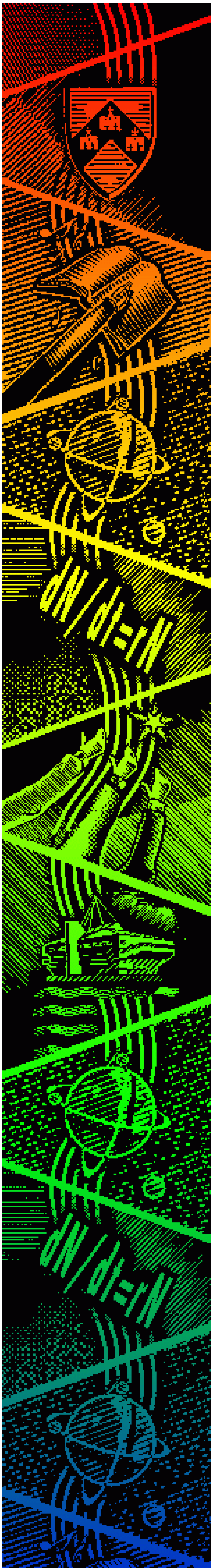

THE UNIVERSITY of York

Discussion Papers in Economics

No. 2003/18

Convertible Subordinated Debt Valuation and “Conversion in Distress”

by

Marco Realdon

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

CONVERTIBLE SUBORDINATED DEBT VALUATION AND "CONVERSION IN DISTRESS"

Dr Marco Realdon
Department of Economics and Related Studies
University of York
Heslington
YO10 5DD
York
UK
Tel: 0044/(0)1904/433750
Email: mr15@york.ac.uk

26/08/03

Abstract

This paper presents new formulae for the valuation of convertible debt and shows how it can be rational for convertible holders to convert not only when the debtor's equity value increases, but also when the debtor approaches distress. "Conversion in distress" can avert costly bankruptcy even when debt cannot be renegotiated. If bankruptcy costs are high, neglecting "conversion in distress" may entail a significant undervaluation of subordinated convertibles. "Conversion in distress" makes convertible debt less sensitive than non-convertible debt to the recovery value of assets in bankruptcy. So convertible financing can reduce the cost of borrowing when lenders are asymmetrically informed about the debtor's assets recovery value.

Key words: subordinated convertible debt, default, bankruptcy costs, "conversion in distress".

JEL classification: G13;G33.

1 INTRODUCTION

This paper analyses the valuation of convertible subordinated debt in the presence of bankruptcy costs when debt renegotiation is not possible, as may be the case for publicly traded convertible bonds. This issue is of interest since the volume of outstanding convertible bonds has grown dramatically since the middle 1990's, while the general creditworthiness of such bonds has significantly decreased. The increased default risk of convertibles is coupled with very high recovery risk since most convertibles are subordinated and the potential bankruptcy costs of convertibles issuers are often high (Moody's Investors Service (2001)).

The literature of structural models or contingent claims analysis has generally valued convertibles without recognising that, especially in the presence of bankruptcy costs, the convertible may be worth converting not only when the debtor's equity value rises, but also when the debtor approaches distress. So this paper derives new closed form solutions for the valuation of subordinated convertible debt which show the impact of "conversion in distress" on the value of a convertible. The reason for converting when the debtor is in the proximity of distress is that conversion reduces the debtor's leverage and may thus avert default and costly bankruptcy. In addition, due to bankruptcy costs or to the subordination covenant, the recovery value in bankruptcy of the non-converted convertible may well be below the value of the fraction of equity into which convertible holders have the right to convert. This argument is particularly important when the convertible is subordinated and entails that ignoring the possibility to convert "in distress" may lead to a significant undervaluation of low grade convertibles, especially when bankruptcy costs are high and when the convertible is very "equity like", ie. when convertible holders receive a large fraction of equity upon conversion. Moreover, the conversion option can increase the bargaining power of subordinated debt holders when debt holders can initiate debt renegotiation with equity holders.

Finally, conversion "in distress" causes the value of convertibles to be less affected by the magnitude of bankruptcy costs and assets "recovery risk" than similar non-convertible debt. This suggests that convertibles may be issued to reduce the cost of borrowing when debt holders are asymmetrically informed about the debtor's assets recovery value in bankruptcy.

The paper is organised as follows. After a review of the most relevant literature, section 2 concerns the valuation of convertible subordinated debt. The case of convertible non-subordinated debt is treated as a special case. Section 3 presents the effects of converting "in distress". Section 4 considers the case in which subordinated debt holders can initiate debt renegotiation. The conclusions follow.

1.1 The most relevant literature

Following the seminal contribution by Black and Scholes, the modern literature on convertible debt valuation starts with Ingersoll (1977) and Brennan and Schwartz (1977). Brennan and Schwartz (1980) and Nyborg (1996) provide models for the valuation of convertible subordinated debt, but make the restrictive assumption that senior debt must have the same maturity as the convertible. Unlike this paper, such contributions assume no conversion "in distress".

Anderson, Pan and Sundaresan (1997 and 2000) (APS) value convertible non subordinated debt in the presence of bankruptcy costs and stochastic interest rates. They assume that debt can be renegotiated and that repeated strategic debt service can take place, highlighting that convertible holders can respond to such hostile strategic debt service by converting. APS cannot provide closed form solutions. Instead, this paper provides closed form solutions for convertible subordinated debt in the presence of bankruptcy costs, corporate taxes and constant interest rates. Unlike in APS, this paper analyses the valuation of convertible subordinated debt when debt renegotiation is not possible, as may be the case for publicly held convertibles.

This paper reaches a complementary conclusion to the one in Brennan Schwartz (1988). Brennan Schwartz suggest that convertibles may be issued to decrease the cost of borrowing when lenders are asymmetrically informed about the debtor's assets volatility, since convertibles are less sensitive to assets volatility than non-convertible debt is. Instead this paper concludes that convertibles may be issued to decrease the cost of borrowing also when lenders are asymmetrically informed about the debtor's assets recovery value in bankruptcy, since convertibles that can be converted also "in distress" are less sensitive to bankruptcy costs than non-convertible debt is.

2 CONVERTIBLE SUBORDINATED DEBT WITH CONVERSION "IN DISTRESS"

This section studies the valuation of convertible subordinated debt when debtor and debt holders cannot renegotiate debt obligations in the proximity of distress. Renegotiation may not be possible because of a high degree of asymmetric information between debtor and debt holders or because the firm's debt is widely held in the public. The case of convertible debt that is not subordinated is encompassed as a special case. The new feature of the present analysis is that convertibles may be converted in order to avoid costly bankruptcy and assets liquidation as the debtor approaches default. But first we need to introduce some assumptions and a "classic" type of convertibles valuation model, i.e. one in which conversion takes place just when the debtor's equity value rises.

2.1 Assumptions and the "classic" convertible valuation model

Without much loss of generality "dynamic market completeness" is assumed. V is the value of the firm's assets, whose risk neutral process follows the geometric Brownian motion

$$dV = (r - b) \cdot V \cdot dt + s \cdot V \cdot dz \tag{1}$$

where r is the default free short rate of interest assumed constant over time, where b is the assets pay-out ratio, where s is the volatility parameter, where dz is the differential of a Wiener process; b and s are also assumed to be constant.

The debtor has issued non-convertible senior debt that has value $D(V,t)$ (t denotes time), that as infinite maturity, that continuously pays coupons at yearly rate C and that has face value F. F and C are constant over time. $D(V,t)$ depends on time even if it has infinite maturity because the default barrier is time-dependent.

The debtor has issued also convertible subordinated debt, whose value is $SD(V,t)$ and that pays a coupon flow at the yearly rate SC, has face value SF and has maturity T. $SD_s(V,t)$ denotes the value of non-convertible subordinated debt with the same coupon, face value and maturity as $SD(V,t)$, thus $SD(V,t) = O(V,t) + SD_s(V,t)$ where $O(V,t)$ is the present value of the conversion option embedded in the convertible. $O(V,t)$ is "European" in that it is assumed to be exercised just at maturity T.

t_x is the corporate tax rate, K denotes fixed bankruptcy costs and aV denotes proportional bankruptcy costs with a constant. Following Kim-Ramaswamy-Sundaresan (1993) and Anderson and Sundaresan (1996), default can be triggered by a liquidity condition at $t_b = \inf\{t < T : V_t = V_b\}$ or at $t_b = \inf\{t > T : V_t = V_b^p\}$ where

$$V_b^p = \frac{C(1-t_x)}{b} \leq V_b = \frac{(SC+C)(1-t_x)}{b}. \tag{2}$$

At maturity T default occurs if $V_b \leq V_T \leq V_{bT}$, where V_{bT} is such that $E(V_{bT}, T) = 0$, $E(V,T)$ denoting the value of the debtor's equity at T. Although not a necessary assumption, in what follows $V_{bT} \geq V_b$ since at T the payment of the face value SF or subordinated convertible debt falls due. This payment is usually a higher hurdle for the solvency of the debtor than the payment of coupons at the yearly rate $(SC+C)$ before T. If $V_b \leq V_T \leq V_{bT}$ default takes place and bankruptcy costs are $BC(V_T,T) = \min(K + aV_T, V_T)$, but if default takes place at $t_b \leq T$, bankruptcy costs are $BC(V_b) = \min(K + aV_b, V_b)$. Debt cannot be renegotiated and that the absolute priority rule (APR) is strictly upheld in bankruptcy.

Finally, for $t \leq T$ equity value is $E(V,t) = Tax(V,t) - BC(V,t) + V - D(V,t) - SD(V,t)$, where $BC(V,t)$ denotes the present value of expected bankruptcy costs and $Tax(V,t)$ denotes the present value of expected debt induced tax savings. Notice that unlike in past analyses of subordinated convertibles (Brennan and Schwartz (1980), Nyborg (1996)), here we do not assume that all the firm's outstanding debt have the same maturity as the convertible. Then, if the debtor is solvent up to T, following Leland (1994) it can be shown that:

$$E(V_T,T) = E(V_T) - SD(V_T,T) \tag{3}$$

$$E(V_T) = V_T + Tax(V_T) - BC(V_T) - D(V_T) \tag{4}$$

$$SD(V_T,T) = SF \tag{5}$$

$$Tax(V_T,T) = Tax(V_T) = \frac{C}{r}t_x\left(1 - \left(\frac{V_T}{V_b^p}\right)^q\right) \tag{6}$$

$$D(V_T,T) = D(V_T) = \frac{C}{r} + \left(-\frac{C}{r} + \max\left(\min\left(F, V_b^p\left(1-a\right) - K\right), 0\right)\right)\left(\frac{V_T}{V_b^p}\right)^q \tag{7}$$

$$BC(V_T,T) = BC\left(V_T\right) = \max\left(aV_b^p + K, V_b^p\right)\left(\frac{V_T}{V_b^p}\right)^q \tag{8}$$

$$q = \frac{-\left(r-b-\frac{1}{2}s^2\right) - \sqrt{\left(r-b-\frac{1}{2}s^2\right)^2 + 2rs^2}}{s^2}. \tag{9}$$

Moreover, if the debtor is solvent up to T, the payoff to the conversion option is

$$O_c\left(V_T,T\right) = \max\left(xE\left(V_T\right) - SD_s(V_T,T), 0\right) \tag{10}$$

with $SD_s(V_T,T) = SF$. $O_c\left(V_T,T\right)$ specifically denotes the value of the conversion option as per the "classic" model, ie. the option that can be exercised just when equity value increases. Thus, in the "classic" model conversion at T only occurs when $xE\left(V_T\right) \geq SF$, i.e. when $V_T \geq V_{cT} > V_{bT} \geq V_b$, where V_{cT} is the "conversion trigger" at T such that

$$xE\left(V_{cT}\right) = SF. \tag{11}$$

Then, if the debtor is solvent up to T, the "classic" model implies the following payoffs to equity holders and to convertible holders at T:

$$E\left(V_T,T\right) = \max\left(E\left(V_T\right) - SF - O_c\left(V,T\right), 0\right) \tag{12}$$

$$SD\left(V_T,T\right) = O_c\left(V,T\right) + SF. \tag{13}$$

But, if the debtor is insolvent by time T, the "classic model" assumes that the recovery value of defaulted convertible subordinated debt is equal to the recovery value of non-convertible subordinated debt, i.e. $R(V) = \min\left(SF, \max\left(V\left(1-a\right) - K - F, 0\right)\right)$, because after default the APR rule is strictly applied in bankruptcy and the conversion option becomes worthless. It follows that the "classic model" for the value of a subordinated convertible $SD\left(V,t\right)$ satisfies the following:

$$\frac{dSD\left(V,t\right)}{dt} + \frac{1}{2}\frac{d^2SD\left(V,t\right)}{dV^2}s^2V^2 + \frac{dSD\left(V,t\right)}{dV}\left(r-b\right)V - rSD\left(V,t\right) + SC = 0 \tag{14}$$

subject to

$$SD\left(V_T,T\right) = \begin{cases} \max\left(xE\left(V_T\right), SF\right) = O_c\left(V,T\right) + SF \text{ if } V_T > V_{bT} \\ R(V_T) = \min\left(SF, \max\left(V_T\left(1-a\right) - K - F, 0\right)\right) \text{ if } V_b \leq V_T \leq V_{bT} \end{cases} \tag{15}$$

$$SD\left(V_b,t_b\right) = R(V_b) = \min\left(SF, \max\left(V_b\left(1-a\right) - K - F, 0\right)\right) \tag{16}$$

$$SD\left(V \to \infty, t\right) \to \infty \tag{17}$$

The model thus far is "classic": "conversion in distress" is not contemplated.

2.2 Conversion in distress

This sub-section presents the convertible valuation model with conversion "in distress". $O(V,t)$ denotes the value of the conversion option when conversion "in distress" is possible. We can view $O(V,t)$ as the sum of (at least) two conversion options: $O_c(V,t)$ to be exercised at T as in the "classic model" when the debtor's equity value rises and $O_d(V,t)$ to be exercised at T if missed conversion causes the debtor to default. The payoff to the latter option is

$$O_d(V,T) = \max(xE(V) - SD_s(V,T), 0) \tag{18}$$

with $SD_s(V,T) = R(V_T)$, since $R(V_T)$ is the bankruptcy recovery value of subordinated debt, be it convertible or not. In other words, as the debtor approaches distress, convertible holders can convert and keep the debtor solvent or not convert and receive bankruptcy proceeds worth $R(V_T)$.

By converting "in distress" convertible holders get less than the face value of their claim SF and so make a spontaneous self-interested concession to the debtor which is not the fruit of any debt renegotiation. Convertible holders do so because, even if by converting they avert the debtor's default and can get $xE(V_T) < SF$, by not converting they let the debtor default and are bound to get even less, i.e. $SD(V_T,T) = R(V_T) < xE(V_T)$.

We define V_{dT} such that

$$SF \geq SD(V_{dT},T) = R(V_{dT}) = xE(V_{dT}) > 0. \tag{19}$$

Thus there can be two conversion triggers at T, namely V_{cT} and V_{dT}, such that

$$V_b \leq V_{dT} \leq V_{bT} < V_{cT}$$

and such that the "classic" conversion option $O_c(V,t)$ is exercised when $V_T \geq V_{cT}$ and the conversion "in distress" option $O_d(V,t)$ when $V_b \leq V_T \leq V_{dT} \leq V_{bT}$. From equation 19 it follows that, ceteris paribus, the higher are the face value of senior debt F or bankruptcy costs a and K, the higher the is V_{dT} and the more likely conversion "in distress" is. But V_{dT} such that $V_b \leq V_{dT} \leq V_{bT}$ can exist even when $F = 0$ or $a = 0$ or $K = 0$. So conversion "in distress" is possible even when the convertible is not subordinated or alternatively even when bankruptcy costs are null (but the tax shield would still be lost in bankruptcy). Raising the percentage of equity x allotted to convertible holders upon conversion makes the convertible more "equity like" because V_{cT} decreases, and at the same time raises V_{dT} so as to make also conversion "in distress" more likely.

Convertible holders may want to convert "in distress" also in case of default at $t_b \leq T$, with $t_b = \inf\{t < T : V_t = V_b\}$, provided $SD(V_b,t_b) = R(V_b) < xE(V_b) < SF$. Again conversion at t_b avoids costly assets liquidation, and it may arguably be possible even if the conversion option is "European". Upon default at t_b all debt becomes immediately due and so conversion should become immediately possible, too. Even if the indenture of a European convertible does not explicitly contemplate conversion at t_b, such conversion is often in the interest of all claim holders. Summarising, if conversion "in distress" is possible also at t_b, we can view the value of the conversion option as the sum of three options, i.e.

$$O(V,t) = O_c(V,t) + O_d(V,t) + O_{t_b}(V,t) \tag{20}$$

where $O_{t_b}(V,t)$ denotes the conversion option to be exercised upon default at $t_b < T$. Each of these three options satisfies the same equation as 14 but without the inhomogenous term SC and the respective payoffs to these options are:

$$O_c(V,T) = \max\left(xE(V) - SD_s(V,T), 0\right), \text{with } SD_s(V,T) = SF \tag{21}$$
$$O_c(V_b,t_b) = 0 \tag{22}$$
$$O_c(V \to \infty, t) \to V \tag{23}$$

$$O_d(V,T) = \max\left(xE(V) - SD_s(V,T), 0\right), \text{with } SD_s(V,T) = R(V) \tag{24}$$
$$O_d(V_b,t_b) = 0 \tag{25}$$
$$O_d(V \to \infty, t) \to 0 \tag{26}$$

$$O_{t_b}(V,T) = 0 \tag{27}$$
$$O_{t_b}(V_b,t_b) = \max\left(xE(V_b) - R(V_b), 0\right) \tag{28}$$
$$O_{t_b}(V \to \infty, t) \to 0. \tag{29}$$

For completeness, when "conversion in distress" at t_b or at T is possible, the payoffs to the various claims are as follows:

$$\begin{cases} \text{If } V_b \le V_T \le V_{dT} \le V_{bT}: & \text{If } V_b \le V_{dT} \le V_T \le V_{bT}: \\ E(V_T,T) = E(V_T)(1-x) & E(V_T,T) = 0 \\ Tax(V_T,T) = Tax(V_T) & Tax(V_T,T) = 0 \\ BC(V_T,T) = BC(V_T) & BC(V_T,T) = K + aV_T \\ D(V_T,T) = D(V_T) & D(V_T,T) = \min(F, \max(V_T(1-a) - K, 0)) \\ SD(V_T,T) = xE(V_T). & SD(V_T,T) = R(V_T). \end{cases}$$

$$\begin{cases} \text{If } O_{t_b}(V_b,t_b) > 0: & \text{If } O_{t_b}(V_b,t_b) = 0: \\ E(V_b,t_b) = E(V_b)(1-x) & E(V_b,t_b) = 0, \\ Tax(V_b,t_b) = Tax(V_b) & Tax(V_b,t_b) = 0 \\ BC(V_b,t_b) = BC(V_b) & BC(V_b,t_b) = BC(V_b) \\ D(V_b,t_b) = D(V_b) & D(V_b,t_b) = \min(F, \max(V_b(1-a) - K, 0)) \\ SD(V_b,t_b) = xE(V_b). & SD(V_b,t_b) = R(V_b). \end{cases} .$$

If "conversion in distress" at T and t_b is possible, the value of a subordinated convertible $SD(V,t)$ satisfies equation 14 subject to condition 17, to condition 30 instead of condition 15 and to condition 31 instead of condition 16:

$$SD\left(V_T,T\right)=\left\{\begin{array}{l}\max\left(xE\left(V_T\right),SF\right)=O_c\left(V,T\right).+SF \text{ if } V_T>V_{bT}\\ \max\left(xE\left(V\right),R\left(V\right)\right) \text{ if } V_b\leq V_T\leq V_{bT}\end{array}\right. \tag{30}$$

$$SD\left(V_b,t_b\right)=\max\left(xE\left(V_b\right),R\left(V_b\right)\right). \tag{31}$$

Then the solution to the subordinated convertible valuation model is provided in the following:

PROPOSITION: Given the above assumptions, given that conversion at t_b is possible, having defined the functions

$$\Omega\left(k_1,k_2,k_3,w\right)=V^{k_3}N\left(k_1\right)-\left(\frac{V}{V_b}\right)^{-2\frac{n(w)}{w\cdot s^2}}\left(V_b\right)^{k_3}N\left(k_2\right)$$

$$\begin{aligned}d\left(z,w\right)&=\frac{w\ln\left(z\right)+\left(n\left(w\right)+\frac{1}{2}s^2w^2\right)\left(T-t\right)}{w\cdot s\sqrt{T-t}}\\ d_2\left(z\right)&=\frac{\ln\left(z\right)+\left(r-b-\frac{1}{2}s^2\right)\left(T-t\right)}{s\sqrt{T-t}}\end{aligned}$$

$$n\left(w\right)=(r-b)w+\frac{1}{2}w\left(w-1\right)s^2$$

the value of convertible subordinated debt is $SD(V,t)=O\left(V,t\right)+SD_s(V,t)$, with

$$O\left(V,t\right)=O_c\left(V,t\right)+O_{t_b}\left(V,t\right)+O_d\left(V,t\right) \tag{32}$$

$$O_c\left(V,t\right)=O_c\left(V,t\right)+O_c\left(Tax\left(V\right),t\right)-O_c\left(BC\left(V\right),t\right)-O_c\left(D\left(V\right),t\right)-O_c\left(SF,t\right) \tag{33}$$

$$O_c\left(V,t\right)=e^{-b(T-t)}\cdot\Omega\left(d\left(\frac{V}{V_{cT}},1\right),d\left(\frac{(V_b)^2}{V_{cT}\cdot V},1\right),1,1\right) \tag{34}$$

$$\begin{aligned}O_c\left(Tax\left(V\right),t\right)=&-\frac{\frac{C}{r}t_x}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot\Omega\left(d\left(\frac{V}{V_{cT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{cT}},q\right),q,q\right)\\ &+\frac{C}{r}t_x\cdot e^{-r(T-t)}\cdot\Omega\left(d_2\left(\frac{V}{V_{cT}}\right),d_2\left(\frac{(V_b)^2}{V_{cT}\cdot V}\right),0,1\right)\end{aligned} \tag{35}$$

$$O_c\left(BC\left(V\right),t\right)=\frac{V_b^p\cdot a+K}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot\Omega\left(d\left(\frac{V}{V_{cT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{cT}},q\right),q,q\right) \tag{36}$$

$$O_c\left(D\left(V\right),t\right)=\frac{\left(-\frac{C}{r}+\max\left(\min\left(F,V_b^p\left(1-a\right)-K\right),0\right)\right)}{\left(V_b^p\right)^q}e^{\left(-r+n(q)\right)\left(T-t\right)}\cdot \tag{37}$$

$$\cdot\Omega\left(d\left(\frac{V}{V_{cT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{cT}},q\right),q,q\right)$$

$$+\frac{C}{r}e^{-r(T-t)}\cdot\Omega\left(d_2\left(\frac{V}{V_{cT}}\right),d_2\left(\frac{\left(V_b\right)^2}{V_{cT}\cdot V}\right),0,1\right)$$

$$O_c\left(SF,t\right)=SF\cdot\Omega\left(d\left(\frac{V}{V_{cT}},1\right),d\left(\frac{\left(V_b\right)^2}{V_{cT}\cdot V},1\right),0,1\right) \tag{38}$$

$$O_{t_b}\left(V,t\right)=\max\left(xE\left(V_b\right)-\max\left(\min\left(SF,V_b\left(1-a\right)-K-F\right),0\right),0\right)\cdot \tag{39}$$

$$\cdot\left(\left(\frac{V}{V_b}\right)^q-\frac{1}{\left(V_b\right)^q}e^{\left(-r+n(q)\right)\left(T-t\right)}\Omega\left(d\left(\frac{V}{V_b},q\right),d\left(\frac{V_b}{V},q\right),q,q\right)\right)$$

$$O_d\left(E\left(V\right),t\right)=O_d\left(V,t\right)+O_d\left(T\left(V\right),t\right)-O_d\left(BC\left(V\right),t\right)-O_d\left(D\left(V\right),t\right)-O_d\left(SD_s\left(V,T\right),t\right) \tag{40}$$

$$O_d\left(V,t\right)=e^{-b(T-t)}\left(\Omega\left(d\left(\frac{V}{V_b},1\right),d\left(\frac{V_b}{V},1\right),1,1\right)-\Omega\left(d\left(\frac{V}{V_{dT}},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}},1\right),1,1\right)\right) \tag{41}$$

$$O_d\left(Tax\left(V\right),t\right)=\frac{\frac{C}{r}t_x}{\left(V_b^p\right)^q}e^{\left(-r+n(q)\right)\left(T-t\right)}\cdot \tag{42}$$

$$\Omega\left(d\left(\frac{V}{V_{dT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}},q\right),q,q\right)-\Omega\left(d\left(\frac{V}{V_b},q\right),d\left(\frac{V_b}{V},q\right),q,q\right)$$

$$+\frac{C}{r}t_x\cdot e^{-r(T-t)}\cdot$$

$$\Omega\left(d_2\left(\frac{V}{V_b}\right),d_2\left(\frac{V_b}{V}\right),0,1\right)-\Omega\left(d_2\left(\frac{V}{V_{dT}}\right),d_2\left(\frac{\left(V_b\right)^2}{V_{dT}\cdot V}\right),0,1\right).$$

$$O_d\left(BC\left(V\right),t\right)=\frac{V_b^p\cdot a+K}{\left(V_p^p\right)^q}e^{\left(-r+n(q)\right)\left(T-t\right)}V^q\cdot \tag{43}$$

$$\left(\Omega\left(d\left(\frac{V}{V_b},q\right),d\left(\frac{V_b}{V},q\right),q,q\right)-\Omega\left(d\left(\frac{V}{V_{dT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}},q\right),q,q\right)\right)$$

$$O_d\left(D\left(V\right),t\right)=\frac{\left(-\frac{C}{r}+\max\left(\min\left(F,V_b^p\left(1-a\right)-K\right),0\right)\right)}{\left(V_b^p\right)^q}e^{\left(-r+n\left(q\right)\right)\left(T-t\right)}\cdot \tag{44}$$

$$\left(\Omega\left(d\left(\frac{V}{V_b},q\right),d\left(\frac{V_b}{V},q\right),q,q\right)-\Omega\left(d\left(\frac{V}{V_{dT}},q\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}},q\right),q,q\right)\right)$$

$$+\frac{C}{r}e^{-r(T-t)}\left(\Omega\left(d_2\left(\frac{V}{V_b}\right),d_2\left(\frac{V_b}{V}\right),0,1\right)-\Omega\left(d_2\left(\frac{V}{V_{dT}}\right),d_2\left(\frac{\left(V_b\right)^2}{V_{dT}\cdot V}\right),0,1\right)\right)$$

$$O_d\left(SD_s\left(V,T\right),t\right)=O_d\left(\max\left(\min\left(SF,V_T\left(1-a\right)-K-F\right),0\right),t\right)= \tag{45}$$

$$\left(1-a\right)e^{-b(T-t)}\cdot\Omega\left(d\left(\frac{V}{\max\left(V_{R=0},V_b\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\max\left(V_{R=0},V_b\right)},1\right),1,1\right)$$

$$-\left(1-a\right)e^{-b(T-t)}\cdot\Omega\left(d\left(\frac{V}{V_{dT}},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}},1\right),1,1\right)$$

$$+\left(K+F\right)e^{-r(T-t)}\cdot\Omega\left(d_2\left(\frac{V}{V_{dT}}\right),d_2\left(\frac{\left(V_b\right)^2}{V\cdot V_{dT}}\right),0,1\right)$$

$$-\left(K+F\right)e^{-r(T-t)}\cdot\Omega\left(d_2\left(\frac{V}{\max\left(V_{R=0},V_b\right)}\right),d_2\left(\frac{\left(V_b\right)^2}{V\cdot\max\left(V_{R=0},V_b\right)}\right),0,1\right)$$

$$
\begin{aligned}
SD_s(V,t) &= SD(V) \qquad (46)\\
&- \frac{\left(-\frac{SC}{r}+R\left(V_b\right)\right)}{\left(V_b\right)^q} e^{(-r+n(q))(T-t)} \cdot \Omega\left(d\left(\frac{V}{V_b},q\right),d\left(\frac{V_b}{V},q\right),q,q\right)\\
&- \frac{SC}{r} e^{-r(T-t)} \cdot \Omega\left(d\left(\frac{V}{V_b},1\right),d\left(\frac{V_b}{V},1\right),0,1\right)\\
&+ SF \cdot \Omega\left(d\left(\frac{V}{\min\left(V_{R=SF},V_{bT}\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\left(\min\left(V_{R=SF},V_{bT}\right)\right)},1\right),0,1\right)\\
&+ (1-a)\, e^{-b(T-t)} \cdot \Omega\left(d\left(\frac{V}{\max\left(V_{R=0},V_b\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\max\left(V_{R=0},V_b\right)},1\right),1,1\right)\\
&- (K+F)\, e^{-r(T-t)} \cdot \Omega\left(d\left(\frac{V}{\max\left(V_{R=0},V_b\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\max\left(V_{R=0},V_b\right)},1\right),0,1\right)\\
&- (1-a)\, e^{-b(T-t)} \cdot \Omega\left(d\left(\frac{V}{\min\left(V_{bT},V_{R=SF}\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\min\left(V_{bT},V_{R=SF}\right)},1\right),1,1\right)\\
&+ (K+F)\, e^{-r(T-t)} \cdot \Omega\left(d\left(\frac{V}{\min\left(V_{bT},V_{R=SF}\right)},1\right),d\left(\frac{\left(V_b\right)^2}{V\cdot\min\left(V_{bT},V_{R=SF}\right)},1\right),0,1\right)
\end{aligned}
$$

with $V_{R=0}$ *such that* $V_{R=0}\,(1-a) = K+F$, $V_{R=SF}$ *such that* $V_{R=SF}\,(1-a) - K - F = SF$ *and* $SD(V) = \frac{SC}{r} + \left(-\frac{SC}{r}+R\left(V_b\right)\right)\left(\frac{V}{V_b}\right)^q$.

See the Appendix for proof of this proposition. The proposition assumes that the conversion option in "European", but the same formulae apply when the conversion option is "American", the debtor distributes no dividends, V_b is exogenously set equal to a fraction α of the face value of outstanding debt $(F+SF)$ and the market is assumed dynamically incomplete rather than complete, so that the expression $(r-b)$ is everywhere substituted by $(m-s\lambda)$, where m is the real drift of the firm's assets value process V and λ is the market price of V-risk.

The above proposition provides the value of the three options making up the conversion option. Past contributions have considered just the "classic" conversion option and neglected the conversion options to be exercised "in distress". Especially when bankruptcy is very costly, neglecting the latter options may lead market participants to undervalue convertibles, subordinated ones in particular.

3 THE VALUE OF CONVERSION "IN DISTRESS"

This section focuses on comparative statics for the value of the conversion "in distress" options. In the base case considered hereafter average realistic parameters are employed and relatively high bankruptcy costs are assumed as may be specific to convertible issuers. Comparative statics reveal that the values of the "conversion in distress" options, $O_{t_b}(V,t)$ and $O_d(V,t)$, can be significant. Figure 1 shows the value of the conversion option with and without conversion "in distress". The prospect of converting "in distress" increases the total value of the conversion option $O(V,t)$ even as the debtor's assets approach the default barrier. The reason is that the value of the conversion option is equal to $O(V,t) = O_c(V,t) + O_d(V,t) + O_{t_b}(V,t)$ and as assets value decreases $O_c(V,t)$ lowers but $O_{t_b}(V,t)$ and $O_d(V,t)$ rise because the probability of converting "in distress" increases.

Figure 2 illustrates how the value of the convertible $SD(V,t)$ can be significantly understated if conversion "in distress" is neglected, especially when the value of the firm's assets is low. Figure 3 shows that the sum of the conversion "in distress" options, $O_d(V,t) + O_{t_b}(V,t)$, has a sort of bell shape for short maturities. The reason is that the payoff to converting at maturity T is higher than the payoff to converting before maturity at $t_b < T$, ie. $O_d(V,T) \geq O_{t_b}(V_b,t_b)$ whenever $V_{bT} \geq V_T \geq V_b$.

The bell shape of the conversion "in distress" option ($O_d(V,t) + O_{t_b}(V,t)$) and the "hump" of the total value of the conversion option $O(V,t)$ imply that these options can get less valuable as well as more valuable as assets volatility s increases. The conversion "in distress" option increases in volatility when the value of the firm's assets is far from the default barrier, because then high volatility increases the chance of distress, the value of equity $E(V)$ and the payoff to $O_d(V,T) > 0$. But when assets value V is near the default barrier V_b, $O_d(V,t)$ decreases in volatility, since higher volatility then increases the probability of default before maturity, which makes $O_d(V,t)$ worthless. It follows that also the total value of the conversion option $O(V,t)$ does not always increase in assets volatility even if $O_c(V,t)$ and $O_{t_b}(V,t)$ always do. Unlike $O_d(V,t)$, $O_{t_b}(V,t)$ always increases with higher assets volatility, because higher volatility increases $E(V_b)$ as well as the probability of default before maturity T.

Figure 3 shows the terminal payoffs and values of the conversion "in distress" option ($O_{t_b}(V,t) + O_d(V,t)$) for different times to maturity. The longer is the time to the maturity of the convertible $(T-t)$ and the flatter is the function $O_{t_b}(V,t) + O_d(V,t)$. So, as convertible maturity gets longer, neglecting conversion "in distress" entails a less significant undervaluation of the convertible for low assets values, but a more significant one for higher assets values.

Figure 4 explains the payoffs to the conversion "in distress" options, $O_{t_b}(V_b,t_b)$ and $O_d(V,T)$, as determined by the value of the fraction of equity $xE(V)$ and by the recovery value of subordinated debt $R(V)$. In the base case $E(V_{bT} = 34) = SF = 20$ so, if $V_b = 15 \leq V_T \leq V_{bT} = 34$, it follows that $O_d(V_T,T) \geq O_{t_b}(V_b,t_b) > 0$. In general a higher default barrier V_b increases the probability of default at some $t_b < T$, increases $xE(V_b)$ and the recovery value $R(V_b)$ and may increase or decrease the value of the conversion options $O_{t_b}(V_b,t_b)$ and $O_{t_b}(V,t)$. But when the levels of outstanding senior debt and bankruptcy costs (F, a and K) are high, the recovery value of subordinated debt may be null even for a high default barrier V_b, in which case $O_{t_b}(V_b, t_b)$ and $O_{t_b}(V, t)$ are more likely to increase in V_b. Figure 4 also implies that, if $V_b \geq V_{bT}$, which may be the case for low b or for high $\frac{C+SC}{F+SF}$ ratios, then $O_d(V_T, T) = 0$.

The conversion "in distress" option ($O_{t_b}(V, t) + O_d(V, t)$) often increases in value when the debtor distributes higher payouts to security holders. Higher payouts b decrease equity value after conversion $E(V)$, increase the level of the default barrier at maturity V_{bT}, decrease the default barriers before and after maturity V_b and V_b^p and, by reducing the drift of the assets value process V, increase the probability of default when V is far from the default barriers. As b rises, $O_d(V, t)$ rises and $O_{t_b}(V, t)$ drops, but the rise of the first is usually greater than the drop of the second.

The effect of leverage on the value of the conversion "in distress" options is mixed. When ceteris paribus the face value of the convertible SF increases, the default barrier at maturity V_{bT} rises as well as the distance ($V_{bT} - V_b$) and $O_d(V, T)$ becomes more valuable. Since also the barrier before maturity V_b rises with SF, $O_{t_b}(V, t)$ often rises too. Instead, when ceteris paribus the face value of senior non-convertible debt F rises, equity value after conversion $E(V)$ decreases and so does the recovery value of subordinated debt in bankruptcy $R(V)$, while the default barriers V_{bT} and V_b increase significantly as the coupon rate on senior non convertible debt is usually higher the coupon on convertible debt. Generally the net effect of a rise in F is to decrease the value of the conversion "in distress" options. When $F = 0$ the convertible is no longer subordinated to senior debt and the payoffs $O_d(V_T, T)$ and $O_{t_b}(V_b, t_b)$ can significantly increase. Thus an increase in leverage due to a rise in F generally decreases the value of the conversion "in distress" options, while an increase in leverage due to a rise in SF enhances the value of such options.

As for the conversion terms set in the convertible indenture, higher x, which makes the convertible more "equity-like", clearly increases the payoffs $O_d(V_T, T)$ and $O_{t_b}(V_b, t_b)$ and the values of the conversion options.

3.1 Conversion "in distress" and bankruptcy costs

Intangible assets and growth opportunities can be compromised or lost in bankruptcy, thus significantly increasing bankruptcy costs. But Figure 4 implies that the payoffs $O_d(V, T)$ and $O_{t_b}(V_b, t_b)$ increase with bankruptcy costs a and especially K, since higher bankruptcy costs depress the recovery value $R(V)$. So, whereas bankruptcy costs always depress the value of non convertible debt $SD_s(V, t)$, they increase the value of the conversion "in distress" options. Hence convertible debt $SD(V, t)$, unlike non-convertible debt, can be partly immunised from the effect of changes in a and K . For example in the base case the subordinated convertible is completely independent of the magnitude of bankruptcy costs, because convertible holders would always convert "in distress" to avert the debtor's bankruptcy.

What above is important when lenders are asymmetrically informed about bankruptcy costs and the debtor's assets recovery value. Such asymmetric information will induce lenders to require higher compensation for lending. Then, especially if the borrower has high default risk, it should consider issuing convertible debt instead of non-convertible debt, since the former is less sensitive to bankruptcy costs. Convertible debt as opposed to non-convertible debt would then make the cost of borrowing less sensitive to uncertainty in bankruptcy costs and recovery risk and effectively decrease the cost of borrowing when lenders are asymmetrically informed about potential bankruptcy costs. Moreover, if lenders believed the recovery value of debt in bankruptcy to be lower (higher) than the debtor's better informed estimate, ceteris paribus the debtor should prefer convertible (non-convertible) debt to non-convertible (convertible) debt.

Finally, since conversion "in distress" is a sort of automatic debt restructuring that makes the bankruptcy prospect more remote and decreases expected bankruptcy costs, issuing convertible debt as opposed to similar non-convertible debt can increase total firm value and equity value. This increase can be significant when the convertible is very "equity-like" (high x ceteris paribus), when bankruptcy costs are high and debt renegotiation to avert bankruptcy is not feasible (e.g. in the case of public debt) or is too costly in terms of money and reputation.

Whereas Anderson-Pan-Sundaresan (2000) reached a similar conclusion for the case in which a convertible can be renegotiated, this conclusion assumes that debt renegotiation is not possible. This conclusion is also consistent with the empirical findings of Essig (1991): convertible issuers are often characterised by high bankruptcy costs.

4 WHEN DEBT HOLDERS CAN INITIATE DEBT RENEGOTIATION

This section highlights further differences between subordinated debt convertible "in distress" and non non-convertible subordinated debt when some debt renegotiation can take place. So far it has been assumed that debt renegotiation was not possible, as it may be the case when senior and subordinated debt are publicly held securities. But if subordinated debt holders could initiate some debt renegotiation with equity holders, it may be objected that upon default at T or at t_b holders of non-convertible subordinated debt may be better off if they spontaneously offered to swap their debt claim for equity, thus again averting immediate costly bankruptcy. For $V_T \leq V_{bT}$, holders of non-convertible subordinated debt could spontaneously offer to swap their claim into a fraction x_o of equity, where x_o is chosen such that

$$R(V_T) \leq x_o E(V_T) \tag{47}$$

$$\max(V(1-a) - K - F - SF, 0) < (1 - x_o) E(V_T) \tag{48}$$

Condition 47 guarantees that the swap be in the interest of subordinated debt holders, and condition 48 guarantees that the swap be also in the interest of equity holders (and possibly also of senior debt holders). The actual value of x_o will depend upon the bargaining power of equity holders and subordinated debt holders. Such swap would avert costly bankruptcy as conversion "in distress" would, but this is not to say that convertible and non-convertible debt are equivalent. In fact, if subordinated debt is convertible into fraction x of equity, then x_o should be such that

$$\max\left(E\left(V_T\right)x, R\left(V_T\right)\right) \leq E\left(V_T\right)x_o \tag{49}$$

$$\max\left(V\left(1-a\right)-K-F-SF,0\right)-O_d\left(V,T\right) < \left(1-x_o\right)E\left(V_T\right) \tag{50}$$

since for x_o such that $R\left(V_T\right) < E\left(V_T\right)x_o \leq E\left(V_T\right)x$ convertible holders would rather convert than swap, thus obtaining $xE\left(V_T\right) \geq x_oE\left(V_T\right)$. So x is like a lower threshold for the fraction of equity which convertible holders can get by spontaneously swapping debt for equity upon default. If subordinated debt is non-convertible, such lower threshold is absent. In other words the conversion option can increase the bargaining power of subordinated debt holders. Again equity holders may concede a higher fraction of equity to convertible holders than contractually agreed (i.e. $x_o \geq x$), since conversion "in distress" would be in the interest of both convertible holders and equity holders. So convertible holders might bargain with equity holders over the percentage of equity they receive upon conversion.

5 CONCLUSIONS

This paper has presented new results and formulae for the valuation of convertible subordinated debt. It has been shown how it can be rational for convertible holders to convert not only when the debtor's equity value increases, but also when the debtor approaches distress.

If debt cannot be renegotiated, convertible holders can make the rational and spontaneous concession to convert "in distress" thus averting costly liquidation of the debtor's assets. This concession seems important since convertibles are often publicly held bonds, i.e. the kind of debt that is more difficult to renegotiate. If debt renegotiation is not possible and if the convertible is subordinated to other debt, convertible holders can find it convenient to convert "in distress" even in the absence of bankruptcy costs. The presence of bankruptcy costs can make converting "in distress" convenient even when the convertible is not subordinated. Since conversion "in distress" can avert immediate bankruptcy, lowers expected bankruptcy costs and increases equity value, it follows that the choice to issue convertibles is more appealing to borrowers with high potential bankruptcy costs. This is the case even when, unlike in Anderson-Pan-Sundaresan (2000), debt cannot be renegotiated.

Overall neglecting the conversion "in distress" option may entail a significant undervaluation of convertible debt, in particular of short maturity low grade one. Convertibles most often are low grade, subordinated and unsecured debts issued by debtors with high bankruptcy costs: all these features contribute to make the conversion "in distress" option more valuable. Furthermore, when subordinated debt holders can initiate debt renegotiation with equity holders, conversion "in distress" can increase the bargaining power of subordinated debt holders.

Finally the conversion "in distress" option makes convertible debt less sensitive than non-convertible debt to the recovery value of assets in bankruptcy. So convertible financing can reduce the cost of borrowing when lenders are uncertain or asymmetrically informed about the debtor's assets recovery value. It follows that convertibles may be issued to reduce the cost of borrowing when lenders are uncertain about the debtor's assets recovery value as well as when lenders are uncertain about the debtor's assets volatility as suggested in Brennan and Schwartz (1988).

Future research may study optimal capital structure in the presence of convertible subordinated debt and "conversion in distress". This may shed more light on the issue of convertible debt design. Further research may also value convertibles assuming other types of debt renegotiation.

Appendix A. DERIVATION OF THE PROPOSITION

Following Leland (1998) we can write $D(V) = \frac{C}{r} + AV^q$, with $A = \left(-\frac{c}{r} + (1-a)\cdot V_b - K\right)\cdot (V_b)^{-q}$. By applying Ito's lemma it can be shown that $d(D(V)) = d(AV^q) = n(q)\cdot AV^q \cdot dt + AV^q \cdot qs \cdot dz$ (above we defined the function $n(w) = (r-b)w + \frac{1}{2}w(w-1)s^2$). Then it is known that the value at time t of a claim (Q) that pays $D(V) = \frac{C}{r} + AV^q$ at time T if $V_T \geq V_{bT}$ and that pays nothing otherwise is:

$$Q(D(V), t, V_{bT}, T) = Q(AV^q, t, V_{bT}, T) + Q\left(\tfrac{C}{r}, t, V_{bT}, T\right),$$

$$Q(AV^q, t, V_{bT}, T) = A \cdot e^{(-r+n(q))(T-t)} \cdot V^q N\left(d\left(\tfrac{V}{V_{bT}}, q\right)\right),$$

$$Q\left(\tfrac{C}{r}, t, V_{bT}, T\right) = e^{-r(T-t)}\tfrac{C}{r} N\left(d\left(\tfrac{V}{V_{bT}}, 1\right) - s\sqrt{T-t}\right),$$

where $N(u)$ is the cumulative of the standard normal density with u as the upper limit of integration (and above we defined the function $d(z,w) = \frac{w\ln(z) + \left(n(w)+\frac{1}{2}s^2w^2\right)(T-t)}{ws\sqrt{T-t}}$).

Using results for valuing down-and-out barrier options (see e.g. Wilmott (1998) at page 192), the value of a claim that pays V at time T if $V_T \geq V_{bT}$ and if $V_t \geq V_b$ for any time $t < T$ and that pays nothing otherwise is:

$$Q(D(V), t, V_b, V_{bT}, T) = Q(AV^q, t, V_b, V_{bT}, T) + Q\left(\tfrac{C}{r}, t, V_b, V_{bT}, T\right),$$

$$Q(AV^q, t, V_b, V_{bT}, T) = A \cdot e^{(-r+n(q))(T-t)} \cdot$$
$$\cdot \left(V^q N\left(d\left(\tfrac{V}{V_{bT}}, q\right)\right) - \left(\tfrac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}} (V_b)^q \cdot N\left(d\left(\tfrac{(V_b)^2}{V\cdot V_{bT}}, q\right)\right)\right),$$

$$Q\left(\tfrac{C}{r}, t, V_b, V_{bT}, T\right) = \tfrac{C}{r}e^{-r(T-t)} \cdot$$
$$\cdot \left(N\left(d\left(\tfrac{V}{V_{bT}}, 1\right) - s\cdot\sqrt{T-t}\right) - \left(\tfrac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\tfrac{(V_b)^2}{V\cdot V_{bT}}, 1\right) - s\cdot\sqrt{T-t}\right)\right).$$

It also follows that $Q\left(D\left(V\right),t,V_b,V_{bT},T\right) = Q\left(D\left(V\right),t,V_{bT},T\right) +$
$-\left(\frac{V}{V_b}\right)^{1-2\frac{n(q)}{s^2}} Q\left(D\left(\frac{(V_b)^2}{V}\right),t,V_{bT},T\right).$

Using the previous result, now we derive the results in Proposition I. To derive a closed formula for $O_d\left(E\left(V\right),t\right)$ we can write:

$O_d\left(E\left(V\right),t\right) = O_d\left(V,t\right) + O_d\left(Tax\left(V\right),t\right) - O_d\left(BC\left(V\right),t\right) - O_d\left(D\left(V\right),t\right) - O_d\left(SD\left(V,T\right),t\right).$

Form what above, we can derive $O_d\left(V,t\right)$ as:

$O_d\left(V,t\right) = Q\left(V,t,T,V_b,V_b \le V_T\right) - Q\left(V,t,T,V_b,V_{dT} \le V_T\right),$

$Q\left(V,t,T,V_b,V_b \le V_T\right) = Q\left(V,t,T,V_b \le V_T\right) - \left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}} V_b N\left(d\left(\frac{V_b}{V},1\right)\right) =$

$= e^{-b(T-t)}\left(VN\left(d\left(\frac{V}{V_b},1\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}} V_b N\left(d\left(\frac{V_b}{V},1\right)\right)\right),$

$Q\left(V,t,T,V_b,V_{dT} \le V_T\right) = e^{-b(T-t)}\left(VN\left(d\left(\frac{V}{V_{dT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{(r-b)}{s^2}} V_b N\left(d\left(\frac{(V_b)^2}{V \cdot V_{dT}},1\right)\right)\right).$

Then we can derive

$O_d\left(Tax\left(V\right),t\right) = Q\left(Tax\left(V\right),t,T,V_b,V_b \le V_T\right) - Q\left(Tax\left(V\right),t,T,V_b,V_{dT} \le V_T\right),$

$Tax\left(V\right) = \frac{C}{r}t_x\left(1 - \frac{1}{\left(V_B^p\right)^q}V^q\right),$

$Q\left(Tax\left(V\right),t,T,V_b,V_b \le V_T\right) = -\frac{\frac{C}{r}t_x}{\left(V_B^p\right)^q} \cdot e^{(-r+n(q))(T-t)}\cdot$

$\cdot\left(V^q N\left(d\left(\frac{V}{V_b},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n}{q \cdot s^2}} (V_b)^q N\left(d\left(\frac{V_b}{V},q\right)\right)\right) +$

$+\frac{C}{r}t_x \cdot e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_b},1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{V_b}{V},1\right)\right)\right),$

$Q\left(Tax\left(V\right),t,T,V_b,V_{dT} \le V_T\right) = -\frac{\frac{C}{r}t_x}{\left(V_B^p\right)^q} \cdot e^{(-r+n(q))(T-t)}\cdot$

$\cdot\left(V^q N\left(d\left(\frac{V}{V_{dT}},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n}{q \cdot s^2}} (V_b)^q N\left(d\left(\frac{(V_b)^2}{V \cdot V_{dT}},q\right)\right)\right) +$

$+\frac{C}{r}t_x \cdot e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_{dT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V \cdot V_{dT}},1\right)\right)\right).$

Then we can derive

$O_d\left(D\left(V\right),t\right) = Q\left(D\left(V\right),t,T,V_b,V_b \le V_T\right) - Q\left(D\left(V\right),t,T,V_b,V_{dT} \le V_T\right),$

$D(V) = \frac{C}{r} + \frac{\left(-\frac{C}{r} + \max\left(\min\left(F,V_b^p(1-a)-K\right),0\right)\right)}{\left(V_b^p\right)^q}V^q,$

$Q\left(D\left(V\right),t,T,V_b,V_b \le V_T\right) = \frac{\left(-\frac{C}{r} + \max\left(\min\left(F,V_b^p(1-a)-K\right),0\right)\right)}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot$

$$\left(V^q N\left(d\left(\frac{V}{V_b},q\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^q N\left(d\left(\frac{V_b}{V},q\right)\right)\right)+$$

$$+\frac{C}{r}e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_b},1\right)\right)-\left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{V_b}{V},1\right)\right)\right),$$

$$Q\left(D\left(V\right),t,T,V_b,V_{dT}\leq V_T\right)=\frac{\left(-\frac{C}{r}+\max\left(\min\left(F,V_b^p(1-a)-K\right),0\right)\right)}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot$$

$$\left(V^q N\left(d\left(\frac{V}{V_{dT}},q\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^q N\left(d\left(\frac{(V_b)^2}{V\cdot V_{dT}},q\right)\right)\right)+$$

$$+\frac{C}{r}e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_{dT}},1\right)\right)-\left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_b)^2}{V\cdot V_{dT}},1\right)\right)\right).$$

Then we can derive

$$O_d\left(BC\left(V\right),t\right)=Q\left(BC\left(V\right),t,T,V_b,V_b\leq V_T\right)-Q\left(BC\left(V\right),t,T,V_b,V_{dT}\leq V_T\right),$$

$$BC(V)=\frac{V_b^p\cdot a+K}{\left(V_b^p\right)^q}V^q,$$

$$Q\left(BC(V),t,T,V_b,V_b\leq V_T\right)=\frac{V_b^p\cdot a+K}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot$$

$$\cdot\left(V^q N\left(d\left(\frac{V}{V_b},q\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^q N\left(d\left(\frac{V_b}{V},q\right)\right)\right),$$

$$Q\left(BC\left(V\right),t,T,V_b,V_{dT}\leq V_T\right)=\frac{V_b^p\cdot a+K}{\left(V_b^p\right)^q}e^{(-r+n(q))(T-t)}\cdot$$

$$\cdot\left(V^q N\left(d\left(\frac{V}{V_{dT}},q\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^q N\left(d\left(\frac{(V_b)^2}{V\cdot V_{dT}},q\right)\right)\right).$$

Then we can derive

$$O_d\left(SD\left(V,T\right),t\right)=O_d\left(\max\left(\min\left(SF,V_T\left(1-a\right)-K-F\right),0\right),t\right)=$$

$$Q\left(SD\left(V,T\right),t,T,V_b,\max\left(V_{R=0},V_b\right)\leq V_T\right)-Q\left(SD\left(V,T\right),t,T,V_b,V_{dT}\leq V_T\right),$$

$$Q\left(SD\left(V,T\right),t,T,V_b,\max\left(V_{R=0},V_b\right)\leq V_T\right)=$$

$$(1-a)\left(VN\left(d\left(\frac{V}{\max(V_{R=0},V_b)},1\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}}(V_b)N\left(d\left(\frac{(V_b)^2}{V\cdot\max(V_{R=0},V_b)},1\right)\right)\right)$$

$$-\left(K+F\right)e^{-r(T-t)}\left(N\left(d\left(\frac{V}{\max(V_{R=0},V_b)},1\right)\right)-\left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_b)^2}{V\cdot\max(V_{R=0},V_b)},1\right)\right)\right),$$

$$Q\left(SD\left(V,T\right),t,T,V_b,V_{dT}\leq V_T\right)=$$

$$(1-a)\left(VN\left(d\left(\frac{V}{V_{dT}},1\right)\right)-\left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}}(V_b)N\left(d\left(\frac{(V_b)^2}{V\cdot V_{dT}},1\right)\right)\right)+$$

$$-\left(K+F\right)e^{-r(T-t)}\left(N\left(\left(d\left(\frac{V}{V_{dT}},1\right)-s\sqrt{T-t}\right)\right)-\left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}}N\left(d\left(\frac{(V_b)^2}{V\cdot V_{dT}},1\right)-s\sqrt{T-t}\right)\right).$$

In a similar way it is possible to derive $O_c\left(E\left(V\right),t\right)$ as:

$O_c(E(V),t) = O_c(V,t) + O_c(Tax(V),t) - O_c(BC(V),t) - O_c(D(V),t) - O_c(SF,t)$,

$$O_c(SF,t) = SFe^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_{cT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{(r-b)}{s^2}} N\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},1\right)\right)\right),$$

$$O_c(V,t) = e^{-b(T-t)}\left(VN\left(d\left(\frac{V}{V_{cT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{(r-b)}{s^2}} V_bN\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},1\right)\right)\right),$$

$$O_c(Tax(V),t) = -\frac{\frac{C}{r}t_x}{(V_b^p)^q}\cdot e^{(-r+n(q))(T-t)}\cdot\left(V^qN\left(d\left(\frac{V}{V_{cT}},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^qN\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},q\right)\right)\right)$$

$$+\frac{C}{r}t_x\cdot e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_{cT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},1\right)\right)\right),$$

$$O_c(D(V),t) = \frac{\left(-\frac{C}{r} + \max\left(\min\left(F, V_b^p(1-a)-K\right),0\right)\right)}{(V_b^p)^q}\cdot e^{(-r+n(q))(T-t)}\cdot$$

$$\cdot\left(V^qN\left(d\left(\frac{V}{V_{cT}},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^qN\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},q\right)\right)\right)+$$

$$+\frac{C}{r}e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_{cT}},1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},1\right)\right)\right),$$

$$O_c(BC(V),t) = \frac{V_b^p\cdot a+K}{(V_b^p)^q}\cdot e^{(-r+n(q))(T-t)}\cdot\left(V^qN\left(d\left(\frac{V}{V_{cT}},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^qN\left(d\left(\frac{(V_b)^2}{V\cdot V_{cT}},q\right)\right)\right).$$

In a similar way we can derive also:

$O_{t_b}(V,t) = O_{t_b}(V) - Q(O_{t_b}(V),t,T,V_b,V_b \le V_T)$,

$O_{t_b}(V) = O_{t_b}(V_b,t_b)\left(\frac{V}{V_b}\right)^q$,

$O_{t_b}(V_b,t_b) = \max(xE(V_b) - \min(SF, \max(V_b(1-a)-K-F,0)),0)$,

$$Q(O_{t_b}(V),t,T,V_b,V_b\le V_T) = \frac{O_{t_b}(V_b,t_b)}{(V_b)^q}\cdot e^{(-r+n(q))(T-t)}\cdot$$

$$\cdot\left(V^qN\left(d\left(\frac{V}{V_b},q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q\cdot s^2}}(V_b)^qN\left(d\left(\frac{V_b}{V},q\right)\right)\right).$$

The above gives us the formula for the total value of the conversion option since $O(V,t) = O_c(V,t) + O_{t_b}(V,t) + O_d(V,t)$.

Then, the value of subordinated debt can be shown to be $SD(V,t) = SD_s(V,t) + O(V,t)$, where $SD_s(V,t)$ is the value of non-convertible debt.

As in Mella-Barral and Tychon (1999), it can be shown that $SD_s(V,t) = SD_s(V) - Q(SD_s(V),t,V_b,V_{bT},T)$, so that:

$$SD_s(V,t) = SD(V) - Q(SD(V),t,T,V_b,V_b\le V_T) + Q(SF,t,T,V_b,V_{bT}\le V_T)$$
$$+Q(\max(\min(SF,V_T(1-a)-K-F),0),t,T,V_b,V_b\le V_T\le V_{bT}),$$

with $SD(V) = \frac{SC}{r} + \left(-\frac{SC}{r} + \max(\min(SF,V_b(1-a)-K-F),0)\right)\left(\frac{V}{V_b}\right)^q$.

Setting $V_{R=0}$ such that $V_{R=0}(1-a) = K+F$, and $V_{R=SF}$ such that $V_{R=SF}(1-a)-K-F = SF$, then when $V_b \leq V_T \leq V_{bT}$:

$$\max\left(\min\left(SF, V_T(1-a)-K-F\right),0\right) = \begin{cases} = SF \text{ if } V_{bT} \geq V_T \geq V_{R=SF}; \\ = \left(V_T(1-a)-K-F\right) \geq 0 \\ \text{if } \min\left(V_{bT}, V_{R=SF}\right) \geq V_T \geq \max\left(V_{R=0}, V_b\right); \\ = 0 \text{ if } V_{R=0} \geq V_T \geq V_b. \end{cases}.$$

Thus

$Q\left(\max\left(\min\left(SF, V_T(1-a)-K-F\right),0\right), t, T, V_b, V_b \leq V_T \leq V_{bT}\right) =$
$Q\left(V_T(1-a)-K-F, t, T, V_b, \max\left(V_{R=0}, V_b\right) \leq V_T\right) +$
$-Q\left(V_T(1-a)-K-F, t, T, V_b, \min\left(V_{bT}, V_{R=SF}\right) \leq V_T\right)$
$+\left[Q\left(SF, t, T, V_b, V_{R=SF} \leq V_T\right) - Q\left(SF, t, T, V_b, V_{bT} \leq V_T\right)\right]$; the term in brackets is necessary only if $V_{bT} \geq V_{R=SF}$, which is not usually the case in practice. It follows that:

$$SD_s(V,t) = SD(V) - Q\left(SD(V), t, T, V_b, V_b \leq V_T\right) + Q\left(SF, t, T, V_b, \min\left(V_{R=SF}, V_{bT}\right) \leq V_T\right)$$
$$+Q\left(V_T(1-a)-K-F, t, T, V_b, \max\left(V_{R=0}, V_b\right) \leq V_T\right) +$$
$$-Q\left(V_T(1-a)-K-F, t, T, V_b, \min\left(V_{bT}, V_{R=SF}\right) \leq V_T\right),$$

$$Q\left(SD(V), t, T, V_b, V_b \leq V_T\right) =$$
$$\frac{\left(-\frac{SC}{r} + \max\left(\min\left(SF, V_b(1-a)-K-F\right),0\right)\right)}{(V_b)^q} e^{(-r+n(q))(T-t)} \cdot$$
$$\cdot\left(V^q N\left(d\left(\frac{V}{V_b}, q\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{n(q)}{q \cdot s^2}} (V_b)^q N\left(d\left(\frac{V_b}{V}, q\right)\right)\right)$$
$$+\frac{SC}{r} e^{-r(T-t)}\left(N\left(d\left(\frac{V}{V_b}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{V_b}{V}, 1\right)\right)\right),$$

$$Q\left(SF, t, T, V_b, \min\left(V_{R=SF}, V_{bT}\right) \leq V_T\right) =$$
$$SF e^{-r(T-t)}\left(N\left(d\left(\frac{V}{\min(V_{R=SF}, V_{bT})}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V \cdot \min(V_{R=SF}, V_{bT})}, 1\right)\right)\right),$$

$$Q\left(V_T(1-a)-K-F, t, T, V_b, \max\left(V_{R=0}, V_b\right) \leq V_T\right) =$$
$$e^{-b(T-t)}(1-a)\left(V N\left(d\left(\frac{V}{\max(V_{R=0}, V_b)}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}} (V_b) N\left(d\left(\frac{(V_b)^2}{V \cdot \max(V_{R=0}, V_b)}, 1\right)\right)\right)$$
$$-(K+F) e^{-r(T-t)}\left(N\left(d\left(\frac{V}{\max(V_{R=0}, V_b)}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V \cdot \max(V_{R=0}, V_b)}, 1\right)\right)\right),$$

$$Q\left(V_T(1-a)-K-F, t, T, V_b, \min\left(V_{bT}, V_{R=SF}\right) \leq V_T\right) =$$
$$e^{-b(T-t)}(1-a)\left(V N\left(d\left(\frac{V}{\min(V_{bT}, V_{R=SF})}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{-2\frac{r-b}{s^2}} (V_b) N\left(d\left(\frac{(V_b)^2}{V \cdot \min(V_{bT}, V_{R=SF})}, 1\right)\right)\right)$$
$$-(K+F) e^{-r(T-t)}\left(N\left(d\left(\frac{V}{\min(V_{bT}, V_{R=SF})}, 1\right)\right) - \left(\frac{V}{V_b}\right)^{1-2\frac{r-b}{s^2}} N\left(d\left(\frac{(V_b)^2}{V \cdot \min(V_{bT}, V_{R=SF})}, 1\right)\right)\right).$$

BIBLIOGRAPHY

[1] Anderson R., Pan Y. and Sundaresan S., 1997, Corporate Bond Yield Spreads and the Term Structure, Columbia PaineWebber Working Paper Series in Money, Economics and Finance: PW/97/15 December 1997, 45.

[2] Anderson R., Pan Y. and Sundaresan S., 2000, Corporate Bond Yield Spreads and the Term Structure, Finance vol. 21, n.2, 15-37.

[3] Anderson R., and Sundaresan S., 1996, "Design and valuation of debt contracts", Review of financial studies 9, n.1, 37-68.

[4] Black F. and Scholes M., 1973, "The pricing of options and corporate liabilities", Journal of political economy 637-659.

[5] Brennan M. and Schwartz E., 1977, "Convertible bonds: valuation and optimal strategies for call and conversion", Journal of finance 32, n.5, 1699-1715".

[6] Brennan M. and Schwartz E., 1980, "Analysing convertible bonds", Journal of financial and quantitative analysis 15, n.4, 907-929.

[7] Brennan M. and Schwartz E., 1988, "The case for convertibles", Journal of applied corporate finance n.1, 55-64.

[8] Ericsson J., 2000, "Asset substitution, debt pricing, optimal leverage and maturity", Finance December, 21, n.2, 40-70.

[9] Essig S., 1991, "Convertible securities and capital structure determinants", PHD dissertation (Grad school of Business, Univ. of Chicago).

[10] Fan H. and Sundaresan S., 2000, "Debt valuation, renegotiation and optimal dividend policy", Review of financial studies 13, n.4, 1057-1099.

[11] Ingersoll J.E., 1977, "A contingent claims valuation of convertible securities", Journal of financial economics 4, n.2, 289-322.

[12] Kim J., Ramaswamy K. and Sundaresan S., 1993, "Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model", Financial Management 117-131.

[13] Leland H., 1994a, "Corporate debt value, bond covenants and optimal capital structure", Journal of finance 49, n.4, 1213-1252.

[14] Leland H., 1994b, "Bond prices, yield spreads, and optimal capital structure with default risk", University of California at Berkeley Research Program in Finance working papers.

[15] Leland H., 1998, "Agency costs, risk management and capital structure", Journal of finance 53, n.4, 1213-1243.

[16] Lewis, Rogalski and Seward, 1998, "Understanding the design of convertible debt" Journal of Applied Corporate Finance 11, 45-53.

[17] Mella-Barral P. and Perraudin W., 1997, "Strategic debt service", Journal of finance 52, n.2, 531-556.

[18] Mella-Barral P. and Tychon P., 1999, "Default Risk in Asset Pricing", Forthcoming in Finance.

[19] Nyborg K.G., 1996, "The use and pricing of convertible bonds", Applied mathematical finance 3, 167-190.

[20] Stump P.M., Ziets K., Marshella T., Rowan M. "Critical issues in evaluating the creditworthiness of convertible debt securities", 2001, "Moody's Investors Service" special comment.

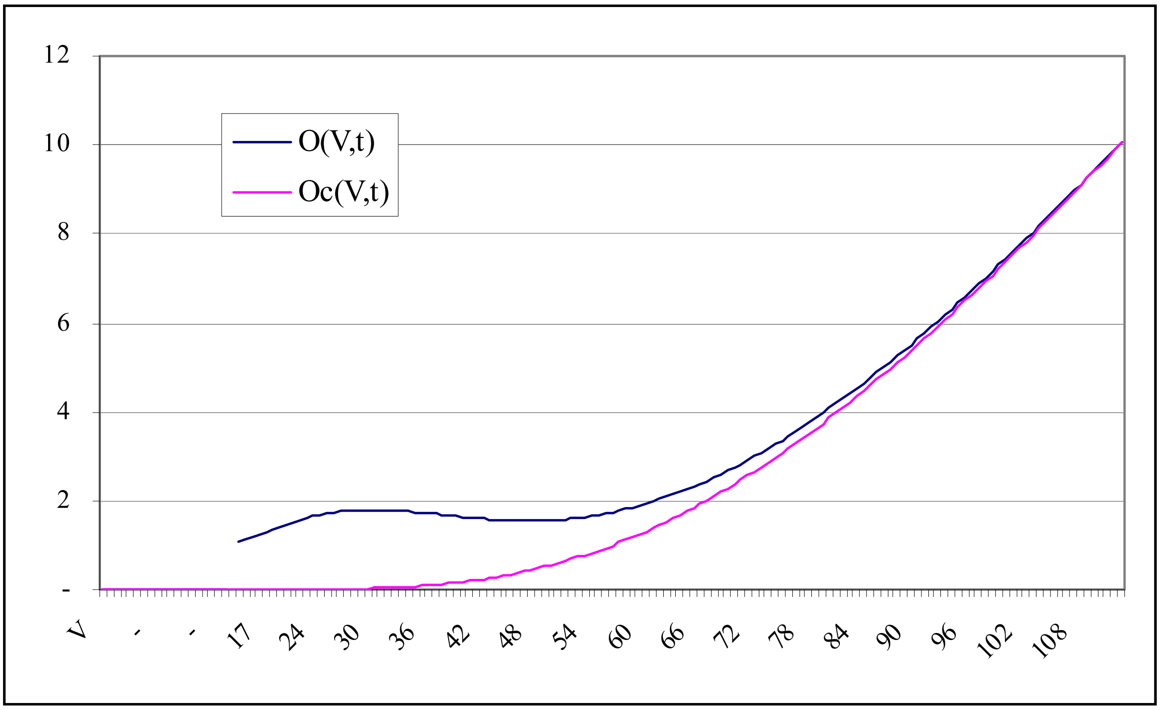

Figure 1: The value of the conversion option with and without "conversion in distress" assuming debt renegotiation is not possible. Base case parameters: $b = 0.06$, $s = 0.2$, $r = 0.04$, $x = 0.35$, $K = 0$, $a = 0.3$, $SC = 0.02 \cdot SF$, $C = 0.05 \cdot F$, $SF = 20$, $F = 20$, $T - t = 5$, $t_x = 0.35$.

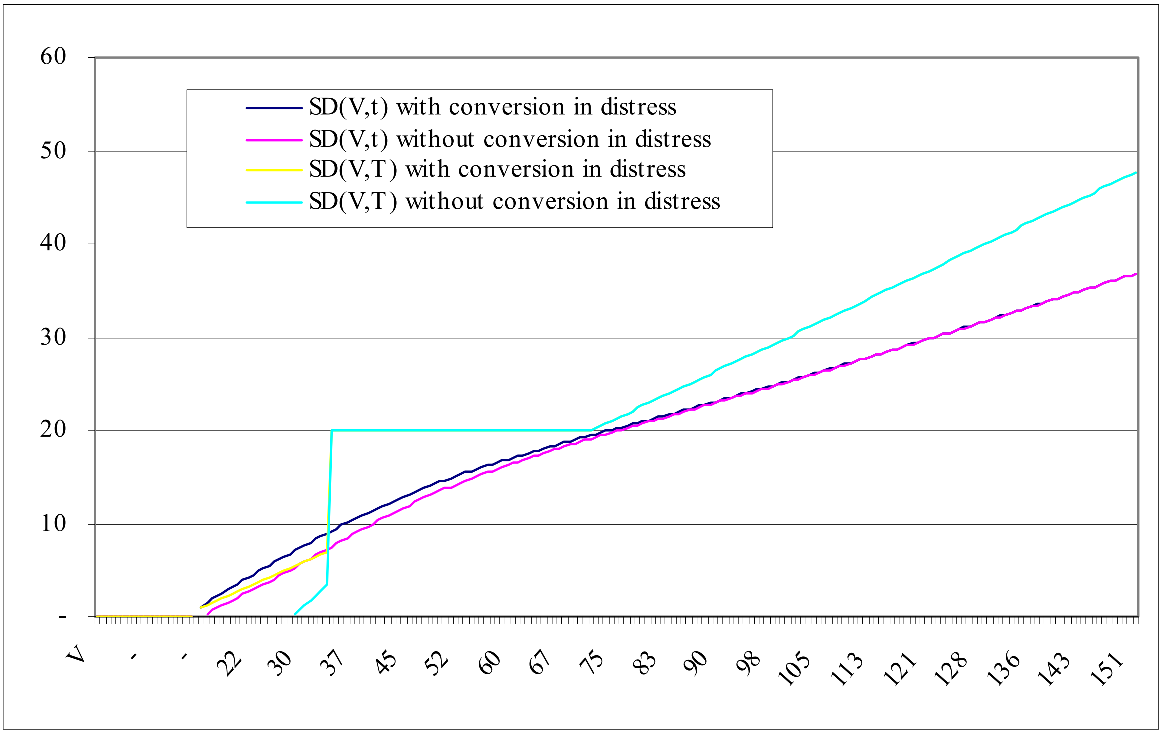

Figure 2: Values and payoff to the convertible in the base case when debt cannot be renegotiated and with or without "conversion in distress"

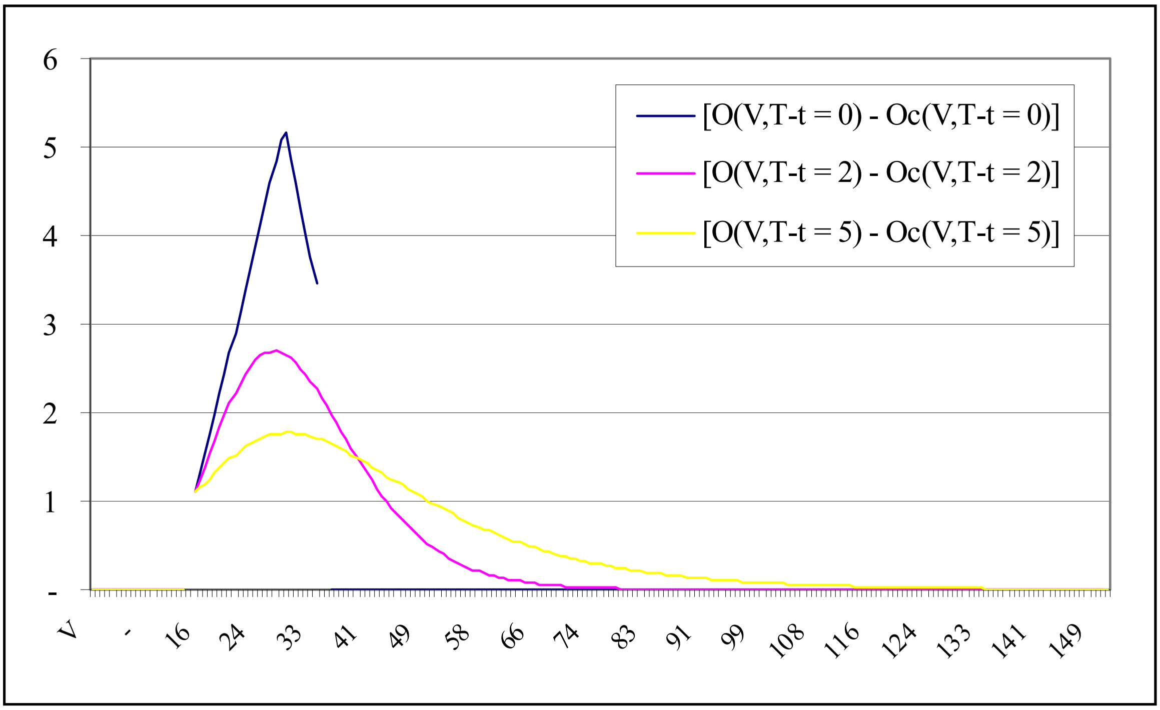

Figure 3: Payoff and values of the conversion "in distress" option $O(V,t) - O_c(V,t) = O_d(V,t) + O_{t_b}(V,t)$ as a function of time to the maturity of the convertible $(T-t)$.

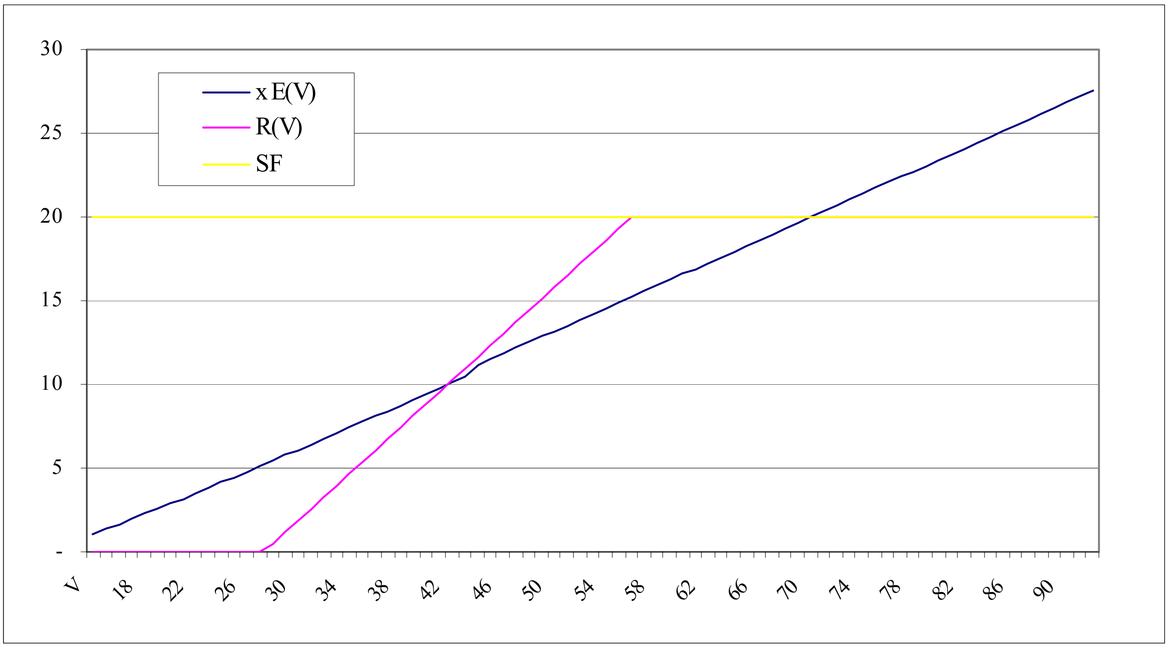

Figure 4: Determinants of the payoffs to the conversion "in distress" options: the fraction of equity $xE(V)$ received upon conversion can be higher than the recovery value of subordinated debt in bankruptcy $R(V)$.