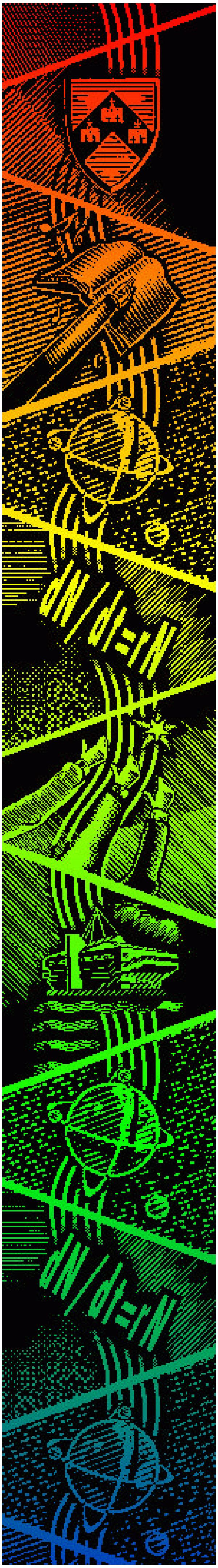

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Desirability of Nominal GDP Targeting Under Adaptive Learning

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Abstract. Nominal GDP targeting has been advocated by a number of authors since it produces relative stability of inflation and output. However, all of the papers assume rational expectations on the part of private agents. In this paper I provide an analysis of this assumption. I use stability under recursive learning as a criterion for evaluating nominal GDP targeting in the context of a model with explicit micro-foundations which is currently the workhorse for the analysis of monetary policy. The results of the paper provide support for such a monetary policy.

Keywords: Nominal GDP, learning, expectational stability.
JEL Classification: E4, E5.

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1. Introduction

Monetary policy rules that utilize as their principal target variable the level or growth rate of some aggregate measure of nominal spending, such as nominal GDP, have had considerable academic support since the early 1980s. More recently, arguments in favor of nominal GDP targeting have also been made, among others, by Hall and Mankiw (1994), Frankel and Chinn (1995), McCallum (1997c) and McCallum and Nelson (1999a) (see Hall and Mankiw (1994) and McCallum and Nelson (1999a) for a more extensive discussion of this literature). Nominal output targeting has two desirable features as a strategy for monetary policy. First, it automatically takes into account movements in both prices and real output, which in practice are the two variables central banks care about most. Second, nominal GDP can serve as a long-run nominal anchor for monetary policy given the common belief that monetary policy can not affect the real economy in the long-run.

Rudebusch (2000) points out that two distinct developments have also boosted an interest in nominal GDP targeting in recent years. The formation of the European Central Bank (ECB) in Europe has encouraged a lively debate about the appropriate strategy for European monetary policy. The announced ECB strategy contains an element of monetary targeting which is closely related to nominal output targeting if there are no large shifts in monetary velocity. In fact, the ECB (1999) has explicitly derived its 4.5% reference value for M3 growth from a desired growth rate for nominal output. The ECB's announced monetary strategy, therefore, provides support for the consideration of nominal output targeting. The second development which has increased interest in nominal GDP targeting has been the behavior of the U.S. economy in recent years. A number of macroeconomic forecasters have been making forecasting errors- both overpredicting inflation and underpredicting output growth (see, for instance, Figure 1 of Brayton, Roberts, and Williams (1999)). In light of this uncertainty about the level of potential output and the dynamics of the U.S. economy, several authors like McCallum (1999) and Orphanides (2000) suggest that monetary policy should focus on nominal GDP growth.

In this paper I study the desirability of nominal GDP targeting in the context of a standard forward looking model which is currently the workhorse in the analysis of monetary policy. I use the "New Phillips curve" model which has explicit micro-economic foundations and has been derived in a number of papers and reviewed, for instance, in Clarida, Gali, and Gertler (1999) and Woodford (1999). I first show that a monetary policy of targeting the growth rate of nominal income generically yields a unique equilibrium when the private sector has rational expectations of inflation and output. Such a question is of utmost importance since it is well known since the paper of Sargent and Wallace (1975) that uniqueness of equilibrium can not be taken for granted in such models. This question takes added importance in the context of nominal GDP targeting since Ball (1999) has argued that this policy would be destabilizing for output and inflation.

The novel contribution of this paper, however, comes from a different angle. Almost the entire literature of targeting nominal GDP has assumed rational expectations on the part of economic agents. By now it is well known that this assumption need neither be innocuous nor realistic. Agents are somehow assumed to be able to coordinate on a particular rational expectations equilibrium (REE). However, it is not obvious whether or how such coordination may arise. In order to complete such an argument, one needs to show the potential for agents to learn the equilibrium of the model being analyzed. An early instance where this was emphasized was Howitt (1992) who showed the instability under learning of interest rate pegging and related rules in flexible price and *ad hoc* IS-LM type models. In his conclusion he explicitly warned that, in general, any rational expectations analysis of monetary policy should be supplemented with an investigation of its stability under learning. He emphasized that the assumption of rational expectations (RE) can be quite misleading in the context of a fixed monetary regime- if the regime is not conducive to learnability, then the consequences can be quite different from those predicted under rational expectations.

Surely, expectational errors may arise in practice from changes in the economic structure or in the practices of policymakers. The assumption that agents somehow have rational expectations immediately after such changes is clearly very strong and need not be correct empirically. The learning analysis, on the other hand, allows for the possibility that expectations might not be initially fully rational, and that, if agents make forecast errors and try to correct them over time, the economy may or may not converge to the rational expectations equilibrium asymptotically. The purpose of this paper is to conduct such an analysis in the context of nominal GDP targeting. This has the potential to determine whether or not the particular rational expectations equilibrium can ever be observed. Bullard and Mitra (2000) recently used learning to evaluate monetary policy rules where the interest rate responds to measures of inflation and output gap- rules which have been popularized since Taylor's (1993) seminal contribution. They show that if agents are assumed to follow adaptive learning rules, then the stability of these Taylor-type monetary policy rules can not be taken for granted *even* in cases when the economy has a determinate equilibrium under rational expectations.

The central result of the paper is that a particular policy of targeting nominal GDP is stable under learning dynamics for all possible parametrizations of the model. This is a policy where the central bank bases the nominal interest rate directly on the expectations of private agents. In such a scenario, agents using adaptive learning mechanisms are indeed able to coordinate on the particular equilibrium under consideration.

The paper is organized as follows. Section 2 discusses the basic model which will be analyzed for determinacy and learnability in section 3 under a monetary policy of targeting the growth rate of nominal GDP. An important weakness of the basic model presented in section 2 is that there are no backward looking elements- consequently, such a model does not capture the persistence in output and inflation commonly observed in the data. Section 4 rectifies this defect and analyzes nominal GDP targeting in a model with endogenous output and inflation persistence. The analysis suggests that the desirability of GDP targeting is essentially unaffected with this extension.

2. The Model

I study a small forward-looking macroeconomic model recently analyzed in Clarida, Gali, and Gertler (1999) which is currently the workhorse for the theoretical analysis of monetary policy (see, for example, McCallum and Nelson (1999b), Rotemberg and Woodford (1999)). It is a dynamic general equilibrium model with temporary nominal price rigidities. Within the model, monetary policy affects the real economy in the short run, as in the traditional IS/LM model. A key difference from the traditional IS/LM model, however, is that the aggregate behavioral equations evolve explicitly from optimization by households and firms.

The basic model analyzed in Section 2 of Clarida, Gali, and Gertler (1999) consists of two structural equations:

$$x_t = -\varphi \left(i_t - \hat{E}_t \pi_{t+1} \right) + \hat{E}_t x_{t+1} + g_t \tag{1}$$

$$\pi_t = \lambda x_t + \beta \hat{E}_t \pi_{t+1} + u_t \tag{2}$$

where x_t is the "output gap" i.e. the difference between actual and potential output, π_t is the period t inflation rate defined as the percentage change in the price level from $t-1$ to t, and i_t is the nominal interest rate; each variable is expressed as a deviation from its long run level. Since our main focus will be on an analysis of learning we use the notation $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ to denote the possibly (subjective) private sector expectations of inflation and output gap next period, respectively, whereas the same notation without the "^" superscript will denote the rational expectations (RE) values.

Equation (1) is the intertemporal IS equation whereas equation (2) is the aggregate supply equation. The IS equation (1) can be derived from log-linearizing the Euler equation associated with the household's saving decision. The aggregate supply equation (the "new Phillips" curve) (2) can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. The parameters φ, λ, and β are structural and are assumed to be positive on economic grounds. In particular, $\beta \in (0, 1)$ is the discount factor and the interest elasticity of the IS curve, φ, corresponds to the intertemporal elasticity of substitution of consumption. The slope of the Phillips curve, λ, depends on the average frequency of price changes and the elasticity of demand faced by suppliers of goods. Prices are more nearly flexible the higher is λ. Finally, the demand shock g_t and the cost push shock u_t are assumed to follow first order autoregressive processes:

$$g_t = \mu g_{t-1} + \hat{g}_t \tag{3}$$

$$u_t = \rho u_{t-1} + \hat{u}_t \tag{4}$$

where $0 < \mu, \rho < 1$ and where both $\hat{g}_t$ and $\hat{u}_t$ are *iid* noise with zero means and variances σ_g^2 and σ_u^2, respectively. The model is closed by assuming that the nominal interest rate i_t is the instrument of monetary policy.

3. Growth Targeting

We assume that the objective of the central bank is to keep nominal income growth equal to a (constant) target value $\Delta\bar{z}$. Thus the central bank sets the interest rate i_t every period so as to make $\Delta z_t = \Delta\bar{z}$, where $z_t = x_t + p_t$, with p_t denoting (log) of the price level and z_t denoting (log) nominal GDP. In terms of inflation this implies

$$\pi_t + x_t - x_{t-1} = \Delta\bar{z} \tag{5}$$

Consequently, equation (5) will be satisfied if the central bank targets a constant growth of GDP every period. Substituting (1) and (2) into (5) yields

$$(1+\lambda)[-\varphi\left(i_t - \hat{E}_t\pi_{t+1}\right) + \hat{E}_t x_{t+1} + g_t] + \beta\hat{E}_t\pi_{t+1} + u_t - x_{t-1} = \Delta\bar{z},$$

and solving this gives a rule for setting the interest rate i_t that will stabilize Δz_t at $\Delta\bar{z}$. This rule is given by

$$\begin{aligned} i_t &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}\hat{E}_t\pi_{t+1} + \varphi^{-1}\hat{E}_t x_{t+1} - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \varphi^{-1}(1+\lambda)^{-1}u_t \\ &\quad + \varphi^{-1}g_t - \varphi^{-1}(1+\lambda)^{-1}\Delta\bar{z} \end{aligned} \tag{6}$$

Plugging this rule (6) into equation (1) yields

$$(1+\lambda)x_t = -\beta\hat{E}_t\pi_{t+1} + x_{t-1} - u_t + \Delta\bar{z} \tag{7}$$

Note that the interest elasticity of the IS equation (1), φ, does not appear in this reduced form IS curve (7). We provide some intuition for this in section 4. For the time being, we note that our complete system, under a policy of targeting a constant growth of nominal GDP every period, is given by (7) and (2), representing the evolution of the endogenous variables x_t and π_t, respectively.

3.1. Determinacy. Ball (1999) has argued that a monetary policy of nominal income targeting would be destabilizing for output and inflation in the sense that there fails to exist a stationary equilibrium in this case. Svensson (1997) has also made similar arguments.

However, as McCallum (1997a) has shown, this is a consequence of the backward looking model that these authors use. In a model similar to the one presented here, McCallum (1997a) has shown that this result can be reversed.[1] I now show that under nominal income targeting equilibrium is generically unique for all parameter configurations.

In the model represented by equations (7) and (2), there is only one predetermined endogenous variable x_{t-1}. We can rewrite our system in matrix form as

$$\begin{bmatrix} 1 & 0 & -(1+\lambda)^{-1} \\ -\lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ x_{t-1} \end{bmatrix}$$

$$= \begin{bmatrix} (1+\lambda)^{-1}\Delta\bar{z} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ x_t \end{bmatrix} + \begin{bmatrix} -(1+\lambda)^{-1} \\ 1 \\ 0 \end{bmatrix} u_t$$

The matrix which is relevant for uniqueness is obtained by pre-multiplying the matrix associated with the expectational variables on the right hand side with the inverse of the left hand matrix. This matrix is given by

$$B_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \beta & \lambda \\ 0 & \beta & 1+\lambda \end{bmatrix} \tag{8}$$

We have two free endogenous variables, x_t and π_t, and one predetermined endogenous variable, x_{t-1}. Consequently, following Farmer (1991, 1999), we need exactly two of the three eigenvalues of B_0 to be inside the unit circle for uniqueness.

Proposition 1. *Under nominal income growth targeting, equilibrium is generically unique.*

Proof. See Appendix A. ■

In general, when the nominal interest rate is the policy instrument of the central bank, determinacy of equilibrium can't be taken for granted. In fact the problem of indeterminacy is particularly acute for *simple* Taylor type rules which respond to future forecasts of output and inflation, that is, for rules of the form $i_t = \varphi_x \hat{E}_t x_{t+1} + \varphi_\pi \hat{E}_t \pi_{t+1}$ (see Bernanke and Woodford (1997) and Bullard and Mitra (2000)). However, Proposition 1 shows that this need not be a problem with nominal income targeting even though such

[1] McCallum's (1997a) model is a special case of the model presented in section 2 in the sense that he normalizes the discount factor β to be 1. In addition, I assume that the nominal interest rate is the policy instrument instead of the real interest rate (as assumed by McCallum (1997a)) since in practice this is a more realistic description of actual policy and it also facilitates the analysis of learning later on.

a policy makes the interest rate rule, (6), forward looking. Ignoring the response of the interest rate to the lagged output gap in (6), this can be understood as follows. Proposition 4 of Bullard and Mitra (2000) provides necessary and sufficient conditions for determinacy of equilibrium for rules of the form $i_t = \varphi_x \hat{E}_t x_{t+1} + \varphi_\pi \hat{E}_t \pi_{t+1}$. Roughly speaking, they show that equilibrium is determinate in a similar model if φ_x is small enough and φ_π is more than one, corresponding to an *active* Taylor rule (the precise magnitudes for determinacy depend on the structural parameters φ, λ, and β; see their Proposition 4). A value of φ_π more than one means that an increase in inflationary pressures causes the nominal interest rate to rise enough to also raise the real interest rate, thereby, reducing the output gap via the IS curve (1) and inflation via the AS curve (2). Even with a small φ_x, one may have indeterminacy if either φ_π is less than one, corresponding to a *passive* Taylor rule, *or* if it is too large. It is interesting to observe that the response to $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ in rule (6) does satisfy the determinacy conditions of Proposition 4 of Bullard and Mitra (2000). In other words, a policy of nominal GDP targeting forces the central bank's response to $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ to lie in the determinate region- the response to $\hat{E}_t x_{t+1}$ is small enough and that to $\hat{E}_t \pi_{t+1}$ is between 1 and 2 (i.e., it is aggressive but not overly aggressive). Note that forward looking interest rate rules have been found to describe the behavior of monetary policy in a number of industrialized countries (see Clarida, Gali, and Gertler (1998)). The analysis above shows that while the problem of indeterminacy may be acute for *ad hoc* Taylor type forward looking rules, the situation is quite different for similar rules which are geared towards maintaining a constant growth rate of GDP over time.

3.2. Learning. I now adapt methods developed by Marcet and Sargent (1989a, 1989b) and Evans and Honkapohja (1999, 2000a) to understand how learning affects these systems.[2] I assume that the agents in the model no longer have rational expectations at the outset. Instead, I replace expected values with adaptive rules, in which the agents form

[2] Some of the recent surveys of the literature on learning in macroeconomic models are Evans and Honkapohja (1999, 2000a), Grandmont (1998), Marimon (1997), and Sargent (1993). A small sample of the literature on learning specifically related to monetary policy includes Bertocchi and Spagat (1993), Ireland (1999), Evans, Honkapohja and Marimon (1998), and Barucci, Bischi, and Marimon (1998).

expectations using the data generated by the system in which they operate. We can imagine the agents to use versions of recursive least squares updating. I use theorems due to Evans and Honkapohja (1999, 2000a) and calculate the conditions for expectational stability (E-stability). Evans and Honkapohja (2000a) have shown that expectational stability, a notional time concept, corresponds to stability under real-time adaptive learning under quite general conditions. In particular, under E-stability of a rational expectations equilibrium (REE), recursive least squares learning is locally convergent to that equilibrium. We may assume that the fundamental disturbances have bounded support since equations (1) and (2) arise out of local linearization of the original nonlinear model. Under this assumption, if a REE is not E-stable, then the probability of convergence of the recursive least squares algorithm to it is zero. Due to this one to one correspondence between the expectational stability of a stationary REE and the stability under real-time adaptive learning, I focus only on expectational stability conditions throughout the paper and the terms "learnability," "expectational stability," and "stability in the learning dynamics" are all used interchangeably.

The analysis of learning is conducted under two different formulations which correspond to different assumptions made about the behavior of private agents by the central bank. In the first formulation, the central bank sets the interest rate in accordance with the policy rule (6) recognizing that the private sector does not have rational expectations. In other words, the interest rate rule is based directly on the subjective expectations of agents and is called the *contemporaneous expectations based policy rule*. This seems to me to be the most realistic assumption to make on the part of the central bank. I also discuss below ways in which such a policy can be implemented.

In the second formulation, the central bank (mistakenly) assumes that the private sector has perfectly rational expectations at every point of time. This means that the policy rule (6) is now based directly on the actual conditional expectations under RE (that is, on $E_t\pi_{t+1}$ and $E_t x_{t+1}$ instead of $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$) and is called the *rational expectations (RE) based policy rule*.[3] While I believe this assumption is not entirely

[3] The terminology is borrowed from Evans and Honkapohja (2000b).

realistic, it does serve as a useful benchmark to judge how useful the assumption of RE is as a guide for monetary policy in the long run. Another reason to study this formulation is that Evans and Honkapohja (2000b) have recently shown that the use of such an interest rate rule by the central bank in the conduct of *optimal* monetary policy leads to *instability* of the REE for *all* structural parameter values. One would like to see whether this instability result continues to be true when the central bank targets a given growth rate of GDP every period.

Contemporaneous Expectations Based Policy Rule. I first turn to a discussion of the *contemporaneous expectations based policy rule.* As mentioned above, in this formulation, I assume that the policy maker sets the nominal interest rate i_t in accordance with the rule (6). This means that the central bank bases its policy on the lagged output gap, the contemporaneous demand and cost push shocks as well as the subjective expectations of private agents. In particular, we are assuming that the bank is able to observe private sector expectations. Before turning to a formal discussion, I first discuss different ways of implementing this rule since it turns out to have desirable features from the point of view of learnability. One may view the commercial forecasts published by various agencies as being the expectations of the private sector. This is discussed extensively in Romer and Romer (2000). These commercial forecasts are often created by firms managing large portfolios so that in a sense these are indeed the forecasts of market participants. It is also the case that market participants often pay for these commercial forecasts; this suggests that they view information processing as difficult and commercial forecasts as valuable. Given this scenario, it is plausible to assume that the private sector will just adopt the commercial forecasts for their own use. Consequently, one way to implement the proposal would be for the central bank to target the predictions of private sector forecasts. This is discussed in Hall and Mankiw (1994). In the United States, there is a published consensus of respected private forecasters, for instance, the Blue Chip Economic Indicators. Around the fifth of each month, Blue Chip surveys economic forecasters at approximately 50 banks, corporations, and consulting firms and then produces a consensus forecast which is a median of the individual forecasts. Romer and Romer (2000) also discuss commercial forecasts prepared by the Survey of Professional Forecasters (SPF), currently being conducted by the Federal Reserve Bank of Philadelphia, which is also based on many commercial forecasts. A third source is commercial forecasts prepared by Data Resources, Inc. (DRI).[4] All in all, this shows that the interest rate rule (6) represents a feasible way to implement monetary policy.

I now proceed step by step to show how the analysis of learning takes place under this rule. I consider learning by private agents of the so called minimum state variable (MSV) solution (see McCallum (1983)). The MSV solution for the model given by (7) and (2) takes the following form:

$$x_t = a_1 + b_1 x_{t-1} + c_1 u_t + d_1 g_t \tag{9}$$

$$\pi_t = a_2 + b_2 x_{t-1} + c_2 u_t + d_2 g_t \tag{10}$$

where $(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2)$ are to be determined by the method of undetermined coefficients.[5] The MSV parameter values will be denoted by $\bar{a}_1$, $\bar{b}_1$ etc.. One computes the expectation

$$\hat{E}_t \pi_{t+1} = a_2 + b_2 x_t + c_2 \rho u_t + d_2 \mu g_t \tag{11}$$

Note that here we are assuming that the private sector is able to observe the contemporaneous output gap (and shocks) in forming its forecasts. We will have something to say about this towards the end of the section. Inserting (11) into the reduced form IS equation (7), one obtains

$$\begin{aligned}(1+\lambda)x_t &= -\beta \hat{E}_t \pi_{t+1} + x_{t-1} - u_t + \Delta \bar{z} \\ &= -\beta(a_2 + b_2 x_t + c_2 \rho u_t + d_2 \mu g_t) + x_{t-1} - u_t + \Delta \bar{z} \\ &= -\beta a_2 + \Delta \bar{z} - \beta b_2 x_t - \beta c_2 \rho u_t - \beta d_2 \mu g_t + x_{t-1} - u_t\end{aligned}$$

Solving for x_t finally yields the actual law of motion (ALM) for output as

$$x_t = \hat{a}_1 + \hat{b}_1 x_{t-1} + \hat{c}_1 u_t + \hat{d}_1 g_t \tag{12}$$

[4] See Romer and Romer (2000) for a more extensive discussion of these issues. Alternatively, as Evans and Honkapohja (2000b) note, if the central bank knows the learning rules of private agents, it can infer the expectations from other observed data.

[5] Actually, the MSV solution will not depend on the shock g_t, that is, $\bar{d}_1 = \bar{d}_2 = 0$ in the RE solution. However, I keep this dependence to allow for generality in the analysis of learning later on.

where

$$\hat{a}_1 = (1+\lambda+\beta b_2)^{-1}(\Delta\bar{z} - \beta a_2) \tag{13}$$

$$\hat{b}_1 = (1+\lambda+\beta b_2)^{-1} \tag{14}$$

$$\hat{c}_1 = -(1+\lambda+\beta b_2)^{-1}(\beta c_2\rho + 1) \tag{15}$$

$$\hat{d}_1 = -(1+\lambda+\beta b_2)^{-1}\beta d_2\mu \tag{16}$$

Similarly inserting (11) into the price adjustment equation (2), one can obtain the ALM for inflation as

$$\begin{aligned}
\pi_t &= \lambda x_t + \beta \hat{E}_t \pi_{t+1} + u_t \\
&= \lambda x_t + \beta(a_2 + b_2 x_t + c_2\rho u_t + d_2\mu g_t) + u_t \\
&= (\lambda + \beta b_2)x_t + \beta a_2 + (\beta c_2 \rho + 1)u_t + \beta d_2 \mu g_t \\
&= (\lambda + \beta b_2)(\hat{a}_1 + \hat{b}_1 x_{t-1} + \hat{c}_1 u_t + \hat{d}_1 g_t) + \beta a_2 + (\beta c_2\rho + 1)u_t + \beta d_2\mu g_t
\end{aligned}$$

Collecting terms finally gives the ALM for π_t as

$$\pi_t = \hat{a}_2 + \hat{b}_2 x_{t-1} + \hat{c}_2 u_t + \hat{d}_2 g_t \tag{17}$$

where

$$\hat{a}_2 = (\lambda+\beta b_2)\hat{a}_1 + \beta a_2 \tag{18}$$

$$\hat{b}_2 = (\lambda+\beta b_2)\hat{b}_1 \tag{19}$$

$$\hat{c}_2 = (\lambda+\beta b_2)\hat{c}_1 + \beta c_2\rho + 1 \tag{20}$$

$$\hat{d}_2 = (\lambda+\beta b_2)\hat{d}_1 + \beta d_2\mu \tag{21}$$

The MSV solution is obtained by solving the set of simultaneous equations $a_1 = \hat{a}_1, b_1 = \hat{b}_1, c_1 = \hat{c}_1, d_1 = \hat{d}_1, a_2 = \hat{a}_2, b_2 = \hat{b}_2, c_2 = \hat{c}_2, d_2 = \hat{d}_2$. As it turns out, for the analysis of expectational stability, we only need to compute the MSV solutions for b_1 and b_2. The two equations involving b_1 and b_2 (given by (14) and (19)) are independent from the rest of the system and they can be solved to get two solutions for b_2 given by

$$b_2^{\pm} = \frac{\beta - \lambda - 1 \pm \sqrt{(1+\lambda-\beta)^2 + 4\beta\lambda}}{2\beta} \tag{22}$$

Correspondingly, the solution for $b_1^{\pm}$ is given by

$$b_1^{\pm} = 1 - b_2^{\pm}$$

It is easy to see that b_2^- (i.e. the solution corresponding to the negative sign outside the square root in (22)) is negative, given our assumptions on the structural parameters, so that $b_1^- > 1$ rendering this solution non-stationary. Consequently, the unique stationary solution corresponds to b_2^+ (henceforth, denoted by $\bar{b}_2$). The corresponding value for b_1^+ (denoted by $\bar{b}_1$) is given by

$$\begin{aligned} \bar{b}_1 &= \frac{1+\beta+\lambda-\sqrt{(1+\lambda-\beta)^2+4\beta\lambda}}{2\beta} \\ &= \frac{2}{1+\beta+\lambda+\sqrt{(1+\lambda-\beta)^2+4\beta\lambda}} \end{aligned} \tag{23}$$

and that for $\bar{b}_2$ is given by

$$\bar{b}_2 = 1 - \bar{b}_1 \tag{24}$$

Observe that $0 < \bar{b}_1 < 1$ and $0 < \bar{b}_2 < 1$.

For the analysis of learning we regard (9) and (10) as a perceived law of motion (PLM) for the agents. Computing expectations, as before in (11), we obtain the corresponding ALM for output and inflation in (12) and (17), respectively. We can then define a mapping from the PLM to the ALM in the parameter space as

$$T(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) = (\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2) \tag{25}$$

The T mapping gives rise to the differential equation defining E-stability, namely

$$\frac{d}{d\tau}(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) = T(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) - \left(\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2\right)$$

The MSV solution corresponds to a fixed point of the T mapping and, hence, an equilibrium of the differential equation. The equilibrium is said to expectationally stable (E-stable) if it is a locally asymptotically stable point of the differential equation.

This model is merely a special case of the one treated in Evans and Honkapohja (1999, 2000a) and we now put our model in their formulation since it is easier to present our analytical results this way. For applying these results we first put the (reduced form) model given by (7) and (2) in matrix form as

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} (1+\lambda)^{-1}\Delta\bar{z} \\ \lambda(1+\lambda)^{-1}\Delta\bar{z} \end{bmatrix} + \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1} \\ 0 & \beta(1+\lambda)^{-1} \end{bmatrix} \begin{bmatrix} \hat{E}_t x_{t+1} \\ \hat{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} (1+\lambda)^{-1} & 0 \\ \lambda(1+\lambda)^{-1} & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -(1+\lambda)^{-1} & 0 \\ (1+\lambda)^{-1} & 0 \end{bmatrix} \begin{bmatrix} u_t \\ g_t \end{bmatrix} \quad (26)$$

To shorten notation, we can write (26) in the form

$$y_t = \alpha + B\hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa w_t \quad (27)$$

where $y_t = [x_t, \pi_t]'$, $w_t = [u_t, g_t]'$ and

$$\alpha = \begin{bmatrix} (1+\lambda)^{-1}\Delta\bar{z} \\ \lambda(1+\lambda)^{-1}\Delta\bar{z} \end{bmatrix} \quad (28)$$

$$B = \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1} \\ 0 & \beta(1+\lambda)^{-1} \end{bmatrix} \quad (29)$$

$$\delta = \begin{bmatrix} (1+\lambda)^{-1} & 0 \\ \lambda(1+\lambda)^{-1} & 0 \end{bmatrix} \quad (30)$$

$$\varkappa = \begin{bmatrix} -(1+\lambda)^{-1} & 0 \\ (1+\lambda)^{-1} & 0 \end{bmatrix} \quad (31)$$

which are the relevant matrices on the right hand side of (26). We also write $w_t = \Phi w_{t-1} + e_t$ where $e_t = [\hat{u}_t, \hat{g}_t]'$ and

$$\Phi = \begin{bmatrix} \rho & 0 \\ 0 & \mu \end{bmatrix} \quad (32)$$

The PLM of the agents takes the same form as the MSV solution given in (9) and (10) and we write this in matrix form as

$$y_t = a + by_{t-1} + cw_t \quad (33)$$

where $a = [a_1, a_2]'$,

$$c = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix} \quad (34)$$

The corresponding ALM is given by (12) and (17) and we write this in matrix form as

$$y_t = \hat{a} + \hat{b}y_{t-1} + \hat{c}u_t \quad (35)$$

where $\hat{a} = [\hat{a}_1, \hat{a}_2]'$,

$$\hat{c} = \begin{bmatrix} \hat{c}_1 & \hat{d}_1 \\ \hat{c}_2 & \hat{d}_2 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} \hat{b}_1 & 0 \\ \hat{b}_2 & 0 \end{bmatrix} \tag{36}$$

Note that the (unique) stationary MSV solution for b, $\bar{b}$, is given by the solution to the system $b = \hat{b}$ with b and $\hat{b}$ being given by (34) and (36). This matrix $\bar{b}$ is given by

$$\bar{b} = \begin{bmatrix} \bar{b}_1 & 0 \\ \bar{b}_2 & 0 \end{bmatrix} \tag{37}$$

where $\bar{b}_1$ and $\bar{b}_2$ are given by (23) and (24). Armed with this notation we are now in a position to prove the following proposition.

Proposition 2. *Suppose the time t information set is $(1, y_t', w_t')'$. The MSV solution under nominal income growth targeting is E-stable for all parameter values.*

Proof. See Appendix B. ■

Proposition 2 shows that for all admissible values of the structural parameters the relevant rational expectations equilibrium (REE) in this economy can be learnt by agents using simple adaptive learning rules. Some intuition can be provided for this result. In the context of *ad hoc* Taylor type rules, Bullard and Mitra (2000) have shown that expectational stability of the REE depends on the structural parameters of the economy as well as the policy parameters of the central bank. In particular, Proposition 5 of Bullard and Mitra (2000) shows that, for simple Taylor rules which respond to $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$, there is expectational stability if the response to $\hat{E}_t\pi_{t+1}$ is more than one (and that to $\hat{E}_t x_{t+1}$ is positive) whereas there is real danger of expectational *instability* if the response to $\hat{E}_t\pi_{t+1}$ is less than one. The interest rule (6) shows that a policy of targeting the growth rate of nominal GDP calls for the interest rate to respond to $\hat{E}_t\pi_{t+1}$ with a coefficient bigger than one, thereby, contributing to stability under learning dynamics. This means that a deviation of $\hat{E}_t\pi_{t+1}$ above its RE value calls for a rise in the real interest rate which reduces x_t via the IS curve (1) and π_t via the inflation equation (2). This in turn reduces $\hat{E}_t\pi_{t+1}$, thereby, driving the economy towards the initial equilibrium. Similarly, the rule (6) tightens monetary policy when $\hat{E}_t x_{t+1}$ rises above its RE value.

In Proposition 2, we have assumed that the private sector and the central bank have information on contemporaneous dated variables (like the output gap) in formulating their forecasts. However, several authors, like McCallum (1993, 1997b, 1999), are especially critical of this assumption since information on the current period output gap is rarely available when the bank (or the private sector) makes a decision. An alternative, suggested by McCallum (1993, 1997b, 1999), would be to assume that the private sector and the bank base their actions only on the last period's information, i.e. on the last quarter's output gap. Even with this more realistic information structure, it is possible to show that the rational expectations equilibrium continues to be learnable. The result about learnability of equilibrium is, therefore, quite robust to different assumptions made about the information structure.

Before concluding this section, we point out some assumptions made (implicitly) in the analysis of stability. The forecasts of agents under learning, given by (11), were directly incorporated in the model- therefore, it is implicitly assumed that the Euler equations represent behavioral rules which describe how private agents respond to their forecasts. In particular, agents respond only to expectations about next period variables and not to expectations further out in the future. This enables us to write the model in a linear setup and use the results of Evans and Honkapohja (1999, 2000a). In addition, the private sector, being monopolistically competitive, is assumed to be populated by large number of "small" agents- strategic behavior in expectations formation and learning is absent. Finally, the central bank is implicitly assumed to have knowledge of the true structure of the economy (given by (1) and (2)) as well as the key structural parameter values β, λ, and φ since it makes use of rule (6) in setting the interest rate. I believe this serves as a reasonable first approximation and, as we have seen, yields a very strong result on the learnability of the REE solution. In addition, if the bank follows the rule (6) with parameters deviating from the specified values by small amounts, then the economy will converge over time to an equilibrium which deviates from the REE by small amounts. I also conjecture that the results on E-stability will be unaltered if the central bank does not have knowledge of the key structural parameters λ and φ and is instead learning about them using least squares, as do the private agents.[6]

It is, however, important that the central bank recognize that expectations of the private sector need not be fully rational during the transition to REE and set the interest rate accordingly. If the central bank mistakenly assumes that agents have rational expectations during the transition and sets the interest rate accordingly then it no longer follows that the MSV solution will be expectationally stable, as we now show in the next section.

Rational Expectations (RE) Based Policy Rule. We now assume that the central bank continues to have knowledge of the true structure of the economy as well as the structural parameters. However, it mistakenly assumes that the private sector has RE at every point of time. Recall that the interest rule given by (6) responds directly to the expectations of the private sector. If the central bank assumes that the private sector has RE at every point of time, then it will use the actual conditional expectations, $E_t\pi_{t+1}$ and $E_t x_{t+1}$, in the rule (6) instead of the subjective expectations, $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$. Such an interest rate rule is called the *RE based policy rule.*

The structure of the economy is still given by (1), (2), and (6). We can show that there exists an MSV solution of the form

$$x_t = a_1 + b_1 x_{t-1} + c_1 u_t + d_1 g_t \tag{38}$$

$$\pi_t = a_2 + b_2 x_{t-1} + c_2 u_t + d_2 g_t \tag{39}$$

where $(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2)$ are to be determined by the method of undetermined coefficients. One computes

$$\hat{E}_t x_{t+1} = a_1 + b_1 x_t + c_1 \rho u_t + d_1 \mu g_t \tag{40}$$

$$\hat{E}_t \pi_{t+1} = a_2 + b_2 x_t + c_2 \rho u_t + d_2 \mu g_t \tag{41}$$

where we have assumed, as before, that the private sector is able to observe the contemporaneous output gap in formulating its forecasts. Inserting into (1), (2), and (6) one

[6] The basis for this conjecture is that in the analysis of E-stability of optimal monetary policy, Evans and Honkapohja (2000b) show that the results on E-stability are unaffected by such simultaneous learning by the central bank and the private sector. Replicating the arguments in the appendix of their paper should generate a similar positive result here since the analysis is essentially similar.

obtains, as before, the MSV values $(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)$. The corresponding interest rate rule under RE can be computed from (6) as[7]

$$\begin{aligned} i_t &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}E_t\pi_{t+1} + \varphi^{-1}E_t x_{t+1} - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \varphi^{-1}(1+\lambda)^{-1}u_t \\ &\quad + \varphi^{-1}g_t - \varphi^{-1}(1+\lambda)^{-1}\Delta\bar{z} \\ &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}(\bar{a}_2 + \bar{b}_2 x_t + \bar{c}_2\rho u_t + \bar{d}_2\mu g_t) + \varphi^{-1}(\bar{a}_1 + \bar{b}_1 x_t + \bar{c}_1\rho u_t + \bar{d}_1\mu g_t) \\ &\quad - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \varphi^{-1}(1+\lambda)^{-1}u_t + \varphi^{-1}g_t - \varphi^{-1}(1+\lambda)^{-1}\Delta\bar{z} \end{aligned}$$

Note that unlike the contemporaneous expectations based policy rule which was based on the subjective expectations of inflation and output, we have now plugged the RE parameter values to get the actual conditional expectations in deriving the RE based policy rule. This is the key difference between the two policy rules. On simplification, the RE based policy rule is given by

$$i_t = \psi_0 + \psi_x x_t - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \psi_u u_t + \psi_g g_t \tag{42}$$

where

$$\begin{aligned} \psi_0 &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}\bar{a}_2 + \varphi^{-1}\bar{a}_1 - \varphi^{-1}(1+\lambda)^{-1}\Delta\bar{z}, \\ \psi_x &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}\bar{b}_2 + \varphi^{-1}\bar{b}_1, \\ \psi_u &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}\bar{c}_2\rho + \varphi^{-1}\bar{c}_1\rho + \varphi^{-1}(1+\lambda)^{-1}, \\ \psi_g &= \{1 + \beta\varphi^{-1}(1+\lambda)^{-1}\}\bar{d}_2\mu + \varphi^{-1}\bar{d}_1\mu + \varphi^{-1}. \end{aligned}$$

We now plug the rule (42) into (1) to get

$$x_t = -\varphi[\psi_0 + \psi_x x_t - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \psi_u u_t + \psi_g g_t] + \varphi\hat{E}_t\pi_{t+1} + \hat{E}_t x_{t+1} + g_t$$

or rearranging terms we get the evolution of output as

$$(1+\varphi\psi_x)x_t = -\varphi[\psi_0 - \varphi^{-1}(1+\lambda)^{-1}x_{t-1} + \psi_u u_t] + \varphi\hat{E}_t\pi_{t+1} + \hat{E}_t x_{t+1} + (1-\varphi\psi_g)g_t \tag{43}$$

We can then get the evolution of inflation from (2) as

$$\begin{aligned} \pi_t &= [\beta + \lambda\varphi(1+\varphi\psi_x)^{-1}]\hat{E}_t\pi_{t+1} + \lambda(1+\varphi\psi_x)^{-1}\hat{E}_t x_{t+1} + \lambda(1+\lambda)^{-1}(1+\varphi\psi_x)^{-1}x_{t-1} \\ &\quad [1 - \lambda\varphi\psi_u(1+\varphi\psi_x)^{-1}]u_t + \lambda(1+\varphi\psi_x)^{-1}(1-\varphi\psi_g)g_t \end{aligned} \tag{44}$$

[7] The explicit values for $(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)$ are not needed here. Later on, however, we will need the explicit values for $\bar{b}_1$ and $\bar{b}_2$ for computing expectational stability.

Note that the actual evolution of output and inflation during the transition to rational expectations (obviously) still depends on the subjective expectations of agents as shown by (43) and (44). We can put (43) and (44) in matrix form as

$$y_t = \alpha_1 + B_1 \hat{E}_t y_{t+1} + \delta_1 y_{t-1} + \varkappa_1 w_t \tag{45}$$

where $y_t = [x_t, \pi_t]'$, $w_t = [u_t, g_t]'$ and

$$B_1 = \begin{bmatrix} (1+\varphi\psi_x)^{-1} & \varphi(1+\varphi\psi_x)^{-1} \\ \lambda(1+\varphi\psi_x)^{-1} & \beta + \lambda\varphi(1+\varphi\psi_x)^{-1} \end{bmatrix} \tag{46}$$

$$\delta_1 = \begin{bmatrix} (1+\lambda)^{-1}(1+\varphi\psi_x)^{-1} & 0 \\ \lambda(1+\lambda)^{-1}(1+\varphi\psi_x)^{-1} & 0 \end{bmatrix} \tag{47}$$

with the forms of α_1 and $\varkappa_1$ being omitted since they are not needed in what follows.

For the analysis of learning, we regard (38) and (39) as the PLM of the agents with corresponding forecasts given by (40) and (41). Plugging the interest rate rule (42) into (1) and (2) and simplifying yields

$$\begin{aligned} x_t &= (1+\varphi\psi_x - \varphi b_2 - b_1)^{-1}[a_1 + \varphi a_2 - \varphi\psi_0 + (1+\lambda)^{-1}x_{t-1} + (\varphi c_2\rho + c_1\rho - \varphi\psi_u)u_t + \\ &\quad (\varphi d_2\mu + d_1\mu + 1 - \varphi\psi_g)g_t] \end{aligned} \tag{48}$$

and

$$\pi_t = (\lambda + \beta b_2)x_t + \beta a_2 + (\beta c_2\rho + 1)u_t + \beta d_2\mu g_t \tag{49}$$

After plugging in the value of x_t from (48) into (49), we can define a map from the PLM, (38) and (39), to the ALM, (48) and (49), and the fixed points of this map give us the MSV values $(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)$. For the analysis of E-stability we need $\bar{b}_1$ and $\bar{b}_2$ which can be shown to be still given by (23), and (24). The MSV solution $(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)$ is E-stable if the eigenvalues of all of the following matrices

$$(I - B_1\bar{b})^{-1}B_1,$$

$$[(I - B_1\bar{b})^{-1}\delta_1]' \otimes [(I - B_1\bar{b})^{-1}B_1],$$

$$\Phi' \otimes (I - B_1\bar{b})^{-1}B_1$$

have real parts less than 1 where B_1, δ_1, Φ, and $\bar{b}$ are given by (46), (47), (32), and (37), respectively (see Evans and Honkapohja ch. 10). The actual E-stability conditions are quite complicated and analytical results do not seem possible. However, we can at least check E-stability for plausible values of structural parameters. The calibrated parameter values for the U.S. economy in Clarida, Gali, and Gertler (2000) are $\varphi = 1, \beta = .99, \lambda = .3, \mu = \rho = .9$. With these values, the REE is expectationally unstable. A similar situation prevails if we use the structural values in Woodford (1999) for the U.S. economy, namely $\varphi = (.157)^{-1} \simeq 6.37, \beta = .99, \lambda = .024$. Even with some other values of structural parameters, we get instability so that this result seems quite robust.

This result is similar in flavor to the one obtained in Evans and Honkapohja (2000b) where it was shown that the optimal monetary policy rule was unstable for all parameter values when the central bank (mistakenly) assumes RE at every point of time for the private sector. In this model too if the central bank targets a nominal variable like GDP and assumes that agents have RE, then it is quite likely that the REE outcome will not emerge as the long-run outcome of the economy. Some intuition for this is as follows. A deviation of $\hat{E}_t\pi_{t+1}$ above its RE value leads, through the reduced form IS curve, (43), to an increase in x_t (recall that from (42), $\psi_x > 0$ since $\bar{b}_1$ and $\bar{b}_2 > 0$) and, through the aggregate supply equation, to an increase in π_t. Over time this leads to upward revisions of both $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$. There is nothing in the interest rate rule (42) to offset this tendency and, over time, the economy moves further away from the REE. It is also quite instructive here to compare the reduced form IS curves under the contemporaneous expectations based policy rule, (7), with that under the RE based policy rule, (43). In the former case, a rise in $\hat{E}_t\pi_{t+1}$ above its RE value causes a *decrease* in x_t (and hence π_t) pushing the economy back towards the REE whereas in the latter case, the same deviation causes an *increase* in x_t and π_t, pushing the economy further away from the REE. I think this is the key to obtaining stability under the contemporaneous expectations based policy rule and instability under the RE based policy rule. The instability result is reminiscent of Howitt's (1992) warning about the assumption of RE leading to misleading results in monetary models. It also points to the importance of targeting the observable expectations of the private sector on the part of the central bank in formulating its interest rate rule.

4. Endogenous Inflation and Output Persistence

A basic problem with the model given by (1) and (2) is that it is entirely forward looking. The only backward looking element that enters the model is through the monetary policy of targeting the growth rate of GDP, namely, via the lagged output gap. As a result, this specification has difficulty capturing the inertia in output and inflation evident in the data (see, for example, Fuhrer and Moore (1995a, 1995b), Rudebusch and Svensson (1999)). Consequently, we now look at a model considered in Clarida, Gali, and Gertler (1999), section 6, which has important backward looking elements. The model consists of the structural equations

$$x_t = -\varphi\left(i_t - \hat{E}_t\pi_{t+1}\right) + (1-\theta)\hat{E}_t x_{t+1} + \theta x_{t-1} + g_t \tag{50}$$

$$\pi_t = \lambda x_t + (1-\phi)\beta\hat{E}_t\pi_{t+1} + \phi\pi_{t-1} + u_t \tag{51}$$

The parameters θ and ϕ capture the inertia in output and inflation, respectively, and are assumed to be between 0 and 1. The shocks g_t and u_t are still assumed to follow the processes (3) and (4). The central bank continues to target the growth rate of GDP so that equation (5) still applies. Substituting (50) and (51) into (5) yields the implied interest rate i_t that will stabilize Δz_t at $\Delta\bar{z}$. This rule is given by

$$\begin{aligned} i_t = {} & \{1+\beta\varphi^{-1}(1+\lambda)^{-1}(1-\phi)\}\hat{E}_t\pi_{t+1} + \varphi^{-1}(1-\theta)\hat{E}_t x_{t+1} + \{\theta\varphi^{-1} - \varphi^{-1}(1+\lambda)^{-1}\}x_{t-1} \\ & +\phi\varphi^{-1}(1+\lambda)^{-1}\pi_{t-1} + \varphi^{-1}(1+\lambda)^{-1}u_t + \varphi^{-1}g_t - \varphi^{-1}(1+\lambda)^{-1}\Delta\bar{z} \end{aligned} \tag{52}$$

Note that we are now assuming that the central bank uses the contemporaneous expectations based policy rule. Plugging the above rule into equation (50) yields

$$x_t = -\beta(1+\lambda)^{-1}(1-\phi)\hat{E}_t\pi_{t+1} + (1+\lambda)^{-1}x_{t-1} - \phi(1+\lambda)^{-1}\pi_{t-1} - (1+\lambda)^{-1}u_t + (1+\lambda)^{-1}\Delta\bar{z} \tag{53}$$

We can then get the reduced form of the price adjustment equation by substituting (53) into (51). This gives us

$$\begin{aligned} \pi_t = {} & \beta(1-\phi)\{1-\lambda(1+\lambda)^{-1}\}\hat{E}_t\pi_{t+1} + \lambda(1+\lambda)^{-1}x_{t-1} + \phi\{1-\lambda(1+\lambda)^{-1}\}\pi_{t-1} \\ & +\{1-\lambda(1+\lambda)^{-1}\}u_t + \lambda(1+\lambda)^{-1}\Delta\bar{z} \end{aligned} \tag{54}$$

Thus, our complete system is now given by (53) and (54), representing the evolution of the endogenous variables x_t and π_t, respectively. We write this in matrix form as

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} (1+\lambda)^{-1}\Delta\bar{z} \\ \lambda(1+\lambda)^{-1}\Delta\bar{z} \end{bmatrix} + \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1}(1-\phi) \\ 0 & \beta(1+\lambda)^{-1}(1-\phi) \end{bmatrix} \begin{bmatrix} \hat{E}_t x_{t+1} \\ \hat{E}_t \pi_{t+1} \end{bmatrix} + \begin{bmatrix} (1+\lambda)^{-1} & -\phi(1+\lambda)^{-1} \\ \lambda(1+\lambda)^{-1} & \phi(1+\lambda)^{-1} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -(1+\lambda)^{-1} \\ (1+\lambda)^{-1} \end{bmatrix} u_t \tag{55}$$

To shorten notation again, we can write (55) in the form

$$y_t = \alpha_2 + B_2 \hat{E}_t y_{t+1} + \delta_2 y_{t-1} + \varkappa_2 w_t.$$

where $y_t = [x_t, \pi_t]'$, $w_t = [u_t, g_t]'$ and

$$B_2 = \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1}(1-\phi) \\ 0 & \beta(1+\lambda)^{-1}(1-\phi) \end{bmatrix} \tag{56}$$

$$\delta_2 = \begin{bmatrix} (1+\lambda)^{-1} & -\phi(1+\lambda)^{-1} \\ \lambda(1+\lambda)^{-1} & \phi(1+\lambda)^{-1} \end{bmatrix} \tag{57}$$

We can immediately make one observation from the system (55). The IS equation parameters, φ and θ, do not enter (55). This generalizes a similar conclusion obtained in section 3. One can understand this phenomenon by getting the reduced form IS equation in a slightly different manner. Equation (5) may be rewritten as

$$\pi_t + x_t = x_{t-1} + \Delta\bar{z} \tag{58}$$

If we now substitute the equation for inflation (51) into (58) we get

$$(1+\lambda)x_t + \beta(1-\phi)\hat{E}_t\pi_{t+1} + \phi\pi_{t-1} + u_t = x_{t-1} + \Delta\bar{z} \tag{59}$$

After rearranging it can be seen that this is *exactly* the reduced form IS equation (53) obtained above, after substituting the interest rate rule (52) into the original IS equation (50). The central bank does not have *direct* control over expected inflation $\hat{E}_t\pi_{t+1}$ and inflation π_t. The interest rate only allows the bank to control directly the demand x_t and through it π_t. The evolution of the growth rate of nominal GDP, on the other hand, depends directly on the evolution of $\hat{E}_t\pi_{t+1}$ as well as the cost push shock u_t and π_t, as must be clear from (58) or (59). Consequently, even though the bank is able to offset the influence of the IS equation parameters and the demand shock g_t from the growth rate of GDP, through suitable control of the interest rate (namely, via the rule (52)), it is unable to do the same as far as the parameters of the aggregate supply equation, (51), are concerned. In particular, the discount factor of the suppliers, β, the persistence in inflation, ϕ, and the degree of price stickiness, λ, continue to affect the growth rate of GDP under a policy of GDP targeting.

We have seen that even though output inertia does not matter for uniqueness and learnability of equilibrium, inflation inertia could still potentially matter. For considering the uniqueness of equilibrium, we write our system as (ignoring the shocks)

$$\begin{bmatrix} 1 & 0 & -(1+\lambda)^{-1} & (1+\lambda)^{-1}\phi \\ 0 & 1 & -\lambda(1+\lambda)^{-1} & -\phi(1+\lambda)^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ x_{t-1} \\ \pi_{t-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\beta(1+\lambda)^{-1}(1-\phi) & 0 & 0 \\ 0 & \beta(1+\lambda)^{-1}(1-\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ x_t \\ \pi_t \end{bmatrix}$$

After pre-multiplying the right hand matrix by the inverse of the left hand matrix, we get the matrix J

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & \beta(1-\phi^{-1}) & -\lambda\phi^{-1} & \phi^{-1} \end{bmatrix}$$

Since there are now two free and two pre-determined endogenous variables, we need exactly 2 eigenvalues of J to be inside the unit circle for the existence of a unique equilibrium. Unfortunately, analytical results on determinacy do not seem possible any more.[8]

For analyzing the E-stability of equilibrium, we proceed as before. The MSV solutions of the system (55) take the form

$$y_t = a + by_{t-1} + cu_t \tag{60}$$

where $y_t = (x_t, \pi_t)'$. The MSV solution for b, $\bar{b}$, is given by solving the matrix quadratic

$$B_2 b^2 - b + \delta_2 = 0$$

[8] When $\phi = 1$, it is possible to show that there are two zero eigenvalues and two complex conjugate eigenvalues with absolute value more than 1; so we have determinacy in this case. By continuity, the same must be true for values of ϕ close to 1. A policy of nominal GDP targeting, therefore, does not cause explosive *instrument instability* even with high levels of inflation (and output) inertia.

where B_2 and δ_2 are given by (56) and (57). For E-stability, we assume that agents have a PLM of the form (60) which leads to an ALM of a similar form. Solving the fixed points of this map will give us the REE solutions which can then be checked for E-stability. Unfortunately, yet again, analytical conditions for E-stability are not available.[9] However, we can still check E-stability numerically for plausible parameter values. For example, we may use the calibrated parameter values for the U.S. economy in Clarida, Gali, and Gertler (2000) (see section 3). We also set $\phi = .71$, which is the value estimated in Rudebusch (2000). The matrix J then has the eigenvalues $\{0, .23, 1.09 + .74I, 1.09 - .74I\}$ so that equilibrium is unique. The unique stationary MSV solution for b is given by

$$\bar{b} = \begin{bmatrix} 0.68 & -0.58 \\ 0.32 & 0.58 \end{bmatrix}$$

and using Proposition 10.3 of Evans and Honkapohja (2000, Ch. 10) it can be checked that the MSV solution $(\bar{a}, \bar{b}, \bar{c})$ is E-stable since all the matrices required for checking this Proposition have real eigenvalues, with the maximal one being .23. Experimenting with some other values of ϕ, we continue to find that there exists a unique stationary MSV solution which is E-stable. We get the same conclusions if we use Woodford's (1999) parameter values for the U.S. economy. Note that these results are to be expected given our previous experience with the basic model in section 3 since the interest rule (52) continues to satisfy the "Taylor principle"- it reacts to $\hat{E}_t\pi_{t+1}$ with a coefficient bigger than one and to $\hat{E}_t x_{t+1}$ with a positive coefficient. Observe also that there continues to be a negative relationship between $\hat{E}_t\pi_{t+1}$ and x_t in the reduced form IS curve (53).

5. Conclusions

A monetary policy of targeting nominal GDP has been advocated by economists for almost two decades now. However, most of the theoretical and empirical defence has taken place in the context of *ad hoc* macroeconomic models. Not much is known about the theoretical behavior of such a policy in models with explicit micro-foundations which are currently being used to give advice to policy makers. In this paper I have shown that

[9] If $\phi = 1$, then B_2 in (56) becomes the null matrix and it is easy to see then that the MSV solution is E-stable. Thus, with a high degree of persistence in inflation and output, we have a determinate equilibrium which is stable under learning dynamics.

such a policy generically leads to a determinate equilibrium in the context of a standard forward looking model which is currently the workhorse for the theoretical analysis of monetary policy- there are no problems of encountering unexpected volatility in inflation and output if a central bank follows such a policy. This is an important consideration since it is well known now that monetary policy rules where the nominal interest rate responds to deviations of inflation and output from target levels (Taylor type rules) may easily lead to indeterminate equilibria (Bernanke and Woodford (1997) and Bullard and Mitra (2000)).

I have gone further than merely analyzing the determinacy of REE under a policy of targeting nominal GDP. It is recognized by a number of economists now that the assumption of rational expectations on the part of private agents is unusually strong. Consequently, I have studied the stability of these macroeconomic systems under learning using methods developed by Evans and Honkapohja (1999, 2000a). In general, determinacy alone is insufficient to induce learnability of a REE in such models. This has been recently shown in Bullard and Mitra (2000) in the context of Taylor type rules.

However, in the context of nominal GDP targeting, I find that determinate equilibria are indeed learnable, even when account is taken of endogenous inflation and output persistence in the model.[10] This is especially heartening since policy rules which lead to unlearnable equilibria are to be avoided. I think this is reasonable since we have already endowed agents with quite a bit of information about the economy in the formulation of adaptive learning in the sense that the perceived law of motion of the agents corresponds to the MSV solution. The agents have the right variables and the right relationship between the variables, as well as initial conditions in the neighborhood of the equilibrium. If agents are unable to learn the MSV solution even under this very favorable assumption, then they are unlikely to learn the equilibrium under more general assumptions.[11] Consequently,

[10] To economize on space, I have only presented the case of the bank targeting a given growth rate of nominal GDP. Some authors like Hall and Mankiw (1994) have, however, advocated a policy of targeting the *level* of GDP. It can be shown that the results obtained for determinacy and learnability also carry over to the case of level targeting, that is, equilibrium is generically unique and E-stable for all possible parametrizations of the model.

[11] An analogy I have in mind are the notions of *weak* and *strong* E-stability used in the learning literature. If a certain equilibrium is not weakly E-stable, then it cannot be strongly E-stable.

learnability of MSV solutions under a particular policy rule should be taken as a minimal requirement in monetary models before being advocated to policy makers.

We have also seen that it may be dangerous for central banks to assume RE on the part of the private agents at every point of time- the economy may diverge from the REE in this case. The central bank should instead base the interest rate directly on the expectations of private agents; something which has been emphasized in Hall and Mankiw (1994). This type of policy rule is conducive to agents being able to coordinate on the unique equilibrium of the economy. This positive result provides an additional argument in favor of nominal GDP targeting and reinforces the views of economists in favor of such a monetary policy. It also provides some support to the announced ECB strategy of monetary targeting.

6. References

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A. Proof of Proposition 1

From the structure of B_0 (given by (8)) it is evident that one of the eigenvalues is zero. The remaining two eigenvalues are given by those of the matrix

$$\begin{bmatrix} \beta & \lambda \\ \beta & 1+\lambda \end{bmatrix}$$

with the following characteristic polynomial

$$p(\phi) = \phi^2 - (1+\beta+\lambda)\phi + \beta$$

Note that $p(0) = \beta > 0$ and $p(1) = -\lambda < 0$ so that one of these eigenvalues is between 0 and 1 and the other is more than 1 (by exploiting the continuity of $p(\phi)$ in ϕ). This shows that exactly 2 eigenvalues of B_0 are inside the unit circle or that equilibrium is unique.

B. Proof of Proposition 2

We have spelled out the PLM, the ALM, and the T map from the PLM to the ALM in equations (33), (35), and (25), respectively. As we noted, our model is merely a special case of the one treated in Evans and Honkapohja (2000, ch. 10). Consequently, we can apply their results (in particular Proposition 10.3) directly here. The MSV solution $(\bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)$ is E-stable if the eigenvalues of all of the following matrices

$$(I - B\bar{b})^{-1}B$$

$$[(I - B\bar{b})^{-1}\delta]' \otimes [(I - B\bar{b})^{-1}B]$$

$$\Phi' \otimes (I - B\bar{b})^{-1}B$$

have real parts less than one. Recall that the matrices B, δ, Φ, and $\bar{b}$ are given, respectively, by (29), (30), (32) and (37). The eigenvalues of $(I - B\bar{b})^{-1}B$ can be shown to be 0 and

$$\gamma = \frac{2\beta}{1+\beta+\lambda+\sqrt{(1+\lambda-\beta)^2 + 4\beta\lambda}} = \beta\bar{b}_1 < 1$$

where $\bar{b}_1$ is given by (23) and is between 0 and 1. The eigenvalues of $\Phi' \otimes (I - B\bar{b})^{-1}B$ are given by $0, 0, \rho\gamma$ and $\mu\gamma$ (by the properties of kronecker products) which are again less than one by our assumptions on ρ and μ. Finally, it can be shown that three of the eigenvalues of $[(I - B\bar{b})^{-1}\delta]' \otimes [(I - B\bar{b})^{-1}B]$ are 0 and the only non-zero eigenvalue is given by

$$\frac{4\beta}{[1+\beta+\lambda+\sqrt{(1+\lambda-\beta)^2+4\beta\lambda}]^2} = \beta\bar{b}_1^2 < 1$$

This proves E-stability of the MSV solution.

```
"July 3, 2001";
"DESIRABILITY OF NOMINAL GDP TARGETING UNDER ADAPTIVE LEARNING";
"PROOF OF PROPOSITION 2";
Clear[β, δ, λ, ρ, μ, B, phi, bbar, b1bar, b2bar, A1, A2, DTa, DTb, DTc]
"The Mathematica routine below computes
  eigenvalues of the matrices required for checking E-stability";
"See Proposition 10.3 of Evans-Honkapohja (2001)";

"B matrix below";
B = {{0, -β / (1 + λ)}, {0, β / (1 + λ)}};
"δ matrix below";
δ = {{1 / (1 + λ), 0}, {λ / (1 + λ), 0}};
"phi matrix below";
phi = {{ρ, 0}, {0, μ}};
"bbar matrix below";
"b1bar=2/(1+β+λ+Sqrt[(1+λ-β)^2+4*β*λ])";
"Note that the value of b1bar is not needed below for the proof";
b2bar = 1 - b1bar;
bbar = {{b1bar, 0}, {b2bar, 0}};

"The function 'kronecker' defines how to compute the
  kronecker product of 2 matrices p and q with f taken to be 'Times'";
kronecker[f_, p_List, q_List] :=
 Flatten[Map[Flatten, Transpose[Outer[f, p, q], {1, 3, 2}], {2}], 1];
 "Matrices for checking EXPECTATIONAL STABILITY";
A1 = Transpose[Inverse[IdentityMatrix[2] - B . bbar] . δ];
A2 = Inverse[IdentityMatrix[2] - B . bbar] . B;

"Eigenvalues of DTa";
DTa = Inverse[IdentityMatrix[2] - B . bbar] . B;
Simplify[Eigenvalues[DTa]]
"Eigenvalues of DTb";
DTb = kronecker[Times, A1, A2];
 Simplify[Eigenvalues[DTb]]
"Eigenvalues of DTc";
DTc = kronecker[Times, Transpose[phi], A2];
Simplify[Eigenvalues[DTc]]
```

$$\left\{0, \frac{\beta}{1+\beta-\text{b1bar}\,\beta+\lambda}\right\}$$

$$\left\{0, 0, 0, \frac{\beta}{(1+\beta-\text{b1bar}\,\beta+\lambda)^2}\right\}$$

$$\left\{0, 0, \frac{\beta\,\mu}{1+\beta-\text{b1bar}\,\beta+\lambda}, \frac{\beta\,\rho}{1+\beta-\text{b1bar}\,\beta+\lambda}\right\}$$