THE UNIVERSITY *of York*

Discussion Papers in Economics

No. 2000/41

Learning About Monetary Policy Rules

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This draft: June 27, 2000‡

Abstract. We study macroeconomic systems with forward-looking private sector agents and a monetary authority that is trying to control the economy through the use of a linear policy feedback rule. A typical finding in the burgeoning literature in this area is that policymakers should be relatively aggressive in responding to available information about the macroeconomy. A natural question to ask about this result is whether policy responses which are too aggressive might actually destabilize the economy. We use stability under recursive learning *a la* Evans and Honkapohja (2000) as a criterion for evaluating monetary policy rules in this context. We find that considering learning can alter the evaluation of alternative policy rules. *JEL Classification* E4, E5.

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*Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System. This paper is available on the world wide web at www.stls.frb.org/research/econ/bullard/.

†This paper was written when the latter author was a postdoctoral fellow at the Research Unit on Economic Structures and Growth (RUESG), Department of Economics, University of Helsinki. Financial support from the Academy of Finland, Osuuspankki, and the Yrjö Jahnsson Foundation is gratefully acknowledged by him. He is also grateful to the Washington University in St. Louis in general and Gaetano Antinolfi in particular for the research support and hospitality during his visit.

‡A version of this paper was originally prepared for the conference "Expectations, Economic Theory and Economic Policy," Perugia, Italy, 23-26 September 1999. The authors thank the Bank of Italy, the European University Institute, and the Centre for Economic Policy Research (CEPR) for sponsoring this event. The authors also thank Graziella Bertocchi, Jordi Gali, Seppo Honkapohja, Ramon Marimon, Mark Salmon, Tom Sargent, and Michael Woodford, as well as participants at seminars, a NBER Monetary Economics conference, a Bank of Finland/CEPR Workshop, and a meeting of Federal Reserve System Committee on Macroeconomics for many helpful comments and discussions. The authors also thank Steve Majesky for research assistance.

1. INTRODUCTION

1.1. Overview. Monetary policy rules have been the subject of a good deal of recent research in the literature on monetary economics and monetary policy.[1] While some of this work has focussed on systems which abstract from or suppress private sector expectations, many of the more recent papers analyze systems where private sector expectations enter the model explicitly.[2] Most of these models involve small, forward-looking representations of the macroeconomy, such as those found in Bernanke and Woodford (1997), Clarida, *et al.*, (1999), Kiley (1998), McCallum and Nelson (1999), and Woodford (1999*a*). In many cases the small model is a log-linearized and simplified version of a larger model derived from optimizing behavior in a dynamic stochastic general equilibrium context.

When private sector expectations enter such models explicitly, recent research has emphasized the possibility that certain policy rules may be associated with indeterminacy of rational expectations equilibrium (REE), and therefore might be viewed as undesirable. Some of the authors that discuss this issue include Bernanke and Woodford (1997), Carlstrom and Fuerst (1998), Christiano and Gust (1999), Clarida, *et al.*, (2000), Rotemberg and Woodford (1998, 1999), and Woodford (1999*a*). In a typical analysis, the authors compute the rational expectations solutions of the system with a given monetary policy rule, and if the rule induces indeterminacy then it is viewed as undesirable. The idea is that if the monetary authorities actually followed such a rule, the system might be unexpectedly volatile as agents are unable to coordinate on a particular equilibrium among the many that exist.[3] In contrast, when equilibrium is determinate, it is normally assumed that the agents can coordinate on that equilibrium.

It is far from clear, however, exactly how or whether such coordination would arise. In order to complete such an argument, one needs to show the potential for agents to learn the equilibrium of the model being analyzed. In this paper, we take on this task.

[1] For a sample of the recent work, see the volumes edited by Taylor (1999) and King and Plosser (1999), and the survey by Clarida, Gali and Gertler (1999).

[2] For a discussion of the differences between forward-looking and backward-looking systems from the perspective of monetary policy rules and optimal control problems, as well as other related issues, see Svensson (1999) and the associated discussion by Woodford (1999*b*).

[3] Alternatively, the agents may be able to coordinate, but the risk exists that the equilibrium achieved may be one with undesirable properties, such as a large degree of volatility. See Woodford (1999*a*, pp. 67-69).

We assume the agents of the model do not initially have rational expectations, and that they instead form forecasts by using recursive learning algorithms—such as recursive least squares—based on the data produced by the economy itself. Our methodology is that of Evans and Honkapohja (1999, 2000).[4] We ask whether the agents in such a world can learn the equilibrium of the system under a range of possible Taylor-type monetary policy feedback rules. We use the criterion of *expectational stability* (a.k.a. *E-stability*) to calculate whether rational expectations equilibria are stable under real time recursive learning dynamics or not. The research of Marcet and Sargent (1989*a*, 1989*b*) and Evans and Honkapohja (1994, 1999, 2000) has shown that the expectational stability of rational expectations equilibrium governs local convergence of real time recursive learning algorithms in a wide variety of macroeconomic models.[5]

We think of learnability as a necessary additional criterion for evaluating alternative monetary policy feedback rules. In particular, in our view economists should only advocate policy rules which induce learnable rational expectations equilibria. Central banks adopting monetary policy rules that are not associated with learnable rational expectations equilibria, under the assumption that private sector agents will coordinate on the equilibrium they are targeting, are making an important mistake. Our analysis suggests that such policymakers will encounter difficulties, as the private sector agents instead fail to coordinate, and the macroeconomic system diverges away from the targeted equilibrium. Learnable equilibria, on the other hand, do not have such problems. Here, the agents can indeed coordinate on the equilibrium the policymakers are targeting, so that the learning dynamics tend toward, and eventually coincide with, the rational expectations dynamics. Learnable equilibria are therefore to be recommended.[6]

[4] Some of the recent surveys of the literature on learning in macroeconomic models are Evans and Honkapohja (1999, 2000), Grandmont (1998), Marimon (1997), and Sargent (1993). A small sample of the literature on learning specifically related to monetary policy includes Howitt (1992), Bertocchi and Spagat (1993), Ireland (1999), Evans, Honkapohja and Marimon (1998), and Barucci, Bischi, and Marimon (1998).

[5] Accordingly, we use the terms "learnability," "expectational stability," "E-stability," and "stability in the learning dynamics" interchangably in this paper.

[6] As Taylor (1993, pp. 203-208) has emphasized, it is also important to consider learning during transitions from one policy rule to a new policy rule.

1.2. Model environment. We consider monetary policy rules which have been suggested by various authors. All of these rules envision the central bank adjusting a short-term nominal interest rate in linear response to deviations of inflation from some target level and to deviations of real output from some target level. We take up four variants of such rules which we believe are representative of the literature: rules where the nominal interest rate set by the central bank responds to deviations of current values of inflation and output (we call this the *contemporaneous data* specification); rules where the interest rate reacts to lagged values of output and inflation deviations (*lagged data* specification); rules where the interest rate responds to future forecasts of inflation and output deviations (*forward looking* rules); and finally, rules which respond to current expectations of inflation and output deviations (*contemporaneous expectations*).

The novel contribution of this paper is to evaluate these policy rules based on the learnability criterion in a standard, small, forward-looking model which is currently the workhorse for the study of such rules. We analyze the stability of equilibria under learning dynamics for these monetary policy rules. We also provide conditions for unique equilibria for these policy rules. Conditions for unique equilibria may be found sporadically for some of these policy rules in the existing literature, and we put these results into a unifying framework. Thus, we are able to evaluate monetary policy rules based not only on whether they induce determinacy but also based on whether they induce learnability.

1.3. Main results. We find that monetary policy rules which react to current values of inflation and output deviations can easily induce determinate equilibria. Moreover, when equilibrium is determinate it is also learnable under this specification. Rules of the contemporaneous data type, considered by Taylor (1993) and argued to explain well the observed actions of the Federal Reserve since the mid-1980s, led to widespread interest in the subject of nominal-interest-rate-based monetary policy rules. However, these rules have often been criticized because they place unrealistic informational demands on the central bank, since precise information on current quarter values of inflation and output is usually not available to policymakers. One of our important findings is that rules which react to *contemporaneous expectations* of inflation and output deviations lead to *exactly* the same regions of determinate *and* learnable equilibria. Consequently, our results suggest that rules where the central bank responds to current expectations of inflation and output deviations are the most desirable in terms of generating both determinacy and learnability. Our reading of the policy rules literature is that such rules have not been given adequate attention[7] and our results suggest more emphasis on them may prove fruitful.

We find that rules which respond to lagged values or to future forecasts of inflation and output deviations do not have the same desirable properties. Rules which respond to lagged data can easily fail to generate determinacy, especially when the central bank responds aggressively to inflation and output. In addition, determinate rational expectations equilibria are not necessarily learnable under the lagged data specification. Forward-looking rules can easily induce equilibrium indeterminacy (see also Bernanke and Woodford (1997)). The danger of indeterminacy rises with more aggressive response to either inflation or output deviations. We find that determinate equilibria are always learnable for forward-looking rules. However, when equilibrium is indeterminate, those equilibria which correspond to the minimum state variable (MSV) solution may also be learnable.[8]

Several authors, such as Taylor (1999) and Clarida, Gali and Gertler (2000), tend to favor "leaning against the wind" policy by the central bank. For example, Taylor (1999) recommends a policy rule which calls for tightening market conditions in response to higher inflation or to increases in production. This would be ensured by a rule where the nominal interest rate responds aggressively to inflation with the coefficient on inflation exceeding one (called *active* rules in the literature) and the coefficient on the output gap is positive. The intuition provided by Taylor (1999) in this case is that a rise in inflation would bring about an *increase* in the real interest rate which would decrease demand and thus reduce inflationary pressures, bringing the economy back towards equilibrium. On the other hand, if the coefficient on inflation is less than one then an increase in inflation would bring about a *decrease* in the real interest rate which would increase demand and

[7] A possible exception is McCallum (1997, p. 358) who does suggest such a policy rule.

[8] We do not examine the learnability of sunspot equilibria, which exist when the equilibrium is indeterminate, in this paper.

add to upward pressures on inflation, pushing the economy away from equilibrium.

In this paper we support the intuition of these authors based on the additional criterion of learnability. If agents do not have rational expectations of inflation and output and instead start with some subjective expectations of these variables, learning recursively using some version of least squares, then a "leaning against the wind" policy on the part of the central bank does indeed push the economy toward the rational expectations equilibrium across the specifications of policy rules we consider. Conversely, a policy rule in which the coefficient on inflation is less than one is likely to be destabilizing in the sense that if agents do not start with rational expectations then they are unlikely to be able to coordinate on the equilibrium induced by the policy rule.

In general, there is no agreement among academicians about the type of policy rules central banks should use, for example, whether they should base their setting of the interest rate on lagged data or future forecasts. We find that across all the types of rules we consider, active rules with little or no reaction to the output gap (or output gap forecasts) generally induce both determinate and learnable rational expectations equilibria. To the extent that both determinacy and learnability are desirable criteria, central banks may want to consider adopting such rules.

1.4. Organization. In the next section we present the model we will analyze throughout the paper. We also discuss the types of linear policy feedback rules we will use to organize our analysis, and a calibrated case which we will employ. In the subsequent sections, we present results on determinacy of equilibrium, and then on learnability of equilibrium, for each of four different classes of policy rules. We conclude with a summary of our findings.

2. The environment

2.1. A baseline model. We study a simple and small forward-looking macroeconomic model analyzed by Woodford (1999a). Woodford (1999a) derived his model from a more elaborate, optimizing framework with sticky prices studied by Rotemberg and Woodford (1998, 1999), and intended it to be a parsimonious description of the U.S. economy, with mechanisms that would remain prominent in nearly any model with complete microfoundations.

We study Woodford's (1999*a*) model without making any changes to it, except for a small notational alteration, namely that we replace Woodford's x with our z. We accordingly write Woodford's (1999*a,* p. 16) system as

$$z_t = z^e_{t+1} - \sigma^{-1}\left(r_t - r^n_t - \pi^e_{t+1}\right) \tag{1}$$

$$\pi_t = \kappa z_t + \beta \pi^e_{t+1} \tag{2}$$

where z_t is the output gap, π_t is the inflation rate, and r_t is the deviation of the short-term nominal interest rate from the value that would hold in a steady state with a given target level of inflation and steady state output growth, and a superscript e represents a (possibly nonrational) expectation. We use this notation for expectations so that we can be flexible in describing our systems under both rational expectations and learning, along with the accompanying informational assumptions in each case. In the nomenclature of this literature, equation (1) is the intertemporal IS equation whereas equation (2) is the aggregate supply equation. The parameters σ, κ, and β are structural, arising from the analysis of the larger dynamic stochastic general equilibrium model, and from the subsequent approximations to that model that produce these equations. In particular, $\beta \in (0,1)$ is the discount factor in the larger model, and $\beta^{-1} - 1$ is the steady state real rate of interest for the economy. The "natural rate of interest" r^n_t is an exogenous stochastic term that follows the process

$$r^n_t = \rho r^n_{t-1} + \epsilon_t \tag{3}$$

where ϵ_t is *iid* noise with variance σ^2_ϵ, and $0 \leq \rho < 1$ is a serial correlation parameter.

We supplement equations (1), (2), and (3), which represent the behavior of the private sector, with a policy rule, which represents the behavior of the monetary authority. We use

$$r_t = r^\star + \varphi_\pi\left(\pi_t - \pi^\star\right) + \varphi_z(z_t - z^\star) \tag{4}$$

where r_t is the short-term nominal interest rate, which is set by the monetary authority in response to deviations of inflation and output from desired or steady state levels.[9] The inflation target $\pi^{\star}$ is a constant, as is the nominal interest rate target $r^{\star}$ and the output gap target $z^{\star}$. In this paper we will assume that $\pi^{\star} = r^{\star} = z^{\star} = 0$, mainly because the dynamics we study will not depend on the values of these constant terms.[10] We will allow several different parameter configurations for φ_{π} and φ_{z} in order to check the robustness of our results across various descriptions of policymaker behavior. We will also consider alternative informational assumptions in the next subsection.

Our complete system for the baseline model is, therefore, given by[11]

$$z_t = z_{t+1}^e - \sigma^{-1}\left(r_t - r_t^n - \pi_{t+1}^e\right), \tag{5}$$

$$\pi_t = \kappa z_t + \beta \pi_{t+1}^e, \tag{6}$$

$$r_t = \varphi_{\pi}\pi_t + \varphi_z z_t, \tag{7}$$

$$r_t^n = \rho r_{t-1}^n + \epsilon_t. \tag{8}$$

2.2. Alternative specifications for setting interest rates. In the model given by equations (5)-(8), only the private sector forms expectations about future values of endogenous variables. The policymakers, whose behavior is embodied in equation (7), only react to information which is observed at time t. In part, this is the nature of systems in which policy feedback rules describe behavior: Taylor's (1993) original motivation for considering such rules was in part that the policymaker respond in a simple and transparent way to available data. McCallum (1993, 1997, 1999) has often argued that such reaction functions are unrealistic, since actual policymakers do not have complete information on variables such as output and inflation in the quarter they must make a decision. We think it is interesting to consider the systems formed when we use alternative informational

[9] Versions of Taylor rules with inertia contain a lagged interest rate term on the right hand side. We analyze only "simple" rules in this paper, and we take up rules with inertia in a companion paper. See Bullard and Mitra (2000).

[10] We note that $r^{\star} = 0$ actually allows for a slightly positive nominal interest rate in steady state, because as Woodford (1999*a*) notes, $r^{\star} = \log \beta$, a slightly negative number, corresponds to zero nominal interest rate in this model. We are not assuming, therefore, that the economy has attained the zero bound on nominal interest rates.

[11] Woodford's model does not have any "cost push" shock as, for example, in Clarida, Gali and Gertler (1999). We could add such shocks to (6) as well as shocks to the monetary authorities' reaction function (7) that are autoregressive of order one without affecting our results in the paper. For simplicity, we omit such shocks from our analysis.

assumptions. Accordingly, we will call the specification embodied in equation (7) our *contemporaneous data* specification, and we will consider other possibilities below.[12]

One alternative is to follow one of McCallum's suggestions and posit that the monetary authorities must react to last quarter's observations on inflation and the output gap, which could possibly be viewed as closer to the reality of central bank practice. This leads to our *lagged data* specification for our interest rate equation,

$$r_t = \varphi_\pi \pi_{t-1} + \varphi_z z_{t-1}. \tag{9}$$

Our complete system for the case of lagged data is, therefore, given by (5), (6), (9), and (8).

Another method of coping with McCallum's criticism is to assume that the authorities set their interest rate instrument in response to their *forecasts* of output and inflation deviations, formed using the information available as of either time $t-1$ or time t, so that the policy rule itself is forward-looking. Some of the authors discussing forward-looking rules include Batini and Haldane (1999) and Bernanke and Woodford (1997). In fact, forward-looking rules have been found to describe the behavior of monetary policy, for instance, in Germany, Japan, and the US since 1979 as described in Clarida, Gali, and Gertler (1998); in its quarterly *Inflation Report*, the Bank of England reports extensively on forecasts of inflation and output. We consider simple versions of such forward-looking policy rules, ones in which the monetary authority looks just one quarter ahead when setting its interest rate instrument. This yields a specification, which we call the *forward expectations* model, in which the interest rate equation is[13]

$$r_t = \varphi_\pi \pi^e_{t+1} + \varphi_z z^e_{t+1}. \tag{10}$$

There are several ways to interpret this equation. When there is learning in the model,

[12] There is an additional problem with the contemporaneous data specification. When private sector expectations are formed using information dated $t-1$ and earlier, a tension is introduced, because the monetary authority is reacting to time t information on inflation and the output gap. Thus the central bank has "superior information" in this specification. In our other specifications, this tension is absent, as the private sector and the central bank use the same information, either for forming expectations or setting the interest rate instrument, or both.

[13] In some recent work on policy rules in this class, the central bank responds to the contemporaneous output gap instead of the future forecast of the output gap. We will have something to say on this related specification in Section 3.3.

it may be *two-sided*, as both policymakers and private sector agents form (identical) expectations of the future. We impute identical learning algorithms to each when we introduce learning. Alternatively, following Bernanke and Woodford (1997), it may be that the central bank simply targets the predictions of private sector forecasters, so that in this interpretation it is only the private sector which is learning. The complete system for the case of *forward expectations* model is given by (5), (6), (10), and (8).

Finally, another way to cope with McCallum's criticism is to assume that the authorities set their interest rate instrument in response to their current *expectations* (as opposed to using current data), formed using the information available as of time $t-1$, of the current period output gap and inflation. This also might be viewed as close to the actual practice of central banks. The interpretations given above for the case when the central bank targets future forecasts of inflation and output carry over to this case. Thus we consider versions of our systems where the policy feedback rule is

$$r_t = \varphi_\pi \pi_t^e + \varphi_z z_t^e. \tag{11}$$

We refer to this system as our *contemporaneous expectations* model and the complete system is given by (5), (6), (11), and (8).

2.3. Methodology. We adapt methods developed by Evans and Honkapohja (1999, 2000) to understand how learning affects these systems. We assume the agents in the model no longer have rational expectations at the outset. Instead, we replace expected values with adaptive rules, in which the agents form expectations using the data generated by the system in which they operate. We imagine that the agents use versions of recursive least squares updating.

We use theorems due to Evans and Honkapohja (1999, 2000) and calculate the conditions for expectational stability (E-stability). Evans and Honkapohja (2000) have shown that expectational stability, a notional time concept, corresponds to stability under real-time adaptive learning under quite general conditions. In particular, under E-stability recursive least squares learning is locally convergent to the rational expectations equilibrium. Moreover, under the assumption that the fundamental disturbances have bounded support, it can also be shown that if a rational expectations equilibrium is not E-stable, then the probability of convergence of the recursive least squares algorithm to the rational expectations equilibrium is zero.

We now define precisely the concept of E-stability. Following Evans and Honkapohja (2000), consider a general class of models[14]

$$y_t = \alpha + BE_t y_{t+1} + \delta y_{t-1} + \varkappa w_t \tag{12}$$

$$w_t = \phi w_{t-1} + e_t \tag{13}$$

where y_t is an $n \times 1$ vector of endogenous variables, α is an $n \times 1$ vector of constants, B, δ, $\varkappa$, and ϕ are $n \times n$ matrices of coefficients, and w_t is an $n \times 1$ vector of exogenous variables which is assumed to follow a stationary VAR, so that e_t is an $n \times 1$ vector of white noise terms. Following McCallum (1983), we focus on the so-called MSV (minimal state variable) solutions which are of the following form

$$y_t = a + by_{t-1} + cw_t \tag{14}$$

where a, b, and c are conformable and are to be calculated by the method of undetermined coefficients to compute the MSV solution. The corresponding expectations are

$$E_t y_{t+1} = a + by_t + c\phi w_t. \tag{15}$$

The MSV solutions consequently satisfy

$$(I - Bb - B)a = \alpha, \tag{16}$$

$$Bb^2 - b + \delta = 0, \tag{17}$$

and

$$(I - Bb)c - Bc\phi = \varkappa. \tag{18}$$

For E-stability we regard equation (14) as the *perceived law of motion* (PLM) of the agents and using (15) one obtains the *actual law of motion* (ALM) of y_t as

$$y_t = (I - Bb)^{-1}[\alpha + Ba + \delta y_{t-1} + (Bc\phi + \varkappa)w_{t-1}]. \tag{19}$$

[14] The class of models discussed in the previous section are special cases of this general class. A similar analysis can also be carried out when expectations are dated time $t-1$ (see Evans and Honkapohja (2000, p. 237)).

Thus the mapping from the PLM to the ALM takes the form

$$T(a,b,c) = \left((I - Bb)^{-1}(\alpha + \beta a\right), (I - Bb)^{-1}\delta, (I - Bb)^{-1}(Bc\phi + \varkappa)). \tag{20}$$

Expectational stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}(a,b,c) = T(a,b,c) - (a,b,c). \tag{21}$$

The fixed points of equation (21) give us the MSV solution. We say that a particular MSV solution $\left(\bar{a},\bar{b},\bar{c}\right)$ is E-stable if equation (21) is locally asymptotically stable at that point. The conditions for E-stability of the MSV solution $\left(\bar{a},\bar{b},\bar{c}\right)$ are given in Proposition 10.3 of Evans and Honkapohja (2000, p. 246). Equation (21) describes a stylized learning process in notional time τ, in which the PLM is partially adjusted towards the ALM generated by the perceptions.

Under real time learning, the PLM is time dependent and takes the form

$$y_t = a_{t-1} + b_{t-1}y_{t-1} + c_{t-1}w_t \tag{22}$$

where the coefficients a_t, b_t, c_t are updated by running recursive least squares on actual data, $x_t' = (1, y_{t-1}', w_t)$. This generates a corresponding ALM for y_t (which is also obviously time dependent). However, as shown in Proposition 10.4 of Evans and Honkapohja (2000, p. 246), the E-stability conditions derived from equation (21) actually govern stability under such adaptive learning. It can also be shown that the recursive least squares algorithm will converge to an E-unstable solution with probability zero (under some regularity conditions). This is the reason why we focus on E-stability conditions throughout the paper.

In the learning literature, an important issue is the so-called "dating of expectations." That is, when an expectation term enters the model, there is a question of what information the agent is able to incorporate when forming expectations. Expectational stability conditions, in general, are influenced by the exact dating of expectations. The convention in the learning literature is to assume all expectations are formed at time t using information available as of time $t-1$. Evans and Honkapohja (1999a) comment that this assumption "... seems more natural in a learning environment." If one assumes instead that time t observations are in the information set, then a simultaneity problem is introduced, where the system is determining time t variables at the same time that agents are using time t variables to form expectations. The problem can usually be handled at the cost of additional complexity. In this paper we generally work out results for both t and $t-1$ dating of expectations and where appropriate report on any differences we find.

We stress that the methodology we employ to analyze the effects of learning imparts a lot of information to the agents in our model. By endowing the agents with a perceived law of motion that coincides with the MSV solution of the system, we are in effect giving the agents the correct specification of the vector autoregression they need to estimate in order to learn the rational expectations equilibrium. The local nature of the analysis further imparts initial expectations which are in the immediate neighborhood of the equilibrium. If, under these circumstances, the system is nevertheless driven away from the rational expectations equilibrium, then we do not hold out too much hope that the system can be rendered stable under some other plausible learning mechanism (although of course that remains an open question). For this reason we think of our learnability criterion as a minimal requirement for a policy rule to meet.[15]

2.4. Parameters. Woodford (1999a) calibrated the parameters σ and κ based on econometric estimates from a larger model contained in Rotemberg and Woodford (1998, 1999); we use these estimated values throughout the paper to illustrate our results. The value of β, which corresponds to the representative household's discount factor in the more general model, is set throughout, following Woodford (1999a), to a value such that $\beta^{-1}-1$ corresponds to the average quarterly real interest rate. Rotemberg and Woodford (1999) respond to the Lucas critique by arguing that the equations describing their economy

[15] Many papers in the learning literature have employed simple overlapping generations models with money as a laboratory for studying learning dynamics. These models have two steady state equilibria, one characterized by low inflation, and another characterized by high inflation. Under least squares learning, the steady state with low inflation can sometimes be stable, but the steady state with high inflation is never stable. Arifovic (1995) showed that learning via an evolutionary dynamic (genetic algorithm learning) could render the low inflation equilibrium stable in some situations in which it was unstable under least squares. But the unstable high inflation equilibrium remained unstable under the evolutionary learning dynamic. Laboratory experiments with human subjects conducted by Marimon and Sunder (1993) confirmed much of thrust of the findings in the theoretical learning literature in this model. Based in part on this experience, we do not think expectational instability is easily reversed with alternative plausible learning models.

Table 1. Parameter configurations.		
Parameter	Controls	Value or range
σ	Intertemporal substitution	.157
κ	Price stickiness	.024
β	Discount factor	.99
ρ	Serial correlation of shock	.35
σ_ϵ	Variance of shock	3.72
φ_π	Policymakers' reaction to inflation	$0 \leq \varphi_\pi \leq 4$
φ_z	Policymakers' reaction to the output gap	$0 \leq \varphi_z \leq 4$

Table 1: Parameter configurations. We illustrate our analytical findings using this calibration from Woodford (1999a).

have coefficients which are not dependent on the parameters in the monetary authority's policy rule; accordingly they study a number of possible policy rules in their paper. We similarly take policy rules as exogenous descriptions of Federal Reserve behavior for our purposes in the present paper, and we study different rules in an effort to understand the robustness of our results to different policy rule specifications. Calibrations of these rules correspond to values for the parameters φ_π and φ_z. For the stochastic process describing the natural rate of interest, we use the serial correlation and the variance suggested by Woodford (1999*a*). Table 1 summarizes our calibration scenarios.

We organize our analysis as follows. We essentially consider four cases corresponding to four different information structures (contemporaneous data, lagged data, forward expectations and contemporaneous expectations) for the policy authority. For later reference, we call rules with $\varphi_\pi > 1$ *active rules* and those with $\varphi_\pi \leq 1$ *passive rules*. In each case, we begin by considering the determinacy of rational expectations equilibrium. We then consider the systems when agents try to learn the MSV solution by calculating the conditions for expectational stability. We maintain the following assumptions on our policy and structural parameters for *all* specifications of monetary policy rules we consider in the paper:[16]

$$\varphi_\pi \geq 0,\ \varphi_z \geq 0 \text{ with at least one strictly positive,} \tag{23}$$

and

$$\kappa > 0;\ \sigma > 0;\ \text{and } 0 < \beta < 1. \tag{24}$$

3. Policy rules and the equilibria they induce

3.1. Contemporaneous data in the policy rule.

Determinacy. In this subsection, we consider the version of the model represented by equations (5)-(8)- this version serves as a baseline for the subsequent analysis. In this case the policy authorities use contemporaneous data in their interest rate rule. We can then substitute the policy rule (7) into (5), and put our system involving the two endogenous variables z_t and π_t (given by equations (5) and (6)) in the following form[17]

$$\begin{bmatrix} 1+\sigma^{-1}\varphi_z & \sigma^{-1}\varphi_\pi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} z_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} z^e_{t+1} \\ \pi^e_{t+1} \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \end{bmatrix} r^n_t. \tag{25}$$

The matrix which is crucial for determining uniqueness of rational expectations equilibrium is obtained by pre-multiplying the right hand side matrix associated with the expectations variables with the inverse of the 2×2 left hand matrix. This matrix, denoted B, is given by

$$B = \frac{1}{\sigma + \varphi_z + \kappa\varphi_\pi} \begin{bmatrix} \sigma & 1 - \beta\varphi_\pi \\ \kappa\sigma & \kappa + \beta(\sigma + \varphi_z) \end{bmatrix} \tag{26}$$

Following Farmer (1991, 1999), in order to determine uniqueness, we have to classify the system in terms of free and predetermined endogenous variables and the exogenous variables. Since neither variable, z_t or π_t, in the system (25) is predetermined we need both of the eigenvalues of B to be inside the unit circle for determinacy; otherwise the equilibrium will be indeterminate. We provide a characterization of the necessary and sufficient condition for determinacy in the following proposition.

[16] While it is possible to obtain analytical conditions for uniqueness and E-stability for negative values of policy and/or structural parameters, we restrict ourselves to what seems to be the economically plausible case.

[17] One obtains a similar system whether expectations are dated time t or time $t-1$.

Proposition 1. *Let* $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$.[18] *Under contemporaneous data interest rate rules the necessary and sufficient condition for a rational expectations equilibrium to be unique is that*

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0. \tag{27}$$

Proof. See Appendix A. ■

A sufficiently aggressive response to inflation and output on the part of the central bank leads to determinacy. In particular, it is easily seen that a sufficient condition for uniqueness is $\varphi_\pi > 1$, corresponding to the activist rule.

Learning. After multiplying equation (25) by the inverse of the left hand matrix, we can write our model in the form

$$y_t = \alpha + By^e_{t+1} + \varkappa r^n_t \tag{28}$$

where $y_t = [z_t, \pi_t]'$, $\alpha = 0$, B is as defined in equation (26), and

$$\varkappa = \frac{1}{\sigma + \varphi_z + \kappa\varphi_\pi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix}. \tag{29}$$

For the moment, we assume t dating of expectations in equation (28), and we will discuss the reason for introducing the constant term α shortly. The MSV solution in this case, therefore, takes the simple form $y_t = \bar{a} + \bar{c}r^n_t$ with $\bar{a} = 0$ and $\bar{c} = (I - \rho B)^{-1}\varkappa$.[19] For the study of learning, we endow agents with a perceived law of motion (PLM) which corresponds to the MSV solution. Agents are assumed to have a PLM of the form

$$y_t = a + cr^n_t. \tag{30}$$

with their time t information set being $(1, y'_t, r^n_t)'$. Using this, we compute $E_t y_{t+1} = a + c\rho r^n_t$ and substituting this into equation (28), we obtain the actual law of motion (ALM) which is followed by y_t as

$$y_t = Ba + (Bc\rho + \varkappa)r^n_t. \tag{31}$$

[18] The condition rules out non-generic cases and is ensured, for example, by ignoring the point $(\varphi_\pi, \varphi_z) = (1, 0)$ (see also the proof of the proposition). In general, we will ignore such non-generic cases.

[19] The solution for $\bar{c}$ is generically unique. We use I to denote a conformable identity matrix throughout the paper.

Using (30) and (31), we can define a map, T, from the PLM to the ALM as

$$T(a,c) = (Ba, Bc\rho + \varkappa). \tag{32}$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}(a,c) = T(a,c) - (a,c). \tag{33}$$

The fixed points of equation (33) give us the MSV solution $(\bar{a}, \bar{c})$.

We say that a particular MSV solution $(\bar{a}, \bar{c})$ is *expectationally stable* if equation (33) is locally asymptotically stable at that point. Thus, our system is merely a special case of the system treated in Evans and Honkapohja (2000, p. 244) since there are no lagged endogenous variables. We can, therefore, apply their results directly to our system.

Before turning to the main result of this section, we emphasize a point which may be of some relevance. In our model (25) or (28) we have suppressed all constant terms normalizing all relevant steady state values to be zero ($\alpha = 0$). However, this is merely a technical (notational) simplification since in reality the model will have constant terms. The model will then take the form $y_t = \alpha + By^e_{t+1} + \varkappa r^n_t$ with $\alpha \neq 0$. The MSV solution would then be of the form $y_t = \tilde{a} + \tilde{c}r^n_t$. We could then endow the agents with a PLM of the form $y_t = a + cr^n_t$ and obtain a map from the PLM to the ALM as before and analyze the E-stability of this system. However, formally the E-stability conditions obtained in this way would be *identical* to the ones we would obtain by allowing the agents to have a constant term in their PLM (as we have done in (30)) although the original model (25) or (28) does not have constant terms. Consequently, our analysis of E-stability is without any loss of generality and covers the case when the original model contains constant terms.[20]

We now compute the necessary and sufficient condition for a MSV solution to equation (28) to be E-stable using the framework spelled out above. This yields an important baseline result, which is that the condition that guarantees E-stability turns out to be identical to the condition which guarantees uniqueness of rational expectations equilibrium.

[20] Similar reasoning holds for all specifications of monetary policy rules considered in later sections. Moreover, even if the original model does not have a constant term, it is quite natural to assume that agents allow for a constant term in their PLM.

Proposition 2. *Let $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$.*[21] *Suppose the time t information set is $(1, y_t', r_t^n)'$. Under contemporaneous data interest rate rules, the necessary and sufficient condition for an MSV solution $(0, \bar{c})$ of (28) to be E-stable is that*

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0. \tag{34}$$

Proof. See Appendix B. ■

If the expectations in equation (28) are instead dated $t-1$, then the MSV solution takes the form $y_t = \hat{a} + \hat{c}r_{t-1}^n$ with $\hat{a} = 0$, $\hat{c} = \rho(I - \rho B)^{-1}\varkappa$. Agents are then assumed to have a PLM of the form $y_t = a + cr_{t-1}^n + \varkappa\epsilon_t$ which in turn leads to an ALM of the form $y_t = Ba + (Bc\rho + \varkappa\rho)r_{t-1}^n + \varkappa\epsilon_t$ and E-stability of the MSV solution $(0, \hat{c})$ can be analyzed as before. Using Proposition 10.1 of Evans and Honkapohja (2000, p. 240) it can be shown that the necessary and sufficient condition for E-stability would be the same as in Proposition 2.

We now show that the expectational stability of the MSV solution is robust to some overparameterizations in the PLM of the agents. For this assume, say with t dating of expectations in equation (28), that agents have a PLM of the form

$$y_t = a + by_{t-1} + cr_t^n \tag{35}$$

and substitute this into equation (28) to derive the corresponding ALM which then takes the form

$$y_t = (I - Bb)^{-1}[Ba - (\rho Bc + \varkappa)r_t^n]. \tag{36}$$

As before we can define a map from the PLM to the ALM as

$$T(a, b, c) = ((I - Bb)^{-1}Ba, 0, (I - Bb)^{-1}(Bc\rho + \varkappa)). \tag{37}$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c). \tag{38}$$

The fixed points of the above equation give us the MSV solution which can be written as $y_t = \bar{a} + \bar{b}y_{t-1} + \bar{c}r_t^n$ with $\bar{a} = \bar{b} = 0, \bar{c} = (I - \rho B)^{-1}\varkappa$. One can analyze E-stability of this

[21] Again this condition rules out non-generic cases. Henceforth, we ignore such non-generic conditions (see also the proof of this proposition).

MSV solution $(0, 0, \bar{c})$ as before. The necessary and sufficient condition for E-stability would again be the same as in Proposition 2.[22] Consequently, we see that E-stability of the MSV solution of equation (28) is governed by the same condition as in Proposition 2 even when agents allow for a lag in the endogenous variables in their PLM. In this sense, the expectational stability of the MSV solution is robust to some overparameterizations in the PLM of the agents.

Propositions 1 and 2 in conjunction show that under contemporaneous data policy rules, the set of parameter values consistent with determinate equilibria are *exactly* the same as the set consistent with expectational stability. Since all determinate REE are E-stable, one could view this as justifying the focus on determinate equilibria in previous studies of policy rules, for the case in which the policy rule reacts to contemporaneous data. We also note that passive rules may lead to instability of equilibria under the learning dynamics—this provides an additional reason for avoiding such rules quite apart from the indeterminacy problems they may cause.

Figure 1 plots the region of determinacy and expectational stability of the MSV solution as a function of φ_π and φ_z, when all other parameters are set at the values given in Table 1. Much of the parameter space is associated with a determinate REE. Our result under learning shows that this entire region is also associated with expectational stability. On the other hand, in the indeterminate region of the parameter space, equilibria corresponding to the MSV solution are always expectationally unstable when agents have a PLM corresponding to the relevant MSV solution.

3.2. Lagged data in the policy rule.

Determinacy. The case of a policy rule with contemporaneous data is probably the least realistic in terms of what policymakers actually know when decisions about interest rates are made. In this subsection, we follow McCallum's (1999) recommendation and consider rules with lagged data, so that policymakers are, in the current quarter, reacting

[22] This result again follows from applying Proposition 10.3 of Evans and Honkapohja (2000, p. 246). The extra matrix that needs to be checked for E-stability of this solution is the null matrix which has eigenvalues zero. A similar result also holds for $t - 1$ dating of expectations in the model.

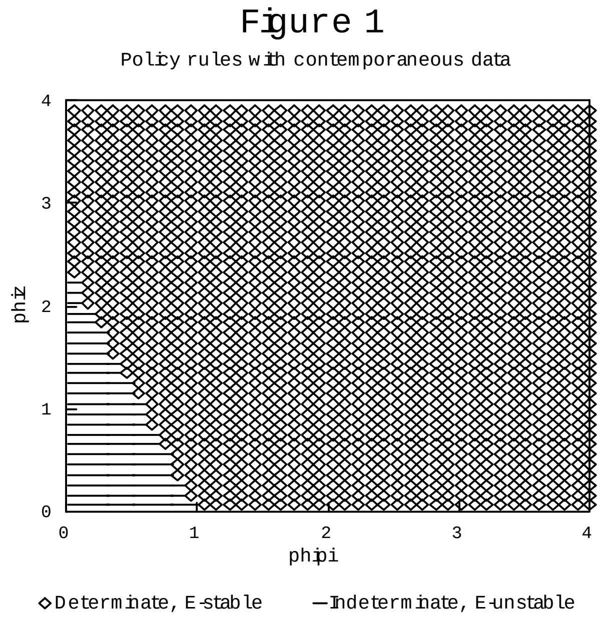

Figure 1: Regions of determinacy and expectational stability for the class of policy rules using contemporaneous data. Parameters other than φ_π (PHIPI) and φ_z (PHIZ) are set at baseline values.

to information about the previous quarter's output gap and inflation rate. The policy rule is, therefore, given by (9) and our complete system is given by (5), (6), (9), and (8).

For the analysis of uniquenes, we move equation (9) one time period forward and rewrite the system of equations (5), (6), and (9) as

$$\begin{bmatrix} 1 & 0 & \sigma^{-1} \\ -\kappa & 1 & 0 \\ \varphi_z & \varphi_\pi & 0 \end{bmatrix} \begin{bmatrix} z_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z^e_{t+1} \\ \pi^e_{t+1} \\ r_{t+1} \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \\ 0 \end{bmatrix} r^n_t. \tag{39}$$

The matrix which is relevant for uniqueness is obtained by pre-multiplying the matrix associated with the expectational variables with the inverse of the left hand matrix. This matrix is given by

$$B = \frac{1}{(\varphi_z + \kappa\varphi_\pi)} \begin{bmatrix} 0 & -\beta\varphi_\pi & 1 \\ 0 & \beta\varphi_z & \kappa \\ \sigma(\varphi_z + \kappa\varphi_\pi) & \varphi_z + (\kappa + \beta\sigma)\varphi_\pi & -\sigma \end{bmatrix}. \tag{40}$$

We now have two free endogenous variables, z_t and π_t, and one predetermined endogenous variable, r_t. Consequently, following Farmer (1991, 1999), we need exactly two of the eigenvalues of B to be inside the unit circle for uniqueness.

In this case it can be shown that a sufficiently aggressive response to inflation and output will necessarily lead to local *explosiveness*, that is, paths that are locally diverging away from the steady state.[23] This has also been observed in the (larger) model of Rotemberg and Woodford (1999). In order to have unique equilibria, we must rule out aggressive response to either inflation or output as is shown in the proposition below.

Proposition 3. *Under lagged data interest rate rules a set of sufficient conditions for unique equilibria are*

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0, \tag{41}$$

and

$$\kappa(\varphi_\pi - 1) + \varphi_z < 2\sigma(1+\beta). \tag{42}$$

Proof. See Appendix C. ■

Proposition 3 shows that active rules with a small response to output can lead to unique equilibria. In the Rotemberg and Woodford (1999) analysis, no further consideration was given to either the explosive region or the region of indeterminacy. Instead, they searched the portion of the parameter space associated with *determinate* equilibria for optimal policy rules—values of φ_π and φ_z—in an effort to identify an optimal policy rule in this class according to a number of criteria. One way to justify this focus is to argue that the determinate equilibria are the only learnable ones, as Proposition 2 established for the case of contemporaneous data policy rules. This baseline result does not carry over to the case of lagged data interest rate rules, however, as we now show.

Learning. For the analysis of learning, we substitute equation (9) into equation (5) and reduce the system to two equations involving the endogenous variables z_t and π_t. Defining $y_t = [z_t, \pi_t]'$, this system can be written as

$$y_t = \beta_1 y^e_{t+1} + \delta y_{t-1} + \varkappa r^n_t \tag{43}$$

[23] It is easy to obtain the necessary and sufficient conditions for local explosiveness- for this we need all the eigenvalues of B to be inside the unit circle. The conditions are somewhat messy but a casual observation shows that an aggressive response of the interest rate to lagged inflation and output will necessarily cause explosiveness.

with

$$\beta_1 = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \kappa\sigma^{-1} + \beta \end{bmatrix}, \tag{44}$$

$$\delta = \begin{bmatrix} -\varphi_z \sigma^{-1} & -\varphi_\pi \sigma^{-1} \\ -\kappa\varphi_z \sigma^{-1} & -\kappa\varphi_\pi \sigma^{-1} \end{bmatrix}, \tag{45}$$

and

$$\varkappa = \begin{bmatrix} \sigma^{-1} \\ \kappa\sigma^{-1} \end{bmatrix}. \tag{46}$$

In this formulation, we assume that expectations of inflation and output of the private sector are formed at time $t-1$. Since the central bank is using last quarter's values of inflation and output in setting the current interest rate, assuming that the private sector has access to current quarter values of inflation and output in forming its expectations is tantamount to assuming that the public has superior information. Assuming $t-1$ dating of expectations removes this tension and puts the monetary authority and the public in a symmetric position. The MSV solution of (43) takes the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} r_{t-1}^n + \varkappa \epsilon_t \tag{47}$$

with the solutions for $\bar{a}, \bar{b}$ and $\bar{c}$ being given by

$$\bar{a} = 0,$$

$$\bar{b} = (I - \beta_1 \bar{b})^{-1} \delta, \tag{48}$$

and

$$\bar{c} = \rho(I - \beta_1 \bar{b} - \rho\beta_1)^{-1} \varkappa.$$

Because equation (48) is a matrix quadratic, there are potentially multiple solutions for $\bar{b}$. The determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ with both its eigenvalues inside the unit circle.[24] For the analysis of learning, we assume that agents have a PLM of the form

$$y_t = a + b y_{t-1} + c r_{t-1}^n + \varkappa \epsilon_t$$

[24] See also Evans and Honkapohja (2000 p. 261) on this point.

corresponding to the MSV solution which leads to an ALM of the form

$$y_t = (\beta_1 + \beta_1 b)a + (\beta_1 b^2 + \delta)y_{t-1} + (\beta_1 bc + \rho\beta_1 c + \rho\varkappa)r_{t-1}^n + \varkappa\epsilon_t.$$

The mapping from the PLM to the ALM takes the form

$$T(a,b,c) = ((\beta_1 + \beta_1 b)a, \beta_1 b^2 + \delta, \beta_1 bc + \rho\beta_1 c + \rho\varkappa)$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}(a,b,c) = T(a,b,c) - (a,b,c). \tag{49}$$

The fixed points of equation (49) give us the MSV solution $(\bar{a}, \bar{b}, \bar{c})$ of equation (47). We say that a particular MSV solution $(\bar{a}, \bar{b}, \bar{c})$ is expectationally stable if equation (49) is locally asymptotically stable at that point.

Our system is in a form where we can apply Proposition 10.1 of Evans and Honkapohja (2000, p. 240).[25] For E-stability of any MSV solution $(\bar{a}, \bar{b}, \bar{c})$ with $t-1$ dating of expectations, we need the eigenvalues of the following three matrices:

$$\bar{b}' \otimes \beta_1 + I \otimes \beta_1 \bar{b}, \tag{50}$$

$$\rho\beta_1 + \beta_1 \bar{b}, \tag{51}$$

and

$$\beta_1 + \beta_1 \bar{b} \tag{52}$$

to have real parts less than one. On the other hand, the MSV solution is not E-stable if any eigenvalue of the matrices (50), (51) or (52) has a real part more than one.

We are unable to obtain analytical results in this case. However, we can illustrate our findings with Figure 2 which depicts determinacy and learnability of the MSV solution for lagged data rules, when all parameters other than φ_π and φ_z are set at the baseline values outlined in Table 1. The determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ in the MSV solution (47) with both its eigenvalues inside the unit circle. We first note that only a subset of the parameter space that is consistent with

[25] Proposition 10.2 of Evans and Honkapohja (2000, p. 243) shows that under recursive least squares learning, the learning algorithm converges locally to a stationary E-stable MSV solution $(\bar{a}, \bar{b}, \bar{c})$.

determinacy is also consistent with learnability in the lagged data case. Passive rules combined with a relatively aggressive response to the output gap may lead to determinate equilibria that are unstable under learning dynamics. While passive rules do not necessarily lead to problems of non-uniqueness of equilibria, they continue to cause instability of equilibria in the learning dynamics.[26] In the indeterminate region of the parameter space we find that there are two stationary solutions which take the form of the MSV solution (47). However, both of these stationary solutions always turn out to be E-unstable.[27]

We find that there continues to be a close connection between active rules and learnability of REE, as an aggressive response to lagged values of inflation deviations leads to learnability. Moreover, a theme that is particularly apparent in this case, and that we will come back to throughout the paper, is that activist rules ($\varphi_\pi > 1$) with little or no reaction to the output gap tend to be associated with rational expectations equilibria which are both determinate and E-stable across all the specifications of monetary policy rules we consider.

3.3. Forward expectations in the policy rule.

Determinacy. With forward expectations in the policy rule, the monetary authority sets its nominal interest rate instrument in response to the *forecasts* of inflation deviations and the output gap, that is, according to the policy rule (10), and our complete model is given by (5), (6), (10), and (8).

There are several ways to interpret the policy rule (10). In practice, following Bernanke and Woodford (1997), there are at least three types of approaches to implement this proposal. First, the central bank could try to "target" the predictions of private sector

[26] In view of Propositions 1 and 3, there may seem to be a contradiction in the statement that passive rules with relatively aggressive response to output lead to indeterminacy under contemporaneous data rules (as illustrated in Figure 1) whereas they lead to determinate equlibria under lagged data rules (as illustrated in Figure 2). However, the contradiction is resolved by noting that Proposition 3 merely gives *sufficient* conditions for unique equlibria in the lagged data case. In particular, inequality (41) is violated in this region so that equlibrium is necessarily indeterminate under contemporaneous data rules (in view of Proposition 1) whereas they could still be determinate (and obviously are as illustrated in Figure 2) in the lagged data case (since this situation is not covered in Proposition 3).

[27] There are no results for the connection between E-stability and convergence of actual real time learning algorithms in explosive cases. Consequently, we do not analyze the expectational stability of explosive situations. We also do not analyze E-stability of sunspot equilibria in the indeterminate region of the parameter space.

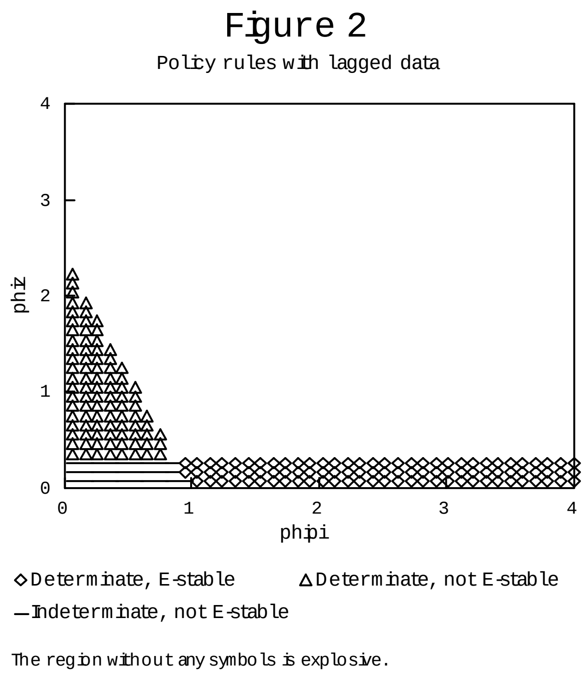

Figure 2: Determinacy and learnability for rules responding to lagged data, with parameters other than φ_π and φ_z set at baseline values. Determinate equilibria may or may not be E-stable. The region of active rules associated with small coefficients on the output gap remains expectationally stable.

forecasters (see Hall and Mankiw (1994)). Second, the central bank could try to target the forecast of inflation implicit in various asset prices. Finally, the monetary authority might try to target its own internal forecasts of inflation (see Svensson (1997)).[28] In the latter two proposals there will be two-sided learning, with both the policymaker and the private sector taking actions based on (identical) expectations of the future path of the output gap and inflation in our scenario. In the first proposal, it will be only the private sector which is learning with the central bank merely reacting to the private sector forecasts. In the case when there is two-sided learning, we assume that both the policymakers and the private agents use identical recursive least squares algorithms to update their expectations.

We can again reduce the system of equations (5), (6), and (10) to two equations

[28] See Bernanke and Woodford (1997) for a more detailed discussion of these various proposals.

involving the endogenous variables (z_t, π_t) by substituting equation (10) into equation (5). The reduced system is then given by

$$\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} z_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1-\sigma^{-1}\varphi_z & \sigma^{-1}(1-\varphi_\pi) \\ 0 & \beta \end{bmatrix} \begin{bmatrix} z_{t+1}^e \\ \pi_{t+1}^e \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \end{bmatrix} r_t^n. \tag{53}$$

The relevant matrix for calculating determinacy (obtained by pre-multiplying the first matrix on the right hand side with the inverse of the matrix on the left hand side), denoted B, is given by

$$B = \begin{bmatrix} 1-\sigma^{-1}\varphi_z & \sigma^{-1}(1-\varphi_\pi) \\ \kappa(1-\sigma^{-1}\varphi_z) & \beta+\kappa\sigma^{-1}(1-\varphi_\pi) \end{bmatrix}. \tag{54}$$

Since the variables z_t and π_t are free, we need both the eigenvalues of B to be inside the unit circle for uniqueness. In this case we are again able to provide a characterization of the necessary and sufficient conditions for determinacy. This is given in the following proposition.

Proposition 4. *Under interest rate rules with forward expectations the necessary and sufficient conditions for a rational expectations equilibrium to be unique are that*

$$\varphi_z < \sigma(1+\beta^{-1}), \tag{55}$$

$$\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_z < 2\sigma(1+\beta), \tag{56}$$

and

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0. \tag{57}$$

Proof. See Appendix D. ■

We first note that, unlike the other specifications, values assigned to φ_z are of primary importance for determining uniqueness. In particular, an aggressive response to output leads to indeterminacy, quite independently of φ_π. Even if the response to output is modest, a sufficiently aggressive response to inflation again leads to indeterminacy. In this respect the picture that emerges from here is akin to the one involving lagged data policy rules. A sufficiently aggressive response to *both* inflation and output leads to non-unique equilibria for interest rate rules that respond either to lagged values or to future forecasts of output and inflation. However, in the case of forward looking rules it is possible to have unique equilibria with active rules coupled with a modest response to output.

In the literature, as for example in Clarida, Gali and Gertler (2000), a forward-looking rule sometimes reacts to the future inflation forecast and the contemporaneous output gap, unlike the specification (10). If the interest rate rule takes this form, that is, of the form

$$r_t = \varphi_\pi \pi^e_{t+1} + \varphi_z z_t, \tag{58}$$

then the problem of indeterminacy is somewhat lessened. In this case, it can be shown that the necessary and sufficient conditions for determinacy are (only) condition (57) and

$$\kappa(\varphi_\pi - 1) - (1+\beta)\varphi_z < 2\sigma(1+\beta). \tag{59}$$

Note that now an aggressive response towards output promotes determinacy. However, a very aggressive response towards inflation still leads to indeterminacy. Consequently, the message we get from forward-looking rules is that greater the forward looking elements in the monetary authority's policy rule, greater is the problem of indeterminacy. In particular, if the monetary authority has a rule of the form of equation (10) then an aggressive response to *either* inflation or output leads to indeterminacy whereas if the bank sets the interest rate according to a rule of the form (58), then it is only an aggressive response to inflation which leads to indeterminacy. Consequently, a central bank which sets interest rates according to a forward-looking rule may want to reduce the number of forward-looking elements in such a rule if it wants to reduce the possibility of self-fulfilling bursts in inflation and output.

Learning. The analysis of E-stability here is akin to the case of rules with contemporaneous data. Equation (53) can be written in the form (after multiplying by the inverse of the left hand matrix)

$$y_t = \alpha + B y^e_{t+1} + \varkappa r^n_t \tag{60}$$

where $y_t = [z_t, \pi_t]'$, $\alpha = 0$, B is as defined in equation (54), and $\varkappa = [\sigma^{-1}, \kappa\sigma^{-1}]'$. For t-dating of expectations, the MSV solution takes the form $y_t = \bar{a} + \bar{c} r^n_t$ with $\bar{a} = 0$,

$\bar{c} = (I - \rho B)^{-1}\varkappa$. The PLM of agents again takes the form of equation (30) and the rest of the analysis proceeds as in Section 3.1. We obtain a complete characterization for E-stability of the MSV solution in the following proposition.

Proposition 5. *Suppose the time t information set is $(1, y_t', r_t^n)'$. Under interest rate rules with forward expectations, the necessary and sufficient condition for an MSV solution $(0, \bar{c})$ of (60) to be E-stable is that*

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0 \tag{61}$$

Proof. See Appendix E. ■

We again obtain an identical stability condition for $t-1$ dating of expectations. The solution is also stable under some overparametrizations in the PLM of the agents. If agents allow for a PLM of the form of equation (35), the conditions for E-stability of the MSV solution $(0, 0, \bar{c})$ would be identical to the one in Proposition 5. Consequently the MSV solution is stable even when agents allow for a lag in the endogenous variables in their PLM.

Propositions 4 and 5 in conjunction show that under policy rules with forward expectations, if an MSV solution is unique, then it must be expectationally stable. In this case, the converse does not hold, as satisfaction of the expectational stability conditions does not imply satisfaction of the determinacy conditions. Consequently, when equilibria are (potentially) indeterminate, as long as agents do not allow for any *sunspot* variable in their PLM (for example), they may still converge to equilibria which correspond to the MSV solution.

We continue to find an intimate connection between active rules and E-stability. In particular, active rules guarantee E-stability. More generally, an aggressive response to both the future forecasts of inflation and output is conducive to learnability.

Figure 3 illustrates the intersections of the regions of determinacy and learnability of the MSV solution at baseline parameter values. Determinate equilibria are always expectationally stable, and again these cases involve active rules with zero or relatively small positive coefficients on φ_z. On the other hand, in the indeterminate region of

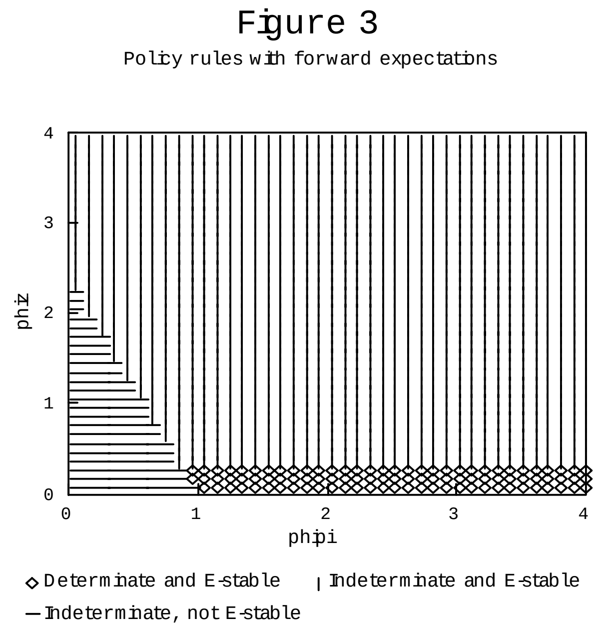

Figure 3: Learnability with forward expectations monetary policy rules, at baseline parameter values. Determinate equilibria are always expectationally stable.

the parameter space, equilibria corresponding to the MSV solution may or may not be learnable when agents have a PLM corresponding to the relevant MSV solution.

3.4. Contemporaneous expectations in the policy rule.

Determinacy. With contemporaneous expectations, the policy rule is given by (11). In this case we assume $t-1$ dating of expectations for the central bank and the private sector so as to put both of them in a symmetric position as far as their information is concerned. Our complete model is given by (5), (6), (11), and (8). We can reduce our system of equations (5), (6), and (11) to two equations by substituting equation (11) into equation (5) and write this system in the following form

$$\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} z_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} z^e_{t+1} \\ \pi^e_{t+1} \end{bmatrix} + \begin{bmatrix} -\sigma^{-1}\varphi_z & -\sigma^{-1}\varphi_\pi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z^e_t \\ \pi^e_t \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \end{bmatrix} r^n_t \ . \tag{62}$$

Defining the vector of endogenous variables by $y_t = [z_t, \pi_t]'$ and after pre-multiplying both sides by the inverse of the matrix on the left hand side, we can write the above system in the form

$$y_t = B_0 y_t^e + B_1 y_{t+1}^e + \varkappa r_t^n \tag{63}$$

where $\varkappa = [\sigma^{-1}, \kappa\sigma^{-1}]'$,

$$B_0 = \begin{bmatrix} -\varphi_z \sigma^{-1} & -\varphi_\pi \sigma^{-1} \\ -\kappa\varphi_z \sigma^{-1} & -\kappa\varphi_\pi \sigma^{-1} \end{bmatrix}, \tag{64}$$

and

$$B_1 = \begin{bmatrix} 1 & \sigma^{-1} \\ \kappa & \beta + \kappa\sigma^{-1} \end{bmatrix}. \tag{65}$$

The situation with contemporaneous expectations is similar to the situation with contemporaneous data at least as far as determinacy is concerned. In particular, the necessary and sufficient condition for a unique REE is given by condition (27) of Proposition 1 (and as portrayed in Figure 1 for baseline parameter values). The easiest way to see this is by replacing the expectations with the actual values (which is required for determining uniqueness following Farmer (1999) or Evans and Honkapohja (2000)) and observing that this yields the matrix, $(I - B_0)^{-1}B_1$, for determining uniqueness which is exactly the same as the one obtained for contemporaneous data, namely, the matrix given in (26). Furthermore, since the variables z_t and π_t are free, we need both eigenvalues of (26) to be inside the unit circle for uniqueness, as in the case of contemporaneous data.

Learning. Given our model of the form of equation (63), the MSV solution takes the form $y_t = \bar{a} + \bar{c} r_{t-1}^n + \varkappa\epsilon_t$ with $\bar{a} = 0$ and $\bar{c} = \rho(I - B_0 - \rho B_1)^{-1}\varkappa$. We assume that agents have the PLM

$$y_t = a + c r_{t-1}^n + \varkappa\epsilon_t \tag{66}$$

from which we compute the expectations $E_{t-1}y_t = a + c r_{t-1}^n$ and $E_{t-1}y_{t+1} = a + cE_{t-1}r_t^n = a + c\rho r_{t-1}^n$. Substituting this into our model (63) yields the ALM

$$y_t = (B_0 + B_1)a + (B_0 c + B_1 c\rho + \varkappa\rho) r_{t-1}^n + \varkappa\epsilon_t. \tag{67}$$

The map from the PLM to the ALM takes the form

$$T(a, c) = ((B_0 + B_1)a, (B_0 c + B_1 c\rho + \varkappa\rho)). \tag{68}$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}(a,c) = T(a,c) - (a,c). \tag{69}$$

We are now in a position to prove the following proposition.

Proposition 6. *The necessary and sufficient condition for the MSV solution* $(0, \bar{c})$ *of (63) to be E-stable under interest rate rules with contemporaneous expectations, is given by inequality (61).*

Proof. See Appendix F. ■

This shows that under condition (61), the MSV solution is E-stable if agents allow for a PLM which corresponds to this solution. In addition, the solution is also E-stable under some overparametrizations (as in the case of contemporaneous data) in the PLM of the agents. For example, if agents allow for a PLM of the form (35), then the MSV solution $(\bar{a}, \bar{b}, \bar{c})$ of equation (63) with $\bar{a} = \bar{b} = 0$ will be E-stable under condition (61). In this case one needs to check further that the eigenvalues of $I \otimes B_0$ have real parts less than one for E-stability (see Evans and Honkapohja (2000, p. 240)). This condition is satisfied since the eigenvalues of B_0, being given by 0 and $-\sigma^{-1}(\kappa\varphi_\pi + \varphi_z)$, are non-positive. Consequently, condition (61) is also the necessary and sufficient condition for E-stability of the MSV solution when agents allow for a PLM of the form of equation (35), that is, when agents allow for a lag in the endogenous variables in their PLM.

Condition (61) is also the necessary and sufficient condition for uniqueness of equilibria under contemporaneous expectations. Hence, Proposition 6 shows that the set of parameters consistent with both unique and E-stable equilibria are *exactly* the same—a conclusion which we also obtained for the specification of contemporaneous data policy rules.

Following our analysis in the previous section, we again obtain the intimate connection between active rules and E-stability, namely, that such rules guarantee E-stability. The situation for the case of baseline parameter values is again summarized in Figure 1, where the determinate region is also the expectationally stable region.

3.5. Remarks and relation to some recent literature. We have noted that the contemporaneous data class of policy rules has been criticized because central banks do not actually have very good current quarter information on the output gap and inflation during the current quarter. The other three classes of policy rules—those involving lagged data, forward expectations, and contemporaneous expectations—all respond to this criticism. Of these, the contemporaneous expectations specification provides the best response according to our analysis. First, the lagged data and forward expectations classes altered the equilibrium configurations dramatically, while the contemporaneous expectations specification left it intact. In particular, with lagged data or forward looking rules, a large portion of the parameter space is associated with either explosiveness or indeterminacy, whereas with contemporaneous expectations a large portion of the parameter space is associated with unique REE. Second, the conditions for expectational stability are such that determinate equilibria are expectationally stable and *vice versa* for the contemporaneous expectations specification, and the condition under which this is true is also the same as with contemporaneous data. This result does not hold for the lagged data or forward expectations specifications: For lagged data only a portion of the parameter space consistent with determinacy is also consistent with learnability, while for forward-looking rules determinate equilibria are always learnable, but in addition equilibria corresponding to the MSV solution may be stable under learning as well in the indeterminate region of the parameter space. Finally, the contemporaneous expectations specification is probably fairly realistic in terms of actual central bank behavior, as policymakers surely compute their expectations of current quarter macroeconomic data when making policy decisions. For these reasons, if we view determinacy and learnability as desirable criteria for a monetary policy rule then the contemporaneous expectations specification seems to be the most desirable.

As mentioned in the introduction, several authors, such as Taylor (1999) and Clarida, Gali and Gertler (2000), tend to favor a "leaning against the wind" policy by the central bank. For example, Taylor (1999) recommends a policy rule which calls for tightening market conditions in response to higher inflation or to increases in production. This would be ensured by an interest rate rule with $\varphi_\pi > 1$ and $\varphi_z > 0$. Taylor (1999) used contemporaneous data in his policy rule. Clarida, Gali and Gertler (2000) reach similar conclusions on the desirability of an aggressive response of the interest rate to inflation with a forward-looking rule.

In this paper we support the recommendations of these authors based on the criterion of learnability. If agents do not have rational expectations of inflation and output and instead start with some subjective expectations of these variables, then a "leaning against the wind" policy on the part of the central bank pushes the economy towards the rational expectations equilibrium. This is true not only when the policy rule reacts to contemporaneous data or to forward-looking variables but also when it reacts to lagged data or to contemporaneous expectations of these variables.[29] For example, a deviation of private sector expected inflation from the rational expectations value leads to an increase in the real interest rate since $\varphi_\pi > 1$. This reduces the output gap through the IS curve which in turn reduces inflation through the aggregate supply equation. The policy, therefore, succeeds in guiding initially nonrational private sector expectations towards the rational expectations value. On the other hand, a policy rule in which $\varphi_\pi < 1$ may be destabilizing in the sense that if agents do not start with rational expectations then they are unlikely to be able to coordinate on the particular equilibrium the policy authorities are targeting. In this case, a deviation of private sector expected inflation from the rational expectations value leads to a decrease in the real interest rate. This increases the output gap through the IS curve which in turn increases inflation through the aggregate supply equation. Over time, this leads to upward revisions of both expected inflation and expected output gap. The interest rate rule is unable to offset this tendency and the economy moves further away from the rational expectations equilibrium. While the literature has already warned against the use of passive rules owing to the indeterminacy problems they easily cause, we find an additional reason for avoiding such rules—namely, that passive rules may lead to unlearnability of rational expectations equilibria.

Because there is no general agreement among authors as to the type of policy rules

[29] For the lagged data case, we saw this numerically in Figure 2.

central banks should follow, we find it useful to recommend a policy rule which responds aggressively to inflation (with a coefficient bigger than one) and mildly to output. Such rules are often associated with rational expectations equilibria which are both determinate and stable under the learning dynamics across all the specifications we consider.[30] We think that this provides an important reason for central banks to consider rules of this type, irrespective of the exact policy rule they want to use. It also provides some foundation for a positive theory of observed monetary policy rules based on interest rate targeting, because estimated policy rules (at least using data since the mid-1980s) tend to have this character.

4. Summary and conclusion

We have studied the stability of macroeconomic systems under learning for various monetary policy rules using methods developed by Evans and Honkapohja (1999, 2000). The systems we study are among those being used to give advice on what central banks might reasonably expect should they adopt certain types of monetary policy rules. A key feature of these economies under rational expectations is that a determinate equilibrium may not exist. However, in virtually all analyses of which we are aware, the agents in the model are simply assumed to be able to coordinate on determinate equilibria when they do exist. In this paper we provide an analysis of this assumption.

In general determinacy alone is insufficient to induce learnability of a rational expectations equilibrium. We conclude that it may be unwise to simply assume that coordination on a unique equilibrium can occur under a reasonable description of agent learning.

We have argued that policy rules which lead to unlearnable equilibria are to be avoided. We think this is reasonable, in part because in the formulation of adaptive learning we have used, we already endowed agents with quite a bit of information about the economy in the sense that the perceived law of motion of the agents corresponds to the MSV solution. The agents, in other words, have the right variables and the right relationship between the variables, as well as initial conditions in the neighborhood of the equilibrium.

[30] However, the degree of activism to inflation should not be too large as that may lead to non-unique equilibria under policy rules which respond to lagged values of inflation and output or to future forecasts of these variables.

If agents are unable to learn the MSV solution even under this very favorable assumption, then they are unlikely to learn the equilibrium under more general assumptions.[31] We have also shown that some of our results are robust to certain overparameterizations in the perceived law of motion of the agents.

In this paper, we have only considered "simple" policy rules, in which policymakers do not respond to the lagged interest rate. In part this was because this is the type of policy rule studied by Taylor (1993) which fueled the current wave of interest in monetary policy rules. However, estimated policy rules usually include a lagged interest rate in order to better capture the interest rate smoothing observed in actual central bank behavior. We expect our results to be valid even when we allow for mild interest rate smoothing on the part of the central bank. We are currently conducting a systematic study of the four variants of policy rules considered here when the central bank also reacts to a lagged interest rate in a companion paper.[32]

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[31] An analogy we have in mind are the notions of *weak* and *strong* E-stability used in the learning literature in univariate models. If a certain equilibrium is not weakly E-stable, then it cannot be strongly E-stable.

[32] See Bullard and Mitra (2000).

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A. PROOF OF PROPOSITION 1

The characteristic polynomial of B, defined by (26), can be shown to be given by

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0 \tag{70}$$

where

$$a_0 = \frac{\beta\sigma}{\sigma + \varphi_z + \kappa\varphi_\pi} \tag{71}$$

and

$$a_1 = \frac{-(\kappa + \sigma + \beta\sigma + \beta\varphi_z)}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{72}$$

Since the variables z_t and π_t are free, equilibrium is determinate if and only if both eigenvalues of B are inside the unit circle. The necessary and sufficient conditions for this are (i) $|a_0| < 1$ and (ii) $|a_1| < 1 + a_0$.[33] Condition (i) implies (after some simplification)

$$\varphi_z + \kappa\varphi_\pi > -(1-\beta)\sigma. \tag{73}$$

Since $0 < \beta < 1$, this condition is trivially satisfied for all $\varphi_\pi \geq 0, \varphi_z \geq 0$. Condition (ii) on the other hand, implies (again after some simplification)

$$\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z > 0 \tag{74}$$

which is the required condition in the proposition. In addition, note that by Descartes' rule of signs (see, for example, Barbeau (1989, p. 171)) the eigenvalues are either both

[33] See J.P. LaSalle (1986, p. 28).

positive or they are a pair of complex conjugates. If the roots are positive, then $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$ rules out an eigenvalue equal to 1 whereas if the eigenvalues are complex, then their product a_0 is not equal to 1, so that we can rule out the case when they are on the unit circle.

B. Proof of Proposition 2

We have spelled out the PLM, the ALM, and the T map from the PLM to the ALM by equations (30), (31), and (32), respectively. As we noted, our model is merely a special case of the one treated in Evans and Honkapohja (2000, p. 246) since we do not have lagged endogenous variables. Consequently, we can apply their results (in particular Proposition 10.3, p. 246) directly here. We need the eigenvalues of the matrix B (given by (26)) and $\rho \otimes B$ ($= \rho B$) to have real parts less than 1 for E-stability. The eigenvalues of ρB are given by the product of the eigenvalues of B and ρ, and since $0 \leq \rho < 1$, it suffices to have only the eigenvalues of B to have real parts less than 1 for E-stability. Consequently, our E-stability conditions are independent of the parameter ρ. On the other hand, the MSV solution will not be E-stable if any eigenvalue of B has a real part more than 1. The characteristic polynomial of $B - I$ is given by $\lambda^2 + a_1\lambda + a_2$ where

$$a_1 = \frac{\kappa(2\varphi_\pi - 1) + (2-\beta)\varphi_z + \sigma(1-\beta)}{\sigma + \varphi_z + \kappa\varphi_\pi} \tag{75}$$

and

$$a_2 = \frac{\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{76}$$

Both eigenvalues of B have real parts less than 1 (that is, both eigenvalues of $B - I$ have negative real parts) if and only if (i) $a_1 > 0$ and (ii) $a_1 a_2 > 0$.[34] Given (i), the latter condition reduces to $a_2 > 0$. Note that

$$a_1 = a_2 + \frac{\kappa\varphi_\pi + \varphi_z + \sigma(1-\beta)}{\sigma + \varphi_z + \kappa\varphi_\pi}. \tag{77}$$

Given our maintained assumptions, $a_2 > 0$ implies $a_1 > 0$. Consequently, the only condition required is $a_2 > 0$ which reduces to (34). On the other hand, if $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z < 0$, then the determinant and trace of $B - I$ is non-zero so that there is no real root equal

[34] These conditions are obtained by applying the *Routh theorem* (see Chiang (1984)).

to zero and in the case of complex eigenvalues, the real parts are non-zero. This shows that condition (34) is necessary and sufficient for E-stability of the MSV solution. Note that $\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z \neq 0$ eliminates the possibility that one of the eigenvalues of B is equal to 1 (recall that for expectational instability we need at least one eigenvalue of B to have real part more than 1).[35]

C. PROOF OF PROPOSITION 3

It is easier to prove our results by working with the inverse of B (where B is given by equation (40)), namely, by working with

$$B^{-1} = \begin{bmatrix} 1+\kappa\beta^{-1}\sigma^{-1} & -\beta^{-1}\sigma^{-1} & \sigma^{-1} \\ -\kappa\beta^{-1} & \beta^{-1} & 0 \\ \varphi_z & \varphi_\pi & 0 \end{bmatrix}. \tag{78}$$

An equilibrium will be determinate if and only if exactly one eigenvalue of B^{-1} is inside the unit circle. The characteristic polynomial of B^{-1}, $p(\lambda)$, is given by $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ where

$$a_1 = -(1 + \beta^{-1} + \kappa\beta^{-1}\sigma^{-1}), \tag{79}$$

$$a_2 = \beta^{-1} - \sigma^{-1}\varphi_z, \tag{80}$$

and

$$a_3 = \frac{\kappa\varphi_\pi + \varphi_z}{\beta\sigma}. \tag{81}$$

Note that $p(0) = a_3$,

$$p(1) = \frac{\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z}{\beta\sigma}, \tag{82}$$

and

$$p(-1) = \frac{\kappa(\varphi_\pi - 1) + (1+\beta)\varphi_z - 2\sigma(1+\beta)}{\beta\sigma}. \tag{83}$$

We have by assumption $a_3 > 0$. Irrespective of whether a_2 is negative or positive, by Descartes' rule of signs there is either one negative root and two positive roots or one negative root and a pair of complex conjugates. We have $p(0) > 0$ and $p(-1) < 0$ by condition (42). So the negative root (λ_1) is inside the unit circle. If all the roots are real, then since $\sum_{i=1}^{3} \lambda_i = 1 + \beta^{-1} + \kappa\beta^{-1}\sigma^{-1}$ we have

$$\lambda_2 + \lambda_3 > 1 + \beta^{-1} + \kappa\beta^{-1}\sigma^{-1} > 2 + \kappa\beta^{-1}\sigma^{-1}. \tag{84}$$

[35] Henceforth, we ignore such non-generic cases.

Consequently, at least one of the positive roots, say λ_2, must be more than 1. By condition (41) we have $p(1) > 0$. This ensures that the third positive root also exceeds 1 by observing that $p(1) > 0$, $p(\lambda_2 + \varepsilon) < 0$ for small positive ε, and that $p(\infty) = \infty$. If the roots are complex, then again we have

$$2\lambda^r > 2 + \kappa\beta^{-1}\sigma^{-1} \tag{85}$$

so that the real part $\lambda^r > 1$.

D. PROOF OF PROPOSITION 4

The characteristic polynomial of B (given by equation (54)) is $\lambda^2 + a_1\lambda + a_0$ where

$$a_0 = \beta(1 - \sigma^{-1}\varphi_z) \tag{86}$$

and

$$a_1 = \kappa\sigma^{-1}(\varphi_\pi - 1) + \sigma^{-1}\varphi_z - 1 - \beta. \tag{87}$$

Since the variables z_t and π_t are free, equilibrium is determinate if and only if both eigenvalues of B are inside the unit circle. The necessary and sufficient conditions for this are (i) $|a_0| < 1$ and (ii) $|a_1| < 1 + a_0$. Condition (i) implies

$$-1 - \beta^{-1} < -\sigma^{-1}\varphi_z < \beta^{-1} - 1. \tag{88}$$

Since $\beta < 1$, the right hand inequality is always satisfied and the other inequality reduces to inequality (55). Condition (ii) implies the inequalities

$$-1 - \beta(1 - \sigma^{-1}\varphi_z) < \kappa\sigma^{-1}(\varphi_\pi - 1) + \sigma^{-1}\varphi_z - 1 - \beta < 1 + \beta(1 - \sigma^{-1}\varphi_z). \tag{89}$$

The right hand inequality reduces after some simplification to inequality (56). The left hand inequality, on the other hand, reduces to

$$\kappa(\varphi_\pi - 1) + (1 - \beta)\varphi_z > 0 \tag{90}$$

which is inequality (57).

E. PROOF OF PROPOSITION 5

For the reasons outlined in the proof of Proposition 2, expectational stability holds if the eigenvalues of B (given by equation (54)) have real parts less than 1 and does not hold if any eigenvalue of B has a real part more than 1. The characteristic polynomial of $(B-I)$ is given by $\lambda^2 + a_1\lambda + a_2$ where

$$a_1 = \frac{\kappa(\varphi_\pi - 1) + \varphi_z + \sigma(1-\beta)}{\sigma} \tag{91}$$

and

$$a_2 = \frac{\kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z}{\sigma}. \tag{92}$$

Again both eigenvalues of B have real parts less than one if and only if (i) $a_1 > 0$ and (ii) $a_2 > 0$. It is easy to show that in this case condition (ii) implies condition (i). Consequently, the only requirement is condition (ii) which implies inequality (61).

F. Proof of Proposition 6

Our system is a special case of the one treated in Evans and Honkapohja (2000, p. 237) since our model does not have any lagged endogenous variables. We can, therefore, apply their Proposition 10.1 (p. 240) directly to our setup. For expectational stability we require the real parts of the eigenvalues of the following matrices to be less than one:[36]

$$\rho B_1 + B_0, \tag{93}$$

and

$$B_0 + B_1. \tag{94}$$

On the other hand, if any eigenvalue of (93) or (94) has real part more than one, then the equilibria are not E-stable. Note that the matrix $B_0 + B_1$ is identical to the matrix B (given by equation (54)) which was crucial for determining uniqueness and E-stability under forward expectations and, therefore, the (necessary and sufficient) condition for the eigenvalues of the real parts of $B_0 + B_1$ to be less than one is given by inequality (61). The matrix $\rho B_1 + B_0 - I$ has the characteristic polynomial $\lambda^2 + a_1\lambda + a_0$ where

$$a_1 = 2 - \rho - \beta\rho + \sigma^{-1}\varphi_z - \kappa\sigma^{-1}(\rho - \varphi_\pi). \tag{95}$$

and

$$a_0 = 1 - \rho - \beta\rho + \beta\rho^2 + \sigma^{-1}\varphi_z(1-\beta\rho) - \kappa\sigma^{-1}(\rho - \varphi_\pi). \tag{96}$$

[36] The matrices B_0 and B_1 are defined in equations (64) and (65), respectively.

Both eigenvalues of $(\rho B_1 + B_0)$ have real parts less than one if and only if (i) $a_0 > 0$ and (ii) $a_1 > 0$. Note that

$$a_1 = a_0 + 1 + \rho(\beta\rho + \beta\sigma^{-1}\varphi_z). \tag{97}$$

Consequently, $a_0 > 0$ implies that $a_1 > 0$. As a result, both eigenvalues of $(\rho B_1 + B_0)$ have real parts less than one if and only if $a_0 > 0$. We can write a_0 as

$$a_0 = \sigma^{-1}[\sigma(1-\rho)(1-\beta\rho) + (1-\beta\rho)\varphi_z + \kappa(\varphi_\pi - \rho)]. \tag{98}$$

The first term within parentheses is positive. Inequality (61), on the other hand, implies that $(1-\beta\rho)\varphi_z + \kappa(\varphi_\pi - \rho) > 0$ since

$$(1-\beta\rho)\varphi_z + \kappa(\varphi_\pi - \rho) = \kappa(\varphi_\pi - 1) + (1-\beta)\varphi_z + \kappa(1-\rho) + \beta(1-\rho)\varphi_z. \tag{99}$$

Inequality (61), therefore, implies $a_0 > 0$. This proves the proposition.