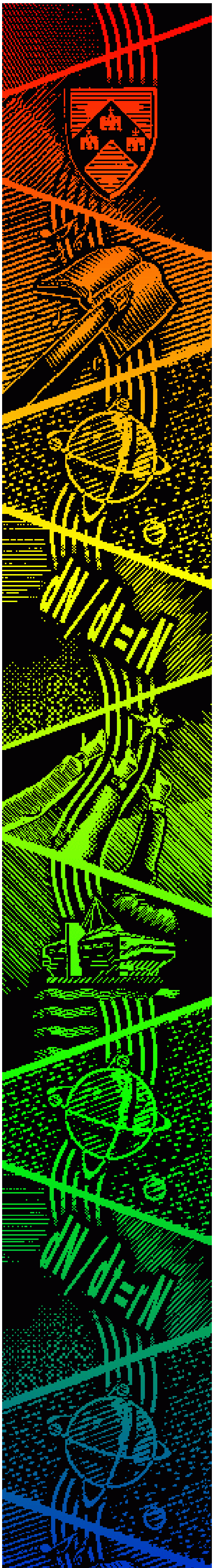

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Mixed Strategies in Simultaneous and Sequential Play of a 2 Player Game

by

Peter Simmons

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

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P Simmons, University of York*

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Abstract

We take a class of games with two players and two actions which only have mixed strategy Nash equilibria. We show that such games can only have hybrid equilibria if played sequentially with one player moving first. The hybrid equilibrium has the leader playing a mixed strategy but the follower playing a pure strategy.

We apply this result to a game between a debtor and a lender following a loan contract. The debtor can have high or low revenues and has to report his state to the lender. The lender can choose whether or not to undertake a costly audit. With simultaneous play there is only a mixed strategy Nash equilibrium with random cheating in reports and auditing. With sequential play if the debtor moves first, there is zero auditing and the debtor cheats as much as possible without giving the incentive to audit. We argue that the setting of the game and the valuable first mover advantage of the debtor mean that we should expect the game to be played sequentially with this hybrid outcome. This is important in the context of the loan contract since the hybrid outcome makes the contract renegotiation proof. Alternatively if the timing allowed the lender to move first, then the equilibrium would have the debtor reporting truthfully and the monitor auditing just sufficiently to ensure truthtelling by the debtor. This has strong links to the optimal debt contract with no commitment (Mookherjee-Png, 1989 and Jost, 1996). However we argue that the natural timing of events makes the debtor the leader.

We then consider other examples and show that the same outcome emerges in matching pennies and in a generic inspection game involving adverse selection in labour markets.

Key words: mixed strategies, loan contracts, first mover advantage.

JEL Nos: C72, C73, D82

We know that every finite game of complete information has a Nash equilibrium but that this may be in mixed strategies, there are also classes of games e.g. inspection games where we know that the only Nash equilibria are in mixed strategies. Also generally whether a game is played sequentially or simultaneously partly depends on the setting of the game: if there is a first mover

*Department of Economics, University of York, York Y01 5DD,UK email: ps1@york.a.uk

advantage to a single player, then if pre-game all players are on an equal footing, we might expect the game to be played sequentially since otherwise that player is giving something up. In this paper we examine two player, two action games in which the only Nash equilibrium is in mixed strategies. In this class we show that if the game is played sequentially then the equilibrium outcome will have the leader randomising in such a way that the follower is induced to play a pure strategy. The equilibrium action of the leader moves as far as possible towards the simultaneous play Nash outcome without also giving the follower the incentive to randomise. Thus we have the result that games like those we take in which the only simultaneous play Nash equilibrium is in mixed strategies, have only hybrid outcomes when played in pure strategies.

Next we examine various games. Our main example is taken from the area of costly state verification loan contracts between two risk neutral parties. There is a risk neutral entrepreneur who seeks finance for a project; the project has uncertain returns with two states: either revenues are high (y_H) or low (y_L). The realised revenues are private information to the entrepreneur, but may be audited by one or more of the risk neutral lenders at a cost of c. The result of any audit is public information to all investors, as is the fact that there has been an audit. All parties are risk neutral there is a safe interest rate r, together with the loan size this determines the expected reservation return of the lender. The problem is only of interest-there are only incentive problems for the firm - if revenues in the low state are insufficient to repay the fair return on the loan. If they were, a loan contract could just specify a constant return in each state and then the firm would have no incentive to cheat on its repayments. Otherwise the repayments must vary by state in such a way as to give an expected return on the loan equal to the fair interest rate. But then since the repayment in the high state must be higher than that in the low state, there is an incentive problem for the firm: it would like to repay the amount corresponding to the low state even if the true state is high.

Given that monitoring the state is costly and that any act of monitoring must be voluntary, in the literature there are two ways of generating the incentive to monitor. Firstly the incentive can be built into the loan contract problem by defining a premium repayment to be made that covers the observation costs whenever there is an audit-then the contract can induce truthtelling by the debtor (Jost, 1996). This is outside the noncooperative scenario we wish to analyse and so we do not consider it here.

Secondly there is the scenario in which monitoring is not contractible and is left to determination at the expost stage (Khalil-Parigi, Persons, Krasa-Villamil). Here the paradigm in the literature that we wish to address has the firm and investor(s) signing a contract that fixes only the loan size and repayments, conditioned on the reported state and the result of any audit. The actual reported state of the firm and the probability of audit are determined as mutual best responses in a *simultaneous* noncooperative game. That is the contract is first signed; next the state is revealed to the firm only and finally there is a Nash equilibrium in the probabilities of the high state firm cheating in its report of the state and in the probability of audit. A general finding of this literature is that this form of contract is not renegotiation proof (Persons, Khalil-Parigi). The idea here is that in the Nash equilibrium, whether it has cheating or not, there is a positive probability of monitoring (at a cost) so that it is possible for one party to make an offer prior to the monitoring that will be a Pareto improvement leading to zero monitoring costs and weakly (at least) increase the returns of both the firm and each monitor.

In this context our point is that there is a "natural" time line for the game after the signing of the contract that means the game is *sequential* rather than simultaneous. Since the debtor learns his state privately first, he has a natural first mover advantage. This matters because the subgame perfect equilibrium of this sequential game actually has zero monitoring and a positive probability of cheating-the firm naturally acts as leader and cheats as much as they can so long as they do not incur the risk of monitoring. But if the game is sequential (which at the renegotiation stage is recognised in Persons) then it is renegotiation proof since, as there is no monitoring, there is no deadweight loss that can be shared between the firm and the investor(s).

Other games that fall in this class are matching pennies and a paradigm for adverse selection in labour markets which we take as a generic example of an inspection game. Also if we allow for communication at the interim stage then the solution is renegotiation proof.

So far as loan contracts with costly state verification are concerned, we conclude that the outcome given the "natural" timing of events and actions, is a contract leading to an equilibrium in which there is no costly monitoring and a a small enough amount of cheating not to trigger monitoring.

The plan of the paper is to outline the abstract game and show the difference between the simultaneous and sequential play solutions in Section 1. In Section 2 we apply these results to the loan contract game arguing that the most plausible outcome is that of a Nash equilibrium in which the debtor randomly cheats as much as possible without inducing the lender to audit. In section 3 we briefly discuss other situations that fall within this class (matching pennies, adverse selection in labour markets), and in section 4 we conclude.

1 The General Result

With two players and only two actions for each player (x for player 1, y for player 2), in general terms write the payoff functions for a potential leader L and the potential follower F as

$$EU_L = \sum_{i,j}^{2} p_i^L p_j^F u_L(x_i, y_j); \quad EU_F = \sum_{i,j}^{2} p_i^L p_j^F u_F(x_i, y_j) \tag{1}$$

or replacing p_1^L by p^L, p_2^L with $1-p^L, p_1^F$ by p^F and p_2^F with $1-p^F$

$$\begin{aligned}
EU_L &= p^L p^F u_L(x_1,y_1) + (1-p^L)p^F u_L(x_2,y_1) \\
&\quad +(1-p^F)p^L u_L(x_1,y_2) + (1-p^L)(1-p^F)u_L(x_2,y_2) \\
EU_F &= p^L p^F u_F(x_1,y_1) + (1-p^L)p^F u_F(x_2,y_1) \\
&\quad +(1-p^F)p^L u_F(x_1,y_2) + (1-p^L)(1-p^F)u_F(x_2,y_2)
\end{aligned} \tag{2}$$

where x_i, y_j are respectively actions of leader and follower; p_i^L, p_j^F are the probabilities with which these actions are chosen in a general mixed strategy.

1.1 Simultaneous Play

The simultaneous play Nash equilibrium is p^{L*}, p^{F*} such that

$$p^{L*} = \arg\max EU_L(p^{L*}, p^{F*}), p^{F*} = \arg\max EU_F(p^{L*}, p^{F*})$$

For there to be no pure strategy Nash equilibrium it must be that
$u_F(x_i,y_j) > u_F(x_i,y_k), k \neq j \quad \Rightarrow \quad there \quad is \quad i' \quad u_L(x_{i'},y_j) > u_L(x_i,y_j)$

Ordering the actions of players so that y_i is a best response to x_i this condition becomes

$$u_F(x_1,y_1) > u_F(x_1,y_2) \quad and \quad u_L(x_2,y_1) > u_L(x_1,y_1) \tag{3}$$

and

$$u_F(x_2,y_2) > u_F(x_2,y_1) \quad and \quad u_L(x_1,y_2) > u_L(x_2,y_2) \tag{4}$$

So when (3)-(4) hold the Nash equilibrium must be in mixed strategies satisfying

$$p^F u_L(x_1,y_1) + (1-p^F)u_L(x_1,y_2) = p^F u_L(x_2,y_1) + (1-p^F)u_L(x_2,y_2) \tag{5}$$

$$p^L u_F(x_1,y_1) + (1-p^L)u_F(x_2,y_1) = p^L u_F(x_1,y_2) + (1-p^L)u_F(x_2,y_2) \tag{6}$$

which we can rewrite as

$$0 = p^F \left[u_L(x_1,y_1) - u_L(x_2,y_1) - u_L(x_1,y_2) + u_L(x_2,y_2)\right] + \left[u_L(x_2,y_1) - u_L(x_2,y_2)\right] \tag{7}$$

$$0 = p^L \left[u_F(x_1,y_1) - u_F(x_2,y_1) - u_F(x_1,y_2) + u_F(x_2,y_2)\right] + \left[u_F(x_2,y_1) - u_F(x_2,y_2)\right] \tag{8}$$

These equations determine the mixed strategy, note that (3) and (4).automatically imply that the equilibrium probabilities from (7) and (8) are in the interior of the unit interval. And of course if (7),(8) fail to hold then one or other player will not randomise-for example suppose that p^L is such that

$$p^L \left[u_F(x_1,y_1) - u_F(x_2,y_1)\right] + u_F(x_2,y_1) > p^L \left[u_F(x_1,y_2) - u_F(x_2,y_2)\right] + u_F(x_2,y_2)$$

then F will set $p^F = 1$.

1.2 Sequential Play

Now consider sequential play with L as leader. If L sets p^L to satisfy (8) then he cannot predict $F's$ response; F is indifferent between any set of probabilities p^F. For this choice of p^L to be optimal for L, it would have to be the case that

$$\frac{\partial EU_L}{\partial p^F} = 0 \tag{9}$$

or

$$\begin{aligned} 0 &= p^L u_L(x_1,y_1) + (1-p^L)u_L(x_2,y_1) - p^L u_L(x_1,y_2) - (1-p^L)u_L(x_2,y_2) \qquad (10) \\ &= p^L\left[u_L(x_1,y_1) - u_L(x_2,y_1) - u_L(x_1,y_2) + u_L(x_2,y_2)\right] + \left[u_L(x_2,y_1) - u_L(x_2,y_2)\right] \end{aligned}$$

Eliminating p^L between (8) and (10) gives

$$\begin{aligned} &[u_F(x_2,y_2) - u_F(x_2,y_1)]\left[u_L(x_1,y_1) - u_L(x_2,y_1) - u_L(x_1,y_2) + u_L(x_2,y_2)\right] \\ &\hfill (11) \\ &= [u_L(x_2,y_2) - u_L(x_2,y_1)]\left[u_F(x_1,y_1) - u_F(x_2,y_1) - u_F(x_1,y_2) + u_F(x_2,y_2)\right] \end{aligned}$$

But (11) and (3)-(4) are mutually contradictory. If the simultaneous play game has no pure strategy equilibrium the LHS is negative whilst the RHS is positive. Thus any two person, two action game that only has a mixed strategy solution when played simultaneously cannot have a mixed strategy solution when played sequentially.

What is the sequential solution if (3)-(4) hold? Since under these conditions $\partial^2 EU_F/\partial p^F \partial p^L > 0$, if $p^L > p^{L*}$, the follower responds with $p^F = 1$ whilst if $p^L < p^{L*}$, the followers response is $p^F = 0$. By setting p^L marginally above p^{L*} the leader gets arbitrarily close to

$$E(U_L|p^F = 1) = p^{L*}u_L(x_1,y_1) + (1-p^{L*})u(x_2,y_1)$$

whilst by setting p^L marginally below p^{L*} the leader gets arbitrarily close to

$$E(U_L|p^F = 0) = p^{L*}u_L(x_1,y_2) + (1-p^{L*})u(x_2,y_2)$$

The difference between these is

$$\begin{aligned} E(U_L|p^F &= 1) - E(U_L|p^F = 0) = p^{L*}[u_L(x_1,y_1) - u_L(x_2,y_1) - u_L(x_1,y_2) + u_L(x_2,y_2)] \\ &\quad + u_L(x_2,y_1) - u_L(x_2,y_2) \end{aligned}$$

The coefficient of p^{L*} is positive but the intercept in this expression is of ambiguous sign. Hence if $u_L(x_2,y_1) > u_L(x_2,y_2)$ for sure the leader sets p^L marginally above p^{L*} and the follower responds with $p^F = 1$. More generally solving for p^{L*} from (4) if

$$\begin{aligned} &[u_F(x_2,y_2) - u_F(x_2,y_1)][u_L(x_1,y_1) - u_L(x_2,y_1) - u_L(x_1,y_2) + u_L(x_2,y_2)] \\ &+ [u_L(x_2,y_1) - u_L(x_2,y_2)][u_F(x_1,y_1) - u_F(x_2,y_1) - u_F(x_1,y_2) + u_F(x_2,y_2)] > 0 \end{aligned}$$

the leader sets p^L marginally above p^{L*}, whilst if it is negative the leader sets p^L marginally below p^{L*}.

Proposition 1 *If a two player two action game only has a simultaneous play NE in mixed strategies, the sequential play solution cannot have both players randomising but will have the leader randomising and the follower playing a pure strategy.*

2 A Loan Game With CSV

The scenario is that the loan contract has been signed, the debtor has borrowed funds from the lender and at repayment time the debtors assets are private information to the debtor and can be either high (y_H) or low (y_L). They are high with exogenous probability p. The low state revenues are insufficient to repay the fair return on the loan but the project that the loan finances is basically profitable in expected value terms. The loan contract stipulates repayments that must be made, and in order to secure the fair return on the loan the high type debtor must make higher expected repayments than the low type. To determine which repayment is due requires the debtor to report his revenues to the lender. As well as this there may be an audit before the repayments are made so that the contract can set repayments that are conditioned both on the report and the results of any costly audit that may be undertaken. The contract specifies that if the debtor reports $\widetilde{H}$ they have to repay R_H if either there is no audit or if there is an audit and the true state is found to be H; if the debtor reports $\widetilde{H}$ but the true state following audit of this report is found to be L the debtor has to repay $R_{LH} = y_L$[1]. If the debtor reports $\widetilde{L}$ and they are audited then if the true state is L they have to repay R_{LL} or if the the true state is H they have to repay $R_{HL} = y_H$. If following a report of $\widetilde{L}$ they are not audited then they have to repay $R_{L.}$ For feasibility $R_{LL}, R_L \square y_L$ and $R_H \square y_H$. We assume that $(1-p)R_{LL} - c \square (1-p)R_L$ and $R_L < R_{LL}$ so that to the lender there is no expected gain from auditing a truthful low state report but at least some contribution to covering monitoring costs through the premium of R_{LL} over R_L. We also assume that $py_H + (1-p)R_{LL} - c - R_L > 0$ so that the expected return to the lender from monitoring a low state report is positive and that $R_H > R_L$ (essentially this follows from the requirement that to yield the fair return, high state repayments must exceed low state repayments in expectation).

2.1 Simultaneous Reporting and Auditing Strategy Choice

Here after the contract is signed, nature determines the borrower type. Then as simultaneous best responses the lender and debtor must select strategies. A strategy for the lender is a decision to monitor or not, A strategy for the debtor is a report to make (high $\widetilde{H}$ or low $\widetilde{L}$) if he has high income and a report to make (high $\widetilde{H}$ or low $\widetilde{L}$) if he has low income. In terms of a payoff matrix we have

[1]We skip over the details but essentially in this scenario the contract must involve maximum punishment for detected cheating (Persons,1996; Khalil-Parigi 1998).

	$Monitor$	$No\ Monitor$
$\widetilde{H}\vert H, \widetilde{H}\vert L$	$pR_H + (1-p)y_L - c$ $(y_H - R_H, 0)$	R_H $(y_H - R_H, y_L - R_H)$
$\widetilde{H}\vert H, \widetilde{L}\vert L$	$pR_H + (1-p)R_{LL} - c$ $(y_H - R_H, y_L - R_{LL})$	$pR_H + (1-p)R_L$ $(y_H - R_H, y_L - R_L)$
$\widetilde{L}\vert H, \widetilde{H}\vert L$	$p(y_H - c) + (1-p)(y_L - c)$ $(0, 0)$	$pR_L + (1-p)R_H$ $(y_H - R_L, y_L - R_H)$
$\widetilde{L}\vert H, \widetilde{L}\vert L$	$p(y_H - c) + (1-p)(R_{LL} - c)$ $(0, y_L - R_{LL})$	$pR_L + (1-p)R_L$ $(y_H - R_L, y_L - R_L)$

In the table the columns are for the lender, the first number in each cell is the expected return that the lender gets if he takes the action of that column and the borrower is following the row strategies. The rows list the strategies that the borrower could follow with the first being the report conditional on his type and (the second row of each cell) the payoff that he would receive if his income is high, the second if his income is low. Then his possible strategies are $(\widetilde{H}|H), (\widetilde{L}|H), (\widetilde{H}|L), (\widetilde{L}|L)$.

The strategy $\widetilde{H}|H, \widetilde{H}|L$ is dominated by $\widetilde{H}|H, \widetilde{L}|L$ for the debtor and the strategy $\widetilde{L}|H, \widetilde{H}|L$ is dominated by $\widetilde{L}|H, \widetilde{L}|L$ for the debtor. Hence the lender can eliminate these strategies of the debtor which leaves us with the reduced matrix

	$Monitor$	$No\ Monitor$
$\widetilde{H}\vert H, \widetilde{L}\vert L$	$pR_H + (1-p)R_{LL} - c$ $(y_H - R_H, y_L - R_{LL})$	$pR_H + (1-p)R_L$ $(y_H - R_H, y_L - R_L)$
$\widetilde{L}\vert H, \widetilde{L}\vert L$	$p(y_H - c) + (1-p)(R_{LL} - c)$ $(0, y_L - R_{LL})$	$pR_L + (1-p)R_L$ $(y_H - R_L, y_L - R_L)$

This game has no pure strategy solution. If the lender monitors the best response of the high type debtor is (strictly) to report $\widetilde{H}$ and the low type debtor always reports $\widetilde{L}$ anyway. But if the debtor plays $\widetilde{H}|H, \widetilde{L}|L$ the lenders best response is not to monitor since $(1-p)(R_{LL} - R_L) < c$. If the lender does not monitor, the best response of the high type debtor is to report $\widetilde{L}$ and again the low type always reports low, but the best response of the lender to $\widetilde{L}, \widetilde{L}$ is to monitor as $p(y_H - c) + (1-p)(R_{LL} - c) > R_L$. There is a mixed strategy solution: let m be the probability of monitoring and l_H the probability that the high type debtor reports low (i.e. the probability the debtor follows $\widetilde{H}|H, \widetilde{L}|L$). The borrowers expected utility becomes conditional on his type; so does his probability of falsely reporting l_H, l_L

$$ED|H = l_H(1-m)(y_H - R_L) + (1-l_H)(y_H - R_H)$$

$$ED|L = m(y_L - R_{LL}) + (1-m)(y_L - R_L)$$

Similarly the lenders expected payoff is

$$EM = p[(1-l_H)R_H + l_H\{m(y_H - c) + (1-m)R_L\}] + (1-p)[m(R_{LL} - c) + (1-m)R_L]$$

When the borrower type is high there is a mixed strategy solution solving

$$\frac{\partial ED|H}{\partial l_H} = 0, \frac{\partial EM}{\partial m} = 0$$

yielding

$$l_H = \frac{(1-p)(R_L - R_{LL}) + c}{p(y_H - R_L)}, m = \frac{R_H - R_L}{y_H - R_L} \tag{12}$$

Thus when the borrower is of low type there is a pure strategy equilibrium; when of high type a mixed strategy equilibrium. If the debtors actually played these strategies it would reveal their type: the lender has to make a best response against either $\widetilde{L}$ for $l_L = 0$ or l_H. One might argue that with simultaneous play the lender does not know what the borrower will declare. If he does not there is one Nash equilibrium with high type which is mixed, and one Nash equilibrium with low type which is $(L^*, L^*), NM$.

2.2 Sequential Reporting and Auditing Strategy Choice

Suppose the game is played sequentially with the borrower as leader. If of high type the borrower would get expected utility $(1-m)(y_H - R_L)$ from declaring $\widetilde{L}$ with $l_H = 1$. If he does this the lenders expected utility is

$$\begin{aligned} EM|\widetilde{L} &= m[\frac{pl_H}{(1-p)+pl_H} y_H + \frac{(1-p)}{(1-p)+pl_H} R_{LL} - c] + (1-m)R_L \\ &= m(py_H + (1-p)R_{LL} - c) + (1-m)R_L \end{aligned}$$

so $\partial EM|\widetilde{L}/\partial m = pl_H y_H + (1-p)R_{LL} - c - R_L = py_H + (1-p)R_{LL} - c - R_L > 0$ and the lender would always monitor. So the debtor would actually get 0. If he declares $\widetilde{H}$ with $l_H = 0$ the lender would get

$$EM|\widetilde{H} = m[pR_H + (1-p)R_{LL} - c] + (1-m)[pR_H + (1-p)R_L]$$

and the lender would never monitor. So the debtor would get $y_H - R_H$. So played sequentially of these two pure strategy options, the debtor will always tell the truth.

However since

$$\frac{\partial EM}{\partial m} = pl_H y_H + (1-p)R_{LL} - (1-p+pl_H)(R_L + c)$$

and

$$\frac{\partial}{\partial l_H}\frac{\partial EM}{\partial m} = p(y_H - R_L - c) > 0$$

the high type debtor could always set $0 < l_H < [(1-p)(R_L - R_{LL}) + c]/[p(y_H - R_L)]$ and then the lender will not monitor. Then this debtors return is $ED|H =$

$y_H - l_H R_L - (1 - l_H)R_H$ which exceeds his return from truthfully reporting so long as l_H is close to $[(1 - p)(R_L - R_{LL}) + c]/[p(y_H - R_L)]$. So we have the result that when played sequentially the high type debtor will cheat as often as possible without inducing the lender to monitor.

On the other hand if played sequentially the low type will still report truthfully and the lender will not monitor so here sequential or simultaneous play lead to the same outcome.

Given the timing of events: nature chooses the debtor type which is private information to the debtor, the debtor has a first mover advantage. There is a gain from the debtor getting in first with his report, and he is in a position to do this. The natural way to play the game is for the debtor to act as leader rather than having either simultaneous noncooperative play leading to the usual Nash equilibrium, or sequential play with the lender acting as leader. If there was some way in which the lender could preempt the report of the debtor - for example by writing the audit strategy into the contract before the debtor knows his type- then the outcome will be identical to the Jost type of contract. There will be random monitoring of just a sufficiently high degree to enforce truthtelling by the high type debtor.

3 Inspection and Related Games

In the literature, inspection games are highlighted as cases in which there is often only a mixed strategy Nash equilibrium outcome when the game is played simultaneously. We consider two cases and show that our main result applies.

3.1 Matching Pennies

In matching pennies the game is zero sum with payoff matrix

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

The payoff of the row player is shown first: he wins if the outcomes match, the column player wins if the outcomes disagree. This game has no pure strategy Nash equilibrium. If we allow for mixed strategies where $p_R, (p_C)$ are the probability that the row (column) player shows H (heads) then the expected payoffs are

$$ER = p_R p_C + (1 - p_R)(1 - p_C) - p_R(1 - p_C) - p_C(1 - p_R)$$

$$EC = p_R(1 - p_C) + p_C(1 - p_R) - p_R p_C - (1 - p_R)(1 - p_C)$$

giving optimal mixed strategies of $p_R = p_C = 0.5$ and a zero expected payoff to each.

Now suppose that this was played as a staged game with R moving first and that R's choice is seen by C and is irrevocable.

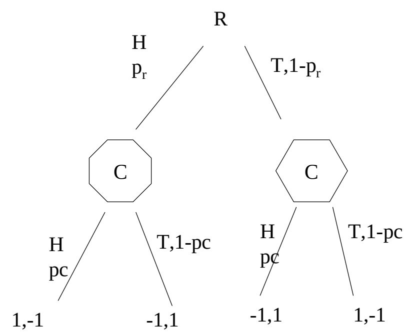

Figure 1:

If R plays a pure strategy (H or T) then C can play the opposite so that for sure R loses and C wins. Whatever R does he cannot prevent this. If R played a mixed strategy i.e. he chooses p_R then if $p_R > 0.5$, C will respond with the pure strategy T for sure which will give payoffs $ER = (1 - p_R) > 0$ and $EC = p_R > 0$. R's best choice is then the lowest $p_R > 0.5$, which of course does not exist. But if R set $p_R = 0.5$ then $C's$ expected payoff is zero whatever p_C he chooses and so is $R's$. Similarly if R sets p_R below 0.5, C responds with H for sure and each player has a positive expected return. So there are multiple ϵ leadership optima with p_R either marginally above or marginally below 0.5.

3.2 An Inspection Game

The generic inspection game presented by Fudenberg and Tirole (1993, example 1.7) is in the context of a shirking worker and a firm who may inspect the worker at a cost. It has a payoff matrix:

	I	NI
S	$0, -h$	$w, -w$
W	$w - g, v - w - h$	$w - g, v - w$

where the row player is a worker who may either shirk (S) or work (W), and the column player is the employer, who can either inspect (I) or not inspect (NI). The worker's payoff is given first out of the two numbers in each cell. If the worker shirks he receives a zero payoff if inspected but otherwise receives the wage w. If he works, it costs him effort $g < w$ but receives the wage whether or not he is inspected. Valuable output is produced for the firm only if the worker works and then its value is $v > w$. Finally, an inspection costs the firm $h < w$. Here $v > w > h$ and $w > g$. This game (which is very similar to the loan contract game) has no pure strategy equilibrium with simultaneous play. If the worker shirks it is best to inspect but if the firm inspects it is best for the worker to work. If the worker works it is best for the firm not to inspect but if the firm does not inspect it is best for the worker to shirk. Of course it has a simultaneous play mixed strategy equilibrium: if ω is the probability the worker works and ι the probability the firm inspects, the expected payoff to the worker is

$$EW = \omega(w - g) + (1 - \omega)(1 - \iota)w$$

and to the firm is

$$EF = \iota[\omega(v - w - h) - (1 - \omega)h] + (1 - \iota)[\omega(v - w) - (1 - \omega)w]$$

and a mixed strategy equilibrium occurs at $\iota = g/w, \omega = (w - h)/w$.

Now suppose that this game is played sequentially with the worker moving first and making an irrevocable choice to work or shirk which is seen by the firm.

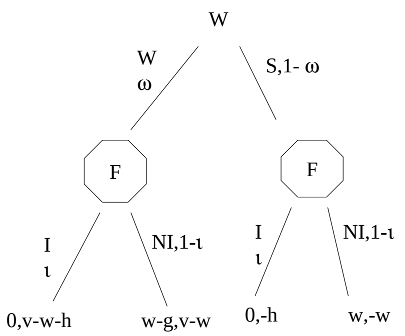

Figure 2:

If the worker shirks the best response of the firm is to inspect; if the worker works the best response of the firm is not to inspect. But suppose the worker sets an irrevocable ω: since $\partial EF/\partial\iota = -h + (1 - \omega)w$, if $\omega > (w - h)/w$ the best response of the firm is not to inspect in which case the expected payoffs are $EW = w - \omega g \square w - (w - h)g/w, EF = \omega(v - w) - (1 - \omega)w$. On the other hand if $\omega < (w - h)/w$ the best response of the firm is to inspect for sure and the expected payoffs of the two parties are $EW = \omega(w - g) \square (w - h)(w - g)/w, EF = \omega(v - w - h) - (1 - \omega)h$. If the worker chooses $\omega = (w - h)/w$, the firm is indifferent between inspecting or not and the workers payoff is $(w - h)(w - g)/w + h(1 - \iota) = w - h - (w - h)g/w + h(1 - \iota) \square w - (w - h)g/w$.

By choosing an ω close to but above $(w-h)/w$, the workers payoff from making the firm not inspect for sure exceeds that from making the firm inspect for sure since

$$[w-(w-h)g/w]-[(w-h)(w-g)/w]=h>0 \tag{13}$$

On the other hand the worker is also better off from making the firm not inspect than than from setting $\omega=(w-h)/w$:

$$\begin{aligned} & [w-(w-h)g/w]-[w-h-(w-h)g/w+h(1-\iota)] && (14)\\ = \quad & \iota h>0 && (15)\end{aligned}$$

So the best strategy for the worker is to choose the lowest $\omega, \omega>(w-h)/w$ and induce the firm not to inspect. This sequential solution has much in common with that of the loan game and matching pennies: the leader never wishes to induce the follower to select a mixed strategy.

4 Conclusions

We have shown that two player, two action games in which the only simultaneous play Nash equilibrium is in mixed strategies cannot have a mixed strategy outcome when the game is played sequentially with one of the players moving first. Instead the outcome will have a hybrid form in which the first mover randomises and the follower responds with a pure strategy. The leader induces a pure response from the follower to the leaders advantage. In the games considered, the leader wants to set his random strategy as close as possible to, but distinct from, his simultaneous play mixed strategy. With sequential play he is strictly better off doing this than playing the mixed strategy, but since there is no closest point to his mixed strategy, his optimal sequential behaviour is ϵ optimal. How the game is played depends on the setting. If the players are an incumbent and a potential entrant then often there is a natural sense in which the incumbent is the leader. If the game is one of one-sided incomplete information where one player is perfectly informed and the other is not then there is often a sense in which the informed party is the leader. In such cases the leader has a valuable first mover advantage which they will wish to retain by acting as leader so we would expect such games to be played sequentially rather than simultaneously. In such games our result is important because it predicts that the equilibrium outcome will have a hybrid form.

We apply the result to a scenario where a loan has been made between two risk neutral parties with a contract that requires a high revenue debtor to make higher repayments than a low revenue debtor. The revenues of the debtor are private information to the debtor. The debtors strategy is a choice of report of his revenues; the high revenue debtor has an incentive to cheat in the report in order to make the low repayment. The lenders strategy is a costly audit choice. With simultaneous play the game only has a mixed strategy Nash equilibrium outcome with random cheating and auditing. However if the debtor is the leader and the game is played sequentially, the equilibrium has a hybrid form in which the debtors randomly cheats as much as he can without inducing the lender to ever audit. Thus the follower is playing a pure strategy-there is no costly auditing and there is as much cheating as possible consistent with zero auditing. Since audit cannot be performed until the report has been made and since the debtor knows his revenues before the report, we argue that it is natural to expect this game to be played sequentially with the debtor moving first. Doing so makes the debtor better off than with simultaneous play. If the lender could somehow preempt the first mover advantage of the debtor and initially declare an audit strategy, then the outcome is that the lender will audit just sufficiently fiercely to deter the debtor from ever cheating in his report. However this has credibility problems: if the lender does this, the debtor responds with truthful reports but then the lender will have no incentive to undertake the costly audit. A way round this might be to try to write the audit strategy into the loan contract (Jost) together with a sequential rationality constraint that ensures that the audit strategy will actually be implemented. However generally this is not renegotiation proof and so is not attainable. The alternative where the debtor is leader *is* renegotiation proof since as there is zero auditing there are no deadweight losses (the audit costs) available for sharing between the players. There are other related games eg in tax avoidance games we predict that the most likely outcome is that tax payers cheat as much as they can without inducing the tax authorities to perform a costly audit. If the tax authorities could preempt this by moving first they would announce an audit policy that is sufficiently fierce to prevent tax payers from ever cheating, but again this would have credibility problems.

We show that matching pennies and a generic form of inspection game also have the property that simultaneous play only has mixed strategy Nash equilibria. Hence our result applies here as well.

One weakness of the approach is that it depends on only two states, with two players and more than two actions/states it fails to hold. It is an open question whether it holds if the number of players is equal to the number of states.

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