THE UNIVERSITY of York

Discussion Papers in Economics

No. 2004/16

Affine Macroeconomic Models of the Term Structure of Interest Rates: The US Treasury Market 1961-99

by

Peter Spencer

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Abstract

JEL Classification: C13, C32, E30, E44, E52.

This paper develops a macroeconomic model of the yield curve and uses this to explain the behaviour of the US Treasury market. Unlike previous macro-finance models which assume a homoscedastic error process, I develop a general affine model which allows volatility to be conditioned by interest rates and other macroeconomic variables. Despite the extensive use of stochastic volatility models in mainstream finance papers and the overwhelming evidence of heteroscedasticity in macroeconomic and asset price data this is the first macro-finance model of the bond market with this feature. My preferred empirical specification uses a single conditioning factor and is thus the macro-finance analogue of the $\mathbb{EA}_1(N)$ specification of the mainstream finance literature. This model performs well in encompassing tests that lead to a decisive rejection of the standard $\mathbb{EA}_0(N)$ macro-finance specification. The resulting specification provides a flexible 10-factor explanation of the behaviour of the US yield curve, keying it in to the behaviour of the macroeconomy.

1 Introduction

This paper develops an econometric model of the US economy and government bond market that allows for a stochastic trend in both first and second moments. Theoretically, this model bridges the gap between the emerging macro-based model of the term structure which assumes a homoscedastic error structure (following the seminal paper of Ang and Piazzesi (2003)) and the conventional finance model, which invariably allows for heteroscedasticity.

Empirically, the model is designed to accommodate the salient characteristics of the historic US data that has been used in the macro-based research. These appear to be influenced by a unit root or near-unit root process associated with the underlying rate of inflation. This behaviour is hard to reconcile with the behaviour of the yield curve because asymptotic forward rates and yields are not well defined if the spot rate is driven by a heteroscedastic process containing a unit root (Campbell, Lo, and MacKinlay (1996)). However, there is now a growing body of evidence suggesting that macroeconomic and financial market volatility is driven by a similar stochastic trend[1]. This observation suggests a way of accommodating unit and near-unit roots in asset pricing models. For as (Campbell, Lo, and MacKinlay (1996)) note, if the stochastic trend drives the volatility as well as the central tendency of the model, its asymptotic characteristics are determined by quadratic rather than linear equations, allowing a regular asymptotic yield curve to emerge even if this trend is non-stationary.

To investigate this possibility, this paper develops a heteroscedastic macro-finance

[1] Milton Friedman argued in his Nobel lecture (Friedman (1977)) that the volatility of inflation, output and other macroeconomic variables could be related to the level of inflation itself. This hypothesis has been developed and tested by Engel (1982), Fischer and Taylor (1981), Ball (1992), Brunner and Hess (1993), Holland (1995), Caporale and McKiernan (1997) and others. There is also an emerging literature on the effect of declining macroeconomic volatility on the equity risk premium (Glosten and Runkle (1993),Scruggs (1998),Brandt and Wang (2003) and Lettau and Wachter (2004)). However, as far as I am aware no one has tried to test this hypothesis on bond market data.

specification which conditions both the central tendency and the variance structure of the model on a stochastic trend variable. In order to ensure that the variance structure remains non-negative definite I employ 'admissibility' restrictions similar to those proposed for the continuous time model by Dai and Singleton (2000). Mathematically, this model has regular variance & asymptotic term structures and provides a plausible description of the relationship between the risk premia and the conditioning variable. It is the macro-finance analogue of the model developed by Dai and Singleton (2000) and Dai and Singleton (2002), which as they say: 'builds upon a branch of the finance literature that posits a short-rate process with a single stochastic central tendency and volatility'. Empirically, this specification encompasses the standard homoscedastic macro-finance model, which it rejects decisively. Despite the extensive use of stochastic volatility models in theoretical and empirical finance papers and the evidence of heteroscedasticity in macroeconomic and asset price data this is the first macro-finance model of the bond market with this stochastic volatility feature.

The paper is set out along the following lines. The next section develops a general affine model of the economy and the bond market and shows how this can be used to derive an affine term structure under the no-arbitrage assumption. This forms the encompassing model for the empirical tests. Section 3 then discusses specializations of this model that are admissible in the sense of Dai and Singleton (2000) and Dai and Singleton (2002), notably the macro-finance analogues of their $\mathbb{EA}_0(N)$ and $\mathbb{A}_1(N)$ and $\mathbb{EA}_1(N)$ specifications. These models are tested against each other and against encompassing specifications that allow a range of macroeconomic variables to condition the price of risk and variance structures. These tests are reported in section 4. As in the Dai and Singleton (2002) tests, the $\mathbb{EA}_1(N)$ model emerges as the preferred specification. Section 5 presents the results for this model. Finally, section 6 offers a conclusion.

2 Modelling the macroeconomy and bond market

This section sets out a general linear dynamic framework for modelling the economy and the bond market and shows how this can be used to derive bond prices under the no-arbitrage assumption. This model is the discrete time analogue of the general affine model developed in a continuous time framework by Duffie and Kan (1996).

2.1 The macroeconomic framework

Suppose that the one-period interest rate $r_{1,t}$ is an element of an $(N \times 1)$ state vector of time $t-$observable variables (or linear combinations of observable variables) $Y_t = (y_{1t}, y_{2t}, ..., y_{Nt},)'$; $t = 1, ..., T$ described by a Vector Auto-Regression or VAR with the state space representation[2]:

$$Y_t = \Theta + \mathcal{K} Y_{t-1} + W_t \tag{1}$$

where Θ is an $N \times 1$ vector and $\mathcal{K}$ an N^2 matrix of known coefficients and parameters to be estimated. The first n of these equations are assumed to be stochastic and the rest identities. $W_t = (w_t', 0_{1,N-n})'$ is an $N \times 1$ error vector, where is w_t is $n \times 1$:

$$\begin{aligned} w_t &= Cv_t; \ where : v_t \sim N(0_{n,1}, \Delta_t | Y_{t-1}); \qquad (2) \\ \Delta_t &= diag[\delta_{1,t}, \delta_{2,t}, ..., \delta_{n,t}]; t = 1, ..., T \end{aligned}$$

C is an n^2 lower triangular matrix with unit diagonals and v_t is a $n-$vector of stochastic error terms.

[2] A higher order system (for example (29) in section 2) can be arranged in this first order state space form (as in (30)).

Following Duffie and Kan (1996) and Dai and Singleton (2000) suppose that any conditional heteroscedasticity in the errors is driven by square root processes in these variables[3]. Suppose that there are $m \leq n$ stochastic volatility terms. The general affine model decomposes each error into components that are conditioned by these terms, plus an orthogonal residual term:

$$v_t = \sum_{s=1}^{m} \vartheta_{s,t} \sqrt{y_{s,t-1}} + \epsilon_t; \tag{3}$$

where $\vartheta_{s,t}$ and ϵ_t are zero-mean stochastic error vectors with the properties:

$$E[\epsilon_t \epsilon_t'] = \Delta_0;$$

$$E[\vartheta_{s,t} \vartheta_{s,t}'] = \Delta_s; \;\; s = 1, 2...m;$$

$$\Delta_s = diag[\delta_{s,1}, \delta_{s,2}, ..., \delta_{s,m}]; \delta_{s,p} \geq 0; \;\; s, p = 0, 1, 2...m;$$

$$E[\vartheta_{s,t} \vartheta_{qt}'] = 0; q \neq s; \quad E[\vartheta_{s,t} \epsilon_t'] = 0.$$

For this model to be valid, we clearly need to ensure that the variables $y_{s,t-1}$entering the square roots in the error structure remain non-negative. This is a difficult technical issue, which is deferred to section 3.2.

The state space representation (1), this error model can be written as:

$$W_t \sim N(0, \Omega_t | Y_{t-1}). \tag{4}$$

[3] Preliminary tests showed no significant evidence of Autoregressive Conditional Heteroscedasticity (ARCH) in this (quarterly) data set.

where:

$$\Omega_t = \begin{bmatrix} C\Delta_t C' & 0_{n,N-n} \\ 0_{N-n,n} & 0_{N-n,N-n} \end{bmatrix}$$
$$= \Omega_0 + \sum_{s=1}^{m} \Omega_s (j_s' Y_{t-1});\ \ t = 1, ..., T.$$

$$\Omega_s = \begin{bmatrix} C\Delta_s C' & 0_{n,N-n} \\ 0_{N-n,n} & 0_{N-n,N-n} \end{bmatrix};\ \ s = 0, 1, ..., m.$$

and where j_s is a unit selection vector:

$$y_{s,t} = j_s' Y_t; s = 1, ..., n. \tag{5}$$

I_m denotes an m^2 identity matrix and $0_{n,m}$ the null matrix of dimension $n \times m$.

2.2 The bond pricing framework

The baseline VAR model is naturally defined under the observed or historical probability measure $\mathcal{P}$ and can be estimated by linear regression methods using historical data. However, the object of this paper is to use this structure to model the macroeconomy and yield curve simultaneously. To develop a consistent yield model we switch to the Risk Neutral (RN) measure $\mathcal{Q}$. The nature of this change of measure will become clearer in the final part of this section, but for now we just note that it rules out arbitrage opportunities.

Define $\tilde{E}_t$ and $\tilde{V}_t$ as the $t-$conditional expectations and variance operators under the RN probability measure $\mathcal{Q}$. Harrison and Kreps (1979) show that absent arbitrage, asset prices are discounted martingales under this measure. Specifically, the price of an $\tau-$period discount bond $P_{\tau,t}$ equals the discounted risk neutral expectation $\tilde{E}_t$ of the $(\tau - 1)$ period bond price (i.e. its own value) in the next period:

$$P_{\tau,t} = \exp[-r_{1,t}]\tilde{E}_t[P_{\tau-1,t+1}];\ \tau = 1, ..., M. \tag{6}$$

The risk neutral pricing measure is obtained by using the adapted process:

$$Y_t = \tilde{\Theta} + \tilde{\mathcal{K}}\ Y_{t-1} + \tilde{W}_t \tag{7}$$

to model the macroeconomic dynamics. As we will see, the coefficients $\tilde{\Theta}$ and $\tilde{\mathcal{K}}$ shift relative to those observed under the observed probability measure $(\Theta, \mathcal{K})$ in a way that reflects the effect of the conditioning variables on volatility and the price of risk. Importantly, the error variances are not affected by the change of measure.

If we adopt the affine log-price trial solution:

$$-p_{\tau,t} = \gamma_\tau + \Psi'_\tau Y_t;\ \tau = 1, ..., M. \tag{8}$$

(where $p_{\tau,t}$ is the natural logarithm of $P_{\tau,t}$) then prices are conditionally lognormal under both measures. This allows us to evaluate expectations like (6) using the well known formula for the expectation of a lognormally distributed variable:

$$P_{\tau,t} = \exp[-r_{1,t} + \tilde{E}_t[p_{\tau-1,t+1}] + \frac{1}{2}\tilde{V}_t[p_{\tau-1,t+1}]]; \tag{9}$$

Differences between the coefficients under measures $\mathcal{P}$ and $\mathcal{Q}$ determine the liquidity premia in an arbitrage free model. To see this, define E_t and V_t as the conditional expectations and variance operators under the historical probability measure, $\mathcal{P}$. Under this measure the expected payoff on the $\tau-$period bond after one period is:

$$E_t[P_{\tau-1,t+1}] = \exp[E_t[p_{\tau-1,t+1}] + \frac{1}{2}V_t[p_{\tau-1,t+1}]]$$

Note that $V_t[p_{\tau-1,t+1}] = \tilde{V}_t[p_{\tau-1,t+1}]$. Dividing by $P_{\tau,t}$ and then using (8), (7),(6) and (9) gives the expected gross return:

$$\frac{E_t[P_{\tau-1,t+1}]}{P_{\tau,t}} = \exp[r_{1,t} + \Psi'_{1,\tau-1}(\tilde{\Theta} - \Theta + (\tilde{\mathcal{K}} - \mathcal{K})Y_t)]$$

Taking logarithms of each side expresses this as a percentage and subtracting $r_{1,t}$ then gives the expected excess return or risk premium:

$$\log E_t[P_{\tau-1,t+1}] - p_{\tau,t} - r_{1,t} = \Psi'_{\tau-1}(\tilde{\Theta} - \Theta + (\tilde{\mathcal{K}} - \mathcal{K})Y_t);\ \tau = 1, ..., M \tag{11}$$

Equation (9) can also be used to derive recursion relationships determining the parameters of the yield equation (8). Since $-p_{\tau,t} = r_{1,t}$ for $\tau = 1$, these parameters must satisfy the starting values:

$$\gamma_1 = 0;\ \Psi_1 = j_r. \tag{12}$$

For $\tau = 2, ..., M$ we take the (negative of the) logarithm of (9) and evaluate means and variances using (1) and (4) to get:

$$\begin{aligned} -p_{\tau,t} &= r_{1,t} - \tilde{E}_t[p_{\tau-1,t+1}] - \frac{1}{2}\tilde{V}_t[p_{\tau-1,t+1}]; \\ &= j_r' Y_t + \gamma_{\tau-1} + \Psi'_{\tau-1}(\tilde{\Theta} + \tilde{\mathcal{K}}\ Y_t) \\ &\quad -\frac{1}{2}\{\Psi_{\tau-1}{}'\Omega_0\Psi_{\tau-1} + \sum_{s=1}^{m}\Psi_{\tau-1}{}'\Omega_s\Psi_{\tau-1}j_s'Y_t\} \end{aligned} \tag{13}$$

Comparing this with (8) and equating coefficients on each state variable gives the restrictions:

$$\Psi_\tau = \tilde{\mathcal{K}}'\Psi_{\tau-1} + j_r - \frac{1}{2}\sum_{s=1}^{m} \Psi_{\tau-1}{}'\Omega_s \Psi_{\tau-1} j_s, \text{ and} \tag{14}$$

$$\gamma_\tau = \gamma_{\tau-1} + \Psi'_{\tau-1}\tilde{\Theta} - \frac{1}{2}\Psi_{\tau-1}{}'\Omega_0 \Psi_{\tau-1};\ \tau = 2, ..., M. \tag{15}$$

Defining the τ-period ahead forward interest rate as $f_{\tau,t} = p_{\tau,t} - p_{\tau+1,t}$, these restrictions give the forward rate structure:

$$\begin{aligned} f_{\tau,t} &= \gamma_{\tau+1} - \gamma_\tau + [\Psi_{\tau+1} - \Psi_\tau]Y_t \qquad (16) \\ &= \Psi'_\tau\tilde{\Theta} - \frac{1}{2}\Psi_\tau{}'\Omega_0\Psi_\tau + [(\tilde{\mathcal{K}} - I_N)\Psi_\tau + j_r - \frac{1}{2}\sum_{s=1}^{m}\Psi_\tau{}'\Omega_s\Psi_\tau j_s]'Y_t \\ \tau &= 1, ..., M. \end{aligned}$$

The asymptotic characteristics of the yield curve follow directly from the first line of this system. This shows that if the factor coefficients Ψ_τ converge upon a constant vector ($lim_{\tau\to\infty}\Psi_\tau = \Psi^*$), then the asymptotic forward rate (and hence the discount and coupon bond yield) is also a constant:

$$f^* = lim_{\tau\to\infty} f_{\tau,t} = \gamma^*_{\tau+1} - \gamma^*_\tau = \Psi^{*\prime}\tilde{\Theta} - \frac{1}{2}\Psi^{*\prime}\Omega_0\Psi^*. \tag{17}$$

Relationships ((14) and (15)) can be solved recursively for the coefficients Ψ_τ and γ_τ, given the starting values (12). These coefficients then determine $p_{\tau,t}$ in (8) and hence the $\tau-$period discount yield:

$$\begin{aligned} r_{\tau,t} &= -p_{\tau,t}(\tilde{\Theta}, \tilde{\mathcal{K}})/\tau \qquad (18) \\ &= \alpha_\tau(\tilde{\Theta}, \tilde{\mathcal{K}}) + b_\tau(\tilde{\Theta}, \tilde{\mathcal{K}})'Y_t;\ where: \\ \alpha_\tau &= \gamma_\tau(\tilde{\Theta}, \tilde{\mathcal{K}})/\tau; \quad b_\tau = \Psi_\tau(\tilde{\Theta}, \tilde{\mathcal{K}})/\tau. \end{aligned}$$

Formally, this is a closed form representation because it is defined in terms of a finite number of elementary operations[4]. The slope coefficients of the yield system $b_\tau(\tilde{\Theta}, \tilde{\mathcal{K}})$ are known as 'factor loadings' and depend critically upon the eigenvalues of the adapted macro system (7). Stacking the M yield equations (18) and adding an error vector e_t gives a multivariate regression system for the M -vector of yields r_t :

$$\begin{aligned} r_t &= \alpha(\tilde{\Theta}, \tilde{\mathcal{K}}) + B(\tilde{\Theta}, \tilde{\mathcal{K}})' Y_t + e_t \\ e_t &\sim N(0, P); \\ P &= diag[\rho_1, \rho_2, ..., \rho_M]. \end{aligned} \tag{19}$$

where e_t is an error vector. The standard assumption in macro-finance models is that this represents measurement error which is orthogonal to the errors W_t in the macro system (1). The encompassing model assumes $m = n$.

3 Specializing the model

The structure defined by ((1), (4) and (19)) provides a general affine modelling framework. This represents the discrete time equivalent of the continuous time affine specification of Duffie and Kan (1996). It is a reduced form and as such it is hard to interpret and contains a large number of parameters. Moreover, as Dai and Singleton (2000) note, the variance structure of the general affine model is endogenous and there is nothing to ensure it remains non-negative. In this section we interpret and specialise the model using the structure provided by Stochastic Discount Factor. Then we look at models that are admissible in the sense of Dai and Singleton (2000).

[4] Although the number of operations implied by the heteroscedastic system is very large for long maturities, this model is linear in variables. This means that when calculating the likelihood (see appendix 3) α_τ and β_τ need only be calculated once for each maturity, irrespective of the number of observations. Moreover these calculations are recursive: the one year calculation feeds into the two year calculation and so on. This system is also recursive in the sense that the slope parameters affect the intercept coefficients, but not vice versa.

3.1 SDF models

If arbitrage is ruled out, the price $P_{\tau,t}$ of an τ-period discount bond must also be described by the pricing kernel (Cochrane (2000), Campbell, Lo, and MacKinlay (1996)):

$$P_{\tau,t} = E_t[M_{t+1}P_{\tau-1,t+1}];\ \tau = 1, ..., M. \tag{20}$$

M_{t+1} is a nominal stochastic discount factor (SDF) with the logarithm m_{t+1}. For the error model (3):

$$-m_{t+1} = \omega_t + r_{1,t} + \lambda_t' C' \epsilon_{t+1} + \sum_{s=1}^{m} \zeta_s' C' \vartheta_{s,t+1} \sqrt{j_s' Y_t} \tag{21}$$

where λ_t and $\zeta_s; s = 1, 2...m$ are $n-$vectors of coefficients related to the prices of different types of risk, the first of which may depend linearly upon Y_t. Define the deficient vectors $Z_s' = [\zeta_s', 0_{n\times(L-1),1}]$ and $\Lambda_t' = [\lambda_t', 0_{n\times(L-1),1}]$. Let:

$$\Lambda_t = \Lambda_0 + \Lambda Y_t = \Lambda_0 + \sum_{s=1}^{n} \Lambda_s j_s' Y_t \tag{22}$$

where (like Λ_t), $\Lambda_s; s = 0, ..., m$ are deficient. Using (2) and (21) we see that M_{t+1} is lognormally distributed conditional upon Y_t.

This model is just identified in the standard homoscedastic case $m = 0$, otherwise it is overidentified. Appendix 1 shows that the general model ((8), (12) (14) and (15), with $m = n$) can be interpreted using the SDF approach by assuming that:

$$\Lambda = 0_{N^2} \tag{23}$$

and defining the risk-adjusted parameters as:

$$\tilde{\mathcal{K}} = \mathcal{K} - \Xi \tag{24}$$

$$\tilde{\Theta} = \Theta - \ \Omega_0 \Lambda_0$$

where Ξ is an N^2 adjustment matrix with the columns:

$$\xi^s = \ \Omega_s \zeta_s \ ; s = 1, ..., n \tag{25}$$

$$= 0_{N,1} \ ; s = n+1, ..., N.$$

Recall that these adjustments determine the risk premia in an arbitrage-free model. Substituting (25) into (11) shows that the premium on a $\tau-$period discount bond is equal to the covariance between the $\tau - 1$ period bond price and the SDF. In the general model these premia depend entirely upon the effect of changes in the macroeconomic variables on volatility. This model is just identified.

When $0 < m < n$, two different types of specification are possible. If we maintain (23) we get the affine class, in which the time variation in the risk premia still depends upon changes in stochastic volatility. Alternatively, we get the 'essentially affine' class, as defined by (Duffee (2002)), in which variations in Y_t can affect the premia. Appendix 1 shows that in this case the shift from the observed probability measure to the risk-neutral pricing measure is effected by using the risk adjustment (24) where now:

$$\xi^s = \ \Omega_0 \Lambda_s \ + \ \Omega_s \zeta_s \ ; s = 1, ..., m \tag{26}$$

$$= \Omega_0 \Lambda_s \ ; s = m+1, ..., n$$

$$= 0_{N,1} \ ; s = n+1, ..., N.$$

The risk parameters in the first line are not separately identified so we set $\Lambda_s = 0_{m,1}; s = 1, ..., m$ to resolve the indeterminacy. Obviously, for $m < n$, the essentially affine specification encompasses the affine one with the same number of volatility terms.

3.2 Admissible models

As noted, the variance structure of the general affine model is endogenous and there is nothing to ensure it remains non-negative. To deal with this problem Dai and Singleton (2000) analyse specializations that are 'admissible' in the sense that they ensure a non-negative definite variance structure when the factors underpinning this structure are continuous random variables. Using their notation, an admissible model with m volatility terms, N state variables and constant Λ_t (23) is denoted $\mathbb{A}_m(N)$. An admissible 'essentially affine' model results if we assume (22) instead of (23) and is denoted $\mathbb{EA}_m(N)$. These models are special cases of the general affine specification developed in the previous section, represented in this paper as $\mathbb{GA}_m(N)$.

The first of these admissible specifications is the standard homoscedastic model $\mathbb{EA}_0(N)$ of the macro-finance literature. In this case the quadratic terms in (14) vanish, making this system linear:

$$\Psi_\tau = \tilde{\mathcal{K}}'\Psi_{\tau-1} + j_r \tag{27}$$

If the roots of the dynamic response matrix $\tilde{\mathcal{K}}$ are less than unity in absolute value then there is an elementary solution for the factor coefficients:

$$\Psi_\tau = (I_N - \tilde{\mathcal{K}}')^{-1}(I_N - (\tilde{\mathcal{K}}')^\tau)j_r \tag{28}$$

In this case the asymptotic factor coefficients are constant $\Psi^* = (I_N - \tilde{\mathcal{K}}')^{-1}j_r$ and so is the rate structure (given (17)). However, it is well known that this specification has irregular asymptotic properties when the RN dynamics are non-stationary (Dewachter and Lyrio (2003)). Campbell, Lo, and MacKinlay (1996) provide a useful discussion of the single factor discrete time case. Obviously, if (7) has a unit root then so does (27). Consequently, the long maturity slope coefficients Ψ^*_τ are time (or maturity) trends not constants and long rates turn negative, tending to minus infinity with maturity. If the system is transformed so that the first variable of the system follows a unit root (or near-unit root) process and the others are stationary (see next section) then it can be shown (by suitably factorising (7) and (27)) that the associated factor loading follows a time trend asymptotically while the others are constant. Consequently, this variable dominates the behaviour of the long term yields and their risk premia. In the case of a near-unit root, the asymptotic forward rate is well defined, but adopts a very large negative value. Again, this root dominates the behaviour of the long maturity yields.

Empirically, as Dewachter and Lyrio (2003) observe, the historical US data which has been used in this literature appears to have a non-stationary common trend related to the underlying rate of inflation, making this problematic[5]. However, Campbell, Lo, and MacKinlay (1996) note that the asymptotic slope parameters are determined by *quadratic* equations in the general affine model and can be positive even if the model has a stochastic trend. To investigate this possibility, we develop an admissible model $\mathbb{EA}_1(N)$ which conditions the central tendency and the variance structure of the model on the stochastic trend variable. Mathematically, this model has regular variance and asymptotic term structures as well as providing a plausible description of the relationship between the risk premia and the conditioning variable.

[5] It may be tempting to regard this as a theroetical curiosity, but with the US Treasury resuming 30 year issuance and the British and French Treasuries issuing 50 year bonds, this asymptotic behaviour is now a practical consideration.

Empirically, unlike the standard model $\mathbb{EA}_0(N)$, this model is accepted as a data-consistent simplification of the general affine model $\mathbb{GA}_m(N)$ described by (7), (12), (14) and (15).

3.2.1 Single conditioning factor models: $\mathbb{A}_1(N); \mathbb{EA}_1(N); \mathbb{GA}_1(N);$

This single factor framework is designed to accommodate the salient characteristics of the historic US data commonly used in this literature. I specialise this in a way that allows regular variance and yield properties to be preserved. As we have seen, interest and inflation rates appear to have a non-stationary common trend related to the underlying rate of inflation and there is now mounting evidence that their volatility is related to a similar trend. This suggests a model with a single factor ($m = 1$) determining mean values and conditioning volatility. This is the macro-finance analogue of $\mathbb{EA}_1(N)$: the preferred model of Dai and Singleton (2002). As Dai and Singleton (2000), note the single factor specification avoids the awkward features of higher order conditioning models like $\mathbb{A}_2(N)$ and $\mathbb{A}_3(N)$ and their essentially affine equivalents.

Suppose that the original data consists of an $n-$vector of time $t-$observable variables $x_t = \{x_{1t}, x_{2t}, ..., x_{nt}\}'$; $t = 1, ..., T$ described by a Vector Auto-Regression or VAR congruent with (1):

$$x_t = \varsigma + \Sigma_{l=1}^{L} \Phi_l x_{t-l} + u_t \tag{29}$$

$$u_t \sim N(0, \Sigma_t); t = 1, ..., T'$$

where $u_t = H\nu_t$ is a $n-$vector of stochastic error terms and $\Sigma_t = H\Delta_t H'$. The state space representation is:

$$X_t = \Upsilon + \Phi X_{t-1} + U_t \tag{30}$$

where: $\Upsilon' = \{\upsilon', 0_{1,N-n}\}$ & $U'_t = \{u'_t, 0_{1,N-n}\}$ are deficient $N = n \times L$ vectors; $X'_t = \{x_t, x_{t-1}, ..., x_{t-L}\}$ and:

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & ... & \Phi_{L-1} & \Phi_L \\ I_n & 0_{n,n} & ... & 0_{n,n} & 0_{n,n} \\ 0_{n,n} & I_n & ... & 0_{n,n} & 0_{n,n} \\ & & & & \\ 0_{n,n} & 0_{n,n} & ... & I_n & 0_{n,n} \end{bmatrix} \tag{31}$$

is an N^2 matrix. Under the measure $\mathcal{Q}$ the dynamics are of the form (30), but with $\tilde{\Upsilon}$ & $\tilde{\Phi}$ replacing Υ & Φ.

To get an admissible model in which there is a single factor conditioning the variance structure ($\mathbb{A}_1(N)$ or $\mathbb{E}\mathbb{A}_1(N)$), we need to transform this model into a recursive system in which the conditioning factor $y_{1,t}$ is linear in X_t:

$$y_{1,t} = \upsilon_1 + \mu_1 X_t$$

and is determined by an independent $I(1)$ or $AR(1)$ model under both measures $\mathcal{P}$ and $\mathcal{Q}$. Under the risk neutral measure used for pricing we model this as:

$$y_{1,t} = \tilde{\kappa}_{11} y_{1,t-1} + w_{1,t} \tag{32}$$

$$w_{1,t} \sim N(0, \delta_{1,1}\ y_{1,t-1});\ \ \delta_{1,1} \geq 0.$$

To conform to (32), the vector of coefficients defining the volatility factor μ_1 must be an eigenvector of $\tilde{\Phi}$ and $\tilde{\kappa}$ must be the associated eigenvalue. The first element of μ_1 is normalised to unity: $\mu_1 = \{1, \mu_1^D\}$. This restriction saves $N-1$ degrees of freedom compared with the unrestricted model $\mathbb{G}\mathbb{A}_1(N)$. Replacing $x_{1,t}$ with $y_{1,t}$ in x_t (keeping the remaining elements, x_t^D unchanged) and partitioning $\mu_1 = \{\mu_0, \mu_1^D, ..., \mu_{L-1}^D\}$ (with $\mu_0 = \{1, \mu_0^D\}$) conformably with X_t gives $y_t = \{y_{1,t}, x_t^D\}$, where:

$$y_t = \upsilon + \sum_{l=0}^{L} \mathcal{M}_l x_{t-l} \tag{33}$$

$$where\ :\ \upsilon = \begin{bmatrix} \upsilon_1 \\ 0_{n-1,1} \end{bmatrix}; \mathcal{M}_0 = \begin{bmatrix} 1 & \mu_0^D \\ 0_{n-1,1} & I_{n-1} \end{bmatrix};$$

$$\mathcal{M}_l = \begin{bmatrix} \mu_l^D \\ 0_{n-1,n} \end{bmatrix}, l = 1, ..., L-1; \mathcal{M}_L = 0_{n,n}$$

with the inverse:

$$x_t = \mathcal{M}_0^{-1}[y_t - \upsilon - \sum_{l=1}^{L} \mathcal{M}_l x_{t-l}\ ]\ ;\ \ \mathcal{M}_0^{-1} = \begin{bmatrix} 1 & -\mu_0^D \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}. \tag{34}$$

Making a similar replacement in X_t gives the transformed state vector:

$$Y_t = \begin{bmatrix} y_{1,t} \\ X_t^D \end{bmatrix} = \begin{bmatrix} \upsilon_1 \\ 0_{N-1,1} \end{bmatrix} + \begin{bmatrix} 1 & \mu_1^D \\ 0_{N-1,1} & I_{N-1} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ X_t^D \end{bmatrix} \tag{35}$$

$$= \mathcal{H} + \mathcal{M} X_t$$

Premultiplying (30) by $\mathcal{M}$ puts this in the form (1): $Y_t = \Theta + \mathcal{K} Y_{t-1} + W_t$, where:

$$\begin{array}{ccc} \mathcal{K} = \mathcal{M}\Phi\mathcal{M}^{-1} & & \Phi = \mathcal{M}^{-1}\mathcal{K}\mathcal{M} \\ \Theta = (I-\mathcal{K})\mathcal{H} + \mathcal{M}\Upsilon & \Leftrightarrow & \Upsilon = \mathcal{M}^{-1}[\Theta - (I-\mathcal{K})\mathcal{H}] \\ W_t = \mathcal{M} U_t & & U_t = \mathcal{M}^{-1} W_t \\ C = \mathcal{M}_0 H & & H = \mathcal{M}_0^{-1} C \end{array} \tag{36}$$

which implies the yield structure (19). This is an invariant transform as defined by Dai and Singleton (2000) because it preserves the dynamic characteristics of the spot rate.

The $\mathbb{A}_1(N)$ and $\mathbb{EA}_1(N)$ models specify the factor dynamics as a recursive system in which the conditioning variable is determined independently of the remaining state variables, but can then influence the means and variances of these variables. Importantly, because the instantaneous volatility of $w_{1,t}$ is proportional to $y_{1,t-1}$, the variance of the shocks to this factor goes to zero if it nears zero, making non-negative values very unlikely. If the conditioning factor is a continuous random variable as in Dai and Singleton (2000) the admissibility conditions ensure that this factor (and hence the variance structure) is non-negative definite. $\mathbb{GA}_1(N)$ provides a test of the admissibility restrictions. It conditions the macro variances on $y_{1,t}$ but drops the exclusion restrictions on the first row of $\mathcal{K}$ and $\tilde{\mathcal{K}}$, increasing the number of parameters by $(N-1)$ $+(n-1)$ compared to the specification $\mathbb{EA}_1(N)$. It thus encompasses $\mathbb{A}_1(N)$ and $\mathbb{EA}_1(N)$ (as well as $\mathbb{EA}_0(N)$ which is specified next).

3.2.2 The homoscedastic model: $\mathbb{EA}_0(N)$

The standard macro-finance model ($\mathbb{EA}_0(N)$) is admissible simply because the variance structure is constant: $\Omega_s = 0; s = 1, ..., m$. To capture time variation in the risk premia, Λ_t is instead assumed to depend upon the conditioning variables (22). In principle, they could be conditioned by any of the N state variables, but to preserve degrees of freedom, it is usually assumed that only a few primary variables are relevant. In this case we use y_{t-1}. This makes the model comparable with the other models and has a greater degree of explanatory power than conditioning on the un-transformed variables x_{t-1}. The model maintains the exclusion restrictions on $\mathcal{K}$ and saves $N-1$ degrees of freedom compared to $\mathbb{GA}_1(N)$. These are in addition to the n saved by the assumption of heteroscedasticity .

3.2.3 The encompassing model: $\mathbb{GA}_n(N)$

This model assumes $m = n$ and thus allows both volatility and the risk premia to depend upon all of the stochastic variables (y_{t-1}). All of the other models are nested within it. Unfortunately admissible specifications of order $m > 1$ have some awkward properties which are unlikely to match those of the data, as noted by Dai and Singleton (2000), so admissibility restrictions are not imposed on this model. Dropping the exclusion restrictions on the first row of $\mathcal{K}$ increases the number parameters by $N-1$ and adding in an extra $N-1$ volatility terms adds another $N(N-1)$ compared to the specification $\mathbb{EA}_0(N)$.

4 Model estimation and evaluation

The specification of the previous section provides various descriptions of the macroeconomy and discount bond markets in an arbitrage-free world. But can any of these provide a plausible parsimonious description of the data generating process? This section describes the data set and empirical results.

4.1 The data

The macromodel used in this research was initially based on the specification developed by Svensson (1999); Rudebusch (2002); Smets (1999) and others. It represents the behaviour of the macroeconomy in terms of the output gap (g_t); the annual CPI inflation rate (π_t) and the 3 month Treasury Bill rate $(r_{1,t})$[6]. The output gap series was the quarterly OECD measure, derived from a Hodrick-Prescott filter, the other data series were provided by Datastream. These macroeconomic data are shown in

[6] In view of the doubts about the short yield interpolation procedure of McCulloch and Kwon (1991) expressed by Ahn and Gao (1999) and others, I used this in preference to their three month maturity yield interpolation.

chart 1.

This specification is often called the 'central bank model' since it provides a basic dynamic description of an economy in which the central bank targets inflation using a Taylor rule. However, it can generate puzzling dynamic responses, because the policy interest rate usually anticipates inflationary developments. To address this problem, we follow Grilli and Roubini (1996) and introduce a long bond yield (r_t^*) into the macromodel. This reflects long term inflation expectations and allows the yield differential against Fed Funds to reflect the stance of monetary policy (Estrella (2005)). I used the 17 year discount bond yield, the longest for which a continuous series is available ($r_t^* = r_{68,t}$). Together with the other yield data, this was taken from McCulloch and Kwon (1991), updated by the New York Federal Reserve Bank[7]. These data have been extensively used in the empirical literature on the yield curve. To represent this curve I use 1,2,3, 5,7 and 10 year maturities. Historical data for longer maturities are sparse and seldom used in empirical yield curve analysis. These yield data are available on a monthly basis, but the macroeconomic data dictated a quarterly time frame (1961Q4-2004Q1, a total of 170 periods). These yield data are shown in chart 2. The 10 year yield is shown at the back of the chart, while the shorter maturity yields are shown at the front.

Table 1 shows the means; standard deviations and first order autocorrelation coefficients of these data. It also shows ADF test results: for this test and sample size, the 5% critical value of the ADF test statistic is 2.83. The ADF test suggest that we reject the null hypothesis of non-stationarity in the case of the output gap. however, it provides clear evidence of stochastic trends in the inflation, short and long rate variables. Importantly, the real yield ($r_t^* - \pi_t$) and the yield gap ($r_t^* - r_{1,t}$) have acceptable ADF statistics (respectively -3.18 and -2.99, but not reported in

[7] I am grateful to Tony Rodrigues of the New York Fed for supplying a copy of this yield dataset.

the table), supporting the hypothesis that there is a single stochastic trend in these data.

4.2 Testing the empirical models

Preliminary work designed to estimate the dimensionality of the model estimated OLS regression equations for the four macro variables. These are in the order: the long rate (r^*), the inflation rate (π), the output gap (g); and the 3-month Treasury bill discount rate (r)[8]. This system was estimated for $L = 2, 3, 4$ and 5 lags, with both fixed and heteroscedastic error structures and the results suggested the use of a three-lag model. This gives a system with a vector X_t of twelve state variables (i.e. current and two lagged values of each macro variable). Existing macro finance models invariably use a first order dynamic specification, without testing this restriction.

Table 2 shows the likelihood statistics for the models specified in section 3. Recall that model M1 (i.e. $\mathbb{A}_1(12)$) conditions the volatility upon $y_{1,t}$ and defines the risk premia consistently with this error specification as in (11). It uses 79 parameters as indicated in the second column of Table 2: θ $(4\times1-1); \mathcal{K}$ $(4\times12-11); \mathcal{M}_1(11); C(6);$ $\lambda_0(4); \lambda_1(4); \Delta_0(4-1);$ $\Delta_1(4); \upsilon(1)$ and $P(6)$. Model M2 is the 'essentially affine' version of this model $\mathbb{EA}_1(12)$ and uses another 9 parameters (for Λ) and M3 is the 'general affine' version ($\mathbb{GA}_1(12)$). We reject M1 in favour of M2. M4 is the standard homoscedastic error version of this model, $\mathbb{EA}_0(12)$. It is nested within models M3 and M5, the general 4-factor stochastic volatility model ($\mathbb{EA}_4(12)$) and decisively rejected against both. The $\mathbb{EA}_1(12)$ specification, M2, is however an acceptable simplification of both M3 and M5. As in Dai and Singleton (2002), it is therefore the preferred specification.

[8] As in a VAR analysis, the ordering of the variables in the vector y_t does not affect the reduced form results, but it does affect the impulse responses, discussed in the next section.

5 Model parameters and properties

These tests strongly support the hypothesis underpinning the $\mathbb{EA}_1(12)$ specification: of a single variable conditioning the variance structure in an admissible way. But what light does this model throw upon the issues raised in the introduction? In particular, how well does handle the unit root problem? What does it say about the efficacy of monetary policy?

5.1 The empirical macro model

We now look at the characteristics of this specification (model M2) in detail. Results for the other models are available upon request from the author. Table 1 reports the basic goodness of fit statistics for the ten equations comprising M2. The first row shows that the model explains 94% of the variance at the short end of the yield curve, rising gradually to 98% in the 10 year area. The parameters of the model are set out in Tables 3 and 4. These are generally well determined, although as we would expect in VAR type analysis, some of the off diagonal dynamic coefficients are insignificant. As in previous studies, some of the risk parameters are poorly determined.

The time variation in the error structure and the risk premia is driven by the transformed variable:

$$\begin{array}{lllll}
y_{1,t} = r_{68,t} & +0.011567 & -0.063663\pi_t & +0.001033 g_t & -0.031542 r_{1,t} \\
 & (31.44) & (-5.59) & (0.08) & (-2.55) \\
 & +0.061046 r_{68,t-1} & +0.055560\pi_{t-1} & -0.065624 g_{t-1} & +0.037318 r_{1,t-1} \\
 & (4.20) & (5.08) & (-5.57) & (2.99) \\
 & + -0.100899 r_{68,t-1} & +0.0300947\pi_{t-2} & +0.026869 g_{t-2} & -0.068729 r_{1,t-2} \\
 & (-6.93) & (2.55) & (2.24) & (5.61)
\end{array}$$

This volatility factor is clearly dominated by the current value of the long bond yield[9]. However the contribution of other variables is significant, particularly the change in the rate of inflation. The $y_{1,t}$ estimates are plotted against the rate of inflation and the spot rate in chart 3. Solving the model recursively conditional upon $y_{1,t}$ shows the steady state effect of a unit increase in $y_{1,t}$ is to raise the steady state rate of inflation by 0.445 and the spot rate by 0.9390 percentage points, implying a rise in the real rate of interest.

We now consider the dynamic properties of the macro model in terms of (a) its eigenvalues and (b) impulse responses. The autoregressive coefficient associated with $y_{1,t}$ is $\kappa_{11} = 0.98809$. This increases to $\tilde{\kappa}_{11} = 0.99044$ under the measure $\mathcal{Q}$, which is very close to unity. Nevertheless, because this drives both first and second moments, the asymptotic forward rate is positive: 3.79%. In contrast the asymptotic forward rate from the homoscedastic equivalent, model M4 is (-)40.05%, reflecting the near-unit root problem discussed in the previous section. It may be shown analytically that with $\tilde{\phi}_{11} > \phi_{11}$, the asymptotic risk premia are positively related to the stochastic trend $y_{1,t}$. Chart 4 shows a times series plot of the risk premia for selected maturities. $\tilde{\kappa}_{11}$ is the principal eigenvalue of the adjustment matrix Φ. The other eigenvalues of Φ are shown in table 5. These indicate a much faster adjustment than in the case of $y_{1,t}$. Three pairs of roots are sinusoidal, reflecting the cyclical nature of the macroeconomic data.

These cyclical effects are seen more clearly in the impulse responses, which show the dynamic effects of innovations in the macro variables on the system. Because these innovations are correlated empirically, we work with orthogonalised innovations using the triangular factorisation of Σ_t (defined in (29)). This is evaluated as $\bar{\Sigma} =$

[9] The models of liquidity premia developed by Glosten and Runkle (1993) and Scruggs (1998) are similar in this respect. The first conditions volatility on the yield gap and the second on a short term interest rate.

$\mathcal{M}_0^{-1}C(\Delta_0+\Delta_1\bar{y}_1)C'(\mathcal{M}_0')^{-1}$ at the mean value ($\bar{y}_1=0.07557$) of the scale variable $y_{1,t}$ and factorised as $G\digamma G'=\bar{\Sigma}$ where:

$$G=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.06546 & 1 & 0 & 0 \\ 0.34674 & -0.02816 & 1 & 0 \\ 0.20532 & 0.28696 & 0.10548 & 1 \end{bmatrix} \tag{37}$$

:

$$\digamma = diag\{1.11386\times 10^{-5}, 6.19261\times 10^{-6}, 8.50651\times 10^{-6}, 2.33839\times 10^{-5}\} \tag{38}$$

This allows us to calculate the perturbations u_t in the macro variables x_t in a way which allows for their contemporaneous correlation using the arrangement:

$$u_t = G\varepsilon_t \tag{39}$$

where ε_t is vector of four shocks which being orthogonal can be varied independently. The orthogonalised impulse responses show the effect on the macro system of increasing each of these shocks by one percentage point for just one period using the Wald representation of the system as described in appendix 3.

This arrangement is affected by the ordering of the macro variables in the vector x_t, making it sensible to order the variables in terms of their likely degree of exogeneity or sensitivity to contemporaneous shocks. The effect of fast-mean reverting macroeconomic shocks on the long yield should in principle be small, leaving this to reflect slow-moving expectational influences, so this is ordered first in the sequence. This means that independent shocks to output inflation and interest rates can then be interpreted as sudden shocks that are not anticipated by the bond market. Following (Hamilton (1994)) inflation is ordered before the output gap, on the Keynesian view that macroeconomic shocks are accommodated initially by output rather than price[10]. Interest rates are placed after these variables on the view that monetary policy reacts relatively quickly to disturbances in output and prices. Thus the variable ordering is: long bond yield; inflation, output gap and spot rate. This means that shocks to the long yield (ε_1) disturb all four variables contemporaneously as indicated by the first column of the matrix shown in (37), independent shocks to inflation (ε_2) affect output and interest rates but not the long yield, and so on.

Chart 5 shows the results of this exercise. The continuous line shows the effect of each independent shock on the spot rate, the dashed line the effect on the long yield, the broken line the effect on inflation and the dotted line the effect on output. Elapsed time is measured in quarters. Panel (i) shows the effect of a shock to the long bond yield (ε_1). This might reflect an increase in the bond market's expected rate of inflation or the underlying real rate of return in the economy. Output and the spot rate increase immediately, but inflation does not (37), meaning that real interest rates increase initially. The long yield acts as a leading indicator for both output and inflation. Output peaks after one year and inflation after three 3 years. The increase in real rates then causes a large fall in output, which brings inflation back close to its initial level after 10 years. After that, there are further cycles in inflation (which are not shown in the charts) but these are heavily damped. In contrast, interest rates adjust downward very slowly, remaining high in real terms, reflecting the near-unit root in the macromodel which is associated with the long bond yield.

Panel (ii) shows the effect of an independent shock to inflation (ε_2), essentially an

[10] Since the contemporaneous correlation betweeen inflation and output is very low (reflected in the coefficient $g_{32} = -0.02816$ in (37)) the ordering of these variables makes no material difference to the impluse responses.

inflationary impulse that is not anticipated by the bond market. The initial effect on the spot rate is only about a quarter of a point, so real interest rates fall. However, output falls back, reaching a trough after falling by 0.8% after two years, reflecting real balance and other contractionary inflationary effects. The fall in output has the effect of reversing the rise in inflation, setting up cycles in these variables. However, these are heavily damped, reflecting the complex eigenvalues shown in table 4. In contrast to the effects of the long bond yield shown in the first panel which are highly persistent in the case of interest rates, the system is close to its initial level after 10 years following this inflationary impulse. The other two panels show similarly fast responses. The spot rate increases by 0.6% in response to an independent output shock, and this together with the contractionary real balance effect of higher inflation moderates the expansionary impulse. The effect of a rise in the spot rate is shown in the final panel. The initial effect is to depress output, but inflation responds with a short lag. Indeed, the lower inflation rate apparently boosts output by positive real balance effects after two years.

Chart 8 shows the results of an ANOVA study which decomposes the conditional forecast variance of each of the 3 macro variables into the separate effects of surprises to the four orthogonal shocks defined in (39). (The variance of the long bond yield is dominated by the innovations in this variable and these results are not reported.) These effects are calculated using the method described in appendix 3. Initially the variances of these variables are strongly influenced by their own innovations. However the influence of the long bond innovations builds up over time, particularly in the case of the spot rate, where this explains over half of the total forecast variance 10 years ahead.

5.2 The empirical yield model

The impulse response patterns for the bond yields are determined using (49) and so depend upon the sensitivity of each yield to the macro factors (the beta coefficients or factor loadings) and the sensitivity of the macro factors to shocks (given by the impulse responses of the previous section). Chart 6 shows the factor loadings as a function of maturity expressed in quarters. These loadings depend upon the risk adjusted dynamics, reflected in the eigenvalues of the matrix $\tilde{\Phi}$ reported in Table 4.

The first panel shows the loadings on $r_{1,t}$ (continuous line) and $r_{68,t}$ (broken line). The spot rate is the link between the macro model and the term structure. Since it *is* the 3 month yield, this variable has a unit coefficient at a maturity of one quarter and other factors have a zero loading (12). The spot rate loadings then tend to decline monotonically with maturity, reflecting the relatively fast adjustment process. In contrast, the slow-moving nature of the long yield means that its loading increases with maturity over most of this range. The next panel shows the loadings on π (dotted line) and g (broken line) which are relatively small and exhibit a humped shape.

The impulse response patterns for the yield model are shown in Charts 8 (a)-(d). These show the effects on the 1 - 10 year yields of the independent shocks described in the previous section. Although the model is fitted using only 6 yield observations in each period, in principle it can be used to compute the yield response at any maturity. The loading pattern means that the impulse response patterns for the short maturity yields are similar to those for the spot rate. Consistent with those results, the effects of macro shocks disappear very quickly, while the expectational and other effects associated with the long yield are very persistent.

Recall that Chart 4 shows the holding period risk premia (annualised one-period ahead expected excess returns) implied by the model. Although Λ_t is flexible in the $\mathbb{EA}_1(N)$ specification, most of the time variation in the risk premia results from variation in the stochastic volatility term. The premia rise and then fall with the degree of macroeconomic volatility. The risk premia tend to increase over the maturity range shown in the chart, largely as a consequence of the increase in the factor loadings on $y_{1,t}$. As noted, this stochastic trend dominates the behaviour of the long maturity premia.

The lower right hand panel of Chart 8 decomposes the conditional forecast variance of the 5 year yield into the separate effects of surprises to the four orthogonal shocks defined in (39). The ANOVA charts for the 7&10 year yields show a similar pattern. These effects are calculated using the method described in appendix 3. Innovations in the three macro have a modest contribution for near-term forecasts, but are increasingly dominated by innovations in the long bond innovations. This explains over 95% of the total forecast variance 10 years ahead.

6 Conclusion

Heteroscedasticity is a common feature of macroeconomic and financial data. The work reported in this paper reflects this feature, providing strong support for the specific hypothesis that the volatilities of US macroeconomic data are influenced by the underlying level of inflation, reflected in long term interest rates and other nominal variables. This finding has major implications for economic policy and the financial markets. The specification developed here extends the new macro-finance model of the yield curve to allow for macroeconomic volatility, bringing it into line with the mainstream finance model with its emphasis on stochastic volatility.

The preferred specification $\mathbb{EA}_1(N)$ is an admissible model which conditions the central tendency and the variance structure of the model on the stochastic trend variable. Although it is of the 'essentially affine' class and allows the risk premia to depend in a flexible way upon variations in macroeconomic variables, these premia are dominated by the underlying stochastic volatility trend. This trend is closely associated with the long bond yield. The use of this yield in the VAR underpinning the system helps to solve the price and other puzzles that have hampered empirical work on the basic central bank model of monetary policy (Grilli and Roubini (1996)). Initial dynamic specification tests suggested a third order system, indicating that the first order specification assumed in the existing macro finance literature is too restrictive.

This VAR gives a plausible description of the macro dynamics, with the long yield apparently acting as a proxy for slow-moving exogenous influences on output and inflation. These influences could reflect autonomous output and inflation expectations (as for example in the model of Dewachter and Lyrio (2003)), or perhaps shifts in the monetary authorities target for inflation . Shocks to the stochastic trend are highly persistent, but the system is back close to its initial values after a five year period following independent shocks to output inflation and interest rates. Three pairs of roots are sinusoidal, reflecting the cyclical nature of macroeconomic data. Short term yields are naturally dominated by short run fluctuations in the macroeconomic variables driving the spot rate. But as maturity increases these effects decay away quickly, reflecting the remarkably fast conditional mean reversion of the macro variables. This leaves the 10 year rate dominated by fluctuations in the long bond yield.

Mathematically, this model has regular variance and asymptotic term structures as well as providing a plausible description of the relationship between the risk premia and the conditioning variable. Empirically, unlike the standard model $\mathbb{EA}_0(N)$, this model is accepted as a data-consistent simplification of the general affine model

$\mathbb{GA}_m(N)$. It provides a 12-factor explanation of the behaviour of the Treasury curve, keying it in to the behaviour of the macroeconomy. It can use a relatively large number of factors because the parameters of the model are informed by the macro data as well as the yield curve (with a total of 1700 data points). This research opens the way to a much richer term structure specification, incorporating the best features of the macro-finance and mainstream finance model.

References

Ahn, D., and B. Gao (1999): "A Parametric Model of Term Structure Dynamics," *Review of Financial Studies*, 12, 721–762.

Ang, A., and M. Piazzesi (2003): "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 43, 75–87.

Ball, L. (1992): "Why Does Inflation Raise Inflation Uncertainty?," *Journal of Monetary Economics*, 29, 371–388.

Brandt, M., and K. Wang (2003): "Time Varying Risk Risk Aversion and Unexpected Inflation," *Journal of Monetary Economics*, 50, 1457–1498.

Brunner, K., and G. D. Hess (1993): "Are Higher Levels of Inflation Less Predicatable?," *Journal of Business and Economic Statistics*.

Campbell, J., A. Lo, and A. MacKinlay (1996): *The Econometrics of Financial Markets*. Princeton University Press, Princeton, N.J.

Caporale, T., and B. McKiernan (1997): "High and Variable Inflation: Further Evidence on the Friedman Hypothesis," *Economics Letters*, 54, 65–68.

Cochrane, J. (2000): *Asset Pricing*. Princeton University Press, Princeton, N.J.

Dai, Q., and K. Singleton (2000): "Specification Analysis of Affine Term Structure Models," *Journal of Finance*, 55, 415–41.

DAI, Q., AND K. SINGLETON (2002): “Expectation Puzzles, Time-Varying Risk Premia and Affine Models of the Term Structure,” *Journal of Financial Economics*, 63, 415–441.

DEWACHTER, H., AND M. LYRIO (2003): “Macro Factors and the Term Structure of Interest Rates,” *mimeo, Leuven, forthcoming, Journal of Money, Credit and Banking.*

DUFFEE, G. (2002): “Term Premia and Interest Rate Forecasts in Affine Models,” *Journal of Finance*, 62, 405–443.

DUFFIE, D., AND R. KAN (1996): “A Yield-Factor Model of Interest Rates,” *Mathematical Finance*, 6, 379–406.

ENGEL, R. (1982): “Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of UK Inflation,” *Econometrica*, 50, 987–1010.

ESTRELLA, A. (2005): “Why Does the Yield Curve Predict Output and Inflation?,” *The Economic Journal*, 115, 722–744.

FISCHER, S HALL, R., AND J. TAYLOR (1981): “Relative Shocks, Relative Price Stability and Inflation,” *Brookings Papers on Economic Activity*, No. 7144, 381–441.

FRIEDMAN, M. (1977): “Nobel Lecture: Inflation and Unemployment,” *Journal of Political Economy*, 85, 451–472.

GLOSTEN, L.R., J. R., AND D. RUNKLE (1993): “On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks,” *The Journal of Finance*, 48, 1779–1801.

GRILLI, V., AND N. ROUBINI (1996): “Liquidity Models in Open Economies: Theory and Empirical Evidence.,” *European Economic Review*, 40, 847–857.

HAMILTON, J. (1994): *Time Series Analysis*. Princeton University Press, Princeton.

HARRISON, J., AND D. KREPS (1979): “Martingales and Arbitrage in Multiperiod

Securities Markets," *Mathematical Finance*, 20, 349–408.

HOLLAND, A. S. (1995): "Inflation and Uncertainty: Tests for Temporal Ordering," *Journal of Money, Credit and Banking*, 27, 827–837.

LETTAU, M. LUDVIGSON, S., AND J. WACHTER (2004): "The Declining Equity Premium: What Role Does Macroeconomic Risk Play?," *NBER and New York Unversity Working Paper Series*.

MCCULLOCH, J., AND H.-C. KWON (1991): "Us Term Structure Data," *Manuscript, Ohio State University*.

RUDEBUSCH, G. (2002): "Assessing Nominal Income Rules for Monetary Policy with Model and Data Uncertainty," *The Economic Journal*, 112, 402–32.

SCRUGGS, J. (1998): "Resolving the Puzzling Intertemporal Relation Between the Market Risk Premium and Conditional Market Variance," *The Journal of Finance*, 53, 575–603.

SMETS, F. (1999): "Output Gap Uncertainty: Does It Matter for the Taylor Rule?," in *Monetary Policy under Uncertainty*, ed. by B. Hunt, and A. Orr, pp. 10–29, Wellington. Reserve Bank of New Zealand.

SVENSSON, L. (1999): "Inflation Targetting as a Monetary Policy Rule," *Journal of Monetary Economics*, 43, 607–54.

7 Appendix 1: The SDF approach

This appendix uses the SDF approach to derive the constraints across the yield coefficients implied by the specifications that are described in section 2. The conditional lognormality of the SDF and prices allows us to write (20) as:

$$p_{\tau,t} = \log P_{\tau,t} = E_t[m_{t+1} + p_{\tau-1,t+1}] + \frac{1}{2}V_t[m_{t+1} + p_{\tau-1,t+1}]; \quad \tau = 1, ..., M. \quad (40)$$

Substituting (1),(4),(8) & (21) into this equation and evaluating means and variances:

$$
\begin{aligned}
-p_{\tau,t} = & \, r_{1,t} + \omega_t + \gamma_{\tau-1} + \Psi'_{\tau-1}(\Theta + \mathcal{K} Y_t) \\
& - \frac{1}{2}\sum_{s=1}^{m}(Z_s + \Psi_{\tau-1})'\Omega_s(Z_s + \Psi_{\tau-1})(j_s' Y_t) \\
& - \frac{1}{2}(\Lambda_t + \Psi_{\tau-1})'\Omega_0(\Lambda_t + \Psi_{\tau-1}).
\end{aligned} \tag{41}
$$

In the special case $\tau = 1$, with $p_0 = 0$:

$$
\begin{aligned}
p_{1,t} &= E_t[m_{t+1}] + \frac{1}{2}V_t[m_{t+1}] \\
&= -(r_{1,t} + \omega_t) - \frac{1}{2}\sum_{s=1}^{m} Z_s'\Omega_s Z_s(j_s' Y_t) - \frac{1}{2}\Lambda_t{}'\Omega_0\Lambda_t \\
&= -r_{1,t}
\end{aligned}
$$

(using (8)). This implies the restrictions:

$$
\omega_t = -\frac{1}{2}\sum_{s=1}^{m} Z_s{}'\Omega_s Z_s j_s' Y_t - \frac{1}{2}\Lambda_t{}'\Omega_0\Lambda_t \tag{42}
$$

The equation (13) defining the general affine yield model can be derived using the SDF approach by adopting (4) and (23) with $m = n$. With (42), this simplifies (41) to:

$$
\begin{aligned}
-p_{\tau,t} = & \, r_{1,t} + \gamma_{\tau-1} + \Psi'_{\tau-1}(\Theta - \Omega_0\Lambda_0 + (\mathcal{K} - \sum_{s=1}^{n}\Omega_s\zeta_s(j_s' Y_t) - \frac{1}{2}\Psi_{\tau-1}{}'\Omega_0\Psi_{\tau-1} \\
& - \frac{1}{2}\Psi_{\tau-1}{}'\Omega_s\Psi_{\tau-1}(j_s' Y_t \;)
\end{aligned}
$$

which is of the form (13) with (24) and (25). To represent the essentially affine model with $m < n$, substitute (4) and (42) into (41), simplifying it to:

$$\begin{aligned} -p_{\tau,t} &= r_{1,t} + \gamma_{\tau-1} + \Psi'_{\tau-1}(\Theta - \Omega_0\Lambda_0 + (\mathcal{K} - \sum_{s=1}^{m}\Omega_s\zeta_s j'_s - \Omega_0\sum_{s=1}^{n}\Lambda_s j'_s)Y_t) \\ &-\frac{1}{2}\Psi_{\tau-1}{}'\Omega_0\Psi_{\tau-1} - \frac{1}{2}\Psi_{\tau-1}{}'\Omega_s\Psi_{\tau-1}(j'_s Y_t\ ) \end{aligned}$$

which is of the form (13) with (24) and (26).

8 Appendix 2: The likelihood function

This appendix derives the likelihood function and describes the numerical optimisation procedure. For simplicity I assume $m = 1$, but the likelihood for the $m = n$ specifications follow straightforwardly.

The first n equations of (36) are stochastic:

$$\begin{aligned} y_t &= \theta + \mathcal{K}_1 y_{t-1} + \Sigma_{l=2}^{L}\mathcal{K}_l x_{t-l} + Cu_t \\ u_t &\sim N(0, \Delta_0 + \Delta_1 y_{1,t-1}) \end{aligned} \tag{43}$$

while the rest are identities.

Admissibility (32) is ensured by imposing exclusion restrictions on all but the first element of the first row of $\mathcal{K}$ to get $\{\kappa_{11}, 0_{N-1,1}\}$. Similarly, the first row of $\mathcal{K}_1$ is $\{\kappa_{11}, 0_{n-1,1}\}$ and for $\mathcal{K}_l$ the first row consists of zero coefficients. Since there is a single conditioning factor, the risk adjustment in (24) and (25) only affects the first column of $\tilde{\mathcal{K}}_1$ in the $\mathbb{A}_1(N)$ specification so these exclusion restrictions are preserved under the measure $\mathcal{Q}$. For $\mathbb{E}\mathbb{A}_1(N)$, I set the first row (and column) of Λ_1 to zero, adding an extra $(N-1)^2$ risk coefficients to model $\mathbb{A}_1(N)$. The restricted macromodel is set up under the measure $\mathcal{P}$ as (43), which can be expressed in terms of the original variables (using (33) and (34)) as:

$$x_t = \varsigma + \Sigma_{l=1}^{L} \Phi_l x_{t-l} + u_t \tag{44}$$

$$where\ :\ \varsigma = \mathcal{M}_0^{-1}[\theta - (I_n - K_1)]\upsilon;\quad u_t = \mathcal{M}_0^{-1} w_t;$$

$$\Phi_1 = \mathcal{M}_0^{-1}[\mathcal{K}_1 \mathcal{M}_0 - \mathcal{M}_1]; \Phi_l = \mathcal{M}_0^{-1}[\mathcal{K}_1 \mathcal{M}_{l-1} + \mathcal{K}_l - \mathcal{M}_l], l = 2, ..., L.$$

The restricted yield model is set up under the measure $\mathcal{Q}$ as (19) and then Y_t is replaced using (33):

$$\begin{aligned} r_t &= \alpha(\tilde{\Theta}, \tilde{\mathcal{K}}) + B(\tilde{\Theta}, \tilde{\mathcal{K}})' Y_t + e_t \\ &= \alpha + B_1 y_{t-1} + \Sigma_{l=1}^{L-1} B_l x_{t-l} + e_t \\ &= \psi + \Sigma_{l=0}^{L-1} \pi_l x_{t-l} + e_t \end{aligned} \tag{45}$$

$$where\ :\ \psi = \alpha + B_1 \varsigma; \pi_1 = B_1 \mathcal{M}_0; \pi_l = [B_1 \mathcal{M}_l + B_l], l = 1, ..., L-1.$$

Because the macro and measurement errors are assumed to be orthogonal, the likelihood of the joint model is the sum of macro and yield components. First, consider the macro component. Using (2) & (44) and the fact that C and $\mathcal{M}_0$ have unit determinants, the loglikelihood for period t can be written as:

$$\begin{aligned} LM_t &= -(n/2)\ln(2\pi) - \ln(|\Delta_t|)/2 - \nu_t{}' \Delta_t^{-1} \nu_t/2 \\ &= -(n/2)\ln(2\pi) - \sum_{i=1}^{n} \ln(\delta_{0i} + \delta_{1i}\ [v + \sum_{l=0}^{L} \mathcal{M}_l x_{t-l-1}])/2 \\ &\quad -\nu_t{}'(\Delta_0 + \Delta_1\ [v + \sum_{l=0}^{L} \mathcal{M}_l x_{t-l-1}])^{-1} \nu_t/2 \end{aligned} \tag{46}$$

where:

$$\nu_t = C^{-1}\mathcal{M}_0(x_t - \varsigma - \Sigma_{l=1}^{L}\Phi_l x_{t-l});$$

and where the restricted coefficient matrices ς, Φ_l are defined in (44). The term in square brackets represents $y_{1,t-1}$ using (33). Summing over T periods gives the loglikelihood for a stand-alone VAR: Similarly, (45) and (19) can be used to represent the likelihood of the yield observation r_t:

$$\begin{aligned} LR_t &= -(M/2)\ln(2\pi) - \sum_{\tau=1}^{M}\ln(\rho_\tau)/2 - e_t{}'P^{-1}e_t/2; \\ where &: \\ e_t &= r_t - \psi - \Sigma_{l=0}^{L-1}\pi_l x_{t-l}; \end{aligned}$$

and where the restricted coefficient matrices ψ, π_l are defined in (45). Adding (46) and summing over T periods gives the loglikelihood of the joint system:

$$\begin{aligned} L = &-(T(n+M)/2)\ln(2\pi) \\ &-\sum_{t=1}^{T}\sum_{i=1}^{n}\ln(\delta_{0i} + \delta_{1i}\ [v + \sum_{l=0}^{L}\mathcal{M}_l x_{t-l-1}])/2 - \sum_{t=1}^{T}\sum_{\tau=1}^{M}\ln(\rho_\tau)/2 \\ &-\sum_{t=1}^{T}\nu_t{}'(\Delta_0 + \Delta_1\ [v + \sum_{l=0}^{L}\mathcal{M}_l x_{t-l-1}])^{-1}\nu_t/2 - \sum_{t=1}^{T}e_t{}'P^{-1}e_t/2 \end{aligned}$$

9 Appendix 3. The impulse responses

Define the lag operator L (where $X_{t-1} = LX_t$) and rewrite (30) setting $\Upsilon = 0$ as $(I - \Phi' L)X_t = U_t$. Since its eigenvalues are less than unity in absolute value, this system can be inverted to give the Wald (or MA) representation:

$$
\begin{aligned}
X_t &= (I - \Phi L)^{-1} U_t \\
&= \sum_{i=0}^{\infty} \Phi^i U_{t-i} \\
&= \sum_{i=0}^{\infty} \Phi^i A \mathcal{E}_{t-i}
\end{aligned}
\tag{47}
$$

where

$$
U_t = A\mathcal{E}_t; \quad \mathcal{E}_t = \{\varepsilon_t'; 0_{1,N-n})'; A = \begin{bmatrix} G & 0_{n,N-n} \\ 0_{N-n,n} & 0_{N-n,N-n} \end{bmatrix}
$$

and where ε_t is a set of orthogonal disturbances defined by (39). Similarly, substituting Y_t in (45) using (35) and (47), omitting the intercept constants:

$$
r_t = B'\mathcal{M} \sum_{i=0}^{\infty} \Phi^i A \mathcal{E}_{t-i}. \tag{49}
$$

This representation shows that the impact of the $n-th$ element of ε_t on the $m-th$ element of X_{t+i} is given by element mn of the matrix $\Phi^i A$, while the impact on the $m-th$ element of r_{t+i} is given by element mn of the matrix $B'\mathcal{M}\Phi^i A$.

Similarly the $t-$conditional covariance matrix for X_{t+i} is:

$$
V_t[X_{t+i}] = \sum_{j=0}^{i} \Phi^{i-j} A V_t[\mathcal{E}_{t+j}] A' (\Phi')^{i-j}
$$

Evaluating $V_t[\mathcal{E}_{t+j}]$ at the mean value:

$$\mathcal{M}\bar{\Omega}\mathcal{M}' = \begin{bmatrix} G\digamma G' & 0 \ldots & 0 \\ 0 & 0 \ldots & 0 \\ & & \\ 0 & 0 \ldots & 0 \end{bmatrix}$$

gives:

$$\begin{aligned} V_t[X_{t+i}] &= \sum_{j=0}^{i} \Phi^{i-j} A\mathcal{M}\bar{\Omega}\mathcal{M}'A'(\Phi')^{i-j} \\ .V_t[r_{t+i}] &= \sum_{j=0}^{i} B'\mathcal{M}V_t[X_{t+i}]\mathcal{M}'B \\ &= \sum_{j=0}^{i} B'\mathcal{M}\Phi^{i-j} A\mathcal{M}\bar{\Omega}\mathcal{M}'A'(\Phi')^{i-j}\mathcal{M}'B. \end{aligned}$$

The $m-th$ diagonal element of each matrix shows the conditional variance of the $m-th$ macro variable or yield maturity respectively. The contribution of the $n-th$ element of ε_t is calculated by setting all but the $n-th$ element of $\digamma$ to zero.

Chart 1: Macroeconomic variables

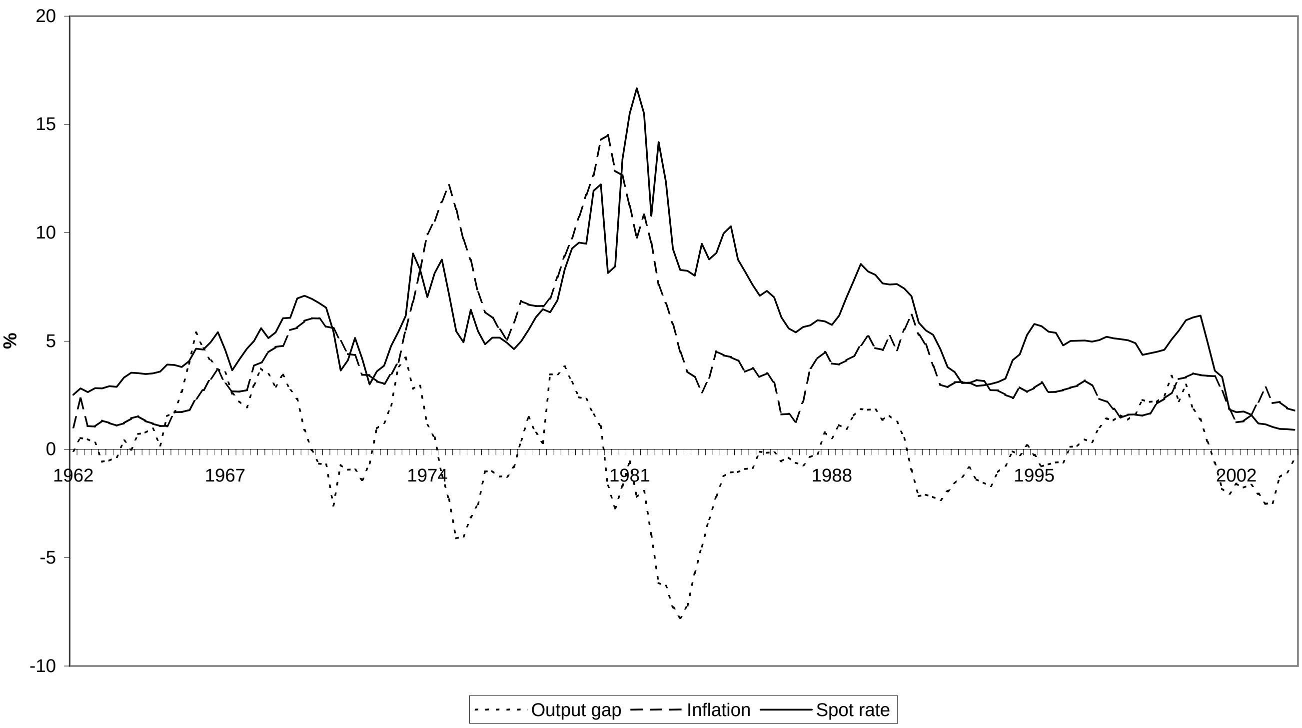


Chart 2: US Treasury discount yields

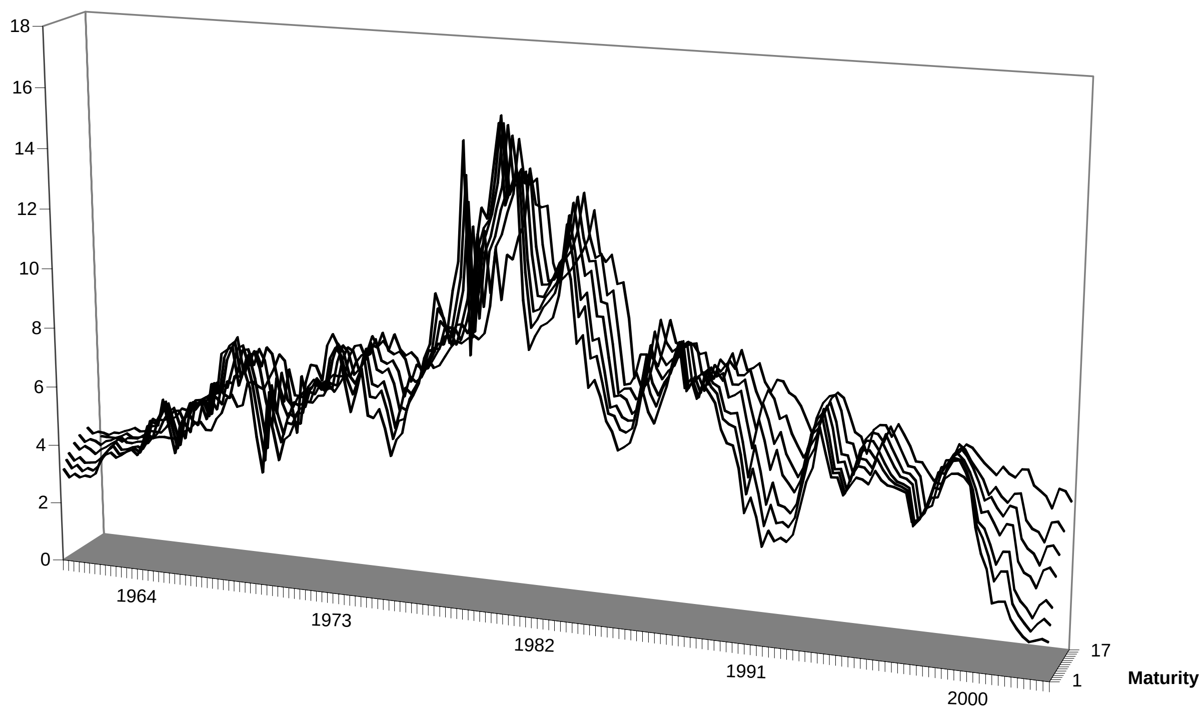


Chart 3: Inflation and the stochastic trend

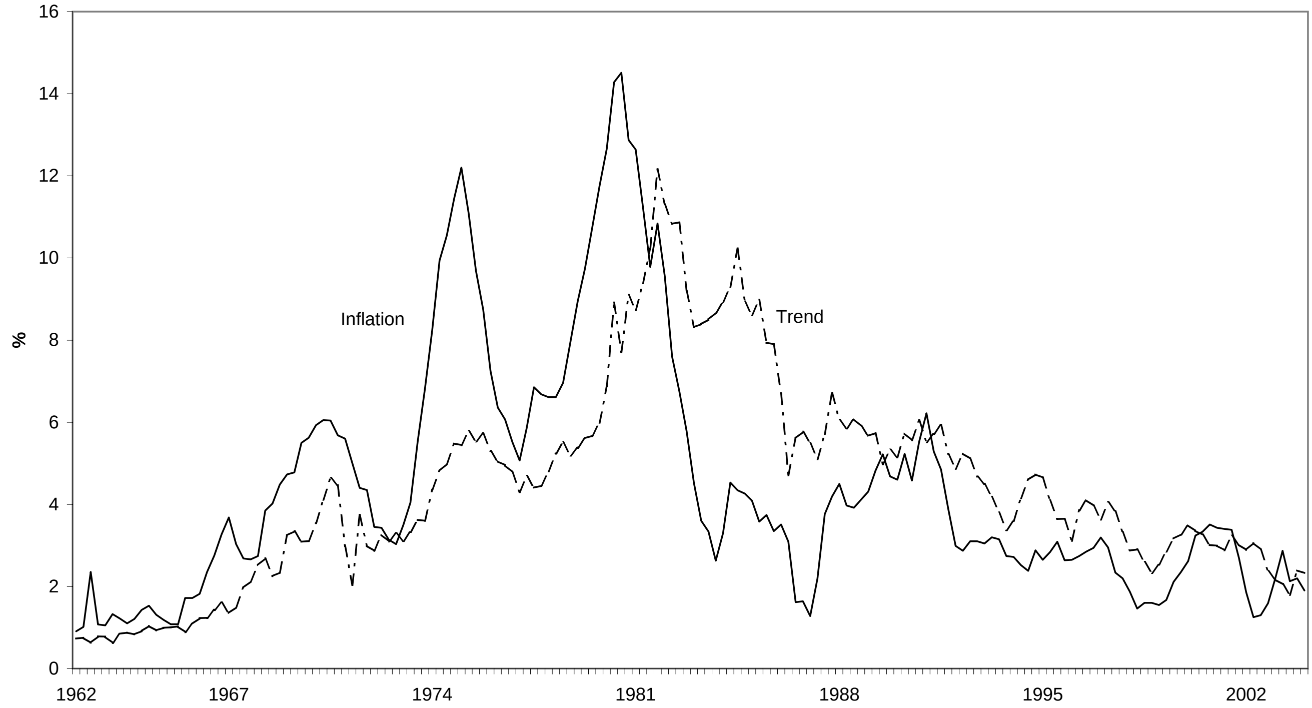


Chart 4: Risk premia

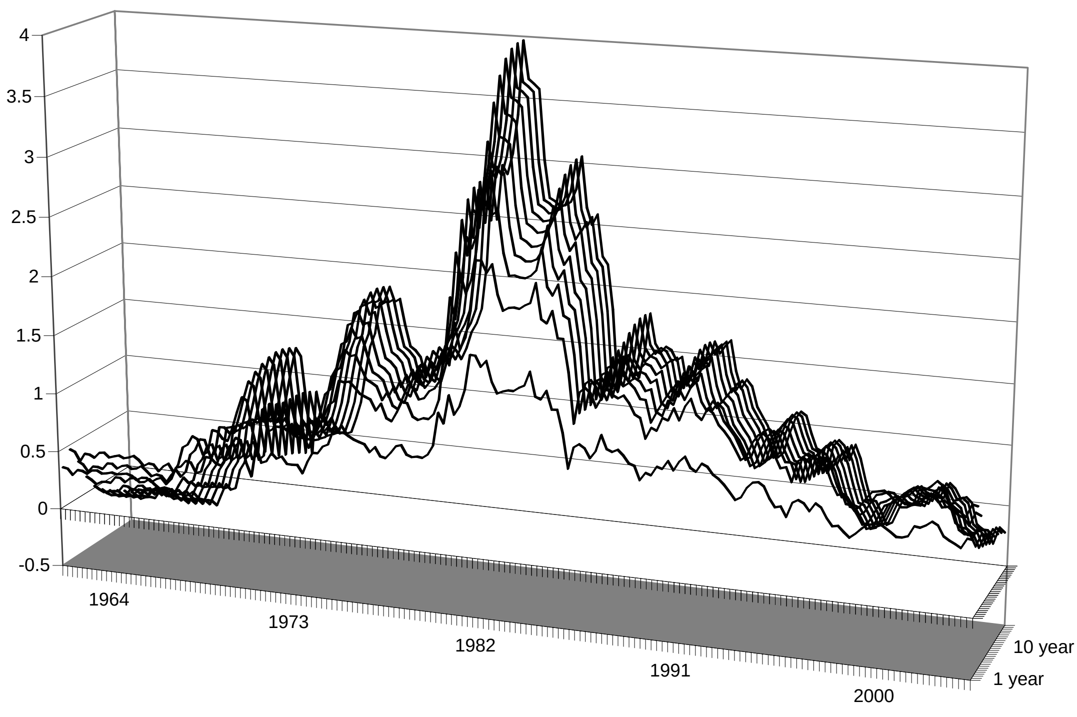
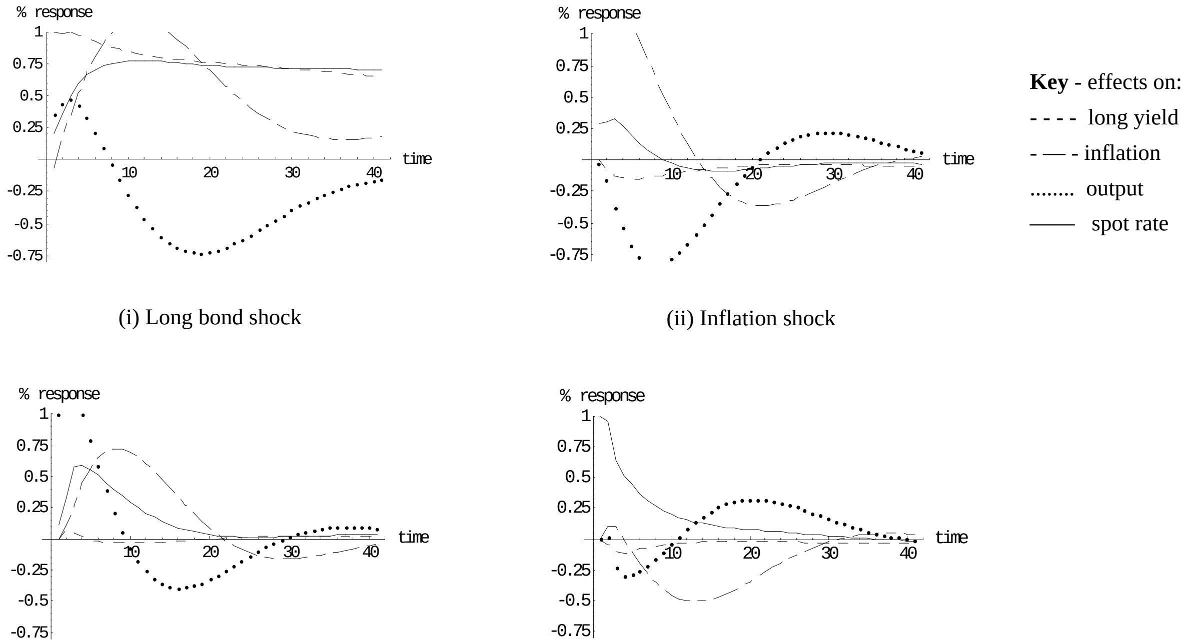


Chart 5: **Model M1 macroeconomic impulse responses**



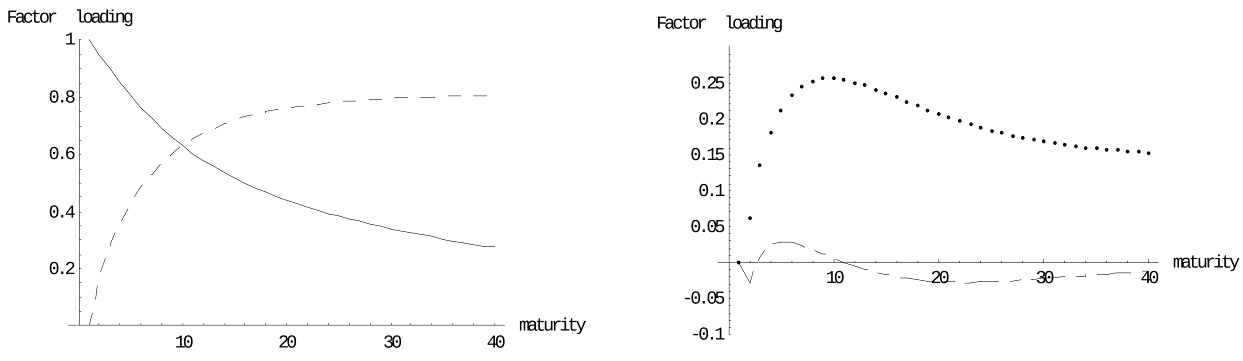
(i) Long bond shock

(ii) Inflation shock

(iii) Output shock

(iv) Spot rate shock

Chart 6 : **Model M2 Factor loadings**



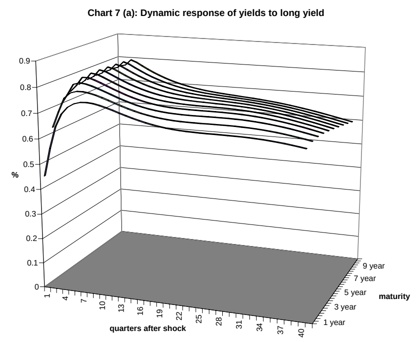
Long yield (- - -) and spot rate (——)

Output (....) and inflation (– – –)

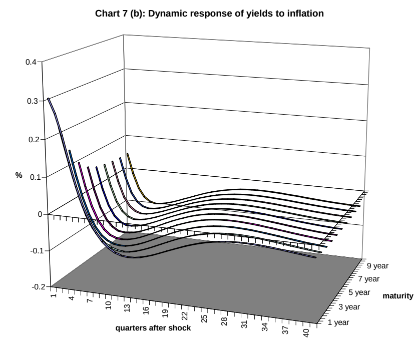
The factor loadings show the cumulative effect (after three years) of changes in the four macro variables on yields at different maturities

Each panel shows the effect of a shock to one the four driving variables. These shocks increase each variable in turn by one percentage point compared to its historical value for just one period. The dynamic effects allow for orthogonality, using the formulae described in appendix 3. For example, a shock to the long rate increases the output, inflation and interest rates immediately, reflecting the empirical correlation between surprises in these variables. The continous line shows the effect on the spot rate, the dashed line the effect on the long yield, the broken line the effect on inflation and the dotted line the effect on output. Elapsed time is measured in quarters.

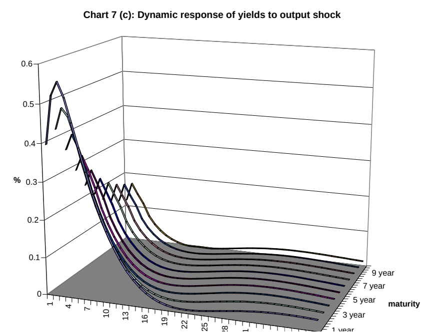
Chart 7: **Model M2 yield impulse responses**



(i) Long bond shock



(ii) Inflation shock



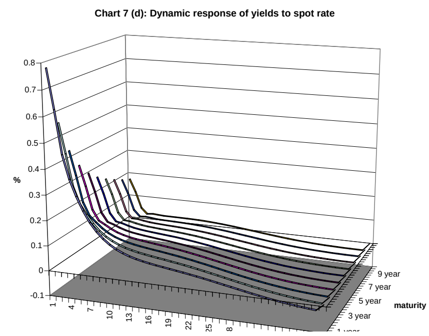
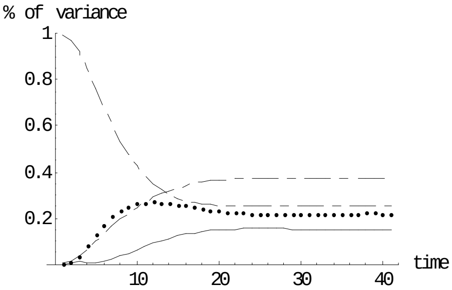
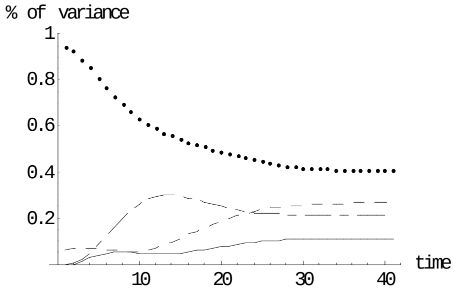


Chart 8: **Model M1 Analysis of Variance**

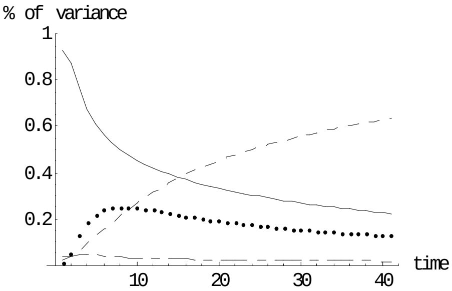


(ii) Inflation variance

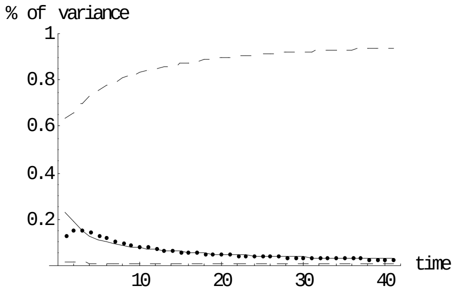


(iii) Output variance

Key - % of variance due
to orthogonal innovations in
- - - - long yield
- — - inflation
.... output
—— spot rate



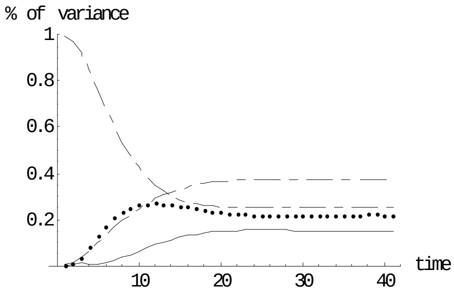
(iv) Spot rate variance



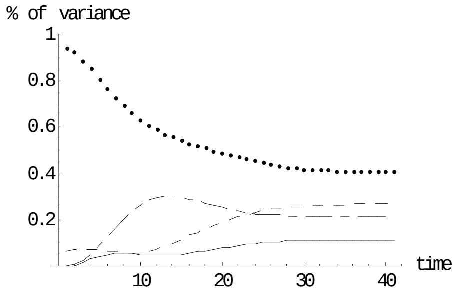
(i) Variance of 5 year yield

Each panel shows the contribution to total variance of innovations in each of the orthogonal shocks representing innovations in each of the four driving variables. These calculations use the formulae described in appendix 3. The continous dashed line shows the effect of innovations in the long yield, the broken line those in inflation the dotted line those in output and the continuous line those in the spot rate. Elapsed time is measured in quarters.

Chart 8: **Model M1 Analysis of Variance**

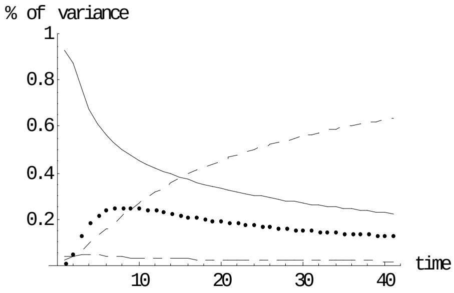


(ii) Inflation variance

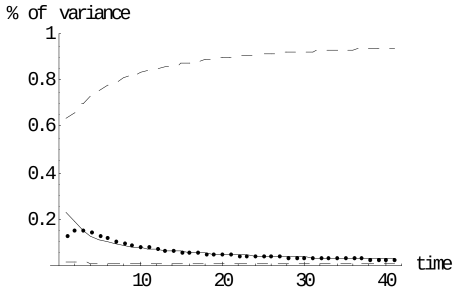


(iii) Output variance

Key - % of variance due
to orthogonal innovations in
- - - - long yield
- — - inflation
.... output
—— spot rate



(iv) Spot rate variance



(i) Variance of 5 year yield

Each panel shows the contribution to total variance of innovations in each of the orthogonal shocks representing innovations in each of the four driving variables. These calculations use the formulae described in appendix 3. The continous dashed line shows the effect of innovations in the long yield, the broken line those in inflation the dotted line those in output and the continuous line those in the spot rate. Elapsed time is measured in quarters.

Table 1a: Data Summary Statistics 1961Q4-2004Q1

	r_{68}	π	g	r_1	r_4	r_8	r_{12}	r_{20}	r_{28}	r_{40}
Mean	7.5567	4.0391	-0.6889	6.3239	6.3954	6.6305	6.7849	7.0021	6.7849	7.2513
Std.	2.28891	2.97537	2.33137	2.76685	2.80915	2.72748	2.64306	2.53722	2.47168	2.41235
Auto.	0.9971	0.99211	0.4632	0.9815	0.9892	0.9923	0.9944	0.9953	0.9963	0.9969
ADF	-2.091	-2.411	-4.133	-2.110	-2.100	-2.063	-2.031	-2.043	-1.991	-1.951

Inflation (π) and interest rates are from Datastream. Output gap (g) is from OECD. Yield data are US Treasury discount bond equivalent data compliled by McCulloch and Kwon (1990), updated by the New York Federal Reserve Bank. *Mean* denotes sample arithmetic mean expressed as percentage p.a.; *Std*. standard deviation and *Auto.* the first order quarterly autocorrelation coefficient. *ADF* is the Adjusted Dickey-Fuller statistic.

Table 1b : Residual Error Statistics M1, 1961Q4-2004Q1

	r_{68}	π	g	r_1	r_4	r_8	r_{12}	r_{20}	r_{28}	r_{40}
R^2	0.944102	0.968076	0.9081	0.908258	0.94466	0.952015	0.958435	0.970747	0.977934	0.981334
RMSE	0.541159	0.531619	0.706754	0.838048	0.660836	0.59747	0.538853	0.433951	0.367156	0.329587

The first row reports the unweighted R^2 and the second the unweighted Root Mean Square Error (RMSE).

Table 2: Model Evaluation

Model (M)	Specification*	**Volatility** affine in:	**Premium** affine in(**):	**Parameters** k(M)	k(2)-k(M)	k(3)-k(M)	k(5)-k(M)	**Loglikelihood** L(M)	2x(L(2)-L(M))	2x(L(3)-L(M))	2x(L(5)-L(M))
M1	$A_1(12)$	$y_{1,t-1}$	$y_{1,t-1}$	79	9	23	48	488.2	19.00	37.80	43.20
	χ_{95}								16.92	35.17	65.17
	p								0.03	0.03	0.67
M2	$EA_1(12)$	$y_{1,t-1}$	y_{t-1}	88		14	39	497.7		18.80	24.20
	χ_{95}									23.68	54.57
	p									0.17	0.97
M3	$GA_1(12)$	$y_{1,t-1}$	y_{t-1}	102			25	507.1			5.40
	χ_{95}										37.65
	p										1.00
M4	$EA_0(12)$	(-)	y_{t-1}	87		15	40	431.1		152.00	157.40
	χ_{95}									25.00	55.76
	p									0.00	0.00
M5	$GA_4(12)$	y_{t-1}	y_{t-1}	127				509.8			

(*) Model specification $S_m(N)$, where:
S denotes specification, m the number of variables conditioning volatility and N the number of state variables.
Specification S=A denotes 'admissible', EA 'essentially affine' and GA 'general affine' models.
The general affine model does not ensure a non-negative variance structure, but the A and EA structures do.
(**) Risk premia depend exclusively upon volatility in the admissible model and on variations in vol Λ_τ
in the essentially affine model.

Table 3: The dynamic structure of Model M2

parameter	estimate	t-value	parameter	estimate	t-value	parameter	estimate	t-value	parameter	estimate	t-value
$\mathcal{K}_1$			$\mathcal{K}_2$			$\mathcal{K}_3$			$\mathcal{M}_1$		
$\kappa_{1,11}$	0.990010	455.99									
$\kappa_{1,21}$	0.200532	8.14	$\kappa_{2,21}$	-0.169998	-8.01	$\kappa_{3,21}$	0.051416	2.21	$\mathcal{M}_{1,2}$	-0.063663	-5.59
$\kappa_{1,22}$	1.157965	55.34	$\kappa_{2,22}$	-0.055608	-2.47	$\kappa_{3,22}$	-0.095410	-3.00	$\mathcal{M}_{1,3}$	0.001033	0.08
$\kappa_{1,23}$	0.085667	3.04	$\kappa_{2,23}$	0.023922	0.86	$\kappa_{3,23}$	-0.007349	-0.28	$\mathcal{M}_{1,4}$	-0.031542	-2.55
$\kappa_{1,24}$	0.116986	6.64	$\kappa_{2,24}$	-0.110861	-5.98	$\kappa_{3,24}$	-0.051768	-2.87	$\mathcal{M}_{1,5}$	0.061046	4.20
$\kappa_{1,31}$	0.062386	18.78	$\kappa_{2,31}$	-0.007737	-2.22	$\kappa_{3,31}$	-0.015809	-4.57	$\mathcal{M}_{1,6}$	0.055560	5.08
$\kappa_{1,32}$	-0.122659	-19.53	$\kappa_{2,32}$	0.004608	0.89	$\kappa_{3,32}$	0.041108	9.00	$\mathcal{M}_{1,7}$	-0.065624	-5.57
$\kappa_{1,33}$	1.068134	98.35	$\kappa_{2,33}$	0.047740	8.38	$\kappa_{3,33}$	-0.185459	-44.96	$\mathcal{M}_{1,8}$	0.037318	2.99
$\kappa_{1,34}$	0.012787	4.75	$\kappa_{2,34}$	-0.239861	-92.46	$\kappa_{3,34}$	0.012787	4.75	$\mathcal{M}_{1,9}$	-0.100899	-6.93
$\kappa_{1,41}$	0.088984	38.22	$\kappa_{2,41}$	0.002609	1.12	$\kappa_{3,41}$	0.000845	0.36	$\mathcal{M}_{1,10}$	0.030094	2.55
$\kappa_{1,42}$	0.040663	12.16	$\kappa_{2,42}$	0.109077	31.88	$\kappa_{3,42}$	-0.099111	-32.70	$\mathcal{M}_{1,11}$	0.026869	2.24
$\kappa_{1,43}$	0.222411	76.30	$\kappa_{2,43}$	0.050708	16.01	$\kappa_{3,43}$	-0.219727	-78.55	$\mathcal{M}_{1,12}$	0.068729	5.61
$\kappa_{1,44}$	0.964865	457.32	$\kappa_{2,44}$	-0.295528	-159.69	$\kappa_{3,44}$	0.223630	114.56			

Table 4: The error and risk structure of Model M2

parameter	estimate	t-value	parameter	estimate	t-value
$\boldsymbol{\Delta}_0$			$\boldsymbol{\Delta}_1$		
			δ_{11}	1.4738×10^{-4}	16.63
δ_{02}	1.012×10^{-6}	27.18	δ_{12}	6.8558×10^{-5}	20.39
δ_{03}	2.7722×10^{-6}	8.64	δ_{13}	3.6184×10^{-5}	4.26
δ_{05}	3.4223×10^{-9}	0.05	δ_{15}	3.0934×10^{-4}	41.20
$\boldsymbol{\lambda}_0$			$\boldsymbol{\lambda}_1$		
$\lambda_{0,1}$	-1.160191	-0.33	$\lambda_{1,1}$	-8.23432	-1.25
$\lambda_{0,2}$	-508.3317	-14.00	$\lambda_{1,2}$	-119.52	-0.38
$\lambda_{0,3}$	-7.826042	-2.28	$\lambda_{1,3}$	0.285029	0.06
$\lambda_{0,4}$	-67.41729	-25.14	$\lambda_{1,4}$	-64.09623	-11.84
$\boldsymbol{\lambda}_{12}$			$\boldsymbol{\lambda}_{13}$		
$\lambda_{12,2}$	-0.078233	-1.32	$\lambda_{13,2}$	-0.000450	-0.10
$\lambda_{12,3}$	-0.006995	-1.62	$\lambda_{13,3}$	0.011791	0.15
$\lambda_{12,4}$	-0.072254	-1.42	$\lambda_{13,4}$	0.003076	0.91
$\boldsymbol{\lambda}_{14}$					
$\lambda_{14,2}$	0.004722	1.42			
$\lambda_{14,3}$	-0.001155	-0.41			
$\lambda_{14,4}$	0.001511	0.73			

Table 5: Eigenvalues of the dynamic responses under the historical (Φ) and RN ($\tilde{\Phi}$) measures

(in order of absolute value)

Φ	$\tilde{\Phi}$
0.99000	0.99090
$0.935648 \pm 0.118287i$	0.89513
0.899875	$0.863725 \pm 0.144256i$
$0.469379 \pm 0.124197i$	$0.507712 \pm 0.126573i$
$-0.00470196 \pm 0.362674i$	$-0.00588641 \pm 0.356612i$
$-0.254779 \pm 0.0674404i$	$-0.259492 \pm 0.0691647i$