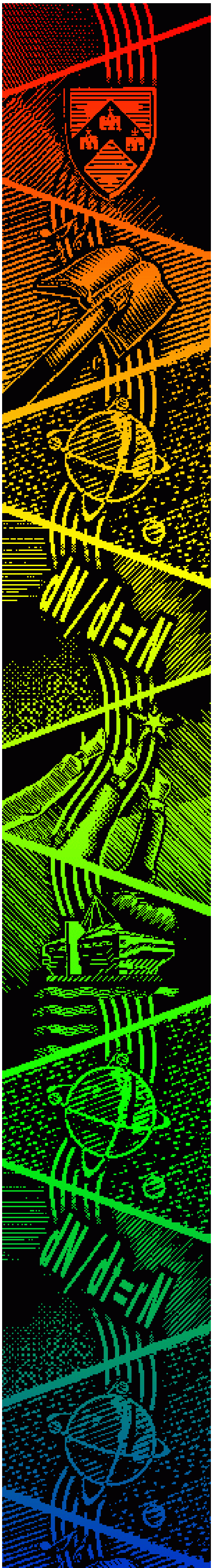

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Bartlett-type Adjustments for Empirical Discrepancy Test Statistics

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Abstract

This paper derives two Bartlett-type adjustments that can be used to obtain higher-order improvements to the distribution of the class of empirical discrepancy test statistics recently introduced by Corcoran (1998) as a generalisation of Owen's (1988) empirical likelihood. The corrections are illustrated in the context of the so-called Cressie-Read goodness-of-fit statistic (Baggerly 1998), and their effectiveness in finite samples is evaluated using simulations.

Keywords and Phrases: Asymptotic expansions, Bartlett and Bartlett-type corrections, Empirical likelihood, Nonparametric likelihood inference.

AMS 1980 Subject Classification: 62E20

1 Introduction

The empirical discrepancy approach to inference developed by the late Steve Corcoran (1998) provides a general unifying framework for analysing different nonparametric likelihood-based test statistics such as the empirical likelihood ratio (Owen, 1988), the Euclidean likelihood ratio (Owen, 1990), the Kullback-Liebler statistic (DiCiccio and Romano, 1990), and others. Empirical discrepancy inference is based on estimating among all the distributions supported on the sample and satisfying a given restriction, the closest to the empirical distribution function. The intuition behind this approach is that without restrictions the empirical distribution function is an optimal estimator (i.e. it is the maximum nonparametric likelihood estimator) of the unknown distribution of the data, but when restrictions are present this is not necessarily true. The estimated probabilities appearing in the resulting constrained estimator of the distribution of the data can then be used to make inference about the restrictions using a χ^2 calibration. Thus the empirical discrepancy inference dispenses with the need for intensive Monte Carlo simulation, as typically required by bootstrap approaches, requiring instead a numerical optimisation.

Confidence regions constructed using empirical discrepancy statistics have coverage error typically of order $O(n^{-1})$ which is the same as for confidence regions based on parametric likelihoods. However, it has been reported (see, for example, Owen (1988), Corcoran, Davison and Spady (1995) and Baggerly (1998)) that in samples of small/moderate size empirical discrepancy regions are often too narrow when the asymptotic χ^2 calibration is used. One possible way to obtain improved confidence regions is to use a bootstrap calibration. The latter was proposed originally by Owen (1988) in the context of empirical likelihood, but can be easily adapted to any empirical discrepancy statistic. It works well (at least for empirical likelihood), but is computationally quite expensive. Another possibility is to use a Bartlett correction. The latter was investigated by a number of authors for specific empirical discrepancy statistics. DiCiccio, Hall and Romano (1991), Chen (1993), Zhang (1996), and others showed that empirical likelihood ratio admits a Bartlett correction. On the other hand, Brown and Chen (1998) and Bravo (1999) showed, respectively, that neither the Euclidean likelihood, nor the Kullback-Liebler and Hellinger statistics admit a Bartlett correction. Baggerly (1998) investigated the issue of Bartlett correctability for the class of empirical discrepancy statistics based on minimising the Cressie-Read goodness-of-fit statistic (Read and Cressie, 1988, Ch. 1). This class is very large and contains, apart from empirical likelihood and Kullback-Liebler, several commonly used test statistics such as Neyman-modified χ^2 and Pearson's χ^2. Baggerly (1998) showed that empirical likelihood is the only member of the Cressie-Read goodness-of-fit statistics to admit a Bartlett correction. More generally, Corcoran (1998) showed that empirical discrepancy statistics admit a Bartlett correction provided that the discrepancy function satisfies two "regularity conditions" defined in (5) below. These conditions are satisfied by the empirical likelihood ratio, but not by *any* of the other above-mentioned empirical discrepancy statistics. Thus a large number of commonly used empirical discrepancy test statistics cannot be Bartlett-corrected, at least in the traditional sense.

The "regularity conditions" (5) ensure that the third and fourth cumulant of the signed square root of an empirical discrepancy test statistic are, respectively, of order $O\left(n^{-3/2}\right)$ and $O\left(n^{-2}\right)$. This, combined with an Edgeworth expansion argument, is sufficient to obtain corrected test statistics that are accurate up to the order $O\left(n^{-2}\right)$, but by no means necessary. Indeed, as is well-known in parametric likelihood inference, it is still possible to improve to third-order (i.e. up to $O\left(n^{-2}\right)$) the accuracy of asymptotic χ^2 tests by means of so-called Bartlett-type corrections. The latter constitute an extension of the traditional Bartlett correction to statistics other than the likelihood ratio, and have been proposed in different forms and context by Chandra and Mukerjee (1991), Cordeiro and Ferrari (1991) and Taniguchi (1991). A detailed review of Bartlett and Bartlett-type corrections can be found in Cribari-Neto and Cordeiro (1996).

In this paper we investigate the possibility of using Bartlett-type corrections for empirical discrepancy statistics. To be specific we derive two Bartlett-type corrections that can be applied to *any* empirical discrepancy statistics. This result is of theoretical importance because it shows that the same corrections developed for fully parametric models can be used in nonparametric settings. It is worth pointing out that although we use the same arguments of Chandra and Mukerjee (1991) and Cordeiro and Ferrari (1991), the actual derivation of the results of this paper does not benefit from these papers since the necesary stochastic expansions are different and involve moments rather than likelihood derivatives. The results of this paper are also of practical importantce because they imply, at least in principle, the possibility of obtaining test statistics with a desirable higher-order accuracy property without resorting to computational intensive methods, such as the bootstrap.

In this paper we also use Monte Carlo simulations to evaluate and compare the effectiveness of the proposed corrections in terms of finite sample accuracy and power. Incidentally, we note here that, with the exception of Chen (1994) in the case of empirical likelihood, most of the simulations studies on the higher-order properties of empirical discrepancy statistics have been focused on their accuracy rather than power properties. Thus, the results of this paper fills, at least partially, this gap since they provide some Monte Carlo evidence on how Bartlett and Bartlett–type corrections affect the power of empirical discrepancy statistics.

The remaining part of the paper is organised as follows: next section reviews briefly the basic theory for empirical discrepancy statistics and recalls the necessary asymptotic expansions. Section 3 derives two general Bartlett-type corrections for empirical discrepancy statistics, whereas Section 4 derives explicitly the corrections for the Cressie-Read goodness-of-fit statistic and reports the results of the Monte Carlo study. Finally, Section 5 contains some concluding remarks and indications for future research. An appendix contains the details of the calculations and proofs of the main results

Notice that throughout the rest of the paper we follow tensor notation and indicate arrays by their elements. Thus, for any index $1 \leq j, k, ... \leq q$, a^j is an $\Re^q$-valued vector, a^{jk} is an $\Re^{q \times q}$-valued matrix, etc. We also follow the summation convention, that is for any two repeated indices, their sum is understood.

2 Empirical discrepancy tests for moment based models

Let $Z_1, ..., Z_n$ be a sequence of independent $\Re^q$-valued random vectors with common unknown nonsingular distribution F_0, and let $\theta \in \Theta \subseteq \Re^q$ be an unknown parameter vector associated with F_0. As in Qin and Lawless (1994), we assume that the information about F_0 and θ is available in the form of the moment restriction

$$E\left[f\left(Z,\theta_0\right)\right]=0, \tag{1}$$

for some specified unique value θ_0 of θ with $f(Z,\theta):\Re^q\times\Theta\rightarrow\Re^s$ $(s\geq p)$ valued vector of known functionally independent functions. For simplicity, we shall consider the class of just-determined moment based models, that is models where $\dim(\Theta)=\dim\{f(Z,\theta)\}$, so that θ_0 may be estimated by solving the sample analogue of (1). Notice that this class of models is very large since it contains all M and most Z type estimators.

For any $a,b\in\Re$, let $h(a,b)$ be a function which satisfies the requirement that $h(a,a)=0$. Let $p_i=F\{Z_i\}$ be a nonparametric likelihood supported on Z_i and let $\widehat{p}_i=1/n$ denote the nonparametric maximum likelihood estimator for p_i. The empirical discrepancy approach for testing the validity of the moment condition (1) (i.e. $H_0:\theta=\theta_0$) is based on the following constrained minimisation

$$\mathcal{ED}\left(\theta_0\right)=\inf_{p_i}\left\{k_h\sum_{i=1}^{n}h\left(p_i,\widehat{p}_i\right)\mid\sum_{i=1}^{n}p_i=1,\quad\sum_{i=1}^{n}p_if\left(Z_i,\theta_0\right)=0\right\} \tag{2}$$

where k_h is a normalising constant which depends on $h(\cdot,\cdot)$ and is chosen so that the test statistic is $O_p(1)$ as $n\rightarrow\infty$. Thus empirical discrepancy effectively reweights the data so that the moment condition (1) holds at θ_0 and the discrepancy function $h(p_i,\widehat{p}_i)$ is minimised.

Let $W(\theta_0)$ denote the solution of (2) and let $\partial^r h:=\partial^r h\left(p_i,\widehat{p}_i\right)/\partial p_i^r\mid_{p_i=\widehat{p}_i}$. The following conditions are assumed to hold with probability 1.

A1 The intersection of the null space of the matrix $\left[\begin{array}{ccc} f(Z_1,\theta_0) & \ldots & f(Z_n,\theta_0)\end{array}\right]$ with the unit simplex is nonempty;

A2 $E\left(\left\|f\left(Z,\theta_0\right)\right\|^{\delta}\right)<\infty$ for δ big enough;

A3 $\limsup_{\|t\|\rightarrow\infty}\left|E\exp\left(\imath t'f\left(Z,\theta_0\right)\right)\right|<1$, for $\imath=(-1)^{1/2}$, $t\in\Re^q$;

A4 $\partial^r h=O_p\left(n^r/\ k_h\right)$ for $r=1,...,4$, and $\partial^2 h\neq 0$.

A1 ensures the uniqueness of $W(\theta_0)$ (as implied in Lemma 2 of Owen (1990)). **A2-A3** are sufficient to ensure that the Edgeworth expansion of $W(\theta_0)$ obtained from the formal delta method is valid in the sense of Bhattacharya and Ghosh (1978).

Note also that the Cramèr condition **A3** implies that F_0 cannot be a distribution supported on lattices. Finally **A4** is the same regularity condition on the derivatives of h assumed by Corcoran (1998).

Let $\sum_0 = E\left[f(Z_i,\theta_0)\, f(Z_i,\theta_0)'\right]$ and let $g^j(Z_i,\theta_0)$ $(j = 1,...,q)$ denote the jth component of $g(Z_i,\theta_0) := \sum_0^{-1/2} f(Z_i,\theta_0)$. Furthermore let

$$\alpha^{j_1 \dots j_k} = E\left[g^{j_1}(Z,\theta_0)\dots g^{j_k}(Z,\theta_0)\right], \quad A^{j_1\dots j_k} = \sum\nolimits_i \left[g^{j_1}(Z_i,\theta_0)\dots g^{j_k}(Z_i,\theta_0) - \alpha^{j_1\dots j_k}\right]/n,$$

denote the standardised moments of $f(Z_i,\theta_0)$ and the discrepancies between sample and true moments, respectively. Note that $\alpha^j = 0$ and $\alpha^{jk} = \delta^{jk}$, where δ^{jk} is the Kronecker delta.

Corcoran (1998) showed that $W(\theta_0)$ admits a stochastic expansion of the form

$$\begin{aligned} n^{-1}W(\theta_0) = \; & k_h \Bigg\{ \frac{b_1}{a_1^2} A^j A^j + \frac{(a_1^2 - a_2)(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2)}{a_1^6} A^j A^j A^k A^k - \\ & \frac{b_1}{a_1^2} A^{jk} A^j A^k + \frac{2a_2 b_1 - a_1 b_2}{a_1^4} \alpha^{jkl} A^j A^k A^l + \frac{a_1 b_3 - 2a_3 b_1}{a_1^5} \alpha^{jklm} A^j A^k A^l A^m + \\ & \frac{b_1}{a_1^2} A^{jl} A^{kl} A^j A^k + \frac{a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} \alpha^{jkl} \alpha^{jmn} A^k A^l A^m A^n + \\ & \frac{3(a_1 b_2 - 2a_2 b_1)}{a_1^4} \alpha^{jkl} A^{lm} A^j A^k A^m + \\ & \frac{2a_2 b_1 - a_1 b_2}{a_1^4} A^{jkl} A^j A^k A^l \Bigg\} + O_p\left(n^{-5/2}\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_1 &= \frac{n^2}{k_h \partial^2 h}, \quad a_2 = -\frac{n^3 \partial^3 h}{2(k_h)^2 (\partial^2 h)^3}, \quad a_3 = \frac{n^4 \left\{3(\partial^3 h)^2 - \partial^2 h \partial^4 h\right\}}{6(k_h)^3 (\partial^2 h)^5}, \\ b_1 &= \frac{a_1^2 (\partial^2 h)}{2n^2}, \quad b_2 = \frac{a_1 a_2 \partial^2 h}{n^2} + \frac{a_1^3 \partial^3 h}{6n^3}, \\ b_3 &= \frac{a_2^2 \partial^2 h}{2n^2} + \frac{a_1 a_3 \partial^2 h}{n^2} + \frac{a_1^2 a_2 \partial^3 h}{2n^3} + \frac{a_1^4 \partial^4 h}{24 n^4}. \end{aligned} \tag{4}$$

Let W^j denote the signed square root of $W(\theta_0)$, and let $\kappa^{j_1\dots j_k}$ denote the kth (multivariate) cumulant of W^j. As shown in the Appendix using W_j and some additional calculations lead to the two regularity conditions derived by Corcoran (1998), namely

$$\partial^3 h + 2n\partial^2 h = 0, \quad \partial^4 h + 3n\partial^3 h = 0, \tag{5}$$

that imply $\kappa_3^{jkl} = O\left(n^{-3/2}\right)$ and $\kappa_4^{jklm} = O\left(n^{-2}\right)$.

If one considers an Edgeworth expansion for the density $f_{W(\theta_0)}(\chi^2)$ of any test belonging to the class $\mathcal{ED}(\theta_0)$, it is shown in the Appendix that they are of the form $f_{W(\theta_0)}(\chi^2) \propto e^{-\chi^2/2}(\chi^2)^{q/2-1}\{1+\psi(\chi^2)/n\}+R_n$, where the coefficient $\psi(\cdot)$ is a polynomial in χ^2 and the remainder R_n is $O(n^{-2})$ by the even-odd property of the polynomials appearing in the Edgeworth expansion for the signed square root of $W(\theta_0)$ (see Barndorff-Nielsen and Hall (1988)). If (5) is not satisfied, then $\psi(\cdot)$ is nonlinear in χ^2 and hence adjusting the statistic through multiplication or division by a constant of the form $1+B/n$ (i.e. the standard Bartlett correction) will not, in general, eliminate the coefficient of order n^{-1} in the adjusted statistic. In the next section we show that, whether $\psi(\cdot)$ is linear in χ^2 or not, it is possible to improve the approximation error of $f_{W(\theta_0)}(\chi^2)$ to the order $O(n^{-2})$ by deriving two Bartlett-type adjustments.

3 Bartlett-type adjustments for empirical discrepancy tests: Theory

In this section we derive two Bartlett-type adjustments that can be used to improve the accuracy of empirical discrepancy test statistics for the null hypothesis $H_0 : \theta = \theta_0$ in (1). The first adjustment is the empirical discrepancy analogue of the one proposed in parametric likelihood theory by Chandra and Mukerjee (1991), and is based on an Edgeworth expansion argument for the signed square root of $W(\theta_0)$. Specifically, consider a perturbed version W^j_{CM} of W^j, where

$$W^j_{CM} = W^j + (\ k_h)^{1/2}\frac{b_1^{1/2}}{a_1}\left(C^{jkl}A^kA^l + C^{jk}A^k/n + C^{jklm}A^kA^lA^m\right), \tag{6}$$

and the C arrays are constants free of n, chosen so that $W_{CM}(\theta_0) = W^j_{CM}W^j_{CM}$ satisfies

$$\Pr\{W_{CM}(\theta_0) \leq u\} = \int_0^u g_q(v)\,dv + O\left(n^{-3/2}\right) \quad \forall u \geq 0, \tag{7}$$

where $g_q(\cdot)$ is the density of a chi squared random variate with q degrees of freedom.

We can prove the following theorem:

Theorem 1 *For any test statistic belonging to $\mathcal{ED}(\theta_0)$, there exist constants $C^{jk}, C^{jkl}, C^{jklm}$ such that (7) holds, where*

$$
\begin{aligned}
C^{jk} &= \frac{b_1^2 - a_1^4}{a_1^2 b_1}\frac{\left(a_1^2 - a_2\right)\left(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2\right)}{a_1^6}\delta^{jk}\left(2+q\right) - \frac{\alpha^{jkll}}{4} - \frac{\alpha^{jll}\alpha^{kmm}}{72} + \\
&\left\{\frac{a_1^4}{4b_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 + \frac{5a_1^2}{2b_1}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) - \right. \\
&\left.\frac{a_1^2}{b_1}\frac{a_2\left(5a_2 b_1 - 3a_1 b_2\right)}{a_1^6} - \frac{11}{18}\right\}\alpha^{jlm}\alpha^{klm} + \left\{\frac{a_1^2}{b_1}\frac{a_2\left(5a_2 b_1 - 3a_1 b_2\right)}{a_1^6} - \right. \\
&\left.\frac{a_1^4}{4b_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 - \frac{5a_1^2}{2b_1}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) + \frac{19}{24}\right\}\alpha^{jkl}\alpha^{lmm}, \\
C^{jkl} &= \frac{a_1^2}{b_1}\left\{\frac{b_1}{3a_1^2} - \frac{1}{2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}\alpha^{jkl}, \\
C^{jklm} &= -\frac{a_1^2}{12b_1}\left[-\frac{b_1}{a_1^2} + 6\left\{\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) + \left(\frac{a_1 b_3 - 2a_3 b_1}{a_1^5}\right)\right\}\right]\alpha^{jklm} - \\
&\frac{\left(a_1^2 - a_2\right)\left(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2\right)}{6a_1^6}\frac{b_1}{a_1^2}\left[3\right]\delta^{jk}\delta^{lm} + \\
&\left\{-\frac{a_1^2}{6b_1}\frac{a_2\left(5a_2 b_1 - 3a_1 b_2\right)}{a_1^6} + \frac{a_1^4}{24b_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 + \right. \\
&\left.\frac{5a_1^2}{12b_1}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) - \frac{7}{54}\right\}\left[3\right]\alpha^{jkn}\alpha^{lmn}. \qquad (8)
\end{aligned}
$$

Proof. See the Appendix ■

One can now verify that the rth cumulant κ^r of $W_{CM}\left(\theta_0\right)$ is $\kappa^r\left\{W_{CM}\left(\theta_0\right)\right\} = 2^{r-1}\left(r-1\right)!q + R_n$, where the remainder R_n is of order $O\left(n^{-2}\right)$ using the same arguments of Barndorff-Nielsen and Hall (1988).

The second type of adjustment is based on the approach developed by Cordeiro and Ferrari (1991). Using (22), proceeding as in the proof of Theorem 1, it follows after some lengthy algebra that the density of $W\left(\theta_0\right)$ is

$$
f_{W(\theta_0)}\left(x\right) = g_q\left(x\right) + \sum_{r=0}^{3}\frac{d_r g_{q+2r}\left(x\right)}{n} + O\left(n^{-3/2}\right), \qquad (9)
$$

where

Define now the modified statistic

$$
W_{CF}\left(\theta_0\right) = W\left(\theta_0\right)\left[1 - \left\{\sum_{r=1}^{3}\frac{c_r\left\{W\left(\theta_0\right)\right\}^{r-1}}{nq\left(q+2\right)\ldots\left(q+2\left(r-1\right)\right)}\right\}\right] \qquad (10)
$$

where the $O(1)$ terms c_r are chosen so that they satisfy:

$$\Pr\left\{W_{CF}(\theta_0) \le u\right\} = \int_0^u g_q(v)\,dv + O\left(n^{-3/2}\right) \quad \forall u \ge 0. \tag{11}$$

We can prove the following theorem:

Theorem 2 *For any test belonging to* $\mathcal{ED}(\theta_0)$*, there exist unique constants* c_1, c_2, c_3 *such that* (11) *holds, where*

$$\begin{aligned}
c_1 &= \frac{(a_1^2 - a_2)(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2)}{a_1^6}\left(1 - 3k_h \frac{b_1}{a_1^2}\right) k_h q(2+q) + \\
&\left[\frac{k_h}{2}\frac{b_1^2}{a_1^4} + 3\left\{\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) + \left(\frac{a_1 b_3 - 2a_3 b_1}{a_1^5}\right)\right\}\left(1 - k_h \frac{b_1}{a_1^2}\right)\right] k_h \alpha^{jjkk} + \\
&\left[\left\{\frac{b_1}{a_1^2} + \frac{2a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} + \frac{5(a_1 b_2 - 2a_2 b_1)}{a_1^4}\right\} k_h - \right. \\
&\left\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} - \right. \\
&\left.\left. 42\frac{b_1}{a_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}\frac{k_h^2}{6} + \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}^2 \frac{k_h^3}{6}\right] \alpha_{jkl}\alpha_{jkl} + \\
&\left[\left\{\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\frac{a_1}{b_1^{1/2}} - \frac{b_1^{1/2}}{a_1}\right\}^2 \frac{k_h}{4} + \left\{\frac{a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} + \right.\right. \\
&\left.\frac{3b_1}{4a_1^2} - \frac{a_1^2}{4b_1}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 - 5\left(\frac{2a_2 b_1 - a_1 b_2}{2a_1^4}\right)\right\} k_h - \\
&\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}\left\{\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) - \frac{b_1}{a_1^2}\right\}\frac{k_h^2}{12} - \\
&\left\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} - \right. \\
&\left.\left. 42\frac{b_1}{a_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}\frac{k_h^2}{12} + \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}^2 \frac{k_h^3}{4}\right] \alpha^{jjk}\alpha^{kll}, \\
c_2 &= \frac{b_1}{a_1^2}\left[-\frac{b_1}{2a_1^2} + 3\left\{\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right) + \left(\frac{a_1 b_3 - 2a_3 b_1}{a_1^5}\right)\right\}\right] k_h^2 \alpha^{jjkk} + \\
&\frac{(a_1^2 - a_2)(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2)}{a_1^6}\frac{b_1}{a_1^2} 3k_h^2 q(2+q) + \\
&\left[\left\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2(5a_2 b_1 - 3a_1 b_2)}{a_1^6} - \right.\right. \\
&\left. 42\frac{b_1}{a_1^2}\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}\frac{k_h^2}{6} - \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2 b_1 - a_1 b_2}{a_1^4}\right)\right\}^2 \frac{k_h^3}{3}\right] \alpha^{jkl}\alpha^{jkl} +
\end{aligned}$$

$$\left[\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}\left\{\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)-\frac{b_1}{a_1^2}\right\}\frac{k_h^2}{12}+\right.$$
$$\left\{12\frac{b_1^2}{a_1^4}+9\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)^2+12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1-3a_1b_2\right)}{a_1^6}-\right.$$
$$\left.\left.42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}\frac{k_h^2}{12}-\frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}^2\frac{k_h^3}{2}\right]\alpha^{jjk}\alpha^{kll}$$
$$c_3 = \frac{k_h^3}{36}\frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}^2\left(9\alpha^{jjk}\alpha^{kll}+6\alpha^{jkl}\alpha^{jkl}\right). \quad (12)$$

Proof. See the Appendix ■

As for the perturbed statistic $W_{CM}(\theta_0)$, one can verify that the rth cumulant κ^r of $W_{CF}(\theta_0)$ is $\kappa^r\{W_{CF}(\theta_0)\}=2^{r-1}(r-1)!q+R_n$ where the remainder R_n is of order $O(n^{-2})$ by the same arguments of Barndorff-Nielsen and Hall (1988).

Remark 1. Both Bartlett-type adjustments (8) and (12) depend on the derivatives of the discrepancy function and on the third and fourth (multivariate) standardised moments of the moment vector $E[f(Z,\theta_0)]=0$ under investigation. In the case of a vector mean, i.e. $E(Z)=\theta_0$, and for a given discrepancy function (or family of), it is possible to give a qualitative characterisation of both adjustments in terms of (multivariate) skewness and kurtosis of the underlying unknown distribution of the data. See next section for an example. For general moment functions, however, a similar characterisation is typically not possible.

Remark 2. Cribari-Neto and Cordeiro (1996) noted that there are alternative definitions of the Bartlett-type correction of Cordeiro and Ferrari (1991) that are all equivalent up to $O(n^{-1})$. Let B_n denote the $O(n^{-1})$ term appearing in the modified statistic (10), and let $T(B_n)$ denote any transformation of B_n such that $T(B_n)=1-B_n+O(n^{-2})$. It then follows that

$$W_{CFT}(\theta_0):=W(\theta_0)T(B_n)=W_{CF}(\theta_0)+O\left(n^{-2}\right).$$

Examples of $W_{CFT}(\theta_0)$ include the scale $1/(1+B_n)$ and exponential $\exp(-B_n)$ transformations which produce, respectively, the scale and exponential Bartlett-type correction, namely

$$W_{CFS}(\theta_0) = W(\theta_0) / (1 + B_n), \text{ and } W_{CFE}(\theta_0) = W(\theta_0) \exp(-B_n). \tag{13}$$

Using simulations Cribari-Neto and Cordeiro (1996) showed that in a number of situations of practical relevance both Bartlett-type corrections in (13) are superior to the original one in terms of finite sample properties. Interestingly, the same conclusion seems to hold in the case of empirical discrepancy statistics; see next section for more details.

Remark 3. As in the case of Bartlett-type corrections for fully parametric models, the Bartlett-type corrections derived in this paper may produce modified statistics that are not necessarily monotonic transformations of the original statistic. Thus it might happen that large values of the original statistic produce small values of the modified statistics, and this can negatively affect the power of the modified statistic. One possible solution to this potential problem is to consider monotonic adjustments of the original statistic, like, for example, those suggested by Kakizawa (1996), and Cordeiro, Ferrari and Cysneiros (1998). Note, however, that even with monotonic adjustments the modified statistic might still be less powerful than the original one. For example, the Bartlett correction for empirical likelihood is a monotonic adjustment, yet as illustrated in Figures 1 and 2 below the Bartlett corrected empirical likelihood ratio is less powerful than the original one.

4 Bartlett-type adjustments for empirical discrepancy tests: Applications

In this section we illustrate Theorems 1 and 2 by deriving the Bartlett-type adjustments for the empirical discrepancy statistic based on the Cressie-Read goodness-of-fit statistic recently introduced by Baggerly (1998). Let $k_h = -2/(\lambda+1)$ and $h(p_i, \widehat{p}_i) = \lambda^{-1}\left\{1 - (p_i/\widehat{p}_i)^{-\lambda}\right\}$ where $-\infty < \lambda < \infty$ is a user-specified parameter. For this choice of the discrepancy function $h(p_i, \widehat{p}_i)$, the constrained minimisation defined in (2) becomes

$$\mathcal{CR}_\lambda(\theta_0) = \inf_{p_i}\left\{-\frac{2}{\lambda(\lambda+1)}\sum_{i=1}^{n}\left\{1 - (p_i/\widehat{p}_i)^{-\lambda}\right\} \mid \sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} p_i f(Z_i, \theta_0) = 0\right\}.$$

Let $W^{\mathcal{CR}_\lambda}(\theta_0)$ denote the solution of $\mathcal{CR}_\lambda(\theta_0)$; Baggerly (1998) showed that unless $\lambda = 0$ (i.e. empirical likelihood) the Cressie-Read goodness of fit statistic $W^{\mathcal{CR}_0}(\theta_0)$ does not admit a traditional Bartlett correction. In fact, it can be verified that, unless $\lambda = 0$, $W^{\mathcal{CR}_\lambda}(\theta_0)$ does not satisfy (5). Thus the Cressie-Read goodness-of-fit statistic provides a natural example of empirical discrepancy statistic where the Bartlett-type corrections are necessary to obtain improved inferences.

Calculations show that the three arrays of constants $C^{jk}, C^{jkl}, C^{jklm}$ (8) of Theorem 1 are

$$\begin{aligned} C^{jk} &= \frac{\lambda^2}{16}\left(-3+2\lambda+\lambda^2\right)\delta^{jk}(2+q) - \frac{\alpha^{jkll}}{4} + \frac{\left(3-\lambda-4\lambda^2\right)}{18}\alpha^{jlm}\alpha^{klm} + \\ &\quad \frac{\lambda(1+4\lambda)}{18}\alpha^{jkl}\alpha^{lmm}, \quad C^{jkl} = -\frac{\lambda}{6}\alpha^{jkl}, \\ C^{jklm} &= \frac{\lambda(3+2\lambda)}{24}\alpha^{jklm} + \frac{\lambda^2(1+\lambda)^2}{96}[3]\,\delta^{jk}\delta^{lm} - \frac{\lambda(1+4\lambda)}{108}[3]\,\alpha^{jkn}\alpha^{lmn}. \end{aligned} \tag{14}$$

Thus, using (14) it follows after some further algebra that the modified test statistic $W_{CM}(\theta_0)$ (7) is

$$\begin{aligned} W_{CM}^{\mathcal{CR}_\lambda}(\theta_0) &= W(\theta_0) + n\Bigg[-\frac{\lambda}{3}\alpha^{jkl}A^jA^kW^l + \left\{\frac{\lambda^2}{8}\left(-3+2\lambda+\lambda^2\right)\delta^{jk}(2+q) - \frac{\alpha^{jkll}}{2} + \right. \\ &\quad \left. \frac{\left(3-\lambda-4\lambda^2\right)}{9}\alpha^{jlm}\alpha^{klm} + \frac{\lambda(1+4\lambda)}{9}\alpha^{jkl}\alpha^{lmm}\right\}A^jA^k/n + \\ &\quad \left\{\frac{\lambda(3+2\lambda)}{12}\alpha^{jklm} + \frac{\lambda^2(1+\lambda)^2}{48}[3]\,\delta^{jk}\delta^{lm} - \frac{\lambda(1+4\lambda)}{54}[3]\,\alpha^{jkn}\alpha^{lmn} + \right. \\ &\quad \left. \frac{\lambda^2}{36}\alpha^{jko}\alpha^{lmo}\right\}A^jA^kA^lA^m\Bigg]. \end{aligned} \tag{15}$$

Turning to the second adjustment, calculations show that the three constants (12) of Theorem 2 are

$$\begin{aligned} c_1 &= \frac{1}{36}\left\{18\alpha^{jjkk} + 5\lambda(\lambda-1)\alpha^{jjk}\alpha^{kll} - 12\alpha^{jkl}\alpha^{jkl} - 18\lambda^2 q(q+2)\right\}, \\ c_2 &= \frac{\lambda}{36}\left\{-9(3+2\lambda)\alpha^{jjkk} + 5(1-\lambda)\alpha^{jjk}\alpha^{kll} + 12(1+\lambda)\alpha^{jkl}\alpha^{jkl} + 27\lambda q(q+2)\right\}, \\ c_3 &= \frac{\lambda^2}{12}\left(3\alpha^{jjk}\alpha^{kll} + 2\alpha^{jkl}\alpha^{jkl}\right), \end{aligned} \tag{16}$$

As for the modified test statistic $W_{CM}^{\mathcal{CR}_\lambda}(\theta_0)$, using (16) gives the second modified empirical discrepancy test statistic $W_{CF}^{\mathcal{CR}_\lambda}(\theta_0)$, namely

$$W_{CF}^{\mathcal{CR}_\lambda}(\theta_0) = W(\theta_0) - \frac{W(\theta_0)}{n}\left[\frac{1}{q}\left\{\frac{\alpha^{jjkk}}{2} - \frac{\alpha^{jkl}\alpha^{jkl}}{3} + \frac{5\lambda(\lambda-1)}{36}\alpha^{jjk}\alpha^{kll} - \right.\right.$$

$$\frac{\lambda^2}{2} q (q+2) \Big\} + \frac{1}{q(q+2)} \Big\{ -\frac{\lambda(3+2\lambda)}{4} \alpha^{jjkk} +$$
$$\frac{\lambda(1+\lambda)}{3} \alpha^{jkl}\alpha^{jkl} + \frac{5\lambda(1-\lambda)\alpha^{jjk}\alpha^{kll}}{36} + \frac{3\lambda^2}{4} q(2+q) \Big\} W(\theta_0) +$$
$$\frac{\lambda^2}{12q(q+2)(q+4)} \left(3\alpha^{jjk}\alpha^{kll} + 2\alpha^{jkl}\alpha^{jkl}\right) W(\theta_0)^2 \Big] . \tag{17}$$

As mentioned in **Remark 1** of the previous section, if the parameter of interest is a vector of means it is possible to characterise the magnitude of the Bartlett-type corrections in terms of skewness and kurtosis $\kappa := \alpha^{jjkk} - q(q+2)$. In particular, in the case of the Cressie-Read statistic considered here the following can be said about the modified statistics (15) and (17) (or equivalently (13)). Symmetric distributions with heavy tails, that is if $\alpha^{jkl} = \alpha^{jjk} = 0$ and $\kappa > 0$ for all j, k, l, produce typically larger Bartlett-type corrections. Note however that for $\lambda < -2/3$ or $\lambda > 0$ the magnitude of the corrections will be reduced. On the other hand, skewed distributions reduce the corrections by $\alpha^{jkl}\alpha^{jkl}/3$, but at the same time because of the nonlinear dependency of the two skewness coefficients on the parameter λ, and of the nonlinear structure of the adjustments themselves it is not possible to assess the overall effect of nonzero skewness on the magnitude of the adjustments.

It is important to note that although both (15) and (17) are asymptotically χ^2_q with an approximation error of order $O(n^{-2})$, the computation of the two modified test statistics is rather different. The modification proposed by Chandra and Mukerjee (1991) involves computation of quantities such as $\alpha^{jkl}A^jA^kW^l$ and $\alpha^{jklm}A^jA^kA^lA^m$; these take, respectively, $O(nq^3)$ and $O(nq^4)$ time to compute. On the other hand, the modification proposed by Cordeiro and Ferrari (1991) requires the computation of most three-fold summations like for example $\alpha^{jjk}\alpha^{kll}$. To further illustrate this point, consider the case of empirical likelihood ($\lambda = 0$). By (15) the resulting modified test statistic is

$$W_{CM}^{\mathcal{CR}_0}(\theta_0) = W(\theta_0) - \left(\frac{\alpha^{jkll}}{2} - \frac{\alpha^{jlm}\alpha^{klm}}{3}\right) A^jA^k, \tag{18}$$

whereas by (17) the resulting modified test statistic is

$$W_{CF}^{\mathcal{CR}_0}(\theta_0) = W(\theta_0) - \frac{1}{qn}\left(\frac{\alpha^{jjkk}}{2} - \frac{\alpha^{jkl}\alpha^{jkl}}{3}\right) W(\theta_0), \tag{19}$$

and coincides with the (original) Bartlett-corrected version of DiCiccio et al. (1991). In the case of univariate problems the two adjustments (18) and (19) coincide since

$A^j A^k = W(\theta_0)/n$. Indeed, in general, the computational difference between (15) and (17) disappears in the case of univariate problems, since both adjustments are functions of the test statistic itself and the unknown moments of the data. This suggests that, unless one is considering univariate problems, the Cordeiro and Ferrari (1991) adjustment (10) and (17) seems preferrable on the grounds of computational simplicity, especially when q is large.

It should also be noted that both (15) and (17) depend on the population moments $\alpha^{j_1 \dots j_k}$ of $f(Z, \theta_0)$ which are usually unknown. In practice, these moments can be replaced by the $n^{1/2}$ consistent estimates

$$\begin{aligned} \widehat{\alpha}^{jkl} &= \sum_{i=1}^{n} \left(\widehat{\Sigma}^{-1/2}\right)^{jm} \left(\widehat{\Sigma}^{-1/2}\right)^{kn} \left(\widehat{\Sigma}^{-1/2}\right)^{lo} f^m\left(Z_i, \widehat{\theta}\right) f^n\left(Z_i, \widehat{\theta}\right) f^o\left(Z_i, \widehat{\theta}\right)/n, \\ \widehat{\alpha}^{jklm} &= \sum_{i=1}^{n} \left(\widehat{\Sigma}^{-1/2}\right)^{jn} \dots \left(\widehat{\Sigma}^{-1/2}\right)^{mq} f^n\left(Z_i, \widehat{\theta}\right) \dots f^q\left(Z_i, \widehat{\theta}\right)/n \end{aligned} \tag{20}$$

where

$$\widehat{\Sigma}^{jk} = \sum_{i=1}^{n} f^j\left(Z_i, \widehat{\theta}\right) f^k\left(Z_i, \widehat{\theta}\right)/n, \quad \widehat{\theta} = \theta_0 + O_p\left(n^{-1/2}\right),$$

without affecting the order of the coverage error of the resulting approximation.

To investigate the finite sample effectiveness of the two modified statistics (15) and (17) we have used simulations. As mentioned in the previous section, there are a number of alternative versions of the modified statistic $W_{CF}^{\mathcal{CR}_\lambda}(\theta_0)$. In the simulations we considered the original as well as the scale and exponential versions defined in (13). While all three corrections reduced the size distortion of the original test statistics (with the scale correction being the most effective) the exponential one was found to be superior in terms of power, and thus we decided to report only the result of the latter[1].We considered three different test statistics all belonging to the Cressie-Read goodness-of-fit statistic $\mathcal{CR}_\lambda(\theta_0)$, namely the Euclidean likelihood $\mathcal{CR}_{-2}(\theta_0)$ $(\lambda = -2)$, the Kullback-Liebler $\mathcal{CR}_{-1}(\theta_0)$ $(\lambda = -1)$, and the empirical likelihood ratio $\mathcal{CR}_0(\theta_0)$ $(\lambda = 0)$, and their modified versions $W_{CM}^{\mathcal{CR}_\lambda}(\theta_0)$ and $W_{CFE}^{\mathcal{CR}_\lambda}(\theta_0)$. Note, however, that in the case of empirical likelihood we used the original modified version $W_{CF}^{\mathcal{CR}_0}(\theta_0)$ as given in (19). We were interested to test a null hypothesis about the population mean $\theta = \mu = E(Z)$ and considered three univariate and two bivariate cases. In the first univariate case samples were drawn from the standard normal distribution; the null hypothesis is $H_0 : \mu = 0$ and the required standardised moments

[1] The full set of simulations' results is available upon request.

are $\alpha^3 = 0$ and $\alpha^4 = 3$. For the second univariate case, samples were drawn from a χ^2_4 (chi-squared distribution with four degrees of freedom); the null hypothesis is $H_0 : \mu = 4$ and the required (standardised) moments are $\alpha^3 = 2^{1/2}$ and $\alpha^4 = 6$. For the third univariate case, samples were drawn for a t_5 (t-distribution with five degrees of freedom); the null hypothesis is $H_0 : \mu = 0$ and the required (standardised) moments are $\alpha^3 = 0$ and $\alpha^4 = 9$. For each combination of the sample size n and nominal α-level Tables 1-3 report the observed size of the three test statistics with and without the theoretical and estimated exponential Bartlett-type adjustments. The latter type adjustments were calculated using (20). The results were obtained from 5000 samples generated by the S-PLUS functions `rnorm` and `rchisq` and `rt`.

Tables 1-3 approx. here

In the first bivariate case samples were drawn from a standard bivariate normal. The null hypothesis was $H_0 : \mu^j = 0$, and the required theoretical moments were $\alpha^{jkl} = 0$ and $\alpha^{jjkk} = 1$. For the second bivariate case we used the same design considered by Chen (1994) and generated a bivariate random vector z_i^j as

$$z_i^1 = x_i^0 + x_i^1, \quad z_i^2 = x_i^0 + x_i^2 \tag{21}$$

where x_i^0, x_i^1, and x_i^2 were drawn independently from the exponential distribution with unit mean. The null hypothesis was $H_0 : \mu^j = 2$ and the required moments were $\alpha^{jjj} = 2\left(\sigma_1 + \sigma_2\right)^3 + 2\left(\sigma_1^3 + \sigma_2^3\right)$, $\alpha^{jkk} = 2\left(\sigma_1 + \sigma_2\right)^3 + 2\sigma_1\sigma_2\left(\sigma_1 + \sigma_2\right)$, $\alpha^{jjjj} = 24\left(\sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2\right)^2$, $\alpha^{jjkk} = 12\left(\sigma_1 + \sigma_2\right)^4$, $\alpha^{jjjk} = 12\left(\sigma_1 + \sigma_2\right)^4 + 12\sigma_1\sigma_2\left(\sigma_1^2 + \sigma_2^2\right)$, where $\sigma_1 = (1/2)\left(1 + 3^{-1/2}\right)$, and $\sigma_2 = (1/2)\left(-1 + 3^{-1/2}\right)$. For each combination of the sample size n and nominal α-level, Tables 4-5 report the observed size of the three test statistics with and without the theoretical and estimated exponential Bartlett-type adjustments. The latter were calculated using the theoretical and the estimated moments as in (20). The results were obtained from 5000 samples generated by the S-PLUS functions `rmvnorm` and `rexp`.

Tables 4-5 approx. here

Bearing in mind that the scale of the simulation study is small, the results of Tables 1-5 indicate the following: Firstly, Bartlett and Bartlett-type corrections are effective in bringing the observed size of the corrected test closer to the nominal value. Secondly, while Bartlett-corrected empirical likelihood ratio statistics are still (slightly) oversized, Bartlett-type corrected Euclidean likelihood and Kullback-Liebler test statistics become (slightly) undersized, in particular with skewed distributions and small sample sizes. This is perhaps not surprising given the nonlinear structure of the Bartlett-type corrections and the curvature exhibited by Q-Q plots[2] of the three test statistics considered. Such curvature, which indicates a somewhat poor χ^2 approximation at the higher quantiles, is the principle way in which empirical discrepancy shows different behaviour from an ordinary parametric likelihood, and implies that Bartlett and Bartlett-type corrected χ^2 calibrations for nonparametric likelihood-based inferences will typically be less effective than those used for parametric likelihood-based inferences. Thirdly, test statistics adjusted with estimated Bartlett-type corrections are typically more accurate (i.e. their actual size is closer to the nominal one) than those adjusted with their theoretical counterpart. This fact can be explained by noting that the sample moments used in the estimated Bartlett and Bartlett-type corrections have a typical (downward) finite sample bias which effectively reduces the magnitude of the estimated corrections. Finally, the Kullback-Liebler $\mathcal{CR}_{-1}(\theta_0)$ performs in general slightly better than the Euclidean likelihood $\mathcal{CR}_{-2}(\theta_0)$ statistic.

It should be mentioned that these corrections are not intended to increase the power of test statistics and can lead to a loss in power. Using the conventional Pitman approach based on the comparison of local (asymptotic) power, Bravo (2003) shows that no member of the Cressie-Read goodness-of-fit statistic is uniformly superior in terms of its second-order local power (i.e. up to the order $o\left(n^{-1/2}\right)$). Using the same approach, it is not difficult to show (see also Cox and Reid (1987)) that empirical discrepancy test statistics and their corrected versions have the same second-order local power, that is they are second-order efficient. Efficiency, however, is an asymptotic property, and thus to assess (and compare) the finite sample power of empirical discrepancy statistics and their corrected versions, we used simulations.

We considered the three test statistics $\mathcal{CR}_\lambda(\theta_0)$ for $(\lambda = -2, -1, 0)$ and their modified versions $W_{CM}^{\mathcal{CR}_\lambda}(\theta_0)$ $W_{CFE}^{\mathcal{CR}_\lambda}(\theta_0)$, and used the five different distributions as

[2]The Q-Q plots are available upon request.

in Tables 1-5, but since the results were fairly similar, and to save space, we report only the results concerning the bivariate normal and exponential (see (21)) cases. In both cases we calculated[3] the finite sample power of the three tests for $H_0 : \mu^j = \mu_0^j$ against $H_n : \mu^j = \mu_0^j + \tau^j$ at the 49 points of $\tau^j = \begin{bmatrix} \tau^1 & \tau^2 \end{bmatrix} := \left(\Sigma_0^{jk}\right)^{1/2} \delta^k$ within the grid $G_\tau = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix} \times \begin{bmatrix} -0.3 & 0.3 \end{bmatrix}$ using 1000 replications for each simulated sample. The nominal level was set to 0.05 and the sample size $n = 25$. All of the three original tests showed good power properties with power increasing along the directions of the alternatives, and peaking at about 0.45 around the edges of G_τ. As expected from Bravo (2003), none of the three test statistic was uniformly superior in G_τ, although empirical likelihood seemed slightly superior for values of the alternative closer to the null hypothesis. In the case of the Bartlett and Bartlett-type corrections, the simulations indicated that the modified statistics still have reasonable power on G_τ, but they are clearly less powerful than the original statistics. Figures 1 and 2 show the power *difference* between the original and their adjusted versions.

Figures 1-2 approx. here

Figures 1 and 2 show that the power differences range from -0.03 to -0.1 which gives power losses between 6 and 20 per cent. Notice that the differences first seem to increase (although not uniformly) according to the direction of the alternatives and then stabilise towards the edge of G_τ -with the possible exception of τ^j approaching $\begin{bmatrix} -0.3 & -0.3 \end{bmatrix}$. Notice also that the magnitude of the differences is bigger for the exponential data (21), and smaller for the Bartlett corrected empirical likelihood ratio. These characteristics of the power difference were found also when considering the other three (univariate) distributions and therefore suggest that, in general, Bartlett and Bartlett-type adjustments affect negatively the power of empirical discrepancy statistics. The magnitude of this negative effect depends on a number of factors including the characteristics of the unknown distribution of the data, the direction of the alternatives and the functional form of the correction itself. Thus, and perhaps not surprisingly, the simulations suggest that the price to pay in order to obtain

[3] Notice that in the case of the original (oversized) test statistics (and of the Bartett-corrected empirical likelihood ratio) the calculations were carried out using Monte Carlo adjusted critical values, whereas in the case of the Bartlett-type corrected Euclidean and Kullback-Liebler statistic we used tabulated critical values

improved inferences is a general, albeit small, loss in power.

5 Conclusions

In this paper we have derived two Bartlett-type adjustments that can be used to obtain improved inferences for the class of empirical discrepancy statistics recently introduced by Corcoran (1998). The finite sample behaviour of the proposed Bartlett-type adjustments has been investigated by means of simulations. The results of the latter are encouraging and suggest that both corrections are effective in bringing the observed size (coverage) of the original test statistics closer to the nominal one. However, they show that the resulting corrected test statistics become in some cases (slightly) undersized (i.e. the resulting coverage regions are larger). The latter point is a simple consequence of relative poor quality of χ^2 approximation to the distribution of the empirical discrepancy statistics, in particular at the higher quantiles (i.e. curved Q-Q plots as mentioned in the previous section), and should not be taken as a criticism of Bartlett-type corrections. As remarked by Corcoran et al. (1995) "[Empirical discrepancy statistics] are a hybrid, where a discrete multinomial distribution is placed on a sample assumed to be from a continuous underlying continuous distribution", and therefore it is perhaps not surprising that although Bartlett and Bartlett-type corrections apply to both parametric and nonparametric likelihoods, they are typically less effective for the latter. Despite this shortcoming, the simulations results show clearly that Bartlett-type corrections do produce improved confidence regions that are accurate enough for many practical purposes, especially if one is willing to accept some losses in terms of powers.

The results of this paper can be used to obtain improved inferences for parameters defined by the class of just-determined moment based models (1) . It would be of some interest to consider the more general case of over-determined moment based models like those considered by Qin and Lawless (1994), since these models are typically characterised by large finite sample size distortions and are often difficult to bootstrap. It would also be of interest to generalise the results of the paper to the so-called smooth functions of means model considered by DiCiccio et al. (1991). We hope to consider these topics in future communications.

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Appendix

The signed squared root decomposotion of $W(\theta_0)$

The signed squared root W^j of $W(\theta_0)$ is a q-dimensional vector such that $W(\theta_0) = nW^jW^j$.Neglecting terms of order $O_p\left(n^{-3/2}\right)$ it follows from (3) that W^j has components

$$\begin{aligned} W^j \;=\;& (\,k_h)^{1/2}\Bigg[\frac{b_1^{1/2}}{a_1}A^j + \frac{A^k}{2}\left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\frac{a_1}{b_1^{1/2}}\alpha^{jkl}A^l - \frac{b_1^{1/2}}{a_1}A^{jk}\right\} + \\ & \frac{a_1}{2b_1^{1/2}}A^k\Bigg\{\frac{(a_1^2 - a_2)(a_1^2b_1 - 5a_2b_1 + 3a_1b_2)}{a_1^6}A^jA^k + \\ & \frac{a_2(5a_2b_1 - 3a_1b_2)}{a_1^6}\alpha^{jkn}\alpha^{lmn}A^lA^m + \frac{3(a_1b_2 - 2a_2b_1)}{a_1^4}\alpha^{jkm}A^{lm}A^l + \\ & \frac{3b_1}{4a_1^2}A^{jl}A^{kl} + \frac{2a_2b_1 - a_1b_2}{a_1^4}A^{jkl}A^l - \frac{a_1^2}{4b_1}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2\alpha^{jkn}\alpha^{lmn}A^lA^m + \\ & \left(\frac{2a_2b_1 - a_1b_2}{2a_1^4}\right)\alpha^{klm}A^{jm}A^l + \frac{a_1b_3 - 2a_3b_1}{a_1^5}\alpha^{jklm}A^lA^m\Bigg\}\Bigg]. \end{aligned}$$

Lengthy calculations show that

$$\begin{aligned} \kappa^j &= (\,k_h)^{1/2}\kappa_1^j/n^{1/2} + O\left(n^{-3/2}\right), \quad \kappa^{j,\kappa} = k_h\left\{\delta^{j\kappa}b_1/a_1^2 + \kappa_2^{j\kappa}/n\right\} + O\left(n^{-2}\right), \\ \kappa^{j,\kappa,l} &= (\,k_h)^{3/2}\kappa_3^{j\kappa l}/n^{1/2} + O\left(n^{-3/2}\right), \quad \kappa^{j,\kappa,l,m} = (\,k_h)^2\kappa_4^{j\kappa lm}/n + O\left(n^{-2}\right), \\ \kappa^{j,\ldots,j_r} &= O\left(n^{1-r/2}\right) \quad \text{for } r \geq 5, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \kappa_1^j &= \frac{1}{2}\left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\frac{a_1}{b_1^{1/2}} - \frac{b_1^{1/2}}{a_1}\right\}\alpha^{jkk}, \\ \kappa_2^{jk} &= \frac{(a_1^2 - a_2)(a_1^2b_1 - 5a_2b_1 + 3a_1b_2)}{a_1^6}\delta^{jk}(2+q) + 3\left\{\frac{2a_2b_1 - a_1b_2}{a_1^4} + \right. \\ & \left.\frac{a_1b_3 - 2a_3b_1}{a_1^5}\right\}\alpha^{jkll} + \left\{\frac{b_1}{a_1^2} + \frac{2a_2(5a_2b_1 - 3a_1b_2)}{a_1^6} + \right. \\ & \left.\frac{5(a_1b_2 - 2a_2b_1)}{a_1^4}\right\}\alpha^{jlm}\alpha^{klm} + \left\{\frac{a_2(5a_2b_1 - 3a_1b_2)}{a_1^6} + \right. \end{aligned}$$

$$\frac{3b_1}{4a_1^2}-\frac{a_1^2}{4b_1}\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)^2-5\left(\frac{2a_2b_1-a_1b_2}{2a_1^4}\right)\Bigg\}\alpha^{jkl}\alpha^{lmm},$$

$$\kappa_3^{jkl} = \frac{b_1^{1/2}}{a_1}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}\alpha^{jkl},$$

$$\kappa_4^{jklm} = \frac{2b_1}{a_1^2}\left\{-\frac{b_1}{a_1^2}+6\left(\frac{2a_2b_1-a_1b_2}{a_1^4}+\frac{a_1b_3-2a_3b_1}{a_1^5}\right)\right\}\alpha^{jklm}+$$

$$4\frac{(a_1^2-a_2)\left(a_1^2b_1-5a_2b_1+3a_1b_2\right)}{a_1^6}\frac{b_1}{a_1^2}[3]\,\delta^{jk}\delta^{lm}+$$

$$\Bigg\{4\frac{b_1^2}{a_1^4}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)^2+4\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1-3a_1b_2\right)}{a_1^6}-$$

$$14\frac{b_1}{a_1^2}\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\Bigg\}[3]\,\alpha^{jkn}\alpha^{lmn}.$$

The last line of (22) follows from the general formulae developed by James and Mayne (1962). Substituting (4) in (22) shows that

$$\kappa_3^{jkl} = -\left(2n\partial^2h+\partial^3h\right)\alpha^{jkl}/\left(n^{3/2}\partial^2h\right)+O\left(n^{-3/2}\right),$$

$$\kappa_4^{jklm} = \left\{-3\left(\partial^3h\right)^2/\partial^2h+\partial^4h-2n\left(2\partial^3h+n\partial^2h\right)\right\}\alpha^{jklm}/\left(n^3\partial^2h\right)+$$

$$64\left\{1/4+\partial^3h/\left(8n\partial^2h\right)\right\}\left\{-\partial^3h/8+\left(\partial^3h+n\partial^2h\right)/4\right\}[3]\,\delta^{jk}\delta^{lm}/\left(n^2\partial^2h\right)+$$

$$\left\{4+4\left(\partial^3h\right)^2/\left(3n\partial^2h\right)^2+14\partial^3h/\left(3n\partial^2h\right)\right\}[3]\,\alpha^{jkn}\alpha^{lmn}/n+O\left(n^{-2}\right)$$

from which one gets (5).

Proof of Theorem 1

Using (22), calculations reveal that the cumulants for the perturbed statistic $n^{1/2}W_{CM}^j$ defined in (6) are

$$\kappa_{CM}^{j} = (\ k_h)^{1/2}\left(\kappa_1^j+C^{jkk}\frac{b_1^{1/2}}{a_1}\right)/n^{1/2}+O\left(n^{-3/2}\right),$$

$$\kappa_{CM}^{j,\kappa} = k_h\left[\delta^{j\kappa}\frac{b_1}{a_1^2}+\left\{\kappa_2^{j\kappa}+\frac{b_1}{a_1^2}\left(2C^{jk}+6C^{jkll}+2C^{jlm}C^{klm}\right)+\right.\right.$$

$$\left.\left.2\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\alpha^{jlm}C^{klm}\right\}/n\right]+O\left(n^{-2}\right),$$

$$\kappa_{CM}^{j,\kappa,l} = (\ k_h)^{3/2}\left(\kappa_3^{j\kappa l}+6\frac{b_1^{3/2}}{a_1^3}C^{jkl}\right)/n^{1/2}+O\left(n^{-3/2}\right),$$

$$\kappa_{CM}^{j,\kappa,l,m} = (\ k_h)^2\left[\kappa_4^{j\kappa lm}+24\frac{b_1^2}{a_1^4}\left\{C^{jklm}+2C^{jkn}C^{lmn}-2\alpha^{jkn}C^{lmn}+\right.\right.$$

$$\left.\left.2\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\frac{a_1^2}{b_1}\alpha^{jkn}C^{lmn}\right\}\right]/n+O\left(n^{-2}\right),$$

$$\kappa_{CM}^{j,\ldots,j_r} \;=\; O\left(n^{1-r/2}\right) \quad \text{for } r \geq 5. \tag{23}$$

Exponentiation of the approximate cumulant generating function implied by (23) and a Taylor expansion of the resulting exponential about $\xi = 0$ yields the approximate moment generating function $\psi_{W_{CM}^j}(\xi)$ of the perturbed statistic W_{CM}^j

$$\begin{aligned}\psi_{W_{CM}^j}(\xi) \;=\; & e^{\frac{\xi^j \xi^j}{2}}\left[1 + \left(\kappa_{CM}^j \xi^j + \frac{\kappa_{CM}^{j,k,l}\xi^j\xi^k\xi^l}{6}\right)/n^{1/2} + \left\{\left(\kappa_{CM}^{j,k} + \kappa_{CM}^j\kappa_{CM}^k\right)\xi^j\xi^k/2 + \right.\right.\\ & \left.\left.\left([4]\,\kappa_{CM}^j\kappa_{CM}^{k,l,m} + \kappa_{CM}^{j,k,l,m}\right)\frac{\xi^j\xi^k\xi^l\xi^m}{24} + \frac{\kappa_{CM}^{j,k,l}\kappa_{CM}^{m,n,o}\xi^j\xi^k\xi^l\xi^m\xi^n\xi^o}{72}\right\}/n\right],\end{aligned}$$

from which, by formal inversion and successive integration of the resulting Edgeworth density over the $\Re^q$-valued sphere of radius $u^{1/2}$, it is easy to see that (7) will hold iff

$$\left(\kappa_{CM}^{j,k} + \kappa_{CM}^j\kappa_{CM}^k\right) = 0, \quad \left([4]\,\kappa_{CM}^j\kappa_{CM}^{k,l,m} + \kappa_{CM}^{j,k,l,m}\right) = 0, \quad \kappa_{CM}^{j,k,l}\kappa_{CM}^{m,n,o} = 0, \tag{24}$$

by the symmetry of the normal distribution. Solving (24) for $C^{jk}, C^{jkl}, C^{jklm}$ gives (8).

Density of W $(_0)$

Lengthy calculations show that the approximate density of $W(\theta_0)$ is given by

$$f_{W(\theta_0)}(x) = g_q(x) + \sum_{r=0}^{3}\frac{d_r g_{q+2r}(x)}{n} + O\left(n^{-3/2}\right),$$

where

$$\begin{aligned} d_0 \;=\; & \left[\left(3k_h\frac{b_1}{a_1^2} - 1\right)\frac{\left(a_1^2 - a_2\right)\left(a_1^2 b_1 - 5a_2 b_1 + 3a_1 b_2\right)}{2a_1^6}k_h q\,(2+q) + \right.\\ & \left[-k_h\frac{b_1^2}{2a_1^4} + 3\left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right) + \left(\frac{a_1b_3 - 2a_3b_1}{a_1^5}\right)\right\}\left(k_h\frac{b_1}{a_1^2} - 1\right)\right] \times \\ & k_h\frac{\alpha^{jjkk}}{2} - \left[\left\{\frac{b_1}{a_1^2} + \frac{2a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} + \frac{5\left(a_1b_2 - 2a_2b_1\right)}{a_1^4}\right\}\frac{k_h}{2} + \right.\\ & \left\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} - \right.\\ & \left.\left. 42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}\frac{k_h^2}{12} - \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}^2\frac{k_h^3}{12}\right] \times \\ & \alpha^{jkl}\alpha^{jkl} + \left[-\left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\frac{a_1}{b_1^{1/2}} - \frac{b_1^{1/2}}{a_1}\right\}^2\frac{k_h}{8} - \left\{\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} + \right.\right. \end{aligned}$$

$$
\begin{aligned}
& \frac{3b_1}{4a_1^2} - \frac{a_1^2}{4b_1}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 - 5\left(\frac{2a_2b_1 - a_1b_2}{2a_1^4}\right)\Bigg\}\frac{k_h}{2} + \\
& \Bigg\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} - \\
& 42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\Bigg\}\frac{k_h^2}{24} + \left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}\times \\
& \left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right) - \frac{b_1}{a_1^2}\right\}\frac{k_h^2}{24} - \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}^2 \\
& \frac{k_h^3}{8}\Bigg]\alpha^{jjk}\alpha^{kll},
\end{aligned}
$$

$$
\begin{aligned}
d_1 \;=\; & \Bigg[\left(-3k_h\frac{b_1}{a_1^2} + \frac{1}{2}\right)\frac{\left(a_1^2 - a_2\right)\left(a_1^2b_1 - 5a_2b_1 + 3a_1b_2\right)}{a_1^6}k_hq\left(2+q\right) + \\
& \left[k_h\frac{b_1^2}{2a_1^4} + 3\left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right) + \left(\frac{a_1b_3 - 2a_3b_1}{a_1^5}\right)\right\}\left(\frac{1}{2} - k_h\frac{b_1}{a_1^2}\right)\right]k_h\alpha^{jjkk} + \\
& \Bigg[\left\{\frac{b_1}{a_1^2} + \frac{2a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} + \frac{5\left(a_1b_2 - 2a_2b_1\right)}{a_1^4}\right\}\frac{k_h}{2} - \\
& \Bigg\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} - \\
& 42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\Bigg\}\frac{k_h^2}{6} + \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}^2\frac{k_h^3}{4}\Bigg]\times \\
& \alpha^{jkl}\alpha^{jkl} + \Bigg[\Bigg\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\frac{a_1}{b_1^{1/2}} - \frac{b_1^{1/2}}{a_1}\Bigg\}^2\frac{k_h}{8} + \Bigg\{\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} + \\
& \frac{3b_1}{4a_1^2} - \frac{a_1^2}{4b_1}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 - 5\left(\frac{2a_2b_1 - a_1b_2}{2a_1^4}\right)\Bigg\}\frac{k_h}{2} - \\
& \Bigg\{12\frac{b_1^2}{a_1^4} + 9\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)^2 + 12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1 - 3a_1b_2\right)}{a_1^6} - \\
& 42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\Bigg\}\frac{k_h^2}{12} - \left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}\times \\
& \left\{\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right) - \frac{b_1}{a_1^2}\right\}\frac{k_h^2}{12} + \frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2} + 3\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right)\right\}^2\times \\
& \frac{3k_h^3}{8}\Bigg]\alpha^{jjk}\alpha^{kll},
\end{aligned}
$$

$$
\begin{aligned}
d_2 \;=\; & \frac{b_1}{a_1^2}\left\{-\frac{b_1}{4a_1^2} + \frac{3}{2}\left(\frac{2a_2b_1 - a_1b_2}{a_1^4}\right) + \frac{3}{2}\left(\frac{a_1b_3 - 2a_3b_1}{a_1^5}\right)\right\}k_h^2\alpha^{jjkk} + \\
& \frac{3\left(a_1^2 - a_2\right)\left(a_1^2b_1 - 5a_2b_1 + 3a_1b_2\right)}{2a_1^6}k_h^2\frac{b_1}{a_1^2}q\left(2+q\right) +
\end{aligned}
$$

$$\left[\left\{12\frac{b_1^2}{a_1^4}+9\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)^2+12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1-3a_1b_2\right)}{a_1^6}-\right.\right.$$
$$\left.\left.42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}2k_h^2-\frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}^2 6k_h^3\right]\times$$
$$\alpha^{jkl}\alpha^{jkl}/24+\left[\left[\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}\left\{\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)-\frac{b_1}{a_1^2}\right\}k_h^2+\right.\right.$$
$$\left\{12\frac{b_1^2}{a_1^4}+9\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)^2+12\frac{b_1}{a_1^2}\frac{a_2\left(5a_2b_1-3a_1b_2\right)}{a_1^6}-\right.$$
$$\left.\left.42\frac{b_1}{a_1^2}\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}k_h^2-\frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}^2 9k_h^3\right]\times$$
$$\alpha^{jjk}\alpha^{kll}/24,$$
$$d_3 = \frac{k_h^3}{72}\frac{b_1}{a_1^2}\left\{-\frac{2b_1}{a_1^2}+3\left(\frac{2a_2b_1-a_1b_2}{a_1^4}\right)\right\}^2\left(9\alpha^{jjk}\alpha^{kll}+6\alpha^{jkl}\alpha^{jkl}\right).$$

Proof ot Theorem 2

Using the recurrence relation $g_{q+2r}(x) = x^r g_q(x)/q(q+2)\ldots(q+2r)$, the density of $f_{W(\theta_0)}(x)$ as given in the asymptotic expansion (9) can be written as

$$f_{W(\theta_0)}(x) = g_q(x)\left\{1+\sum_{r=0}^{3}\frac{d_r' x^r}{n}\right\}+O\left(n^{-3/2}\right),$$

where $d_r' = d_r/(q)_r$. As in Cordeiro and Ferrari (1991), we can define the modified statistic

$$W_{CF}(\theta_0) = W(\theta_0)\left\{1-\sum_{r=1}^{3}\frac{c_r\left\{W(\theta_0)\right\}^{r-1}}{n}\right\}+O\left(n^{-3/2}\right),$$

and note that the moment generating function $\psi_{W_{CF}(\theta_0)}(\xi)$ of $W_{CF}(\theta_0)$ can be expressed as

$$\psi_{W_{CF}(\theta_0)}(\xi) = \psi_{\chi_q^2}(\xi)+\frac{\mathcal{J}(\xi)(1-2\xi)^{-q/2}}{n2^{q/2}\Gamma(q/2)}+O\left(n^{-3/2}\right)$$

where

$$\mathcal{J}(\xi) = \int_0^\infty \exp\{-y/2\}\, y^{q/2-1}\left\{\sum_{j=1}^{3}\frac{\left(d_j'-c_3\xi\right)}{(1-2\xi)^j}y^j+d_0'\right\}dy.$$

Solving for c_1, c_2, c_3 the equation $\mathcal{J}(\xi)=0$ gives (12) after some algebra.

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Tables and figures

Table 1. Observed size of a nominal $\alpha\%$ -level Euclidean likelihood $\mathcal{CR}_{-2}$,Kullback-Liebler $\mathcal{CR}_{-1}$ and empirical likelihood $\mathcal{CR}_0$ test with theoretical and estimated Bartlett-type adjustments for $N(0,1)$ data.

n	Statistic	10	5	Statistic	10	5	Statistic	10	5
15	$\mathcal{CR}_{-2}$	13.90	8.41	$\mathcal{CR}_{-1}$	13.71	7.39	$\mathcal{CR}_0$	13.83	7.53
	W^a_{CM}	11.12	6.23	W^a_{CM}	9.92	5.73^c	W^a_{CM}	12.12	6.22
	$\widehat{W}^b_{CM}$	11.29	6.54	$\widehat{W}^b_{CM}$	10.44^c	5.97	$\widehat{W}^b_{CM}$	12.36	7.01
	W^a_{CFE}	8.98	3.97	W^a_{CFE}	8.89	4.03	W^a_{CF}	12.12	6.62
	$\widehat{W}^b_{CFE}$	9.05	4.01	$\widehat{W}^b_{CFE}$	8.97	4.12	$\widehat{W}^b_{CF}$	12.36	7.01
25	$\mathcal{CR}_{-2}$	13.44	7.43	$\mathcal{CR}_{-1}$	13.03	7.02	$\mathcal{CR}_0$	13.12	7.05
	W^a_{CM}	12.37	5.97	W^a_{CM}	11.91	5.53^c	W^a_{CM}	12.01	6.32
	$\widehat{W}^b_{CM}$	12.44	6.09	$\widehat{W}^b_{CM}$	12.10	5.69^c	$\widehat{W}^b_{CM}$	12.33	6.51
	W^a_{CFE}	10.69^c	4.12	W^a_{CFE}	9.74^c	4.77^c	W^a_{CF}	12.01	6.32
	$\widehat{W}^b_{CFE}$	10.78^c	4.35^c	$\widehat{W}^b_{CFE}$	9.83^c	4.88^c	$\widehat{W}^b_{CF}$	12.33	6.51
50	$\mathcal{CR}_{-2}$	11.82	6.31	$\mathcal{CR}_{-1}$	1166	5.83^c	$\mathcal{CR}_0$	11.54	5.71^c
	W^a_{CM}	11.06^c	5.71^c	W^a_{CM}	9.72^c	5.53^c	W^a_{CM}	11.03^c	5.35^c
	$\widehat{W}^b_{CM}$	11.15^c	5.79^c	$\widehat{W}^b_{CM}$	9.89^c	5.65^c	$\widehat{W}^b_{CM}$	11.26^c	5.56^c
	W^a_{CFE}	10.81^c	4.15	W^a_{CFE}	9.65^c	4.81^c	W^a_{CF}	11.03^c	5.35^c
	$\widehat{W}^b_{CFE}$	10.89^c	4.25^c	$\widehat{W}^b_{CFE}$	9.92^c	4.92^c	$\widehat{W}^b_{CF}$	11.26^c	5.56^c

a Adjusted test with the theoretical Bartlett-type correction, *b* Adjusted test with estimated Bartlett-type correction
c Difference between observed and nominal size is not statistically significant at 0.01 level.

Table 2. Observed size of a nominal $\alpha\%$ -level Euclidean likelihood $\mathcal{CR}_{-2}$, Kullback-Liebler $\mathcal{CR}_{-1}$ and empirical likelihood $\mathcal{CR}_0$ test with theoretical and estimated Bartlett-type adjustments for t_5 data

n	Statistic	10	5	Statistic	10	5	Statistic	10	5
15	$\mathcal{CR}_{-2}$	3.47	7.86	$\mathcal{CR}_{-1}$	13.33	7.23	$\mathcal{CR}_0$	13.80	8.21
	W^a_{CM}	8.69	4.03	W^a_{CM}	9.03^c	4.10	W^a_{CM}	8.99	4.17
	$\widehat{W}^b_{CM}$	8.97	4.11	$\widehat{W}^b_{CM}$	9.27^c	4.12	$\widehat{W}^b_{CM}$	9.08	4.04
	W^a_{CFE}	8.15	3.94	W^a_{CFE}	8.89	4.05	W^a_{CF}	8.99	4.17
	$\widehat{W}^b_{CFE}$	8.50	4.15	$\widehat{W}^b_{CFE}$	8.53	4.10	$\widehat{W}^b_{CF}$	9.08	4.04
25	$\mathcal{CR}_{-2}$	12.53	6.71	$\mathcal{CR}_{-1}$	12.91	6.85	$\mathcal{CR}_0$	12.61	7.24
	W^a_{CM}	8.79	4.08	W^a_{CM}	9.27^c	4.15	W^a_{CM}	9.69^c	4.26^c
	$\widehat{W}^b_{CM}$	9.05	4.15	$\widehat{W}^b_{CM}$	9.54^c	4.21^c	$\widehat{W}^b_{CM}$	9.44^c	4.35^c
	W^a_{CFE}	8.94	4.05	W^a_{CFE}	9.01	4.08	W^a_{CF}	9.69^c	4.26^c
	$\widehat{W}^b_{CFE}$	9.14^c	4.12	$\widehat{W}^b_{CFE}$	9.15^c	4.18^c	$\widehat{W}^b_{CF}$	9.44^c	4.35^c
50	$\mathcal{CR}_{-2}$	12.15	6.42	$\mathcal{CR}_{-1}$	12.26	6.71	$\mathcal{CR}_0$	12.05	6.95
	W^a_{CM}	9.38^c	4.31^c	W^a_{CM}	9.51^c	4.44^c	W^a_{CM}	10.35^c	5.57^c
	$\widehat{W}^b_{CM}$	9.51^c	4.46^c	$\widehat{W}^b_{CM}$	9.69^c	4.63^c	$\widehat{W}^b_{CM}$	10.57^c	5.38^c
	W^a_{CFE}	9.11^c	4.25^c	W^a_{CFE}	9.31^c	4.39^c	W^a_{CF}	10.35^c	5.57^c
	$\widehat{W}^b_{CFE}$	9.26^c	4.51^c	$\widehat{W}^b_{CFE}$	9.42^c	4.51^c	$\widehat{W}^b_{CF}$	10.57^c	5.38^c

a Adjusted test with the theoretical Bartlett-type correction, b Adjusted test with estimated Bartlett-type correction
c Difference between observed and nominal size is not statistically significant at 0.01 level.

Table 3. Observed size of a nominal $\alpha\%$ -level Euclidean likelihood $\mathcal{CR}_{-2}$, Kullback-Liebler $\mathcal{CR}_{-1}$ and empirical likelihood $\mathcal{CR}_0$ test with theoretical and estimated Bartlett-type adjustments for χ^2_4 data

n	Statistic	10	5	Statistic	10	5	Statistic	10	5
15	$\mathcal{CR}_{-2}$	15.25	10.05	$\mathcal{CR}_{-1}$	14.55	8.93	$\mathcal{CR}_0$	14.84	9.35
	W^a_{CM}	12.35	8.52	W^a_{CM}	12.41	7.43	W^a_{CM}	12.04	7.55
	$\widehat{W}^b_{CM}$	12.64	8.81	$\widehat{W}^b_{CM}$	12.78	7.59	$\widehat{W}^b_{CM}$	12.38	7.89
	W^a_{CFE}	8.42	3.78	W^a_{CFE}	8.25	3.59	W^a_{CF}	12.04	7.55
	$\widehat{W}^b_{CFE}$	8.78	4.02	$\widehat{W}^b_{CFE}$	8.19	3.77	$\widehat{W}^b_{CF}$	12.38	7.89
25	$\mathcal{CR}_{-2}$	13.08	7.74	$\mathcal{CR}_{-1}$	12.64	7.21	$\mathcal{CR}_0$	12.58	7.52
	W^a_{CM}	12.31	6.98	W^a_{CM}	11.95	6.57	W^a_{CM}	11.69	6.41
	$\widehat{W}^b_{CM}$	12.04	7.20	$\widehat{W}^b_{CM}$	12.21	6.74	$\widehat{W}^b_{CM}$	11.90	6.74
	W^a_{CFE}	8.69	3.95	W^a_{CFE}	8.90	4.00	W^a_{CF}	11.69	6.41
	$\widehat{W}^b_{CFE}$	8.83	4.09	$\widehat{W}^b_{CFE}$	9.06^a	4.16^c	$\widehat{W}^b_{CF}$	11.90	6.74
50	$\mathcal{CR}_{-2}$	11.34	6.49	$\mathcal{CR}_{-1}$	11.27	6.32	$\mathcal{CR}_0$	11.07	6.05
	W^a_{CM}	10.93^c	6.12	W^a_{CM}	10.74^c	5.97	W^a_{CM}	9.95^c	5.89
	$\widehat{W}^b_{CM}$	11.09^c	6.32	$\widehat{W}^b_{CM}$	10.89^c	6.16	$\widehat{W}^b_{CM}$	9.73^c	5.99
	W^a_{CFE}	9.03^c	4.06	W^a_{CFE}	9.42^c	4.09	W^a_{CF}	9.95^c	5.89
	$\widehat{W}^b_{CFE}$	9.19^c	4.27^c	$\widehat{W}^b_{CFE}$	9.27^c	4.34^c	$\widehat{W}^b_{CF}$	9.73^c	5.99

a Adjusted test with the theoretical Bartlett-type correction, *b* Adjusted test with estimated Bartlett-type correction
c Difference between observed and nominal size is not statistically significant at 0.01 level.

Table 4. Observed size of a nominal $\alpha\%$ -level Euclidean likelihood $\mathcal{CR}_{-2}$, Kullback-Liebler $\mathcal{CR}_{-1}$, and empirical likelihood $\mathcal{CR}_0$ test with theoretical and estimated Bartlett-type adjustments for bivariate $N(0, I)$ data

n	Statistic	10	5	Statistic	10	5	Statistic	10	5
15	$\mathcal{CR}_{-2}$	14.25	9.76	$\mathcal{CR}_{-1}$	16.55	10.25	$\mathcal{CR}_0$	16.48	11.97
	W_{CM}^a	8.41	3.88	W_{CM}^a	8.34	3.94	W_{CM}^a	12.32	8.59
	$\widehat{W}_{CM}^b$	8.56	3.95	$\widehat{W}_{CM}^b$	8.15	4.01	$\widehat{W}_{CM}^b$	12.75	8.91
	W_{CFE}^a	8.32	3.42	W_{CFE}^a	8.24	3.63	W_{CF}^a	12.24	8.46
	$\widehat{W}_{CFE}^b$	8.45	3.51	$\widehat{W}_{CFE}^b$	8.05	3.72	$\widehat{W}_{CF}^b$	12.53	8.65
25	$\mathcal{CR}_{-2}$	12.49	7.15	$\mathcal{CR}_{-1}$	12.33	6.84	$\mathcal{CR}_0$	13.76	8.11
	W_{CM}^a	8.62	3.96	W_{CM}^a	8.44	3.95	W_{CM}^a	10.73	6.55
	$\widehat{W}_{CM}^b$	8.73	4.04	$\widehat{W}_{CM}^b$	8.52	4.07	$\widehat{W}_{CM}^b$	10.58	6.89
	W_{CFE}^a	8.35	3.79	W_{CFE}^a	8.56	3.89	W_{CF}^a	10.86	6.49
	$\widehat{W}_{CFE}^b$	8.44	3.87	$\widehat{W}_{CFE}^b$	8.64	3.96	$\widehat{W}_{CF}^b$	10.72	6.37
50	$\mathcal{CR}_{-2}$	11.74	6.42	$\mathcal{CR}_{-1}$	12.01	5.90	$\mathcal{CR}_0$	12.54	6.12
	W_{CM}^a	9.09^c	4.06	W_{CM}^a	9.29^c	4.10^c	W_{CM}^a	10.93^c	5.31^c
	$\widehat{W}_{CM}^b$	9.23^c	4.15^c	$\widehat{W}_{CM}^b$	9.45^c	4.21^c	$\widehat{W}_{CM}^b$	11.05^c	5.42^c
	W_{CFE}^a	8.55	3.99	W_{CFE}^a	9.11^c	4.05	W_{CF}^a	10.88^c	5.22^c
	$\widehat{W}_{CFE}^b$	8.72	4.08	$\widehat{W}_{CFE}^b$	9.27^c	4.16^c	$\widehat{W}_{CF}^b$	10.97^c	5.31^c

a Adjusted test with the theoretical Bartlett-type correction, *b* Adjusted test with estimated Bartlett-type correction
c Difference between observed and nominal size is not statistically significant at 0.01 level.

Table 5. Observed size of a nominal $\alpha\%$ -level Euclidean likelihood $\mathcal{CR}_{-2}$, Kullback-Liebler $\mathcal{CR}_{-1}$ and empirical likelihood $\mathcal{CR}_0$ test with theoretical and estimated Bartlett-type adjustments for z_i^1, z_i^2 data as in (21).

n	Statistic	10	5	Statistic	10	5	Statistic	10	5
15	$\mathcal{CR}_{-2}$	17.18	11.48	$\mathcal{CR}_{-1}$	20.24	13.18	$\mathcal{CR}_0$	21.98	15.18
	W^a_{CM}	11.48	3.25	W^a_{CM}	7.99	3.21	W^a_{CM}	15.23	9.91
	$\widehat{W}^b_{CM}$	12.21	3.51	$\widehat{W}^b_{CM}$	8.76	3.53	$\widehat{W}^b_{CM}$	16.43	11.06
	W^a_{CF}	8.51	3.16	W^a_{CF}	7.05	3.11	W^a_{CF}	15.55	10.09
	$\widehat{W}^b_{CF}$	8.39	3.37	$\widehat{W}^b_{CF}$	7.84	3.99	$\widehat{W}^b_{CF}$	16.21	10.89
25	$\mathcal{CR}_{-2}$	13.54	8.04	$\mathcal{CR}_{-1}$	15.94	9.08	$\mathcal{CR}_0$	16.52	10.56
	W^a_{CM}	8.99	3.62	W^a_{CM}	8.19	3.71	W^a_{CM}	12.33	8.77
	$\widehat{W}^b_{CM}$	9.17	3.79	$\widehat{W}^b_{CM}$	8.96	3.84	$\widehat{W}^b_{CM}$	13.62	9.53
	W^a_{CF}	8.94	3.49	W^a_{CF}	9.05	3.57	W^a_{CF}	12.73	8.51
	$\widehat{W}^b_{CF}$	8.82	3.63	$\widehat{W}^b_{CF}$	8.93	3.68	$\widehat{W}^b_{CF}$	13.84	8.32
50	$\mathcal{CR}_{-2}$	11.45	6.24	$\mathcal{CR}_{-1}$	11.23	6.17	$\mathcal{CR}_0$	12.68	7.84
	W^a_{CM}	9.52^c	3.85	W^a_{CM}	9.15^c	3.99^c	W^a_{CM}	11.18	6.15
	$\widehat{W}^b_{CM}$	9.48^c	3.97	$\widehat{W}^b_{CM}$	9.34^c	4.11^c	$\widehat{W}^b_{CM}$	11.48	6.42
	W^a_{CF}	9.05	3.81	W^a_{CF}	9.09	3.87	W^a_{CF}	11.29	6.06
	$\widehat{W}^b_{CF}$	9.23^c	3.93	$\widehat{W}^b_{CF}$	9.13^c	4.01	$\widehat{W}^b_{CF}$	11.49	6.37

a Adjusted test with the theoretical Bartlett-type correction, b Adjusted test with estimated Bartlett-type correction
c Difference between observed and nominal size is not statistically significant at 0.01 level.

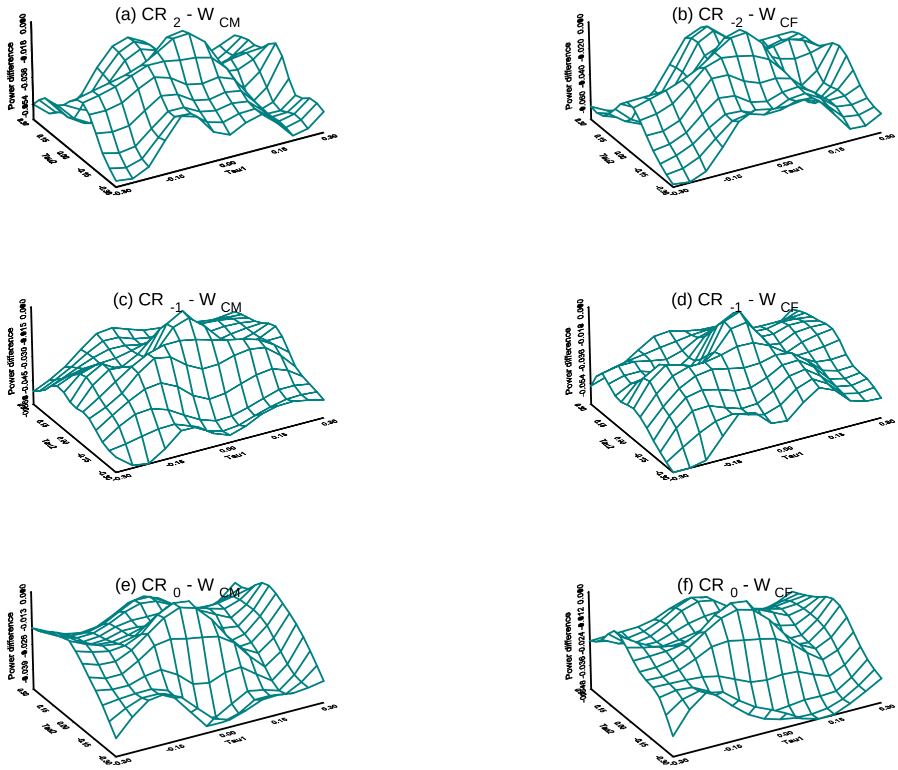

Figure 1: Observed power difference between the original Euclidean likelihood $\mathcal{CR}_{-2}$ (a-b), Kullback-Liebler $\mathcal{CR}_{-1}$ (c-d), and empirical likelihood $\mathcal{CR}_{0}$ (e-f) and their corrected versions W_{CM} (left column) and W_{CF} (right column) for $N(0, I)$ data.

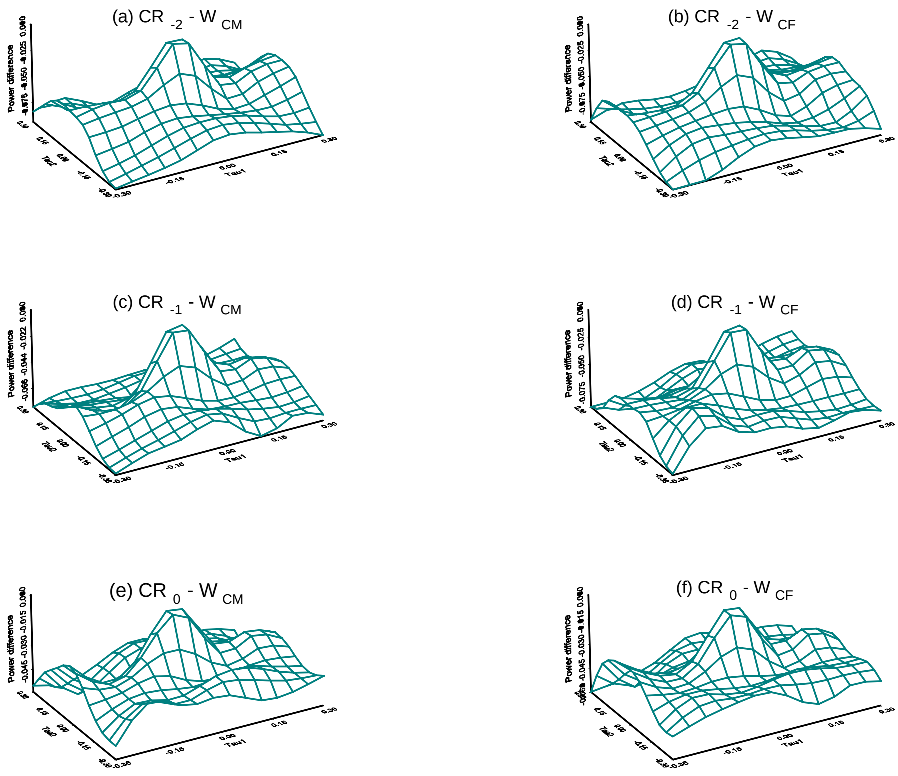

Figure 2: Observed power difference between the original Euclidean likelihood $\mathcal{CR}_{-2}$ (a-b), Kullback-Liebler $\mathcal{CR}_{-1}$ (c-d), and empirical likelihood $\mathcal{CR}_0$ (e-f) and their corrected versions W_{CM} (left column) and W_{CF} (right column) for z_i^1 and z_i^2 data as in (21).