15, n.1, 243-288.

[2] Ang A. and Piazzesi M., 2003, "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variable", Journal of Monetary Economics 50, 745-787.

[3] Beaglehole D. and Tenney M., 1991, "General solutions of some contingent claim pricing equations, Journal of Fixed Income 1, 69-83.

[4] Chen L., Filipovic D. and Poor V., 2004, "Quadratic term structure models for risk-free and defaultable rates", Mathematical Finance, 14, n.4, 515-536.

[5] Constantinides G., 1992, "A theory of the nominal term structure of interest rates", The Review of Financial Studies 5, n.4, 531-552.

[6] Cox J. and Ingersoll J.E.jr and Ross S.A., 1985, "A theory of the term structure of interest rates", Econometrica 53, n.2, 384-408.

[7] Dai Q. and Singleton K., 2003, "Term structure dynamics in theory and reality", The Review of financial studies 16, n.3, 631-678.

[8] Dai Q., Le A. and Singleton K, 2005, "Discrete-time dynamic term structure models with generalised market prices of risk", working paper.

[9] Duffie D. and Singleton K., 1999, "Modeling term structures of defaultable bonds", The Review of financial studies 12, n.4, 687-720.

[10] Dufresne P. and Goldstein R., 2001, "Do credit spreads reflect stationary leverage rations?", Journal of Finance 56, 1929 - 1957.

[11] Elton E., Gruber M., Agrawal D. and Mann C., 2001, "Explaining the rate spread on corporate bonds", Journal of Finance 56, 247 - 277.

[12] Ericsson J. and Reneby J., 2005, "Estimating structural bond pricing models", Journal of Business 78.

[13] Leippold M. and Wu L., 2002, "Asset pricing under the quadratic class", Journal of Financial and Quantitative Analysis 37, n.2, 271-294.

[14] Leippold M. and Wu L., 2003, "Design and estimation of quadratic term structure models", European Finance Review 7, 47-73.

[15] Longstaff F., Mithal S. and Neis E., 2004, "Corporate yield spreads: default risk or liquidity? New evidence from the credit default swap market", forthcoming in Journal of finance.

[16] Sun T., 1992, "Real and nominal interest rates: a discrete time model and its continuous time limit", The Review of Financial Studies 5, n.4, 581-611.