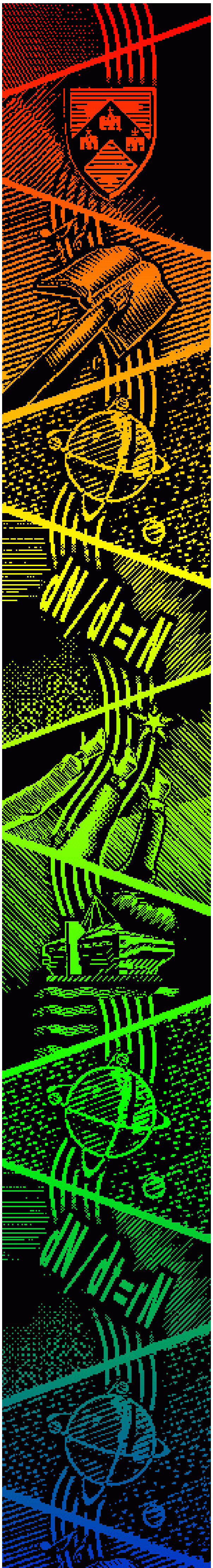

THE UNIVERSITY *of York*

Discussion Papers in Economics

No. 2003/06

Comparative Statics with Consumption Externalities

by

Peter Simmons and Emanuela Randon

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Comparative Statics with Consumption Externalities

P Simmons and E Randon*

Abstract

We consider the comparative statics of consumer demand when there are consumption externalities in one commodity between two individuals.We show that the externaliy can switch goods which would naturally be normal into inferior goods and as a result the externality can also lead to Giffen goods. In addition the externality can transform complementarity relations between goods. Thus substitutes can become complements or vice versa once the feedback effects of the externality are taken into account. Next we consider the effect of externalities on Slutsky symmetry and negativity restrictions With consumption externalities there are generalised forms of such restrictions. We derive these both for the two individual case and for cases in which either there are two individuals but all goods may cause externalities or there is a single externality good but H individuals. We relate the generalised symmetry restrictions to the rank conditions of Browning and Chiappori. Finally we consider the effects of consumption externalities on consumer surplus analysis.

JEL Nos: D1, D6, R2

Externalities in consumption have often been stressed in the conspicuous consumption literature where the motivation is primarily one of envy. There are also commodities where there are technological reasons for consumption externalities. Two obvious examples are consumption activities that are subject to congestion e.g. many recreational activities ranging from cycling, through beach holidays to watching a football match, and consumption activities that have a network dimension like telephone calls or ownership of mobile phones. Of course there are also explicit group activities like card games or chess. There is not really a well developed theory of consumer behaviour or of testable re-

*Department of Economics, University of York,York Y01 5DD

strictions arising in this context to match the theory of consumption without externalities.

In this paper we want to address three questions:

- What are the effects on the comparative statics of consumer demand of having consumption externalities between consumers?
- In an environment where there are consumption externalities can we find conditions we can test empirically for the existence of individual consumer preferences with consumption externalities?
- Can we see how welfare analysis e.g. consumer surplus analysis is affected by consumption externalities?

The answer that we find to the first point is that particular types of consumption externality patterns can reverse the usual comparative static predictions of price or income changes so that for example even if in private terms no good is inferior for any individual, once the external effects are taken into account, goods can appear empirically inferior or even Giffen. In a variety of contexts of different generality (in terms of the number of consumers, the number of goods and the ways that externality effects arise) we answer the second question by finding testable restrictions on individual and market demand that will allow us to distinguish situations where externalities are important. Finally we show that the presence of external effects may actually make consumers welcome price increases rather than reductions so that the usual predictions of consumer surplus analysis can be reversed in the presence of consumption externalities. More precisely our results are that:

- The externality can reverse the sign of the comparative static effects of income or price changes;

- In general for any given individual budget constraints there may be multiple preference maximising individual demands once the externalities have been taken into account;

- There is a form of the Slutsky symmetry restriction as a necessary condition for preference maximisation which is related to the Browning-Chiappori (B-C) idea of adding rank one matrices to the usual Slutsky matrix but that these rank one matrices represent the effect of externalities on a given individuals preferences not the effect of price-income dependence in a group preference function. In B-C the rank one matrices are "arbitrary" in as much as their form depends both on the way in which the function of prices and/or income enters preferences and on the form of the function of prices and income. The effects have the interpretation of representing the way in which the distributional power between a group of individuals varies with prices and income. In our case the matrices are arbitrary because they depend on the way in which externalities enter individual utilities in general. We also have more of these matrices since each externality between a pair of individuals or goods gives a distinct channel through which there are additional price/income effects on demand. However the functional form for each externality effect has some restriction in principle since this must represent the Nash equilibrium demand of another individual for some good;

- When there are only two individuals and a single good that has external effects between the two then the form of the rank one matrix that should be added is fully determined and there is a simple empirical test for both the presence of externalities and the existence of individual preferences which generalises the usual Slutsky symmetry condition. With H individuals and a single good through which there are general externalities for all individuals, $H-1$ rank one matrices must be added to the pseudoSlutsky matrix to give a form that is symmetric. The nature of these rank one matrices is derived and is observable so in principle testing is possible.

- A special case of some interest is what we term common popular externalities: with H individuals, each individual utility depends on the total consumption by all other individuals of one externality-inducing good. This is like a pure congestion case. Here we show that although there are H individuals, only a single rank one matrix must be added to the pseudoSlutsky matrix to result in a symmetric matrix;

- Similarly we have a generalised form of the negative semidefiniteness condition for these cases;

- The externality may be of a form which makes the consumer benefit from price increases rather than price decreases and the usual consumer surplus will reveal this.

The general framework we use has H individuals with the hth having preferences defined by $u_h(x_h; x_{-h})$ where the notation x_{-h} indicates the consumption of each good by each individual other than h. Since this is very general often we specialise it to two individuals and/or cases in which only some goods have externalities. We envisage an environment in which each individual has a private budget m_h and all individuals face the same market prices. Each individual takes decisions privately to maximise their own utility conditional on the choices of other individuals i.e. given their budget constraint each individual makes a best response to the consumption activities of others. We then analyse what we call a Nash equilibrium in which these best responses are mutually consistent[1]. Our analysis is all static so we do not address questions of how external effects evolve, nor questions of dynamic adjustment (with a given set of interdependent preferences) of one individuals consumption to that of others.

There are links between some of our results and those of models with price dependent preferences or which are concerned with particular aspects of intrahousehold decision-making-the literature dealing with the collective model. So the plan of the paper is to start by recalling three main results from this literature and then move on to consider the two individual case with a single externality inducing good in Section 2 where we analyse both the comparative statics and the generalised Slutsky symmetry restrictions. In Section 3 we allow all goods to have externality effects and look at the form of the Slutsky symmetry restrictions, in Section 4 we take H individuals and a single externality good also specialising this to the case in which the externality works through the total consumption of other individuals of that good and in Section 5 we analyse the effects of the externality on consumer surplus.

1 Some Prior Literature

There is a prior literature focusing more on the related topics of price dependent preferences and the collective model of group e.g. household decisionmaking. The earlier literature on externalities e.g. McKenzie (1955) looks more at normative properties showing that with general consumption externalities, person-

[1] In terms of formal game theory, each individual has a strategy space consisting of choice of an n vector x_h which satisfies his budget constraint and a continuous payoff function depending on the actions of all. The game is played simultaneously by all individuals.

alised pricing is necessary to decentralise a Pareto optimum-this is equivalent to a system of Pigovian taxes. As we will see the price dependent literature is close to the questions we wish to analyse. Pollak(1977) shows that in a world in which individual preferences have the form $u(x,p)$ where x and p are respectively vectors of quantities consumed and prices paid per unit in competitive markets, preferences are not uniquely defined by demand behaviour. For example two sets of preferences of the form $F(v(x),p)$ and $G(v(x),p)$ where $F()$ and $G()$ are increasing in $v()$ will generate exactly the same demand behaviour since both sets of demands solve the problem $\max\{v(x)|p'x \leq m\}$ where m is the consumers income. On the other hand he also shows that *any* system of demand equations $x_i = f_i(p,m)$ which are homogeneous of degree zero and satisfy the budget constraint can be used to construct a fixed coefficients utility function that will then have generated those demands. More precisely *any* demand system $x_i = f_i(p,m)$, homogeneous of degree zero in p,m which satisfies the budget constraint and which has no inferior goods can be rationalised by the utility

$$V(x,p) = \min\{x_1, h_2(x_2,p), ...h_n(x_n,p)\}$$

where $h_i(x_i,p)$ is defined by $x_i = g_i(p, h_i(x_i,p))$ for $i > 1$, and $g_i(p,x_1) = f_i(p, g_1(x_1,p))$ for $i > 1$ and $g_1(x_1,p)$ is defined by $x_1 = f_1(p, g_1(x_1,p))$. His idea is that we compute the income $g_1()$ that would make the individual buy the fixed quantity x_1 at any prices p. Then calculate the quantity of each other good ($g_i()$ for $i > 1$) that the individual would buy at any prices with this income $g_1(x_1,p)$, and finally compute the quantity of good 1 that the individual will buy if they buy x_i of good i at prices p, when income is set at a level such that they buy $x_1 = h_i(x_i,p)$ units of good 1.Having fixed coefficients guarantees that the individual will choose $x_1 = h_i(x_i, p)$ for $i > 1$ and so replicates the actual demand behaviour. The implication here is that if we allow utility to depend on prices and income in an arbitrary way except for homogeneity then there are no restrictions on demand behaviour beyond adding up and homogeneity of degree zero.

On the other hand broadly in the collective scenario, Browning and Chiappori (1998) show that if the group acts so as to spend its household budget so as to achieve a particular Pareto optimum, the choice of which itself depends on prices and income, then it is as if the household has a utility function which depends on prices and income. For example take 2 individuals each with preferences given by $u_h(x_h), h = A, B$. In general these individuals consume the same goods[2]. If goods are purchased in competitive markets and the group has a total budget m to allocate then any Pareto optimal allocation between the individuals will lead to a form of group preferences given by $v(x, \lambda) = \max\{\lambda u_1(x_1) + (1-\lambda)u_2(x_2) | x_1 + x_2 \leq x\}$. The solution to this problem defines the allocation of each commodity between the two individuals $x_i = X_i(x, \lambda)$. As $0 \leq \lambda \leq 1$ varies over the unit interval different Pareto optima are defined. If the group always selects a Pareto optimal outcome then group decisions solve $\max\{v(x, \lambda) | p'x \leq m\}$ giving a two stage representation of the demands: optimal aggregate quantities are given by $x = X(p, m, \lambda)$ and these are then allocated between the individuals. If we add to this specification the assumption that $\lambda = \Lambda(p, m, ..)$ then the group decisions are taken to maximise a price dependent preference function. Browning and Chiappori go on

[2] In fact they allow for intrahousehold externalities and also a particular type of intrahousehold public good since they take $u_h(x_A, x_B, X)$ with total household purchases of good i being $x_i = x_{Ai} + x_{Bi} + X_i$. We abstract from this since our interest is in interhousehold externalities ie in the function $v(.)$ below.

to show that if group decisions are taken in this way then the usual individual Slutsky symmetry condition which is a necessary condition for individual utility maximisation (and with the addition of negative semidefiniteness of the Slutsky matrix and some regularity conditions also a sufficient condition for utility maximisation) is generalised. The observable analogue to the usual Slutsky matrix (they call this the pseudo-Slutsky matrix) is replaced by the pure Slutsky matrix which abstracts from the effect of prices on preferences (and is symmetric) plus a rank one matrix. With a number of individuals greater than two but less than the number of goods the pseudo-Slutsky matrix is equal to the sum of a symmetric matrix and a number of rank one matrices. In general with H individuals $H-1$ rank one matrices are required. Finally they show that with two individuals restrictions of this type on the pseudo-Slutsky matrix only have bite if there are at least five goods or if the number of individuals is no greater than one less than the number of goods.

Lechene and Preston (2000) take a two individual model where each individual's utility depends on consumption of m goods private to them and on consumption of n public goods. The public goods are purchased on markets in varying quantities by each individual. Each spends his exogenous private income on his private goods and quantities of the public goods so as to maximise his utility taking the purchase of public goods by the other individual as fixed. They then look for a Nash equilibrium in which each individuals choices in public and private goods are a best response to the choices of the other individual within the budget constraint. They go on to consider the case in which each individual has preferences that are separable between the private and public goods and the subutility preferences for the public goods are identical everywhere for the two individuals. Within this framework they derive comparative static restrictions on the demands for each individual and show, in particular, that there is a form of B-C restriction on cross price and income responses. This paper is closest to the approach that we take but we consider more general forms of preferences than they do and more than two individuals. The reason that they restrict individual preferences in this way is that they argue that a Nash equilibrium in which both individuals have positive spending on each public good is only possible if individual preferences over public goods are at least locally identical since they argue that each individual in Nash equilibrium must want the same aggregate purchase by both of them of each public good. However whilst sufficient this is not necessary- take a simple example with a single public good which satisfies separability but which does not have identical preferences. For example let

$$\begin{aligned} u_A(q_A, Q_A + Q_B) &= \sum_i \beta_{Ai} \ln(q_{Ai} - \alpha_{Ai}) + \gamma_A \ln(Q_A + Q_B - \delta_A) \\ u_B(q_B, Q_A + Q_B) &= \sum_i \beta_{Bi} \ln(q_{Bi} - \alpha_{Bi}) + \gamma_B \ln(Q_A + Q_B - \delta_B) \end{aligned}$$

where $\sum \beta_{Ai} + \gamma_A = \sum \beta_{Bi} + \gamma_B = 1$ for individuals A and B and for well defined preferences quantities are restricted so that $(q_{Ai} - \alpha_{Ai}) > 0$; $(q_{Bi} - \alpha_{Bi}) > 0$; $(Q_A + Q_B - \delta_A) > 0$ and $Q_A + Q_B - \delta_B > 0$. Each individual has n private goods and one good Q subject to externalities where it is the total consumption that matters. Each individual maximises their utility within their budget constraint

$$\sum p_i q_{hi} + \pi Q_h \leq m_h$$

and so has reaction curves given by

$$\begin{aligned} q_{hi} &= \alpha_{hi} + \beta_{hi}[m_h - \sum \alpha_{hi} p_i - \pi(\delta_h - Q_k)]/p_i \\ Q_h &= \delta_h - Q_k + \gamma_h[m_h - \sum \alpha_{hi} p_i - \pi(\delta_h - Q_k)]/\pi \end{aligned}$$

The Nash equilibrium is determined just from the last equation for each individual. Rewriting these

$$Q_h = \delta_h + (\gamma_h - 1)Q_k + \gamma_h[m_h - \sum \alpha_{hi}p_i - \pi\delta_h]/\pi$$

and solving these for Q_h

$$Q_h = \frac{\delta_h + \gamma_h[m_h - \sum \alpha_{hi}p_i - \pi\delta_h]/\pi + (\gamma_h - 1)\{\delta_k + \gamma_k[m_k - \sum \alpha_{ki}p_i - \pi\delta_k]/\pi\}}{1 - (\gamma_A - 1)(\gamma_B - 1)}$$

Then the Nash equilibrium total purchase of the public good is

$$\begin{aligned} Q &= \frac{\delta_A\gamma_B(1-\gamma_A) + \delta_B\gamma_A(1-\gamma_B) + \gamma_A\gamma_B\sum m_h/\pi}{1-(\gamma_A-1)(\gamma_B-1)} \\ &\quad - \frac{\gamma_A\gamma_B}{1-(\gamma_A-1)(\gamma_B-1)}(\sum p_i(\alpha_{Ai} + \alpha_{Bi}) \end{aligned} \tag{1}$$

which, as Lechene and Preston say, depends only on total income of the two. The quantity Q in (1) is also identical to that which each individual would choose if they selected their private consumptions and the total public good consumption Q using the combined income of the two individuals net of the cost of buying the other individuals private goods. The point is that there so many arbitrary functions involved (each individuals demand for each private and public good) that requiring them to want the same aggregate of public good spending in Nash equilibrium or that (1) be an interior solution for each individuals public good purchase does not require that they have identical preferences.

2 Two Individuals; One Externality Inducing Good

To clarify the notation in this case let the two individuals be A, B and let good 1 cause the externality so that preferences for the two are given by $u_A(x_{1A}, ..x_{nA}, x_{1B})$, $u_B(x_{1B}, ..x_{nB}, x_{1A})$ where for example x_{iA} denotes the quantity of good i consumed by individual A. The individual best responses or reaction curves are defined by

$$\{X_{1A}(p, m_A, x_{1B})...X_{nA}(p, m_A, x_{1B})\} = \arg\max\{u_A(x_{1A}, ..x_{nA}, x_{1B})|p'x_A \leq m_A, x_{iA} \geq 0\}$$

$$\{X_{1B}(p, m_A, x_{1A})...X_{nB}(p, m_A, x_{1A})\} = \arg\max\{u_B(x_{1B}, ..x_{nB}, x_{1A})|p'x_B \leq m_B, x_{iB} \geq 0\}$$

and a consistent or Nash equilibrium requiring mutual best responses is defined by the system $F_{iA}(p, m_A, m_B), F_{iB}(p, m_A, m_B)$ $i = 1..n$ which solves the equations

$$F_{iA}(p, m_A, m_B) = X_{iA}(p, m_A, F_{1B}(p, m_A, m_B)) \tag{2}$$

$$F_{iB}(p, m_A, m_B) = X_{iB}(p, m_B, F_{1A}(p, m_A, m_B)) \tag{3}$$

This system of equations can be solved in two steps: first solve the two equations

$$F_{1A}(p, m_A, m_B) = X_{1A}(p, m_A, F_{1B}(p, m_A, m_B)) \tag{4}$$

$$F_{1B}(p, m_A, m_B) = X_{1B}(p, m_B, F_{1A}(p, m_A, m_B)) \tag{5}$$

for the Nash equilibrium demands for the first good: $F_{1A}(p, m_A, m_B)$, $F_{1B}(p, m_A, m_B)$ and then substitute these demands into the remaining equations of (2),(3) for $i = 2...n$ to define the Nash equilibrium demands for the remaining goods. It follows that much of the analysis can be undertaken just by looking at the first good.

We assume that individual preferences are such that each wishes to purchase a positive quantity of every good and that the reaction curves are continuously differentiable. Nash equilibria involving corners (where there is zero consumption by some individuals of a good with external effects) are of interest but not central to our concern of comparative statics[3].

In terms of the reaction functions we define

Definition 1 *Strong externalities for any good i to be the case in which*

$$\partial X_{iA}/\partial x_{1B}\partial X_{iB}/\partial x_{1A} > 1$$

Definition 2 *Symmetric externalities for any good i where*

$$sign(\partial X_{iA}/\partial x_{1B}) = sign(\partial X_{iB}/\partial x_{1A})$$

Definition 3 *Positive externalities for any good i to be the case in which*

$$sign(\partial X_{iA}/\partial x_{1B}) > 0; sign(\partial X_{iB}/\partial x_{1A}) > 0$$

negative externalities externalities where

$$sign(\partial X_{iA}/\partial x_{1B}) < 0; sign(\partial X_{iB}/\partial x_{1A}) < 0$$

and mixed externalities where $sign(\partial X_{iA}/\partial x_{1B}) = -sign(\partial X_{iB}/\partial x_{1A})$

[3] To sketch out how to extend our analysis, let

$$\overline{U}_h(x_{1h}, x_{1k}, p_2, ..p_n, m_h - p_1 x_{1h}) = \max_{x_h i>1} \{u_h(x_h, x_{1k}) | \sum_{i=2} p_i x_{ih} \leq m_h - p_1 x_{1h}\}$$

and let $\widetilde{x}_{1k}$ solve $\frac{\partial \overline{U}_h(0, \widetilde{x}_{1k})}{\partial x_{1h}} = 0$ and x^*_{1h} solve $\frac{\partial \overline{U}_h(x_{1h}, 0)}{\partial x_{1h}} = 0$. Here $\widetilde{x}_{1k}$ is the level of $k's$ consumption of good 1 which makes h wish to consume 0 of good 1. And x^*_{1h} is the optimal level of consumption of good 1 for h if $k's$ consumption of good 1 is zero.

The complete analysis is tedious as there are many cases but to illustrate take the case where the reaction curves in good 1 for each individual have negative slope and $\overline{U}_h$ is concave in x_{1h}. The best response for say individual A, x_{1A}, is then $x_{1A} = 0$ for $x_{1B} \geq \widetilde{x}_{1B}$
$x^*_{1A} \geq x_{1A} > 0$ for $0 \leq x_{1B} \leq \widetilde{x}_{1B}$.

We then have four cases:

(1) $x^*_{1A} > \widetilde{x}_{1A}, x^*_{1B} > \widetilde{x}_{1B}$

Here there is an interior Nash equilibrium with both individuals consuming positive quantities of good 1

(2) $x^*_{1A} > \widetilde{x}_{1A}, x^*_{1B} < \widetilde{x}_{1B}$

Here there is a unique Nash equilibrium with $x_{1A} = 0, x_{1B} > 0$

(3) $x^*_{1A} < \widetilde{x}_{1A}, x^*_{1B} > \widetilde{x}_{1B}$

Here the unique Nash equilibrium has $x_{1A} > 0, x_{1B} = 0$

(4) $x^*_{1A} < \widetilde{x}_{1A}, x^*_{1B} < \widetilde{x}_{1B}$

Here there are three Nash equilibria: one at $(0, \widetilde{x}_{1B})$, one at $(\widetilde{x}_{1A}, 0)$ and one with both individuals consuming positive quantities of good 1.

Strong externalities restrict the relative slopes of the reaction curves; Figs 1, 2 show cases of strong and weak externalities where they are positive and Figs 3, 4 show strong and weak externalities where they are negative. If preferences exhibit mixed externalities at any prices and incomes then there is at most one Nash equilibrium, we could ensure there is exactly one by imposing boundary conditions on the reaction curves. When there are symmetric either positive or negative externalities there may be multiple Nash equilibria or a corner Nash equilibrium.

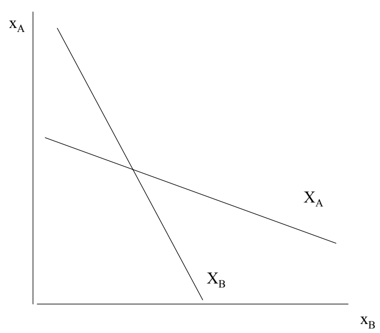

Fig 1.Weak symmetric negative externalities

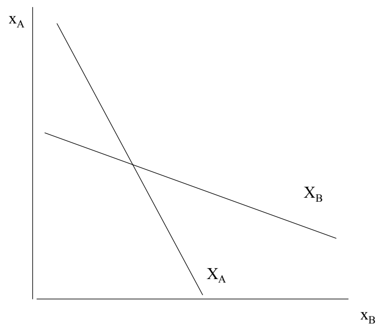

Fig 2 Strong symmetric negative externalities

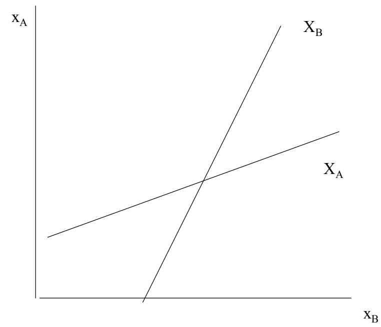

Fig 3. Weak symmetric positive externalities

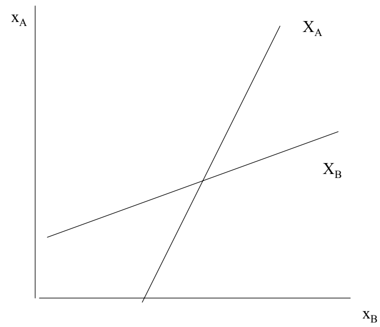

Fig 4. Strong symmetric positive externalities

2.1 Comparative Static Effects of Price and Income Changes

The first point we wish to make is that the interdependence of preferences can reverse the signs of comparative static effects. In general many Nash equilibria may exist and there are many possible configurations. In the present framework if both reaction curves in good 1 have the same slope always or if at least one of the reaction curves for good 1 changes slope, there may be any number (finite or inifinite) of Nash equilibria although generically there is always an odd number of equilibria. But we can make our point in contexts in which there is a single Nash equilibrium. So we assume[4]:

- There is a unique interior Nash Equilibrium for any (p, m_h).

Suppose that there no inferior goods in the system of individual reaction curves so that $\partial X_{ih}/\partial m_h > 0$ for $h = A, B$. Then if the externality is sufficiently strong it can reverse the comparative static effects of price or income changes in the reaction curves. Differentiating (4), (5) gives

$$\frac{\partial F_{iA}}{\partial p_j} = \frac{\partial X_{iA}}{\partial p_j} + \frac{\partial X_{iA}}{\partial x_{1B}} \frac{\partial F_{1B}}{\partial p_j} \tag{6}$$

$$\frac{\partial F_{iA}}{\partial m_A} = \frac{\partial X_{iA}}{\partial m_A} + \frac{\partial X_{iA}}{\partial x_{1B}} \frac{\partial F_{1B}}{\partial m_A} \tag{7}$$

$$\frac{\partial F_{iA}}{\partial m_B} = \frac{\partial X_{iA}}{\partial x_{1B}} \frac{\partial F_{1B}}{\partial m_B} \tag{8}$$

$$\frac{\partial F_{iB}}{\partial p_j} = \frac{\partial X_{iB}}{\partial p_j} + \frac{\partial X_{iB}}{\partial x_{1A}} \frac{\partial F_{1A}}{\partial p_j} \tag{9}$$

$$\frac{\partial F_{iB}}{\partial m_B} = \frac{\partial X_{iB}}{\partial m_B} + \frac{\partial X_{iB}}{\partial x_{1A}} \frac{\partial F_{1A}}{\partial m_B} \tag{10}$$

[4] With several Nash equilibria the idea of comparative statics of the Nash equilibrium becomes quite ambiguous-which equilibrium do we compare with which before and after a price or income change?

$$\frac{\partial F_{iB}}{\partial m_A} = \frac{\partial X_{iB}}{\partial x_{1A}} \frac{\partial F_{1A}}{\partial m_A} \tag{11}$$

Using (6) and (9) for $i = 1$ we derive

$$\frac{\partial F_{1A}}{\partial p_j} = \left[\frac{\partial X_{1A}}{\partial p_j} + \frac{\partial X_{1A}}{\partial x_{1B}} \frac{\partial X_{1B}}{\partial p_j} \right] / \left[1 - \frac{\partial X_{1A}}{\partial x_{1B}} \frac{\partial X_{1B}}{\partial x_{1A}} \right] \tag{12}$$

$$\frac{\partial F_{1B}}{\partial p_j} = \left[\frac{\partial X_{1B}}{\partial p_j} + \frac{\partial X_{1B}}{\partial x_{1A}} \frac{\partial X_{1A}}{\partial p_j} \right] / \left[1 - \frac{\partial X_{1A}}{\partial x_{1B}} \frac{\partial X_{1B}}{\partial x_{1A}} \right] \tag{13}$$

so long as the denominator is non-zero. If we take $j = 1$ we know that since there are no inferior goods, $\partial X_{1h}/\partial p_1 < 0$. The sign of $\partial F_{1h}/\partial p_1$ is then dictated by the strength and the sign of the marginal externality effects: let $D = 1 - \partial X_{1A}/\partial x_{1B} \cdot \partial X_{1B}/\partial x_{1A}$. If $D > 0$ so that external effects are weak then when external effects are positive the sign of $\partial F_{1h}/\partial p_1 < 0$ and the comparative static effects are preserved despite the externality. But if external effects are strong ($D < 0$) and positive then the first good appears as a Giffen good ($\partial F_{1h}/\partial p_1 > 0$). The intuition is that with strong positive external effects if the price of good 1 rises there is a gain from increasing the purchase of good 1 because this leads the other individual to increase their purchase of good 1 which more than compensates for the reduction on spending on other goods. This gives us

Proposition 1 *(i) if $D > 0, \partial X_{1h}/\partial x_{1k} > 0$ then $\partial F_{1h}/\partial p_1 < 0$*

(ii) if $D < 0, \partial X_{1h}/\partial x_{1k} > 0$ then $\partial F_{1h}/\partial p_1 > 0$

(iii) if $\partial X_{1h}/\partial x_{1k} < 0$ then $\partial F_{1h}/\partial p_1$ is of ambiguous sign with either strong or weak externalities.

The sign of D is given by the asymmetry/symmetry and strength of the external effects between the two individuals. If the externality is one-way so that either $\partial X_{1A}/\partial x_{1B} = 0$ or $\partial X_{1B}/\partial x_{1A} = 0$ then $D = 1$ and unless there is a strong symmetric externality, $D > 0$. If $D > 0$ then the signs of comparative statics are preserved if the externality is positive. If the externality is asymmetric so that $\partial X_{1A}/\partial x_{1B} \cdot \partial X_{1B}/\partial x_{1A} < 0$ then $D > 0$. Thus to reverse the comparative statics of price changes for good 1 requires either a strong positive externality or strong symmetric externalities between the individuals.

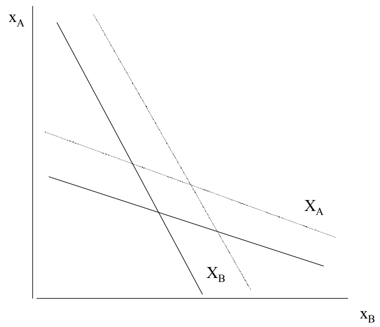

Fig 5. Weak or strong negative externalities; a fall in p_1

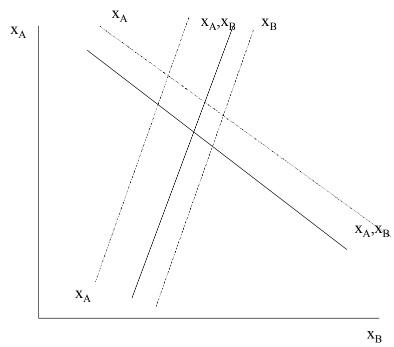

Fig 6. Asymmetric externalities; a fall in p_1

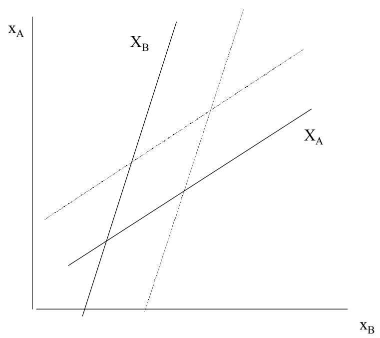

Fig 7. Weak positive externalities: a fall in p_1

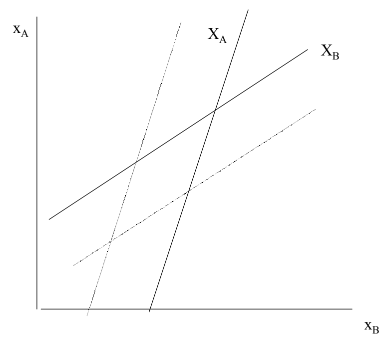

Fig. 8. Strong Positive externalities: a fall in p_1

We can see this geometrically. Since the price change affects both individuals, both reaction curves shift. When both reaction curves have positive slope but externalities are weak, a fall in p_1 by itself raises consumption of good 1 by each individual; this consumption increase further raises the consumption of each individual through the external effect and so that result is unambiguous: for each individual $\partial F_{ih}/\partial p_1 < 0$ when the reaction curves both have positive slope. But if externalities are positive and strong then the fall in p_1 raises individual B's consumption of good 1 by so much that in Nash equilibrium individual A reduces his consumption of good 1. On the other hand if either the reaction curves both have negative slope, or have differing slopes then the price fall in good 1 serves to raise the consumption for each individual of that good but for at least one individual the externality effect is then working in a negative direction so that the overall effect for the individual of the price fall is ambiguous: for at least one h in these cases $\partial F_{ih}/\partial p_1$ is of ambiguous sign.

For $j > 1$ similar forces are at work. The sign of $\partial F_{1h}/\partial p_j$ may be opposite to that of $\partial X_{1h}/\partial p_j$. For example suppose that for both individuals goods 1 and j are either complements or substitutes. Then if $D < 0$ and marginal external effects are positive in the Nash equilibrium, goods 1 and j appear with the opposite relationship - substitutes become complements and complements become substitutes.

Proposition 2 *For $j > 1$ in Nash equilibrium goods 1 and j are complements if externalities are strong and positive and goods j and 1 are naturally substitutes.*

For $i > 1$ we can use (6), (9), (13) and (12) to deduce that

$$\frac{\partial F_{iA}}{\partial p_j} = \frac{\partial X_{iA}}{\partial p_j} + \frac{\partial X_{iA}}{\partial x_{1B}} \left[\frac{\partial X_{1B}}{\partial p_j} + \frac{\partial X_{1B}}{\partial x_{1A}} \frac{\partial X_{1A}}{\partial p_j} \right] / \left[1 - \frac{\partial X_{1A}}{\partial x_{1B}} \frac{\partial X_{1B}}{\partial x_{1A}} \right] \qquad (14)$$

so that for example if i and j are substitutes in the reaction curve of A, goods 1 and j are complements for both A and B, external effects are strong and positive, then i and j are substitutes for A in the Nash equilibrium. On the other hand if all these goods $i, j, 1$ are complements for both A and B and there are strong positive externalities, then it is possible that in the Nash equilibrium goods i and j are substitutes for A.

For income changes the situation is simpler. From (7) and (11) for $i = 1$

$$\frac{\partial F_{1A}}{\partial m_A} = \frac{\partial X_{1A}}{\partial m_A} / D \tag{15}$$

$$\frac{\partial F_{1A}}{\partial m_B} = \frac{\partial X_{1B}}{\partial m_B} / D \tag{16}$$

So for good 1 we see that it will appear inferior if external effects are strong. From (8), and (11) we also get the sign of the effect of the other individuals income on the demand for good 1. It is given by the sign of the marginal external effect.

Proposition 3 *Good* 1 *is inferior in the Nash equilibrium if external effects are strong*

For $i > 1$ we have

$$\begin{aligned} \frac{\partial F_{iA}}{\partial m_A} &= \frac{\partial X_{iA}}{\partial m_A} + \frac{\partial X_{iA}}{\partial x_{1B}} \frac{\partial F_{1B}}{\partial m_A} & (17) \\ &= \frac{\partial X_{iA}}{\partial m_A} + \frac{\partial X_{iA}}{\partial x_{1B}} \frac{\partial X_{1A}}{\partial m_A} / D & (18) \end{aligned}$$

so that if external effects are positive but weak then good i is normal despite the externality. But if either external effects are strong and positive or are weak but negative and the marginal propensity to consume good 1 is high then good $i > 1$ may appear as inferior in the Nash equilibrium.

Proposition 4 *Good* $i > 1$ *can be inferior in the Nash equilibrium if*

(i) external effects are strong and positive

or if

(ii) external effects are weak but negative and the marginal propensity to consume good 1 *is high*

(i) is especially interesting: the externality itself is positive in the sense that taken on its own in the reaction curves an increase in B's consumption of a good leads A to increase his consumption of the good. So although each individual benefits from the consumption of the other, the fact that in equilibrium individuals must be making mutual best responses means that we may observe an increase in A's income causing a decrease of A's consumption of good 1.

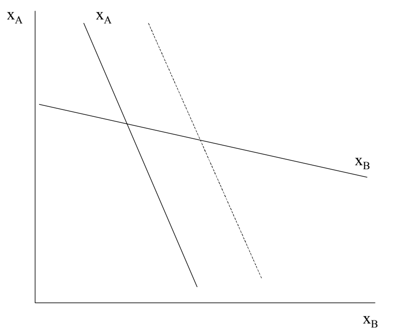

Fig 9. an increase in m_A

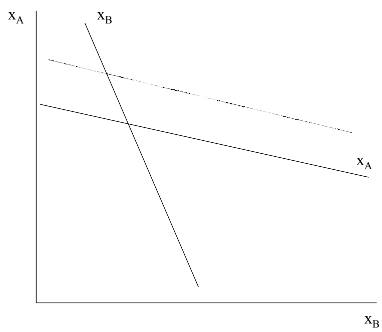

Fig 10. an increase in m_A

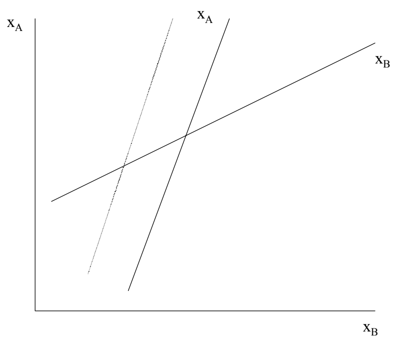

Fig 11 an increase in m_A

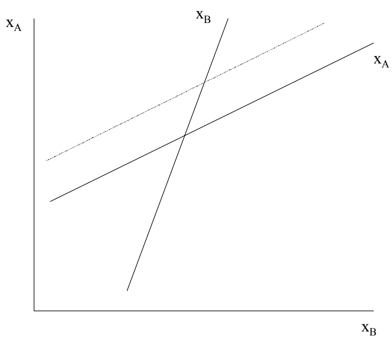

Fig 12 an increase in m_A

As always fundamental properties stemming from the individual budget constraints are preserved in the Nash equilibrium. Thus

Proposition 5 *(a) $X_{ih}(.)$ and $F_{ih}(.)$ are each homogeneous of degree zero in p and the relevant incomes* *(b)* $\sum p_i F_{ih} \equiv m_h \equiv \sum p_i X_{ih}$

from which Engel aggregation properties such as

$$\sum p_i \partial F_{ih}/\partial m_h \equiv 1 \equiv \sum p_i \partial X_{ih}/\partial m_h$$

follow as do conditions like

$$\sum p_i \partial F_{ih}/\partial m_k \equiv 0 \equiv \sum p_i \partial X_{ih}/\partial x_{1k}$$

2.2 Symmetry Conditions

We can use (4), (5) to develop a symmetry restriction in this case. We know that

$$\frac{\partial X_{ih}}{\partial p_j} + X_{jh}\frac{\partial X_{ih}}{\partial m_h} \tag{19}$$

forms a symmetric and negative semidefinite matrix. But if decisions are taken as we hypothesise and the market is always in Nash equilibrium then econometrically all we can estimate are the functions $F_{ih}(p, m_A, m_B)$ and so we wish to express the usual Slutsky matrix in (19) in terms of derivatives of $F_{ih}()$. Differentiating (4),(5) gives

$$\frac{\partial F_{iA}}{\partial p_j} = \frac{\partial X_{iA}}{\partial p_j} + \frac{\partial X_{iA}}{\partial x_{1B}}\frac{\partial F_{1B}}{\partial p_j} \tag{20}$$

$$\frac{\partial F_{iA}}{\partial m_A} = \frac{\partial X_{iA}}{\partial m_A} + \frac{\partial X_{iA}}{\partial x_{1B}}\frac{\partial F_{1B}}{\partial m_A} \tag{21}$$

$$\frac{\partial F_{iA}}{\partial m_B} = \frac{\partial X_{iA}}{\partial x_{1B}}\frac{\partial F_{1B}}{\partial m_B} \tag{22}$$

From (22) we can solve for the externality derivative

$$\frac{\partial X_{iA}}{\partial x_{1B}} = \frac{\partial F_{iA}}{\partial m_B} \Big/ \frac{\partial F_{1B}}{\partial m_B}$$

In a similar way

$$\frac{\partial X_{iB}}{\partial x_{1A}} = \frac{\partial F_{iB}}{\partial m_A} / \frac{\partial F_{1A}}{\partial m_A}$$

However from (20), (21)

$$\begin{aligned} \frac{\partial X_{iA}}{\partial p_j} + x_{jA}\frac{\partial X_{iA}}{\partial m_A} &= \frac{\partial F_{iA}}{\partial p_j} - \frac{\partial X_{iA}}{\partial x_{1B}}\frac{\partial F_{1B}}{\partial p_j} + x_{jA}(\frac{\partial F_{iA}}{\partial m_A} - \frac{\partial X_{iA}}{\partial x_{1B}}\frac{\partial F_{1B}}{\partial m_A}) \\ &= \frac{\partial F_{iA}}{\partial p_j} + x_{jA}\frac{\partial F_{iA}}{\partial m_A} - (\frac{\partial F_{iA}}{\partial m_B}/\frac{\partial F_{iB}}{\partial m_B})(\frac{\partial F_{1B}}{\partial p_j} + x_{jA}\frac{\partial F_{1B}}{\partial m_A}) \end{aligned}$$

This gives us

Proposition 6 *With only two individuals,* A, B*, and only good* 1 *having an externality effect*

$$\frac{\partial F_{iA}}{\partial p_j} + x_{jA}\frac{\partial F_{iA}}{\partial m_A} - (\frac{\partial F_{iA}}{\partial m_B}/\frac{\partial F_{iB}}{\partial m_B})(\frac{\partial F_{1B}}{\partial p_j} + x_{jA}\frac{\partial F_{1B}}{\partial m_A})$$

$$\frac{\partial F_{iB}}{\partial p_j} + x_{jB}\frac{\partial F_{iB}}{\partial m_B} - (\frac{\partial F_{iB}}{\partial m_A}/\frac{\partial F_{iA}}{\partial m_A})(\frac{\partial F_{1A}}{\partial p_j} + x_{jB}\frac{\partial F_{1A}}{\partial m_B})$$

are symmetric in i *and* j

and each of these forms a negative semidefinite matrix.

Note that if there is no externality effect so that $\frac{\partial F_{iA}}{\partial m_B} = \frac{\partial F_{iB}}{\partial m_A} = 0$ *this reduces to symmetry of the regular Slutsky matrix.*

We can relate this condition to the Browning-Chiappori rank result. For example the pseudo-Slutsky matrix for B has the form

$$\begin{aligned} \left[\frac{\partial F_{iB}}{\partial p_j} + x_{jB}\frac{\partial F_{iB}}{\partial m_B}\right] &= [S_{ij}] + \left[\frac{\partial F_{iB}}{\partial m_A}/\frac{\partial F_{iA}}{\partial m_A}\right]\left[\frac{\partial F_{1A}}{\partial p_j} + x_{jB}\frac{\partial F_{1B}}{\partial m_A}\right] && (23) \\ &= [S_{ij}] + [A_i][B_j] && (24) \end{aligned}$$

where $S_{ij} = S_{ji}$. and the product $[A_i]\,[B_j]$ forms a rank one matrix. Taking the difference between i, j and j, i gives a rank two matrix in general:

$$\left[\frac{\partial F_{iB}}{\partial p_j} + x_{jB}\frac{\partial F_{iB}}{\partial m_B}\right] - \left[\frac{\partial F_{jB}}{\partial p_i} + x_{iB}\frac{\partial F_{jB}}{\partial m_B}\right] = A_i B_j - A_j B_i \qquad (25)$$

Since the terms in this equation are observable in principle then the restriction is testable in the same way as in Browning-Chiappori. The sign restriction on the matrix in Proposition 6 tells us that externality corrected "pseudo-compensated" demands slope downwards.

3 Two Individuals and General Externalities

In this section we generalise the preceding results to the case in which all goods may have an externality inducing effect. Here A and B have preferences given by $u_h(x_A, x_B), h = A, B$ where x_h is an n vector. The links between the reaction curves and the equilibrium demands become

$$F_{iA}(p, m_A, m_B) = X_{iA}(p, m_A, F_{1B}(p, m_A, m_B), ..F_{nB}(p, m_A, m_B)) \qquad (26)$$

$$F_{iB}(p, m_A, m_B) = X_{iB}(p, m_B, F_{1A}(p, m_A, m_B), ..F_{nA}(p, m_A, m_B)) \qquad (27)$$

3.1 Comparative Statics of Price and Income Changes

The logic of calculating the slopes of Nash equilibrium demands or Engel curves is preserved. From (26),(27)

$$\frac{\partial F_{iA}}{\partial p_j} = \frac{\partial X_{iA}}{\partial p_j} + \sum_k \frac{\partial X_{iA}}{\partial F_{kB}}\frac{\partial F_{kB}}{\partial p_j} \qquad (28)$$

$$\frac{\partial F_{iB}}{\partial p_j} = \frac{\partial X_{iB}}{\partial p_j} + \sum_k \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial F_{kA}}{\partial p_j} \qquad (29)$$

which in matrix form is

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ --- \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_{iA}}{\partial p_j} \\ --- \\ \frac{\partial X_{iB}}{\partial p_j} \end{bmatrix} + \begin{bmatrix} 0 & | & \frac{\partial X_{iA}}{\partial F_{kB}} \\ --- & --- & --- \\ \frac{\partial X_{iB}}{\partial F_{kA}} & | & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ --- \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix}$$

where

$$\left[\frac{\partial F_{iA}}{\partial p_j}\right] = \begin{bmatrix} \frac{\partial F_{1A}}{\partial p_1} & . & . & \frac{\partial F_{1A}}{\partial p_n} \\ . & & & . \\ . & & & . \\ \frac{\partial F_{nA}}{\partial p_1} & . & . & \frac{\partial F_{nA}}{\partial p_n} \end{bmatrix} \text{ and similarly for } \left[\frac{\partial F_{iB}}{\partial p_j}\right], \left[\frac{\partial X_{iA}}{\partial p_j}\right], \left[\frac{\partial X_{iB}}{\partial p_j}\right]$$

and

$$\left[\frac{\partial X_{iA}}{\partial F_{kB}}\right] = \begin{bmatrix} \frac{\partial X_{1A}}{\partial F_{1B}} & . & . & \frac{\partial X_{1A}}{\partial F_{nB}} \\ . & & & . \\ . & & & . \\ \frac{\partial X_{nA}}{\partial F_{1B}} & . & . & \frac{\partial X_{nA}}{\partial F_{nB}} \end{bmatrix} \text{ and similarly for } \left[\frac{\partial X_{iB}}{\partial F_{kA}}\right].$$

In the same way

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial m_A} \\ --- \\ \frac{\partial F_{iB}}{\partial m_A} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_{iA}}{\partial m_A} \\ --- \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & | & \frac{\partial X_{iA}}{\partial F_{kB}} \\ --- & --- & --- \\ \frac{\partial X_{iB}}{\partial F_{kA}} & | & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F_{iA}}{\partial m_A} \\ --- \\ \frac{\partial F_{iB}}{\partial m_A} \end{bmatrix}$$

where $\left[\frac{\partial F_{iA}}{\partial m_A}\right] = \begin{bmatrix} \frac{\partial F_{1A}}{\partial m_A} \\ . \\ . \\ \frac{\partial F_{nA}}{\partial m_A} \end{bmatrix}$.

This gives us

Proposition 7 *So long as the matrices* $\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]$ *and* $\left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]$ *are each nonsingular,*

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ --- \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix} = A \begin{bmatrix} \frac{\partial X_{iA}}{\partial p_j} \\ --- \\ \frac{\partial X_{iB}}{\partial p_j} \end{bmatrix} \text{ where}$$

$$A = \begin{bmatrix} I + \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right) & | & \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1} \\ -------------------- & -- & -------------------- \\ \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)[I - \frac{\partial X_{iA}}{\partial F_{kB}}\frac{\partial X_{iB}}{\partial F_{kA}}]^{-1} & | & I + \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)[I - \frac{\partial X_{iA}}{\partial F_{kB}}\frac{\partial X_{iB}}{\partial F_{kA}}]^{-1}\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right) \end{bmatrix} \tag{30}$$

and

$$\left[\begin{array}{c} \frac{\partial F_{iA}}{\partial m_A} \\ --- \\ \frac{\partial F_{iB}}{\partial m_A} \end{array}\right] = \left[\begin{array}{c} I + \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)]^{-1}\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial m_A}\right) \\ ------------------- \\ \left[\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]^{-1}\left(\frac{\partial X_{iA}}{\partial m_A}\right) \end{array}\right] \tag{31}$$

As in the case with a single good causing the externality, the Nash equilibrium comparative static effects of price changes are combinations of changes coming from every reaction curve and the externality effects. It is then possible for signs of changes to be reversed between the reaction curve and the Nash equilibrium demand. In (30) there are two sorts of feedback effects of a price change working through the externalities. Firstly the price change causes A to change consumption of all goods which in itself shifts $A's$ reaction curve. Secondly the price change leads B to change consumption of all goods which then leads to a further change by A. Either of these effects can result in a sign reversal eg we may have $\partial F_{iA}/\partial p_j > 0$ but $\partial X_{iA}/\partial p_j < 0$ if for example terms in $\left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]^{-1}$ are predominantly negative or if terms in $\left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1}$ tend to be of opposite sign to those in $\partial X_{iB}/\partial p_j$. It is perhaps easier to see this in the marginal propensities to consume in (31).

3.2 Symmetry Conditions With Two Individuals and General Externalities

One approach to exploring symmetry restrictions would be to use (28),(29) directly. Define Slutsky and pseudo-Slutsky functions

$$S_{ij}^h = \frac{\partial X_{ih}}{\partial p_j} + x_{jh}\frac{\partial X_{ih}}{\partial m_h}$$

$$\Sigma_{ij}^{hh} = \frac{\partial F_{ih}}{\partial p_j} + x_{jh}\frac{\partial F_{ih}}{\partial m_h}$$

$$\Sigma_{ij}^{hk} = \frac{\partial F_{ih}}{\partial p_j} + x_{jk} \frac{\partial F_{ih}}{\partial m_k}$$

Using these expressions we deduce

Proposition 8

$$\left[\frac{\partial F_{iA}}{\partial p_j}\right] + \left[\frac{\partial F_{iA}}{\partial m_A}\right] F_A^{'} + \left[\frac{\partial F_{iA}}{\partial m_B}\right] F_B^{'} =$$
$$= \left[I + \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right) [I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right) \frac{\partial X_{iA}}{\partial F_{kB}}]^{-1} \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right] [S_{ij}^A]$$
$$+ \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right) \left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right) \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]^{-1} [S_{ij}^B]$$

Thus with two individuals and general externalities, there must be two "arbitrary" matrices

$$\left[I + \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right) [I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right) \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)]^{-1} \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]$$

and

$$\left[\frac{\partial X_{iB}}{\partial F_{kA}}\right] \left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right) \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]^{-1}$$

which make the LHS observable matrix a combination of two symmetric matrices. However for empirical application this is cumbersome, although theoretically it is attractive because it stresses the importance of both individual's incomes in "compensating" one individual.

A more useful empirical approach comes from the equations[5]

$$S_{ij}^A = \Sigma_{ij}^{AA} - \sum_k \frac{\partial X_{iA}}{\partial x_{kB}} \Sigma_{kj}^{BA} \tag{32}$$

5

$$\frac{\partial F_{iA}}{\partial p_j} = \frac{\partial X_{iA}}{\partial p_j} + \sum_k \frac{\partial X_{iA}}{\partial x_{kB}} \frac{\partial F_{kB}}{\partial p_j}$$

$$\frac{\partial F_{iA}}{\partial m_A} = \frac{\partial X_{iA}}{\partial m_A} + \sum_k \frac{\partial X_{iA}}{\partial x_{kB}} \frac{\partial F_{kB}}{\partial m_A}$$

from which it follows.

$$S^B_{ij} = \Sigma^{BB}_{ij} - \sum_k \frac{\partial X_{iB}}{\partial x_{kA}} \Sigma^{AB}_{kj} \tag{33}$$

On the other hand we have

$$\frac{\partial F_{iA}}{\partial m_B} = \sum_k \frac{\partial X_{iA}}{\partial x_{kB}} \frac{\partial F_{kB}}{\partial m_B} \tag{34}$$

and

$$\frac{\partial F_{iB}}{\partial m_A} = \sum_k \frac{\partial X_{iB}}{\partial x_{kA}} \frac{\partial F_{kA}}{\partial m_A} \tag{35}$$

We can use (34),(35) to solve for one of the marginal externality effects in terms of the other and hence derive a symmetry result.

Proposition 9 *Consumer demands are consistent with general externalities in a 2 person world if there are* $2(n-1)$ *functions* $\partial X_{iA}/\partial x_{kB}, \partial X_{iB}/\partial x_{kA}$ *which make*

$$\frac{\partial F_{1B}}{\partial m_B} \Sigma^{AA}_{ij} - \Sigma^{BA}_{1j} \frac{\partial F_{iA}}{\partial m_B} - \sum_{k=2} \frac{\partial X_{iA}}{\partial x_{kB}} [\frac{\partial F_{1B}}{\partial m_B} \Sigma^{BA}_{kj} - \Sigma^{BA}_{1j} \frac{\partial F_{kB}}{\partial m_B}]$$

$$\frac{\partial F_{1A}}{\partial m_A} \Sigma^{BB}_{ij} - \Sigma^{AB}_{1j} \frac{\partial F_{iB}}{\partial m_A} - \sum_{k=2} \frac{\partial X_{iB}}{\partial x_{kA}} [\frac{\partial F_{1A}}{\partial m_A} \Sigma^{AB}_{kj} - \Sigma^{AB}_{1j} \frac{\partial F_{kA}}{\partial m_A}]$$

each be symmetric and negative semidefinite matrices.

This proposition bears a family resemblance to the Browning-Chiappori result but there are two major differences. Here the factors refer to marginal externality effects across goods and individuals whereas in Browning-Chiappori the arbitrary factors refer to effects of prices on preferences. However we have $2(n-1)$ rank one matrices instead of just two since each pair of goods are connected by an externality which generates one channel by which there are additional price-income effects on demand. Secondly we have differences between the Slutsky term for goods i and j and the Slutsky term for goods i and 1 appearing.

3.3 An Externality Aggregate

A special case of general externalities in the two individual world is that in which each individuals utility depends on their own consumption and on some aggregate function of the consumption of the other individual. Thus $U_h = u_h(x_h, V(x_k))$ for $h = A, B, k \neq h$. An even more special case of this would be one-way externalities where say only individual A has an externality and he is affected by the level of utility of individual B either through a caring relation or through envy. In this case the externality function for individual A is just the level of utility of individual B and B has no externality function. In the case of general one way externalities so that only A has externality effects in their utility function, Proposition 7 would simplify to

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ --- \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix} = \begin{bmatrix} I & | & \frac{\partial X_{iA}}{\partial F_{kB}} \\ --- & --- & --- \\ 0 & | & I \end{bmatrix} \begin{bmatrix} \frac{\partial X_{iA}}{\partial p_j} \\ --- \\ \frac{\partial X_{iB}}{\partial p_j} \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial m_A} \\ --- \\ \frac{\partial F_{iA}}{\partial m_B} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_{iA}}{\partial m_A} \\ --- \\ \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\left(\frac{\partial X_{iB}}{\partial m_B}\right) \end{bmatrix} \tag{37}$$

and if the externality in A's utility comes through a single aggregate function $V(.)$ then $\left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]$ is a rank one matrix $\left[\frac{\partial X_{iA}}{\partial V}\right]\left[\frac{\partial V}{\partial F_{kB}}\right]'$.

With just a one way externality from B to A the symmetry condition of Proposition 9 for individual A reduces to

$$\frac{\partial F_{1B}}{\partial m_B}\Sigma_{ij}^{AA} - \frac{\partial F_{1B}}{\partial p_j}\frac{\partial F_{iA}}{\partial m_B} - \sum_{k=2}\frac{\partial X_{iA}}{\partial x_{kB}}\left[\frac{\partial F_{1B}}{\partial m_B}\frac{\partial F_{kB}}{\partial p_j} - \frac{\partial F_{1B}}{\partial p_j}\frac{\partial F_{kB}}{\partial m_B}\right]$$

must be a symmetric and negative semidefinite matrix, while for individual B we just recover the standard conditions that

$$\frac{\partial F_{1A}}{\partial m_A}S_{ij}^{B} = \frac{\partial F_{1A}}{\partial m_A}\Sigma_{ij}^{BB}$$

must be a symmetric and negative semidefinite matrix. When B has no external effects in his utility, his Slutsky and pseudoSlutsky matrices coincide and so the symmetry condition for B is just that of the regular theory. It is obvious but still interesting that with one way externalities, those individuals without externalities will obey traditional theory-their best response is unique whatever the action of others. It is as if their reaction curve is vertical. But those individuals affected by externalities will capture all of the effects of the externalities.

4 H individuals and a single externality good

Here the situation is analogous to that with two individuals. In Nash equilibrium we have nH equations:

$$F_{ih}(p, m_A, ...m_H) = X_{iA}(p, m_h, F_{1-h}(p, m_A, ...m_H)) \tag{38}$$

where $F_{1-h}(p, m_A, ...m_H)$ represents the $H-1$ list of equilibrium demands for good 1 of all individuals other than h.

This system of equations can be solved in two steps: first solve the H equations involving good 1

$$F_{1h}(p, m_A, ...m_H) = X_{1A}(p, m_h, F_{1-h}(p, m_A, ...m_H)) \tag{39}$$

for the functions $F_{1h}()$. Substituting these into (38) gives $F_{ih}()$ for $i > 1$.

Differentiating (38) gives

$$\frac{\partial F_{i\eta}}{\partial p_j} = \frac{\partial X_{i\eta}}{\partial p_j} + \sum_{h \neq \eta} \frac{\partial X_{i\eta}}{\partial x_{1h}} \frac{\partial F_{1h}}{\partial p_j} \tag{40}$$

$$\frac{\partial F_{i\eta}}{\partial m_\eta} = \frac{\partial X_{i\eta}}{\partial m_\eta} + \sum_{h \neq \eta} \frac{\partial X_{i\eta}}{\partial x_{1h}} \frac{\partial F_{1h}}{\partial m_\eta} \tag{41}$$

$$\frac{\partial F_{i\eta}}{\partial m_k} = \sum_{h \neq \eta} \frac{\partial X_{i\eta}}{\partial x_{1h}} \frac{\partial F_{1h}}{\partial m_k} \tag{42}$$

4.1 Comparative Static Effects of Income and Price Changes

Writing these in matrix notation for fixed i and j we have

$$\left[\frac{\partial F_{i\eta}}{\partial p_j}\right] = \left[\frac{\partial X_{i\eta}}{\partial p_j}\right] + \left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right]\left[\frac{\partial F_{1\eta}}{\partial p_j}\right]$$

where $\left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right] = \begin{bmatrix} 0 & \frac{\partial X_{i1}}{\partial x_{12}} & . & . & \frac{\partial X_{i1}}{\partial x_{1H}} \\ \frac{\partial X_{i2}}{\partial x_{11}} & 0 & & & . \\ . & & . & & . \\ . & & & . & . \\ \frac{\partial X_{in}}{\partial x_{11}} & & & & 0 \end{bmatrix}$, $\left[\frac{\partial F_{i\eta}}{\partial p_j}\right] = \begin{bmatrix} \frac{\partial F_{i1}}{\partial p_j} \\ \frac{\partial F_{i2}}{\partial p_j} \\ . \\ . \\ \frac{\partial F_{iH}}{\partial p_j} \end{bmatrix}$, $\left[\frac{\partial X_{i\eta}}{\partial p_j}\right] = \begin{bmatrix} \frac{\partial X_{i1}}{\partial p_j} \\ \frac{\partial X_{i2}}{\partial p_j} \\ . \\ . \\ \frac{\partial X_{iH}}{\partial p_j} \end{bmatrix}$,

$$\left[\frac{\partial F_{1\eta}}{\partial p_j}\right] = \begin{bmatrix} \frac{\partial F_{11}}{\partial p_j} \\ \frac{\partial F_{12}}{\partial p_j} \\ . \\ . \\ \frac{\partial F_{1H}}{\partial p_j} \end{bmatrix}$$

Similarly for income effects we have

$$\left[\frac{\partial F_{i\eta}}{\partial m_h}\right] = \left[\frac{\partial X_{i\eta}}{\partial m_h}\right] + \left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right]\left[\frac{\partial F_{1\eta}}{\partial m_h}\right]$$

where now $\left[\frac{\partial F_{i\eta}}{\partial m_h}\right] = \begin{bmatrix} \frac{\partial F_{i1}}{\partial m_1} & . & . & \frac{\partial F_{i\eta}}{\partial m_H} \\ . & & & . \\ . & & & . \\ \frac{\partial F_{iH}}{\partial m_1} & . & . & \frac{\partial F_{iH}}{\partial m_H} \end{bmatrix}$, $\left[\frac{\partial X_{i\eta}}{\partial m_h}\right] = \begin{bmatrix} \frac{\partial X_{i1}}{\partial m_1} & 0 & . & 0 & \\ . & & . & & . \\ . & & & . & . \\ 0 & & . & 0 & \frac{\partial X_{iH}}{\partial m_H} \end{bmatrix}$

Using the equations for the first commodity ($i = 1$) and solving for the effects of price and income changes gives

Proposition 10 *With H individuals and a single externality good so long as $\left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]$ is nonsingular*

$$\left[\frac{\partial F_{i\eta}}{\partial p_j}\right] = \left[\frac{\partial X_{i\eta}}{\partial p_j}\right] + \left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right]\left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1}\left[\frac{\partial X_{1\eta}}{\partial p_j}\right]$$

$$\left[\frac{\partial F_{1\eta}}{\partial m_\eta}\right]=\left[I-\left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1}\left[\frac{\partial X_{1\eta}}{\partial m_\eta}\right]$$

$$\left[\frac{\partial F_{i\eta}}{\partial m_h}\right]=\left[\frac{\partial X_{i\eta}}{\partial m_h}\right]+\left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right]\left[I-\left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1}\left[\frac{\partial X_{1\eta}}{\partial m_h}\right]$$

Proposition 10 represents a matrix generalisation of (12),(13),(15) and (16) and has the same interpretation as those equations. In Nash equilibrium the marginal effect of one individuals income on the quantity demanded of good i by another individual depends on the feedback effects of that individual's income via changes in the demand for good 1 by all other individuals and similarly for prices. Hence again strong externalities can reverse the sign of comparative static effects.

4.2 Symmetry Conditions for H Individuals and a Single Externality Good

In Nash equilibrium the own income effect has the form $\frac{\partial F_{ih}}{\partial m_h}=\frac{\partial X_{ih}}{\partial m_h}+\sum_{k\neq h}\frac{\partial X_{ih}}{\partial x_{1k}}a_{kh}\frac{\partial X_{1h}}{\partial m_h}$ where a_{kh} is the khth element of $\left[I-\left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1}$. We can use this to expess the pseudo Slutsky matrix in terms of Slutsky matrices similarly to Proposition 8. So

$$\frac{\partial F_{ih}}{\partial p_j}+\frac{\partial F_{ih}}{\partial m_h}F_{jh}=\frac{\partial X_{ih}}{\partial p_j}+\frac{\partial X_{ih}}{\partial m_h}F_{jh}+\sum_{k\neq h}\frac{\partial X_{ih}}{\partial x_{1k}}a_{kh}\left[\frac{\partial X_{1h}}{\partial p_j}+\frac{\partial X_{1h}}{\partial m_h}F_{jh}\right] \tag{43}$$

which is the sum of a symmetric matrix and $H-1$ terms each of which is of rank 1. This gives us

Proposition 11 $\sum_{ij}^h=S_{ij}^h+\sum_{k\neq h}\frac{\partial X_{ih}}{\partial x_{1k}}a_{kh}S_{1j}^h$

However to derive testable restrictions it is more useful to express the Slutsky matrix in terms of the pseudo-Slutsky matrix. We want to use (42) to solve for the marginal externality effects: the unknowns are the $n(H-1)^2$ terms $\partial X_{i\eta}/\partial x_{1h}$. In fact the equations can be solved in blocks of $H-1$ equations that involve the $H-1$ terms $\partial X_{1\eta}/\partial x_{1h}$ for fixed values of η and i as k varies through its possible $H-1$ values.

For example with three individuals A, B, C we have

$$\begin{aligned}
\begin{bmatrix} \partial X_{iA}/\partial x_{1B} \\ \partial X_{iA}/\partial x_{1C} \end{bmatrix} &= \begin{bmatrix} \partial F_{1B}/\partial m_B & \partial F_{1C}/\partial m_B \\ \partial F_{1B}/\partial m_C & \partial F_{1C}/\partial m_C \end{bmatrix}^{-1} \cdot \begin{bmatrix} \partial F_{iA}/\partial m_B \\ \partial F_{iA}/\partial m_C \end{bmatrix} \qquad (44) \\
&= \begin{bmatrix} \partial F_{iA}/\partial m_B \cdot \partial F_{1C}/\partial m_C - \partial F_{1C}/\partial m_B \cdot \partial F_{iA}/\partial m_C \\ \partial F_{1B}/\partial m_C \cdot \partial F_{iA}/\partial m_C - \partial F_{iA}/\partial m_B \cdot \partial F_{1B}/\partial m_C \end{bmatrix} / \\
& \quad / \left(\partial F_{iB}/\partial m_B \cdot \partial F_{iC}/\partial m_C - \partial F_{iC}/\partial m_B \cdot \partial F_{iB}/\partial m_C\right)
\end{aligned}$$

and similar systems for individuals B and C. We can use the solution for the externality effects to deduce a symmetry restriction.

Proposition 12 *For each* η, $\sum^{hh} - \sum_{k \neq h} \begin{bmatrix} \frac{\partial X_{1h}}{\partial F_{1k}} \\ \cdot \\ \cdot \\ \frac{\partial X_{nh}}{\partial F_{1k}} \end{bmatrix} \left[\sum_{11}^{kh} .. \sum_{1n}^{kh}\right]$ *must form a symmetric negative semidefinite matrix where* $\begin{bmatrix} \frac{\partial X_{1h}}{\partial F_{1k}} \\ \cdot \\ \cdot \\ \frac{\partial X_{nh}}{\partial F_{1k}} \end{bmatrix}$ *is a function of observable partial derivatives of the* $F()$ *functions so long as* $\begin{bmatrix} \frac{\partial F_{i1}}{\partial m_A} & \cdot & \frac{\partial F_{ih-1}}{\partial m_A} & \frac{\partial F_{ih+1}}{\partial m_A} & \cdot & \frac{\partial F_{iH}}{\partial m_A} \\ \cdot & & & & & \\ \frac{\partial F_{i1}}{\partial m_{h-1}} & & & & & \\ \frac{\partial F_{i1}}{\partial m_{h+1}} & & & & & \cdot \\ \cdot & & & & & \cdot \\ \frac{\partial F_{i1}}{\partial m_H} & & \frac{\partial F_{ih-1}}{\partial m_H} & \frac{\partial F_{ih+1}}{\partial m_H} & \cdot & \frac{\partial F_{iH}}{\partial m_H} \end{bmatrix}$ *is nonsingular for each* h, i.

4.3 Common Popular Single Channel Effects

The formulation in (38) is quite general in that the externality depends both on the aggregate consumption of the first good and on the distribution of its consumption. A special case of this with some appeal is that in which only the total consumption of good 1 by all other individuals affects the preferences of any one individual. This is similar to the model that Lechene-Preston study although we are thinking more of a good with network externalities than a public good.

Here

$$F_{i\eta}(p, m_A, ..m_H) = X_{i\eta}(p, m_\eta, \sum_{l\neq\eta} F_{1l}(p, m_A, ..m_H))$$

Again we find the Nash equilibrium by first solving the H equations for the Nash equilibrium functions for good 1, then substituting these back for the other goods.

The comparative statics work through

$$\frac{\partial F_{i\eta}}{\partial p_j} = \frac{\partial X_{i\eta}}{\partial p_j} + \frac{\partial X_{i\eta}}{\partial x_1} \sum_{h\neq\eta} \frac{\partial F_{1h}}{\partial p_j} \tag{45}$$

$$\frac{\partial F_{i\eta}}{\partial m_\eta} = \frac{\partial X_{i\eta}}{\partial m_\eta} + \frac{\partial X_{i\eta}}{\partial x_1} \sum_{h\neq\eta} \frac{\partial F_{1h}}{\partial m_\eta} \tag{46}$$

$$\frac{\partial F_{i\eta}}{\partial m_k} = \frac{\partial X_{i\eta}}{\partial x_1} \sum_{h\neq\eta} \frac{\partial F_{1h}}{\partial m_k} \tag{47}$$

From (47) we derive the externality effect

$$\frac{\partial X_{i\eta}}{\partial x_1} = \frac{\partial F_{i\eta}}{\partial m_k} / \sum_{h\neq\eta} \frac{\partial F_{1h}}{\partial m_k} \tag{48}$$

and using this in the symmetry condition gives

$$S^\eta_{ij} = \Sigma^\eta_{ij} - [\frac{\partial F_{i\eta}}{\partial m_k} / \sum_{h\neq\eta} \frac{\partial F_{1h}}{\partial m_k}] \sum_{h\neq\eta} \Sigma^h_{1j} \tag{49}$$

which must form a symmetric and negative semidefinite matrix.

(48) itself imposes some restriction on the form of the Nash equilibrium demands since the RHS must be independent of k.

5 Welfare Effects of Price Changes

A traditional approach to welfare analysis of price changes uses either the compensating or equivalent variation measured as the change in the consumers expenditure function due to the price change. In the two individual case with only good 1 having an externality effect, the consumer surplus measure of the cost of price change from p to p' is

$$CS_A(p, p', x_{1B}, u_A) = g_A(p', x_{1B}, u_A) - g_A(p, x_{1B}, u_A) \tag{50}$$

where $g_A(p', x_{1B}, u_A)$ measures the minimum cost at p', x_{1B} to the consumer of attaining a utility level of u_A. For u_A either the original or new utility level can be taken. This is often approximated by

$$\widetilde{CS}_A(p, p', x_{1B}, u_A) = \frac{\partial g_A(p, x_{1B}, u_A)}{\partial p_j}(p_j^{'} - p_j) \tag{51}$$

if only the jth price changes, which we can interpret as the area beneath the compensated demand curve for good j. If a price rises then, if $B's$ consumption of good 1 is fixed, A is unambiguously worse off. If A, B are always in Nash equilibrium instead we have

$$CS_A(p, p', m_A, m_B, u_A) = g_A(p', F_{1B}(p', m_A, m_B), u_A) - g_A(p, F_{1B}(p, m_A, m_B), u_A) \tag{52}$$

and when only the jth price changes the approximation would become

$$\widetilde{CS}_A(p, p', m_A, m_B, u_A) = \left[\frac{\partial g_A(p, x_{1B}, u_A)}{\partial p_j} + \frac{\partial g_A}{\partial F_{1B}}\frac{\partial F_{1B}}{\partial p_j}\right](p_j^{'} - p_j) \tag{53}$$

We know that $\partial g_A(p, x_{1B}, u_A)/\partial p_j > 0$ but we can have the result that A is actually better off from a price rise if the term $\partial g_A/\partial F_{1B} \cdot \partial F_{1B}/\partial p_j$ is sufficiently negative. Thus either if there are strong positive externalities so that $\partial g_A/\partial F_{1B} < 0$ and goods 1 and j are substitutes for B in the Nash equilibrium, or if there are negative externalities so that $\partial g_A/\partial F_{1B} > 0$ and goods 1 and j are complements for B in the Nash equilibrium, then A may be better off from a rise in p_1. The intuition is that the price rise induces B to change his consumption of good 1 to raise $A's$ welfare by more than cost increasing effect of the price rise reduces $A's$ welfare.

6 Conclusion

In this paper we have analysed the effects of consumption externalities on the comparative static properties of consumer demands. Two particular properties are important in determining these- the strength of the externality and its sign. If externalities are strong they can overturn the usual comparative static effects and in directions that are sometimes surprising. Thus with strong positive externalities between two individuals, goods which are basically normal can become inferior under the presence of the externality. We have also found that there are generalised forms of Slutsky symmetry restrictions so that in contrast with the general price dependent literature (Pollak) there are some restrictions on demand when we know something about the source of the price dependence of utility. The externalities can also reverse the usual welfare implications of price changes so that for example it is possible that consumers will prefer price increases to price falls.

A Appendix

Proof of Proposition 7. We have

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_{iA}}{\partial p_j} \\ \frac{\partial X_{iB}}{\partial p_j} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial X_{iA}}{\partial F_{kB}} \\ \frac{\partial X_{iB}}{\partial F_{kA}} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F_{iA}}{\partial p_j} \\ \frac{\partial F_{iB}}{\partial p_j} \end{bmatrix} = \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & \frac{\partial X_{iA}}{\partial F_{kB}} \\ \frac{\partial X_{iB}}{\partial F_{kA}} & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} \frac{\partial X_{iA}}{\partial p_j} \\ \frac{\partial X_{iB}}{\partial p_j} \end{bmatrix}$$

However $\begin{bmatrix} I & -\frac{\partial X_{iA}}{\partial F_{kB}} \\ -\frac{\partial X_{iB}}{\partial F_{kA}} & I \end{bmatrix}^{-1} =$

$$\begin{bmatrix} I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}} & \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1} \\ \frac{\partial X_{iB}}{\partial F_{kA}}[I - \frac{\partial X_{iA}}{\partial F_{kB}}\frac{\partial X_{iB}}{\partial F_{kA}}]^{-1} & I + \frac{\partial X_{iB}}{\partial F_{kA}}[I - \frac{\partial X_{iA}}{\partial F_{kB}}\frac{\partial X_{iB}}{\partial F_{kA}}]^{-1}\frac{\partial X_{iA}}{\partial F_{kB}} \end{bmatrix}$$

For the income effects

$$\begin{aligned}
\begin{bmatrix} \frac{\partial F_{iA}}{\partial m_A} \\ \frac{\partial F_{iB}}{\partial m_A} \end{bmatrix} &= \begin{bmatrix} \frac{\partial X_{iA}}{\partial m_A} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial X_{iA}}{\partial F_{kB}} \\ \frac{\partial X_{iB}}{\partial F_{kA}} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F_{iA}}{\partial m_A} \\ \frac{\partial F_{iB}}{\partial m_A} \end{bmatrix} \\
&= \begin{bmatrix} I & -\frac{\partial X_{iA}}{\partial F_{kB}} \\ -\frac{\partial X_{iB}}{\partial F_{kA}} & I \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial X_{iA}}{\partial m_A} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[\frac{\partial X_{iA}}{\partial m_A}\right] \\ \left[\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[I - \left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\right]^{-1}\left[\frac{\partial X_{iA}}{\partial m_A}\right] \end{bmatrix}
\end{aligned}$$

We have

$$\begin{aligned}
[\Sigma_{ij}^{AA}] &= \left[\frac{\partial F_{iA}}{\partial p_j}\right] + \left[\frac{\partial F_{iA}}{\partial m_A}\right] F_A^{'} \\
&= \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[\frac{\partial X_{iA}}{\partial p_j}\right] + \\
&\quad \left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1}\left[\frac{\partial X_{iB}}{\partial p_j}\right] + \\
&\quad \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[\frac{\partial X_{iA}}{\partial m_A}\right] F_A^{'} \\
&= \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right]\left[\left[\frac{\partial X_{iA}}{\partial p_j}\right] + \left[\frac{\partial X_{iA}}{\partial m_A}\right] F_A^{'}\right] \\
&\quad + \left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1}\left[\frac{\partial X_{iB}}{\partial p_j}\right] \\
&= \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right][S_{ij}^{A}] \\
&\quad + \left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1}\left[\frac{\partial X_{iB}}{\partial p_j}\right]
\end{aligned}$$

where S^A is a symmetric matrix. Adding $\left[\frac{\partial F_{iA}}{\partial m_B}\right] F_B^{'} = \left[\frac{\partial X_{iA}}{\partial F_{kB}}\right] \left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1} \left[\frac{\partial X_{iB}}{\partial m_B}\right] F_B^{'}$ to both sides yields

$$\begin{aligned}
\left[\frac{\partial F_{iA}}{\partial p_j}\right] + \left[\frac{\partial F_{iA}}{\partial m_A}\right] F_A^{'} + \left[\frac{\partial F_{iA}}{\partial m_B}\right] F_B^{'} \quad &= \quad \left[I + \frac{\partial X_{iA}}{\partial F_{kB}}[I - \frac{\partial X_{iB}}{\partial F_{kA}}\frac{\partial X_{iA}}{\partial F_{kB}}]^{-1}\frac{\partial X_{iB}}{\partial F_{kA}}\right][S_{ij}^A] \\
&+ \left[\frac{\partial X_{iA}}{\partial F_{kB}}\right]\left[I - \left(\frac{\partial X_{iB}}{\partial F_{kA}}\right)\left(\frac{\partial X_{iA}}{\partial F_{kB}}\right)\right]^{-1}[S_{ij}^B]
\end{aligned}$$

where S^B is also symmetric. ■

Proof of Proposition 9. We have

$$\frac{\partial X_{iA}}{\partial x_{1B}} = \frac{\partial F_{iA}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B} - \sum_{k=2}^{n}\frac{\partial X_{iA}}{\partial x_{kB}}\frac{\partial F_{kB}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B}$$

$$\frac{\partial X_{iB}}{\partial x_{1A}} = \frac{\partial F_{iB}}{\partial m_A}/\frac{\partial F_{1A}}{\partial m_A} - \sum_{k=2}^{n}\frac{\partial X_{iB}}{\partial x_{kA}}\frac{\partial F_{kA}}{\partial m_A}/\frac{\partial F_{1A}}{\partial m_A}$$

so that

$$\begin{aligned}
\sum_{k=1}^{n}\frac{\partial X_{iA}}{\partial x_{kB}}\Sigma_{kj}^{BA} \quad &= \quad \Sigma_{1j}^{BA}[\frac{\partial F_{iA}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B} - \sum_{k=2}^{n}\frac{\partial X_{iA}}{\partial x_{kB}}\frac{\partial F_{kB}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B}] + \sum_{k=2}\frac{\partial X_{iA}}{\partial x_{kB}}\Sigma_{kj}^{BA} \\
&= \quad \Sigma_{1j}^{BA}\frac{\partial F_{iA}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B} + \sum_{k=2}^{n}\frac{\partial X_{iA}}{\partial x_{kB}}[\Sigma_{kj}^{BA} - \Sigma_{1j}^{BA}\frac{\partial F_{kB}}{\partial m_B}/\frac{\partial F_{1B}}{\partial m_B}]
\end{aligned}$$

and similarly

$$\sum_{k=1}^{n}\frac{\partial X_{iB}}{\partial x_{kA}}\Sigma_{kj}^{AB} = \Sigma_{i1}^{AB}\frac{\partial F_{iB}}{\partial m_A}/\frac{\partial F_{1A}}{\partial m_A} + \sum_{k=2}^{n}\frac{\partial X_{iB}}{\partial x_{kA}}[\Sigma_{kj}^{AB} - \Sigma_{i1}^{AB}\frac{\partial F_{kA}}{\partial m_A}/\frac{\partial F_{1A}}{\partial m_A}]$$

■

Putting these back in (33),(34) gives

$$\frac{\partial F_{1B}}{\partial m_B}S_{ij}^A = \frac{\partial F_{1B}}{\partial m_B}\Sigma_{ij}^{AA} - \Sigma_{1j}^{BA}\frac{\partial F_{iA}}{\partial m_B} - \sum_{k=2}\frac{\partial X_{iA}}{\partial x_{kB}}[\frac{\partial F_{1B}}{\partial m_B}\Sigma_{kj}^{BA} - \Sigma_{1j}^{BA}\frac{\partial F_{kB}}{\partial m_B}] \tag{54}$$

$$\frac{\partial F_{1A}}{\partial m_A}S_{ij}^B = \frac{\partial F_{1A}}{\partial m_A}\Sigma_{ij}^{BB} - \Sigma_{1j}^{AB}\frac{\partial F_{iB}}{\partial m_A} - \sum_{k=2}\frac{\partial X_{iB}}{\partial x_{kA}}[\frac{\partial F_{1A}}{\partial m_A}\Sigma_{kj}^{AB} - \Sigma_{1j}^{AB}\frac{\partial F_{kA}}{\partial m_A}] \tag{55}$$

Here the right hand sides of each equation must be symmetric and negative semidefinite matrices. On the RHS's there are $n-1$ arbitrary factors in each equation given by the marginal externality terms $\partial X_{iA}/\partial x_{kB}, \partial X_{iB}/\partial x_{kA}$.

Proof of Proposition 10. Using the equations for the first commodity and solving out for price effects

$$\left[\frac{\partial F_{1\eta}}{\partial p_j}\right] = \left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1} \left[\frac{\partial X_{1\eta}}{\partial p_j}\right]$$

We can then insert this into the expression for commodities $i > 1$ to get

$$\left[\frac{\partial F_{i\eta}}{\partial p_j}\right] = \left[\frac{\partial X_{i\eta}}{\partial p_j}\right] + \left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right] \left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1} \left[\frac{\partial X_{1\eta}}{\partial p_j}\right]$$

Taking $i = 1$ and solving for $\left[\frac{\partial F_{1\eta}}{\partial m_h}\right]$ leads to

$$\left[\frac{\partial F_{1\eta}}{\partial m_h}\right] = \left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1} \left[\frac{\partial X_{1\eta}}{\partial m_h}\right]$$

and using this in the equations for $i > 1$ gives

$$\left[\frac{\partial F_{i\eta}}{\partial m_h}\right] = \left[\frac{\partial X_{i\eta}}{\partial m_h}\right] + \left[\frac{\partial X_{i\eta}}{\partial x_{1h}}\right] \left[I - \left[\frac{\partial X_{1\eta}}{\partial x_{1h}}\right]\right]^{-1} \left[\frac{\partial X_{1\eta}}{\partial m_h}\right]$$

■

Proof of Proposition 12. We want to use (42) to solve for the marginal externality effects: the unknowns are the $n(H-1)^2$ terms $\partial X_{i\eta}/\partial x_{1h}$. In fact the equations can be solved in blocks of $H-1$ equations that involve the $H-1$ terms $\partial X_{1\eta}/\partial x_{1h}$ for fixed values of η and i as k varies through its possible $H-1$ values.

We have

$$F_{ih}(p, m_A, ...m_H) = X_{ih}(p, m_h, F_{1-h}(p, m_A, ...m_H)) \qquad (56)$$

All H incomes enter F_{ih} but only $H-1$ functions $F_{1\eta}$ enter the RHS. Differentiate this equation wrt each $m_k, k \neq h$ and write the result in matrix notation as a system of $H-1$ equations in $H-1$ variables $\partial X_{ih}/\partial F_{1k}$ for $k \neq h$:

$$f_h = \digamma_h \begin{bmatrix} \frac{\partial X_{ih}}{\partial F_{11}} \\ . \\ \frac{\partial X_{ih}}{\partial F_{1h-1}} \\ \frac{\partial X_{ih}}{\partial F_{1h+1}} \\ . \\ \frac{\partial X_{ih}}{\partial F_{1H}} \end{bmatrix}$$

where $\digamma_{ih} = \begin{bmatrix} \frac{\partial F_{i1}}{\partial m_1} & . & \frac{\partial F_{ih-1}}{\partial m_1} & \frac{\partial F_{ih+1}}{\partial m_1} & . & \frac{\partial F_{iH}}{\partial m_1} \\ . & & & & & \\ \frac{\partial F_{i1}}{\partial m_{h-1}} & & & & & \\ \frac{\partial F_{i1}}{\partial m_{h+1}} & & & & & . \\ . & & & & & . \\ \frac{\partial F_{i1}}{\partial m_H} & & \frac{\partial F_{ih-1}}{\partial m_H} & \frac{\partial F_{ih+1}}{\partial m_H} & . & \frac{\partial F_{iH}}{\partial m_H} \end{bmatrix}$ and $f_{ih} = \begin{bmatrix} \frac{\partial F_{ih}}{\partial m_1} \\ . \\ \frac{\partial F_{ih}}{\partial m_{h-1}} \\ \frac{\partial F_{ih}}{\partial m_{h+1}} \\ . \\ \frac{\partial F_{ih}}{\partial m_H} \end{bmatrix}$

So long as $\digamma_{ih}$ is nonsingular for each h, i we can solve to get $\begin{bmatrix} \frac{\partial X_{ih}}{\partial F_{11}} \\ . \\ \frac{\partial X_{ih}}{\partial F_{1h-1}} \\ \frac{\partial X_{ih}}{\partial F_{1h+1}} \\ . \\ \frac{\partial X_{1h}}{\partial F_{1H}} \end{bmatrix} =$ $\digamma_{ih}^{-1} f_{ih} = A_{ih} f_{ih}$ for each h and i.

On the other hand we have

$$\sum_{ij}^{hh} = S_{ij}^{hh} + \sum_{k \neq h} \frac{\partial X_{ih}}{\partial F_{1k}} \sum_{1k}^{kh}$$

To relate this to rank one matrices we can write it as

$$\sum^{hh} = S^{hh} + \sum_{k \neq h} \begin{bmatrix} \frac{\partial X_{1h}}{\partial F_{1k}} \\ . \\ . \\ \frac{\partial X_{nh}}{\partial F_{1k}} \end{bmatrix} \left[\sum_{11}^{kh} .. \sum_{1n}^{kh} \right]$$

where $\sum^{hh}$ and S^{hh} are each nxn. Now if we use our earlier results and form the Hxn matrix

$$\begin{bmatrix} \frac{\partial X_{1h}}{\partial F_{11}} & \cdot & \cdot & \frac{\partial X_{nh}}{\partial F_{11}} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial X_{1h}}{\partial F_{1H}} & & & \frac{\partial X_{nH}}{\partial F_{1H}} \end{bmatrix} = \begin{bmatrix} F_{1h}^{-1} f_{1h} | & \cdot & \cdot & | F_{nh}^{-1} f_{nh} \end{bmatrix}$$
$$= \left[\alpha_{lk}^{h} \right]$$

we can write

$$\sum^{hh} = S^{hh} + \sum_{k \neq h} \begin{bmatrix} \alpha_{1k}^{h} \\ \cdot \\ \cdot \\ \alpha_{Hk}^{h} \end{bmatrix} \left[\sum_{11}^{kh} .. \sum_{1n}^{kh} \right]$$

Since we know that S^{hh} is symmetric and negative semidefinite, it follows that

$$\sum^{hh} - \sum_{k \neq h} \begin{bmatrix} \alpha_{1k}^{h} \\ \cdot \\ \cdot \\ \alpha_{Hk}^{h} \end{bmatrix} \left[\sum_{11}^{kh} .. \sum_{1n}^{kh} \right]$$

must be symmetric and negative semidefinite. This relates to the rank one restriction idea except now that for each good and each individual h, $H - 1$ rank one symmetric matrices have to be added to $\sum^{hh}$ to ensure its symmetry. Of course these additional rank one matrices are defined in terms of observable derivatives of the Nash equilibrium demands. ■

References

[1] M Browning & PA Chiappori, Efficient Intra-Household Allocation: A General Characterisation and Empirical Tests, *Econometrica*, **1998**, **66**, 1241-1278

[2] V Lechene & I Preston, Departures from Slutsky Symmetry in Noncooperative household Models, 2000, *WP University of Oxford, 2000, ref. 052*

[3] L McKenzie, Competitive Equilibrium with Depenedent Consumer Preferences, *Second Symposium on Linear Programming, National Bureau of Standards and Department of the Air Force*, 1955, 277-294

[4] R Pollak, Price Dependent Preferences, *American Economic Review*, 1977,**67**,**3**,64-75