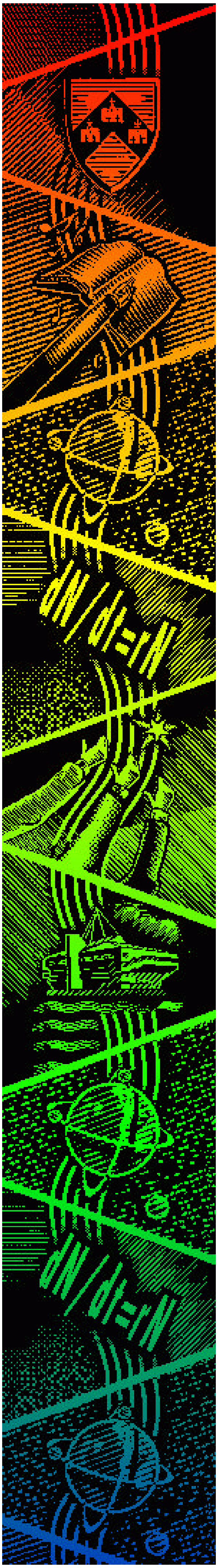

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The Distribution of a Ratio of Quadratic Forms in Noncentral Normal Variables

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Abstract

An expression for the exact cumulative distribution function of a ratio of quadratic forms in noncentral normal variable is derived in terms of infinite series of top order invariant polynomials.

Key Words: ratio of quadratic forms, quadratic forms in normal variables

1 Introduction

The distribution of a ratio of quadratic forms in normal variables has attracted considerable attention in statistics over the last few decades. Imhof (1961), Davies (1973) and Shively, Ansley, and Kohn (1990) have devised algorithms for the computation of the densities and cumulative distribution functions (CDF) of ratios of quadratic forms which are efficient and easy to implement. However exact results are available for specific cases only. Precisely von Neumann (1941) has characterized the distribution of the von Neumann ratio in terms of the derivative of its density. Gurland (1948) and (1953) has given inversion formulae for the CDF and the density of a ratio of quadratic forms. Koopmans (1942), L.R Anderson (1942) and T.W. Anderson (1971) have studied the serial correlation coefficient in the circular case and have derived formulae for its density and CDF. Recently, Hillier (2001) has obtained the density of a quadratic form uniformly distributed on the unit n-sphere, and Forchini (2001) has derived the CDF of a ratio of quadratic forms in central normal variables. Lieberman (1994) and Marsh (1998) have derived saddlepoint approximations for the density of a ratio of quadratic forms in normal variables for the central and noncentral case respectively.

In this paper we generalized the results of Forchini (2001) to the noncentral case, and obtain a representation of the CDF of a ratio of quadratic forms at a particular point by writing it as the CDF evaluated at zero of the difference of two independent positive definite quadratic forms in noncentral normal variables which are constructed by separating these eigenvalues into positive and negative. The resulting expression in term of the invariant polynomials of two matrix arguments introduced by Davis (1979) has not appeared before in the literature.

The relevant related literature is now summarised. The density of a positive definite quadratic form of normal random variables is given by Gurland (1956), Ruben (1962), James (1964) for the central case and by Phillips (1986) for the noncentral case. The distribution of indefinite quadratic forms is given by Gurland (1955) and Robinson (1965) for the case of central normal random variables and by Shah (1963) for the noncentral case. However, we will not use these results because they give expressions which are not convergent everywhere or contain unsolved integrals. Imhof (1961), Davies (1973) and Shively, Ansley, and Kohn (1990) give algorithms for the numerical calculation of the density and CDF of a quadratic form.

2 Preliminary results

Suppose that Y is a $(n \times 1)$ random vector having a multivariate normal distribution with mean vector η and (positive definite) covariance matrix Ω,

$$\text{pdf}_Y(y) = (2\pi)^{-\frac{n}{2}} |\Omega|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(y-\eta)' \Omega^{-1}(y-\eta)\right\}. \tag{1}$$

It is required to find the CDF of

$$R = \frac{Y'AY}{Y'BY}, \tag{2}$$

where A and B are $(n \times n)$ symmetric matrices and B is positive semidefinite. The distribution of R is the same as the distribution of $(V'AV)/(V'BV)$ where $V = Y(Y'Y)^{\frac{1}{2}}$ is a vector distributed on the unit n-sphere. Therefore, the results below hold for scale-mixtures of normals. Let $\text{F}_R(r)$ be the CDF of R at the point r given that $Y \sim N(\eta, \Omega)$,

$$\begin{aligned}
\text{F}_R(r) &= \Pr\left\{\frac{Y'AY}{Y'BY} \le r \mid Y \sim N(\eta,\Omega)\right\} \\
&= \Pr\left\{\frac{Y'A^*Y}{Y'B^*Y} \le r \mid Y \sim N\left(\Omega^{-\frac{1}{2}}\eta, I_n\right)\right\} \\
&= \Pr\left\{Y'(A^* - rB^*)Y \le 0 \mid Y \sim N\left(\Omega^{-\frac{1}{2}}\eta, I_n\right)\right\},
\end{aligned} \tag{3}$$

where $A^* = \Omega^{\frac{1}{2}} A \Omega^{\frac{1}{2}}$ and $B^* = \Omega^{\frac{1}{2}} B \Omega^{\frac{1}{2}}$. Let $Y = H'X$, and $\mu = H'\eta$, where H is an orthogonal matrix which diagonalizes $A^* - rB^*$,

$$H'(A^* - rB^*)H = \begin{pmatrix} \Sigma_1(r) & 0 & 0 \\ 0 & -\Sigma_2(r) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

and $\Sigma_1(r)$ and $-\Sigma_2(r)$ are diagonal matrices containing the $n_1 \ge 0$ positive and the $n_2 \ge 0$ negative eigenvalues of $A^* - rB^*$ respectively. Note that n_1 and n_2 vary as r varies. By partitioning Y and μ conformably to $H'(A^* - rB^*)H$, $X = (X_1', X_2', X_3')'$, $\mu = (\mu_1', \mu_2', \mu_3')$ we obtain

$$\text{F}_R(r) = \Pr\{X_1'\Sigma_1(r)X_1 - X_2'\Sigma_2(r)X_2 \le 0 | X_1 \sim N(\mu_1, I_{n_1}), X_2 \sim N(\mu_2, I_{n_2})\}, \tag{5}$$

where X_1, X_2 and X_3 are independent, and $n_1 + n_2 = \text{rank}(A^* - rB^*) = \text{rank}(A - rB) \le n$. The above results allow us to write the CDF of R as

$$\text{F}_R(r) = \Pr\{Q_1 - Q_2 \le 0\}, \tag{6}$$

where $Q_1 = X_1'\Sigma_1(r)X_1 > 0$ and $Q_2 = X_2'\Sigma_2(r)X_2 > 0$ are independent noncentral quadratic forms in normal variables. Note that $\mathrm{F}_R(r) = 0$ for values of r for which $n_2 = 0$, and $\mathrm{F}_R(r) = 1$ for values of r for which $n_1 = 0$. If $n_1 > 0$ and $n_2 > 0$ we can find the joint density of (Q_1, Q_2) as a product of the marginal densities of Q_1 and Q_2 (since they are independent). Thus the CDF of R at $r = 0$ is

$$\begin{aligned}\mathrm{F}_R(r) &= \int_{q_2>0}\int_{0<q_1<q_2} \mathrm{pdf}_{Q_1}(q_1)\,\mathrm{pdf}_{Q_2}(q_2)\,dq_1dq_2 \\ &= \int_{q_2>0}\int_{0<x<1} \mathrm{pdf}_{Q_1}(q_2x)\,dx\,\mathrm{pdf}_{Q_2}(q_2)\,q_2dq_2.\end{aligned} \tag{7}$$

This integral can be evaluated by expanding the densities of Q_1 and Q_2 as infinite series and by integrating term by term. This procedure leads to an expression for the CDF of a ratio of two quadratic forms in normal variables which does not seem to have been derived before in the statistical literature.

The relationship between equation (6) and lack of differentiability of the CDF of R at some points in its domain is studied in Forchini (2001).

3 The exact density function of a quadratic form in noncentral normal variables

This section derives the exact density of a quadratic form in noncentral normal variables which is analogous to that derived by Phillips (1986) but is more suitable to be used in the procedure outlined above.

Let $X \sim N(\mu, I_n)$ and consider $Y = \Sigma^{\frac{1}{2}}X \sim N\left(\Sigma^{\frac{1}{2}}\mu, \Sigma\right)$, then

$$\mathrm{pdf}_Y(y) = (2\pi)^{-\frac{n}{2}}|\Sigma|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\mu'\Sigma^{-1}\mu\right\}\exp\left\{-\frac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-\frac{1}{2}}y\right\}. \tag{8}$$

By writing

$$\Sigma^{-1} = \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n + \Sigma^{-1} \tag{9}$$

we have

$$\mathrm{pdf}_Y(y) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mu'\Sigma^{-1}\mu\right\} \exp\left\{-\frac{1}{2}y'y\left(\mathrm{tr}\,\Sigma^{-1}\right)\right\}$$

$$\exp\left\{-\frac{1}{2}y'\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)y + \mu'\Sigma^{-\frac{1}{2}}y\right\}. \tag{10}$$

Decomposing y to polar coordinates $y = q^{\frac{1}{2}}v$ where $q = y'y$ and $v = y/(y'y)^{1/2}$ (the Jacobian is $q^{n/2-1}$) and integrating out v we obtain the density of $Q = Y'Y$ as

$$\mathrm{pdf}_Q(q) = \frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mu'\Sigma^{-1}\mu\right\} \exp\left\{-\frac{1}{2}q\left(\mathrm{tr}\,\Sigma^{-1}\right)\right\} q^{n/2-1}$$

$$\int_{v'v=1} \exp\left\{-\frac{1}{2}qv'\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)v + q^{1/2}\mu'\Sigma^{-\frac{1}{2}}v\right\}(dv), \tag{11}$$

where (dv) denotes the normalized Haar measure on the unit n-sphere (Muirhead (1982)). Noting that the integral over $v'v = 1$ is invariant to the transformation of μ to $h\mu$, where $h \in O(1)$, we have that the density of Q is

$$\mathrm{pdf}_Q(q) = \frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mu'\Sigma^{-1/2}\mu\right\} \exp\left\{-\frac{1}{2}q\left(\mathrm{tr}\,\Sigma^{-1}\right)\right\} q^{n/2-1}$$

$$\int_{v'v=1} \exp\left\{-\frac{1}{2}qv'\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)v\right\} \tag{12}$$

$${}_0F_1\left(\frac{1}{2}; \frac{1}{4}q\mu'\Sigma^{-1/2}vv'\Sigma^{-1/2}\mu\right)(dv).$$

The vector v can be seen as the first column of an orthogonal matrix $H \in O(n)$. So we can write

$$\begin{aligned} v'\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)v &= \mathrm{tr}\left(\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)vv'\right) \\ &= \mathrm{tr}\left(\left(\Sigma^{-1} - \left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)HE_nH'\right) \end{aligned}$$

and the argument of the hypergeometric function becomes $\frac{1}{4}q\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}HE_nH'$ (note that by so doing it becomes a matrix argument hypergeometric function), where E_n is an $n \times n$

with all components equal to zero apart from the element in position (n,n) which is one. So

$$\mathrm{pdf}_Q(q) = \frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mu'\Sigma^{-1/2}\mu\right\} \exp\left\{-\frac{1}{2}q\,\mathrm{tr}\left(\Sigma^{-1}\right)\right\}$$

$$q^{n/2-1}\int_{O(n)} \exp\left\{\frac{1}{2}q\,\mathrm{tr}\left(\left(\Sigma^{-1}-\left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)HE_nH'\right)\right\}$$

$${}_0F_1\left(\frac{1}{2};\frac{1}{4}q\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}HE_nH'\right)(dH).$$

Expanding the exponential and the hypergeometric functions and integrating term by term we have

$$\begin{aligned}\mathrm{pdf}_Q(q) &= \frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\mu'\Sigma^{-1/2}\mu\right\} \exp\left\{-\frac{1}{2}q\,\mathrm{tr}\left(\Sigma^{-1}\right)\right\} \\ & q^{n/2-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j!\,(1/2)_i\,i!}\left(\frac{1}{2}\right)^{j+2i} q^{i+j} \\ & \int_{O(n)} C_{[j]}\left(\left(\Sigma^{-1}-\left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right)HE_nH'\right) \\ & C_{[i]}\left(\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}HE_nH'\right)(dH).\end{aligned}$$

The last integral can now be evaluated using equation (1.2) of Davis (1979)

$$\sum_{\phi\in[j]\cdot[i]} \frac{C_\phi^{[j],[i]}\left(\left(\Sigma^{-1}-\left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right),\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}\right)C_\phi^{[j],[i]}\left(E_n,E_n\right)}{C_\phi\left(I_n\right)}, \tag{13}$$

where $[k]$ denotes the top order partition of k. The notation employed is explained in Davis (1979).

Note that $C_\phi^{[j],[i]}(E_n,E_n) = 1$ for $\phi = [j+i]$ but it is zero for every other partition ϕ of $j+i$. Moreover, $C_{[j+i]}(I_n) = (n/2)_{j+i} / (1/2)_{j+i}$, so the last term is just

$$\frac{\left(\frac{1}{2}\right)_{j+i}}{\left(\frac{n}{2}\right)_{j+i}} C_{[j+i]}^{[j],[i]}\left(\left(\Sigma^{-1}-\left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right),\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}\right) \tag{14}$$

and the density of the quadratic form Q is

$$\mathrm{pdf}_Q(q) = \frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}|\Sigma|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\mu'\Sigma^{-1/2}\mu\right\}\exp\left\{-\frac{1}{2}q\,\mathrm{tr}\left(\Sigma^{-1}\right)\right\}$$

$$q^{n/2-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{j+i}}{(1/2)_i\left(\frac{n}{2}\right)_{j+i}j!i!}\left(\frac{1}{2}\right)^{j+2i}q^{i+j}$$

$$C_{[j+i]}^{[j],[i]}\left(\left(\Sigma^{-1}-\left(\mathrm{tr}\,\Sigma^{-1}\right)I_n\right),\Sigma^{-1/2}\mu\mu'\Sigma^{-1/2}\right). \tag{15}$$

The form of the density of a quadratic form given in the above display is similar to that of equation (13) of Phillips (1986). The existence of a term in which q is in the exponent in equation (15) makes it easier to derive the CDF of a ratio of quadratic forms.

4 The exact CDF of a ratio of quadratic forms in non-central normal variables

As indicated in Section 2 the CDF of a ratio of quadratic forms in normal variables can be written as the CDF of the difference of two positive definite quadratic forms in noncentral normal variables $Q_1 = Y_1'\Sigma_1 Y_1$ and $Q_2 = Y_2'\Sigma_2 Y_2$ where $Y_1 \sim N(\mu_1, I_{n_1})$ and $Y_2 \sim N(\mu_2, I_{n_2})$, and $\Sigma_1 = \Sigma_1(r)$, and $\Sigma_2 = \Sigma_2(r)$, and we use (7) to find the CDF of $Q_1 - Q_2$ at zero. To do this we need to evaluate

$$\int_{0<x<1}\mathrm{pdf}_{Q_1}(q_2 x)\,dx = \frac{\pi^{\frac{n_1}{2}}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)}|\Sigma_1|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\mu_1'\Sigma_1^{-1/2}\mu_1\right\}$$

$$\int_{0<x<1}\exp\left\{-\frac{1}{2}xq_2\,\mathrm{tr}\left(\Sigma_1^{-1}\right)\right\}$$

$$\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{j_1+i_1}}{(1/2)_{i_1}\left(\frac{n_1}{2}\right)_{j_1+i_1}j_1!i_1!}\left(\frac{1}{2}\right)^{j_1+2i_1}(xq_2)^{\frac{n_1}{2}+i_1+j_1-1}$$

$$C_{[j_1+i_1]}^{[j_1],[i_1]}\left(\left(\Sigma_1^{-1}-\left(\mathrm{tr}\,\Sigma_1^{-1}\right)I_{n_1}\right),\Sigma_1^{-1/2}\mu_1\mu_1'\Sigma_1^{-1/2}\right)dx.$$

Integrating term by term we just need to find

$$\int_{0<x<1} \exp\left\{-\frac{1}{2}xq_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right\} x^{\frac{n_1}{2}+i_1+j_1-1} dx$$

$$= \frac{\Gamma\left(\frac{n_1}{2}+i_1+j_1\right)}{\Gamma\left(\frac{n_1}{2}+i_1+j_1+1\right)} \exp\left\{-\frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right\}$$

$${}_1F_1\left(1; \frac{n_1}{2}+i_1+j_1+1; \frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right)$$

$$= \frac{\Gamma\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}\right)_{i_1+j_1}}{\Gamma\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}+1\right)_{i_1+j_1}} \exp\left\{-\frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right\}$$

$${}_1F_1\left(1; \frac{n_1}{2}+i_1+j_1+1; \frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right). \tag{16}$$

The desired integral is

$$\int_{0<x<1} \operatorname{pdf}_{Q_1}(q_2 x)\, dx = \frac{\pi^{\frac{n_1}{2}}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}+1\right)} |\Sigma_1|^{-\frac{1}{2}}$$

$$\exp\left\{-\frac{1}{2}\mu_1'\Sigma_1^{-1/2}\mu_1\right\} \exp\left\{-\frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right\}$$

$$\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{j_1+i_1}}{(1/2)_{i_1}\left(\frac{n_1}{2}+1\right)_{i_1+j_1} j_1! i_1!} \left(\frac{1}{2}\right)^{j_1+2i_1}$$

$$C_{[j_1+i_1]}^{[j_1],[i_1]}\left(\left(\Sigma_1^{-1} - \left(\operatorname{tr}\Sigma_1^{-1}\right) I_{n_1}\right), \Sigma_1^{-1/2}\mu_1\mu_1'\Sigma_1^{-1/2}\right)$$

$$q_2^{\frac{n_1}{2}+i_1+j_1-1}\, {}_1F_1\left(1; \frac{n_1}{2}+i_1+j_1+1; \frac{1}{2}q_2 \operatorname{tr}\left(\Sigma_1^{-1}\right)\right). \tag{17}$$

The CDF of R at r is can thus be written as

$$F_R(r) = \frac{\pi^{\frac{n_1+n_2}{2}} \exp\left\{-\frac{1}{2}\mu_1' \Sigma_1^{-1/2}\mu_1 - \frac{1}{2}\mu_2'\Sigma_2^{-1/2}\mu_2\right\}}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1}{2}+1\right)|\Sigma_1|^{\frac{1}{2}}|\Sigma_2|^{\frac{1}{2}}}$$

$$\int_{q_2>0} \exp\left\{-\frac{1}{2}q_2\left(\mathrm{tr}\left(\Sigma_1^{-1}\right)+\mathrm{tr}\left(\Sigma_2^{-1}\right)\right)\right\}$$

$$\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{j_1+i_1}\left(\frac{1}{2}\right)_{j_2+i_2}\left(\frac{1}{2}\right)^{j_1+j_2+2(i_1+i_2)}}{\left(\frac{1}{2}\right)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{n_1}{2}+1\right)_{i_1+j_1}\left(\frac{n_2}{2}\right)_{j_2+i_2} j_1!i_1!j_2!i_2!}$$

$$C_{[j_1+i_1]}^{[j_1],[i_1]}\left(\left(\Sigma_1^{-1}-\left(\mathrm{tr}\,\Sigma_1^{-1}\right)I_{n_1}\right), \Sigma_1^{-1/2}\mu_1\mu_1'\Sigma_1^{-1/2}\right)$$

$$C_{[j_2+i_2]}^{[j_2],[i_2]}\left(\left(\Sigma_2^{-1}-\left(\mathrm{tr}\,\Sigma_2^{-1}\right)I_{n_2}\right), \Sigma_2^{-1/2}\mu_2\mu_2'\Sigma_2^{-1/2}\right)$$

$$q_2^{\frac{n_1+n_2}{2}+i_1+j_1+i_2+j_2-1}\,{}_1F_1\left(1; \frac{n_1}{2}+i_1+j_1+1; \frac{1}{2}q_2\,\mathrm{tr}\left(\Sigma_1^{-1}\right)\right)dq_2 \qquad (18)$$

So that evaluating the Laplace transform term by term and defining

$$\Sigma_1^{*-1} = \frac{1}{\mathrm{tr}\left(\Sigma_1^{-1}\right)+\mathrm{tr}\left(\Sigma_2^{-1}\right)}\Sigma_1^{-1}$$

and

$$\Sigma_2^{*-1} = \frac{1}{\mathrm{tr}\left(\Sigma_1^{-1}\right)+\mathrm{tr}\left(\Sigma_2^{-1}\right)}\Sigma_2^{-1},$$

we find the CDF of R in (2) is

$$F_R(r) = \frac{\pi^{\frac{n_1+n_2}{2}} \Gamma\left(\frac{n_1+n_2}{2}\right) \exp\left\{-\frac{1}{2}\mu_1' \Sigma_1^{-1/2} \mu_1 - \frac{1}{2}\mu_2' \Sigma_2^{-1/2} \mu_2\right\}}{\Gamma\left(\frac{n_2}{2}\right) \Gamma\left(\frac{n_1}{2}+1\right) |\Sigma_1^*|^{\frac{1}{2}} |\Sigma_2^*|^{\frac{1}{2}}}$$

$$\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{j_1+i_1} \left(\frac{1}{2}\right)_{j_2+i_2} \left(\frac{n_1+n_2}{2}\right)_{i_1+j_1+i_2+j_2}}{\left(\frac{1}{2}\right)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{n_1}{2}+1\right)_{i_1+j_1} \left(\frac{n_2}{2}\right)_{j_2+i_2} j_1! i_1! j_2! i_2!}$$

$$C_{[j_1+i_1]}^{[j_1],[i_1]}\left(\Sigma_1^{*-1} - \left(\operatorname{tr}\Sigma_1^{*-1}\right) I_{n_1}, \frac{1}{2}\Sigma_1^{*-1/2}\mu_1\mu_1'\Sigma_1^{*-1/2}\right)$$

$$C_{[j_2+i_2]}^{[j_2],[i_2]}\left(\Sigma_2^{*-1} - \left(\operatorname{tr}\Sigma_2^{*-1}\right) I_{n_2}, \frac{1}{2}\Sigma_2^{*-1/2}\mu_2\mu_2'\Sigma_2^{*-1/2}\right)$$

$${}_2F_1\left(\frac{n_1+n_2}{2} + i_1 + j_1 + i_2 + j_2, 1; \frac{n_1}{2} + i_1 + j_1 + 1; \operatorname{tr}\left(\Sigma_1^{*-1}\right)\right). \tag{19}$$

This series can be shown to be absolutely convergent by majorization for all values of r. When $\mu_1 = 0$ and $\mu_2 = 0$ this reduces to the formula given in Theorem 4 of Forchini (2001).

Note that the CDF of R depends only on Σ_1^{*-1}, Σ_2^{*-1} (i.e. it depends only on the normalised n_1 positive and the n_2 negative eigenvalues of $A^* - rB^*$) and the noncentrality parameters $\Sigma_1^{*-1/2}\mu_1\mu_1'\Sigma_1^{*-1/2}$ and $\Sigma_2^{*-1/2}\mu_2\mu_2'\Sigma_2^{*-1/2}$.

The numerical evaluation of the top order invariant polynomials can be done using the results of Chikuse (1987) and Smith (1993).

5 Discussion: convergence problems

Although the infinite series representation for the CDF of R given in (19) is convergent, its convergence is very slow. To understand how serious the problem is, a simple example will be considered. Let $Y \sim N(0, I_n)$, and consider the quadratic form

$$R = \frac{Y' i (i'i)^{-1} i' Y}{Y'\left(I_T - i (i'i)^{-1} i'\right) Y/(n-1)}$$

where i is a n-dimensional vector of ones. Note that $R \sim F(1, n-1)$, and has a CDF which can be easily and accurately calculated numerically using Imhof (1961)'s procedure. Noting that $D_1 = 1$ and $D_2 = \frac{r}{n-1} I_{n-1}$, after a straightforward but tedious simplification, equation (19) yields

$$\mathrm{F}_R(r) = \frac{2\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)\left(\frac{r}{n-1}\right)^{\frac{n-1}{2}}}\left(1+\frac{(n-1)^2}{r}\right)^{-\frac{n}{2}}$$
$$\sum_{p=0}^{\infty}\frac{\left(\frac{n}{2}\right)_p}{p!}\left(\frac{\frac{(n-1)(n-2)}{r}}{1+\frac{(n-1)^2}{r}}\right)^p {}_2F_1\left(\frac{n}{2}+p, 1; \frac{3}{2}, \frac{1}{1+\frac{(n-1)^2}{r}}\right). \quad (20)$$

A plot of the exact CDF of R (dashed line) and of (20), with the summation in the above display replaced by $\sum_{p=0}^{P}$, for various values of P (solid lines), is shown in Figure 1 for $n = 10$. Even for this very simple example and this very small sample size, the number of terms to include in the summation is very high and a reasonable approximation can be obtain with $P \geq 100$. The number of terms required by a good approximation of the CDF of R increases considerably with the sample size. In more complicated examples the convergence problems are more serious.

[Figure 1 approximately here]

6 Conclusion

This paper has derived for the first time an exact expression for the CDF of a ratio of quadratic forms in noncentral normal variables, and has extended the results available in the literature. Although the series representation in equation (19) solves the problem by finding in a rigorous exact expression for such a CDF, its practical use is limited by its very slow convergence rate.

References

Anderson, L. R. (1942): "The Distribution of the Serial Correlation Coefficient," *The Annals of Mathematical Statistics*, 13, 1–13.

Anderson, T. W. (1971): *The Statistical Analysis of Time Series*. John Wiley and Sons, New York.

Chikuse, Y. (1987): "Methods for Constructing Top Order Invariant Polynomials," *Econometric Theory*, 3, 195–207.

Davies, R. B. (1973): "Numerical Inversion of a Characteristic Function," *Biometrika*, 60, 415–417.

Davis, A. W. (1979): "Invariant Polynomials with Two Matrix Arguments Extending the Zonal Polynomials: Applications to Multivariate Distribution Theory," *Annals of the Institute of Statistical Mathematics*, 31, 465–485.

Forchini, G. (2001): "The Exact Cumulative Distribution Function of a Ratio of Quadratic Forms in Normal Variables, with Application to the AR(1) Model," *Discussion Paper Series No. 01/02, Department of Economics and Related Studies, University of York.*

Gurland, J. (1948): "Inversion Formulae for the Distribution of Ratios," *The Annals of Mathematical Statistics*, 19, 228–237.

——— (1953): "Distribution of Quadratic Forms and Ratios of Quadratic Forms," *The Annals of Mathematical Statistics*, 24, 416–427.

——— (1955): "Distribution of Definite and Indefinite Quadratic Forms," *The Annals of Mathematical Statistics*, 26, 122–127.

——— (1956): "Quadratic Forms in Normally Distributed Random Variables," *Sankhya*, 17, 37–50.

Hillier, G. H. (2001): "The Density of a Quadratic Form in a Vector Uniformly Distributed on the n-Sphere," *Econometric Theory*, 17, 1–28.

Imhof, J. P. (1961): "Computing the Distribution of Quadratic Forms in Normal Variables," *Biometrika*, 48, 419–426.

James, A. T. (1964): "Distributions of Matrix Variates and Latent Roots Derived from Normal Samples," *Annals of Mathematical Statistics*, 35, 475–501.

Koopmans, T. C. (1942): "Serial Correlation and Quadratic Forms in Normal Variables," *The Annals of Mathematical Statistics*, 12, 14–33.

Lieberman, O. (1994): "Saddlepoint Approximation for the Distribution of a Ratio of Quadratic Forms in Normal Variables," *Journal of the American Statistical Association*, 89, 924–928.

Marsh, P. W. N. (1998): "Saddlepoint Approximations for Noncentral Quadratic Forms," *Econometric Theory*, 14, 539–559.

Muirhead, R. J. (1982): *Aspects of Multivariate Statistical Theory.* John Wiley and Sons, New York.

Phillips, P. C. B. (1986): "The Exact Distribution of the Wald Statistic," *Econometrica*, 54, 881–895.

Robinson, J. (1965): "The Distribution of a General Quadratic Form in Normal Variates," *Australian Journal of Statistics*, 7, 110–114.

Ruben, H. (1962): "Probability Content of Regions under Spherical Normal Ditributions, IV: The Distribution of Homogeneous and Non-Homogeneous Quadratic Functions of Normal Variables," *The Annals of Mathematical Statistics*, 33, 542–570.

Shah, S. B. (1963): "Distribution of Definite and of Indefinite Quadratic Forms for a Non-Central Normal Distribution," *The Annals of Mathematical Statistics*, 34, 186–190.

Shively, T. S., C. F. Ansley, and R. Kohn (1990): "Fast Evaluation of the Distribution of the Durbin-Watson and Other Invariant Test Statistics in Time Series Regression," *Journal of the American Statistical Association*, 85, 676–685.

Smith, M. D. (1993): "Expectation of Ratios of Quadratic Forms in Normal Variables: Evaluating some Top-Order Invariant Polynomials," *Australian Journal of Statistics*, 35, 271–282.

von Neumann, J. (1941): "Distribution of the Ratio of the Mean Square Successive Difference to the Variance," *The Annals of Mathematical Statistics*, 12, 367–395.

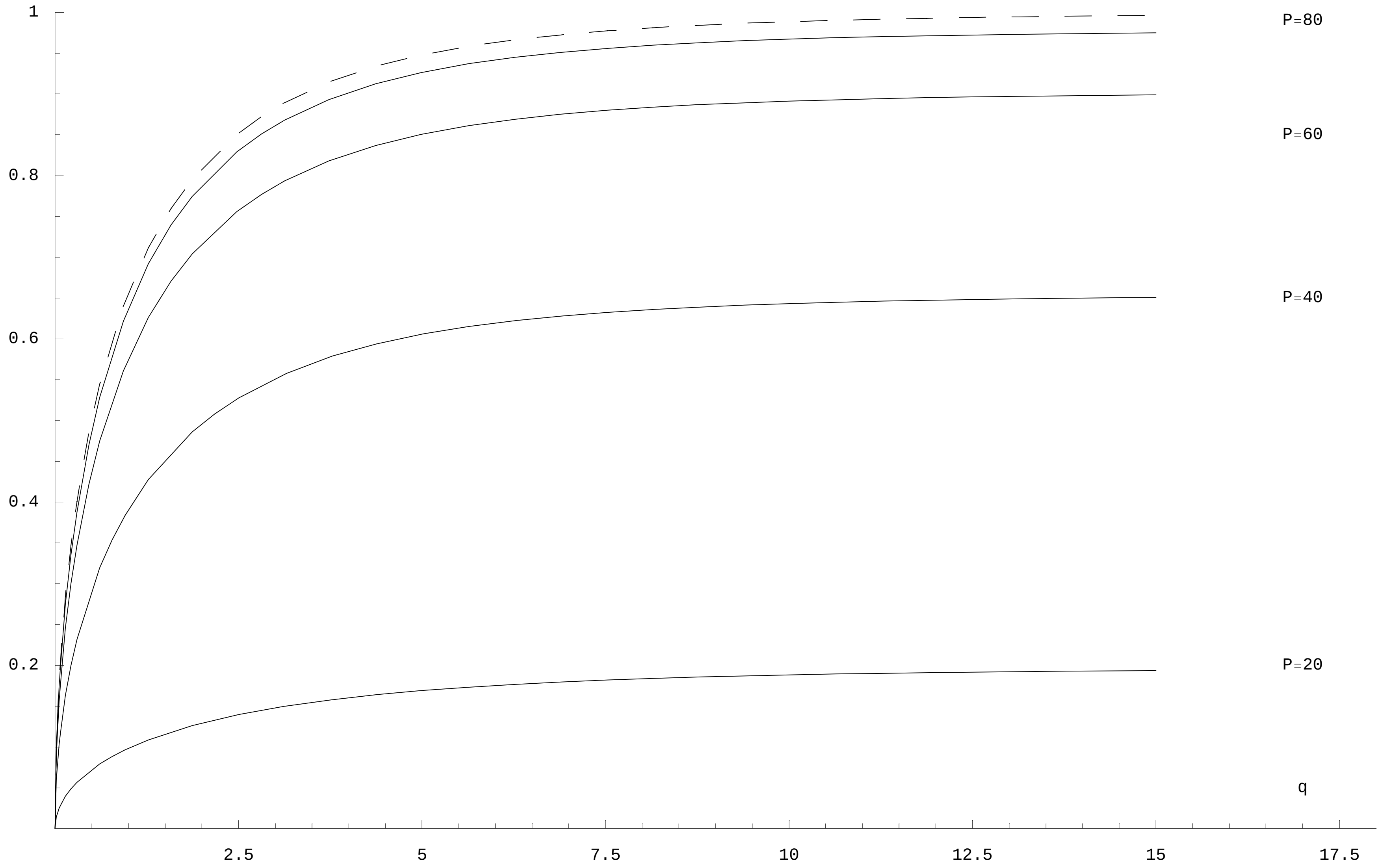

Figure 1. Exact (dashed line) CDF of F(1,9) and approximations (solid lines) for different values of P.