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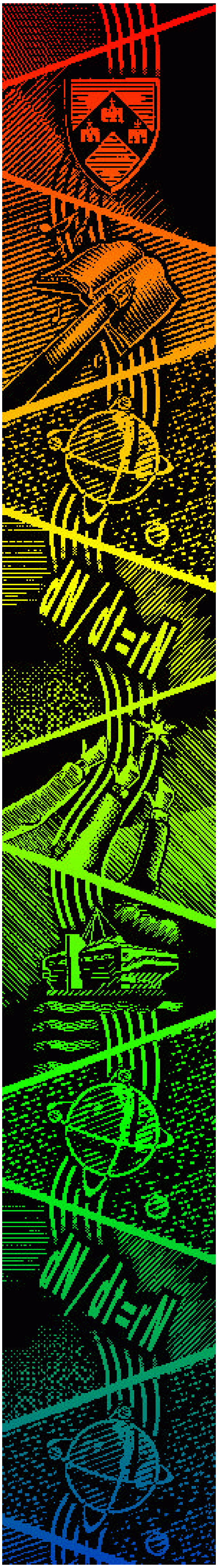

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The Meaning of Death: Some Numerical Simulations of a Model of Healthy and Unhealthy Consumption

by

Martin Forster

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

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Abstract

This paper presents some numerical simulations of a model of healthy and unhealthy consumption to investigate the impact of various terminal conditions on an individual's life-span, pathways of consumption and health. A 'benchmark' model, in which both the life-span and the individual's 'death' stock of health are fixed, is compared to (i) a version in which the 'death' stock of health is freely chosen; (ii) a version in which life-span is freely chosen; (iii) a version in which both the 'death' stock of health and life-span are freely chosen. Results show how the choice of terminal conditions has a striking impact on both optimal plans and comparative static/dynamic predictions and raise questions about how to model 'death' in deterministic demand for health models. Results also illustrate the application of iterative processes to determine an optimal life-span in continuous time models, the role of the marginal value of health capital in determining optimal plans, and the importance of checking the second-order conditions for the optimal choice of life-span in such models.

JEL codes: I1 H0

Keywords: the demand for health; healthy and unhealthy consumption; terminal and transversality conditions; optimal life-span

*Department of Economics and Related Studies, University of York, Heslington, York YO10 5DD, United Kingdom, e-mail: mf8@york.ac.uk. Funding from the ESRC postgraduate training programme is acknowledged. Thanks to Isaac Ehrlich and Michael Grossman for extensive comments. Also to Michael Caputo, Tony Culyer, Andrew Jones, Peter Lambert, John Mooney, John Robinson, Walter Ried, Peter Simmons, participants in the January 1997 Health Economists' Study Group meeting, and the summer 2000 seminar series at Brunel University. Thanks to Michael Kofler for providing the mapleps program. Mistakes are the responsibility of the author.

1 Introduction

There now exists an extensive theoretical literature on the demand for health, motivated by Grossman's (1972a, 1972b) seminal work. A focus of much of this literature, especially in recent years, is on the modelling of 'optimal' life-span, time paths of health and health investment, and how these are affected by changes in parameters such as the inherited stock of health, rate of health capital depreciation and level of education.

Simplifications and generalisations of Grossman's model have produced models which adopt different assumptions and methods of solution and which, in some cases, yield conflicting results. Further, the focus of many of these models has remained narrowly on 'medical care' type investments which affect health only, rather than on the consumption of goods that yield utility and also impact on health. This paper focuses on how such models have dealt with conditions for the marginal value placed on the terminal, or 'death', stock of health and the terminal time (or life-span). Treatment of terminal conditions was first recognised as being an important issue in demand for health models by B. Forster (1989) and Ehrlich and Chuma (1990) and has been addressed more recently in the work of Ried (1998) and Grossman (1998).

The paper presents a series of numerical simulations of a model of healthy and unhealthy consumption which differ in the way these terminal conditions are modelled. A 'benchmark' model, in which both the terminal stock of health and life-span are fixed, and no restriction is placed on the marginal value of health at the point of death, is compared to: (i) a version in which the terminal stock of health is freely chosen, implying that the marginal value of health stock equals zero at the point of death; (ii) a version in which life-span is freely chosen; (iii) a version in which both the terminal stock of health and life-span are freely chosen. These comparisons are made in a model in which individuals choose an optimal 'mix' of healthy and unhealthy consumption: healthy goods yield utility and increase health stock; unhealthy goods yield utility but decrease health stock. This reflects growing evidence of the role that 'lifestyles' play in determining health and life-span, over and above the impact of medical care (Fuchs, 1986; Contoyannis, 1999).

Results show how the choice of terminal conditions has a striking impact on both optimal plans and comparative static/dynamic predictions. They also highlight the role played by restrictions on utility and earnings that some, but not all, demand for health models impose. The continuous time approach used in this paper is shown to have the advantage over discrete time approaches in that it can make full use of transversality conditions to determine optimal terminal points. This avoids the use of iterative processes to determine an optimal life-span that characterises existing discrete time models. Further insight is provided on the role of the marginal value of health capital in determining optimal plans, and results stress the importance of checking the second-order conditions for the optimal choice of life-span in continuous time models.

The paper makes heavy use of phase diagrams to illustrate results, and emphasis is placed on explaining the intuition behind the forces that drive an individual's optimal plans in each version of the model. The main aim of the paper is therefore to lend additional insight and intuition to what has become quite a complex analytical literature, rather than to propose hypotheses that might be subject to empirical testing.

As far as possible in the simulations, features of existing demand for health models are retained: health provides what Grossman terms 'investment' and 'consumption' benefits; the individual inherits an initial stock of health and dies when health reaches a 'death' stock, and the evolution of health stock over time is determined by a stock-adjustment equation. The model also allows for decreasing returns to healthy consumption and increasing 'damage' to unhealthy consumption, addressing concerns about 'bang-bang' solutions first noted by Ehrlich and Chuma (1990) and discussed more recently by Ried (1998) and Grossman (1998). However, the emphasis on phase diagrams necessitates two simplifications. Firstly, there is only one state variable modelled (health), which rules out the inclusion of a capital market. Secondly, the rate of depreciation of health capital is time invariant to ensure that solutions to the model are *autonomous* ones, i.e., they do not contain an independent time trend. These simplifications are discussed in full in the paper, and do not affect its main messages.

Section 2 of the paper considers the previous demand for health literature. Section 3 presents the 'benchmark' model and section 4 presents numerical simulations of the 'benchmark' model and its three variants. Section 5 discusses the results in more detail and section 6 concludes.

2 Previous literature

Table 1 summarises the demand for health models that motivate this paper. The models are chosen because, firstly, they are all *deterministic*, rather than *stochastic* (the model presented in this paper is deterministic). Secondly, they illustrate the different ways researchers have modelled the terminal conditions that are the focus of this paper.

Column 2 of table 1 classifies each model as belonging to one of three categories, A, B or C, described in more detail below. Column 3 notes whether the authors use a discrete or continuous time model. Column 4 notes whether the models assume that the 'death' stock of health ($H_{\min}$) is fixed or is freely chosen and whether its marginal value equals zero. Column 5 notes whether the terminal time, or life-span (T), is fixed or is freely chosen. Column 6 notes whether earnings are restricted to equal zero at the point of death, and column 7 does the same for utility. Some models restrict earnings to equal zero at the point of death because they assume the individual has no time available for market activities at this point. For the same reason, some models restrict utility to equal zero at this point. Finally, column 8 details the methods used to determine the optimal endpoint for the problem and, specifically, whether any transversality conditions or 'iterative processes' are used. Transversality conditions are the additional necessary conditions required for the optimal choice of boundary points such as the terminal stock of health or the terminal time (see Léonard and van Long 1992, pages 221-262 for a full treatment). They are used in the continuous time models, and in one of the discrete time models. Iterative processes to determine optimal life-span are features of the discrete time models.

The three categories listed in table 1 are:

Category A *Discrete time models which place a zero marginal value on* $H_{\min}$ *and which use an iterative process to model the optimal, free choice of* T (Grossman, 1972a; Grossman, 1972b; Grossman, 1998; Ried, 1998). Grossman (1998), acknowledging that his original 1972 paper does not satisfactorily address the determination of an optimal life-span, discusses at length how it might be established. He presents an iterative process (Grossman, 1998, pages 501-504) which increments or decrements the time horizon by 1, depending on whether the terminal stock of health is lower than or higher than the (exogenously set) 'death' stock. Ried (1998) also suggests that an iterative process be used to establish the optimal terminal time, citing Canon et al. (1970) and commenting: 'the optimal final period ... has to be determined through the analysis of a sequence of fixed terminal time problems with terminal time varying over a feasible domain' (Ried, 1998, page 389). Both authors assume that the healthy time available to the individual equals zero at the (exogenously set) 'death' stock and, further, that this 'death' stock has zero marginal value. Ried imposes this latter restriction explicitly via a transversality condition that sets the costate variable for health to zero at the point of death. Grossman does not impose a transversality condition but comments: '... optimal gross investment in health is positive except in the very last year of life' (Grossman 1998, page 500), implying that the marginal value of the 'death' stock equals zero. Ried also imposes the explicit restriction that utility equals zero at the point of death.

Category B *Continuous time models with free choice of* $H_{\min}$ *(implying that the marginal value of* $H_{\min}$ *equals zero) and a fixed* T (B. Forster, 1989; Eisenring, 1999). B. Forster (1989) presents a continuous time model in which the terminal time is fixed and the terminal stock of health is freely chosen, implying that health stock is used up until its marginal value equals zero. This optimal terminal stock of health is established by using the transversality condition that sets the costate variable for health to zero at the terminal point of the problem (B. Forster, 1989, pages 50 and 52) in the

Table 1: Some demand for health models and their treatment of terminal conditions

Author and year	Category	Discrete or continuous time?	'death stock' ($H_{\min}$) free or fixed?, marginal value	life-span (T) free or fixed?	Endpoint restriction on: earnings?	Endpoint restriction on: utility?	Determination of endpoint of problem
Grossman (1972, 1998)	A	discrete	fixed, $= 0$	free	$= 0$	not clear	iterative process for choice of T
Forster (1989)	B	continuous	free,$= 0$	fixed	none	none	transversality condition for choice of $H_{\min}$
Ehrlich and Chuma (1990)	C	continuous	fixed, not clear	free	not clear	not clear	transversality condition for choice of T
Ried (1998)	A	discrete	fixed, $= 0$	free	$= 0$	$= 0$	iterative process for choice of T, transversality condition for choice of $H_{\min}$
Eisenring (1999)	B	continuous	free, $= 0$	fixed	none	none	transversality condition for choice of $H_{\min}$

same way that Ried (1998) does. Using Oniki's (1973) methods, Eisenring (1999) presents a full set of comparative dynamic results for B. Forster's model. Neither B. Forster nor Eisenring place restrictions on earnings or utility at the point of death.

Category C *Continuous time model with a freely chosen T and a fixed $H_{\min}$* (Ehrlich and Chuma, 1990). In Ehrlich and Chuma's continuous time model the terminal stock of health is fixed and the terminal time is freely chosen. The free choice of the terminal time imposes a different transversality condition to that used by B. Forster and Eisenring - instead of the costate variable for health being set to zero at T, it is the Hamiltonian that is set to zero (Ehrlich and Chuma, 1990, page 767). It is unclear whether Ehrlich and Chuma impose restrictions on the marginal value of health stock, earnings and utility at the point of death. The authors argue that, because Grossman's (1972a, 1972b) analysis does not deal with the appropriate transversality condition for the free choice of T, its subsequent treatment of time paths for health, health investment and comparative dynamics must be treated with caution.

2.1 Comment

The models listed in table 1 differ significantly in the way they deal with terminal conditions for the stock of health, the terminal time, earnings and utility. This range of approaches motivates the models that are solved in this paper.

Firstly, the Category B models, which fix the terminal time and allow the individual to choose the terminal stock of health freely, are clearly not suitable for modelling the optimal choice of life-span. Nevertheless, it is useful to consider them, because the transversality condition used in these models, which sets the marginal value of the 'death' stock of health to zero, also appears in Ried's (1998) model and has important implications for paths of health investment. To date, Ehrlich and Chuma's model remains the only Category C model - a continuous time model which models the optimal choice of the terminal time. Table 1 also shows that additional terminal conditions are imposed on earnings and utility in some (Grossman 1972a, 1972b, 1998 and Ried 1998), but not all (B. Forster 1989 and Eisenring 1999), of the models.

Secondly, the models in table 1 differ according to whether they adopt a discrete or continuous time approach. The continuous time approach is preferred here for a number of reasons. Firstly, it yields a full set of transversality conditions that uniquely determine optimal paths, avoiding the need for iterative processes. Secondly, it can be used to illustrate a continuous-time iterative process to determine an optimal T for comparison with the discrete time approaches. Thirdly, there have been some recent exchanges on comparative dynamic effects of marginal and non-marginal changes in parameter values in discrete time models (see the editorial by Grossman 1998, page 504 and Ried 1998, pages 388-389) that do not apply in a continuous time setting.[1]

In order to isolate the effects of the various terminal conditions listed in table 5 on optimal plans, the simulations in this paper consider only one or two such conditions in turn. Initially, no restrictions are imposed on utility and earnings at the point of death, and we explore the effects on optimal plans of setting the marginal value of health to zero and modelling a free choice of terminal time. We then consider imposing both of these conditions in a model, along with an additional restriction for the level of utility at the point of death.

3 The 'benchmark' model

In the 'benchmark' model, an individual maximises lifetime utility, V, over a planning period $[0, T]$, $0 < T < \infty$, where T is the individual's life-span and is exogenous. The individual's utility function $U(C(t), Z(t), H(t))$, is a continuous, twice differentiable and strictly concave function of the individual's own stock of health, $H(t)$ (the state variable) and two health-affecting goods, $C(t)$ and $Z(t)$ (the control variables). Utility is discounted at the rate of time preference, ρ. Consumption of goods is such that $U_C > 0, U_{CC} < 0, U_Z > 0, U_{ZZ} < 0, U_{CZ} > 0, U_{CH} > 0, U_{ZH} > 0$.

As in previous models of the demand for health, the individual starts the planning period with an initial stock of health H_0 and dies when health reaches a fixed 'death' stock $H_{\min} > 0$. Health stock depreciates at the time-invariant rate of depreciation δ, $0 < \delta < 1$. As well as providing 'consumption' benefits via the utility function, health yields 'investment' benefits by providing an income stream $w(t) = w(H(t))$. $w(H(t))$ is a continuous, twice-differentiable function such that $w(0) = 0, w_H > 0, w_{HH} < 0$ on the interval $H \in [0, H_0]$. $w(t)$ is allocated to consuming $C(t)$ and $Z(t)$, whose time invariant prices are pc and pz respectively.

C is an unhealthy good, consumption of which reduces the individual's stock of health at an increasing rate via the relationship $f(C(t))$. $f(C(t))$ is a continuous, twice-differentiable function such that $f(0) = 0, f_C < 0, f_{CC} < 0$ on the interval $C \in [0, C_{\max}(t)]$, where $C_{\max}(t)$ is the maximum level of C that the individual may consume at any instant (the level of C that could be consumed if all available instantaneous income w is spent on C). Z is a healthy good, consumption of which increases the individual's stock of health

[1] Ried and Grossman discuss whether, in a discrete time model where the terminal time T is freely chosen, T is invariant to changes in parameter values because of the discrete nature of the model. This issue does not arise in a continuous time setting because the domain for T is continuous and so marginal changes in a parameter value will result in marginal changes in T.

at a decreasing rate via the relationship $g(Z(t))$. $g(Z(t))$ is a continuous, twice-differentiable function such that $g(0) = 0, g_Z > 0, g_{ZZ} < 0$ on the interval $Z \in [0, Z_{\max}(t)]$, where $Z_{\max}(t)$ is the maximum level of Z that the individual may consume at any instant (the level of Z that could be consumed if all available instantaneous income w is spent on Z).

A time constraint, in which the individual allocates time to producing various commodities (as well as suffering 'unhealthy time'), is omitted to keep the numerical simulations manageable and to isolate the impact of the transversality conditions on optimal plans. This means that, initially, no restrictions are placed on earnings or utility at the point of death. The implication of introducing 'unhealthy time' is investigated by exploring the meaning of 'zero utility' at the point of death in section 5.2.

The problem is to choose the levels of the control variables $C(t)$ and $Z(t)$ to maximise:

$$V \equiv \int_0^T U(C(t), Z(t), H(t))e^{-\rho t}dt, \tag{1}$$

subject to:

$$\dot{H}(t) = f(C(t)) + g(Z(t)) - \delta H(t), \tag{2}$$

$$w(H(t)) = pzZ(t) + pcC(t), \tag{3}$$

$$H(0) = H_0,\ H(t) > H_{\min}\ \forall t \in [0, T),\ H(T) = H_{\min}. \tag{4}$$

Eq. (2) is the stock adjustment equation for health, showing that health depreciates at the rate δ but may be supplemented by consuming the healthy good Z, or further reduced by consuming C. Eq. (3) is the budget constraint, showing that instantaneous income is spent on consuming the goods Z and C. Finally, Eq. (4) gives the boundary conditions for health capital: the individual starts the planning period at time 0 with a stock of health H_0, and must finish the planning period at time T with the 'death' stock $H_{\min}$.

To solve the problem set up the current-value Hamiltonian:

$$\mathcal{H}(t) = U(C(t), Z(t), H(t)) + \psi(t)(f(C(t)) + g(Z(t)) - \delta H(t)), \tag{5}$$

where $\psi(t) \geq 0$ is the current-value costate variable, or marginal value of health capital. $\mathcal{H}(t)$ is maximised subject to the budget constraint in Eq. (3) by forming the current-value Lagrangian, $\mathcal{L}(t)$:

$$\mathcal{L}(t) = \mathcal{H}(t) + \lambda(t)(w(H(t)) - pzZ(t) - pcC(t)), \tag{6}$$

where $\lambda(t) \geq 0$ is the current-value Lagrange multiplier for income. Where convenient in the rest of the paper, the time argument (t) is omitted.

3.1 Necessary and sufficient conditions

Applying Pontryagin's maximum principle (Pontryagin et al., 1962) to Eq. (6) gives the following first-order necessary conditions for an optimal solution:

$$\mathcal{L}_C = U_C + \psi f_C - \lambda pc = 0, \tag{7}$$

$$\mathcal{L}_Z = U_Z + \psi g_Z - \lambda pz = 0, \tag{8}$$

$$\dot{\psi} = -U_H + (\delta + \rho)\psi - \lambda w_H. \tag{9}$$

Additionally, the boundary conditions for health in Eq. (4) must be satisfied, and the individual must choose a path of consumption and health that links H_0 to $H_{\min}$ in exactly T units of time.

Now, let $(C^*(t), Z^*(t), H^*(t))$ be a path satisfying Eq. (7) to (9) and the boundary conditions in Eq. (4). Let $\psi^*(t)$ and $\lambda^*(t)$ be, respectively, the values of the costate variable and Lagrange multiplier associated with these conditions. Then, by Theorem 7.9.1 of Léonard and van Long (1992, page 251), the above necessary conditions are sufficient for a global maximum providing the Lagrangian $\mathcal{L} = U(\cdot) + \psi^*(t)(f(C(t)) + g(Z(t)) - \delta H(t)) + \lambda^*(t)(w(H(t)) - pzZ(t) - pcC(t))$ is concave in the variables $(C(t), Z(t), H(t))$. The assumptions embodied in the functional forms adopted for this problem, namely that, given $\psi^*(t) \geq 0, \lambda^*(t) \geq 0$, $U(C(t), Z(t), H(t))$, $f(C(t)) + g(Z(t)) - \delta H(t)$ and $w(H(t)) - pzZ(t) - pcC(t)$ are each concave in $(C(t), Z(t), H(t))$ satisfy Corollary 6.5.1 of Léonard and van Long (1992, page 214), and ensure the concavity of the Lagrangian.

3.2 Optimal 'mix' of healthy and unhealthy consumption

The Hamiltonian and first-order conditions provide important information that serves as a reference point in future discussion. For a small time interval Δ beginning at time t, the Hamiltonian $\mathcal{H}(\cdot)\Delta$ measures the total utility yielded from current health and consumption, $U(C, Z, H)\Delta$, plus the contribution of the change in health stock to future utility, $\psi \dot{H} \Delta$. This is simply the change in health stock $\dot{H}\Delta$, valued at its marginal value, ψ. The Hamiltonian is thus measuring the *overall utility prospect* of the individual at any instant (see Dorfman 1969, page 822 or Chiang 2000, page 207 for further details), and is what the individual seeks to maximise in any Δ. Consuming Z yields utility directly in Δ and also stores more health stock for future periods. However, consuming *too much* Z during Δ will result in too little utility available from consuming C, even though consuming C reduces health stock. So the individual must choose the optimal 'mix' of healthy and unhealthy consumption so as to balance current and future utility prospects.

How is this optimal 'mix' arrived at? Imagine that neither Z nor C influence health stock. Then the optimal 'mix' occurs when: $U_C/U_Z =$

pc/pz. However, the individual also takes into account the marginal effect of Z and C on health capital, valued at $\psi g_Z > 0$ and $\psi f_C < 0$. Z is therefore consumed above the level that would occur if it did not influence health, and C below the level, with the optimal levels given by:

$$\frac{U_C + \psi f_C}{U_Z + \psi g_Z} = \frac{pc}{pz}, \tag{10}$$

obtained from Eqs. (7) and (8).

The marginal value of health stock evolves according to the relationship in Eq. (9). Rearranged this gives:

$$\psi \left[\delta + \rho - \frac{\dot{\psi}}{\psi} \right] = U_H + \lambda w_H. \tag{11}$$

That is, health stock equates the current marginal benefits, measured in utility terms (the right hand side of Eq. (11)) with the current marginal costs (the left hand side). This is similar to Grossman's (1972a) Eq. (1-13) and Ehrlich and Chuma's (1990) Eq. (13), except that health stock is measured in current-value, rather then present-value terms (this accounts for the appearance of the rate of time preference ρ in Eq. (11)), and there is no interest rate because there is no capital market.[2]

3.3 Phase diagram

Since the problem involves only one state variable, H, and C may be expressed as a function of Z and H by rearranging the budget constraint in Eq. (3), the problem can be reduced to one containing a single state and a single control variable and analysed using a phase diagram in (Z, H) space. This is shown in figure 1. The features of the phase diagram are:

Upper and lower bounds on consumption of Z. $Z_{\max}$ is the upper bound on Z and is increasing and convex (see the appendix). $Z_{\max}$ is the locus of points where the individual spends all available income on consuming Z. It is upward-sloping because at higher levels of health the individual generates more income via $w(H)$. In the numerical simulations, the Cobb-Douglas

[2]Noting that the present-value costate variable, π, is defined by $\pi = \psi e^{-\rho t}$, and the present-value Lagrange multiplier, μ, by $\mu = \lambda e^{-\rho t}$, we can write the present-value equivalent of Eq. (11) as:

$$\begin{aligned} \pi e^{\rho t} \left[\delta + \rho - \frac{e^{\rho t}(\dot{\pi} + \pi\rho)}{\pi e^{\rho t}} \right] &= U_H + \mu e^{\rho t} w_H \\ \Rightarrow \pi \left[\delta - \frac{\dot{\pi}}{\pi} \right] &= \frac{U_H}{e^{\rho t}} + \mu w_H, \end{aligned}$$

which is this model's equivalent to Grossman's (1972a) Eq. (1-13) and Ehrlich and Chuma's (1990) Eq. (13).

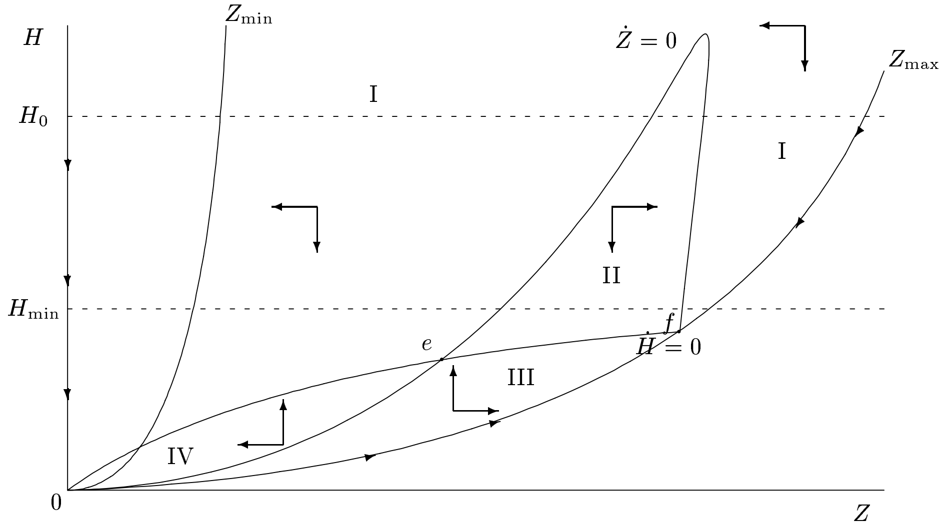

Figure 1: Phase diagram for the model

form assumed for the utility function implies an interior solution for Z and so the individual will not consume at a point on $Z_{\max}$.

$Z_{\min}$ is the lower bound on Z and the locus is derived by considering the restrictions placed on the marginal value of health stock, $\psi \geq 0$. When $\psi = 0$ the individual places no value on a marginal increase in health stock and consumes Z only for its consumption benefits, i.e. for its effect on utility. Define the implicit function:

$$\Psi(H, Z) = U_C pz - U_Z pc \equiv 0, \tag{12}$$

obtained by letting $\psi = 0$ and rearranging Eq. (10). This defines the lowest possible level of consumption of Z: the level that would occur if health had no value. The sign of the slope of the function $\Psi(H, Z)$ is ambiguous and so the simulation presented in section 4 is used to guide its shape. It is drawn in figure 1 as the upward-sloping, convex function $Z_{\min}$. At all points on $Z_{\min}$, $\psi = 0$; to the left $\psi < 0$ and to the right, $\psi > 0$.

In figure 1, the feasible region for consumption of Z therefore lies on and to the right of $Z_{\min}$ and to the left of $Z_{\max}$.

The $\dot{H} = 0$ locus. The slope of this locus is derived by considering the function:

$$\Omega_1(H, Z) = f\left(\frac{1}{pc}[w(H) - pzZ]\right) + g(Z) - \delta H, \tag{13}$$

obtained by substituting the equation for C from Eq. (3) into Eq. (2). The appendix shows that this locus passes through the origin and slopes upwards.

In the simulation it is concave. To establish the direction of movement of H above and below the $\dot{H} = 0$ locus, consider a point lying on the locus, $\bar{H}$, and the function:

$$\Omega_1(\bar{H}, Z) = f\left(\frac{1}{pc}[w(\bar{H}) - pzZ]\right) + g(Z) - \delta\bar{H}. \tag{14}$$

If Z increases by a small amount ϵ, $\Omega_1(\bar{H}, Z+\epsilon) > \Omega_1(\bar{H}, Z)$ and so to the right of the $\dot{H} = 0$ locus H is rising. If Z decreases by a small amount ϵ, $\Omega_1(\bar{H}, Z-\epsilon) < \Omega_1(\bar{H}, Z)$ and so to the left of the $\dot{H} = 0$ locus H is falling. These directions of movement are marked by the vertical arrows in figure 1.

The $\dot{Z} = 0$ locus. There are two $\dot{Z} = 0$ loci. When $Z = 0$ for all t, that is, where the individual spends all income on C for all t, $\dot{Z} = 0$. Since this lies to the left of the $Z_{\min}$ locus it is ruled out of the analysis. The second locus is defined by:

$$\dot{Z} = \frac{-\frac{\dot{H}}{pc}\theta_1 - \dot{\psi}\theta_2}{\theta_3}, \tag{15}$$

where $\theta_1 = [pcU_{ZH} - pzU_{CH} + w_H U_{ZC} - ((w_H pz)/pc)(U_{CC} + f_{CC}\psi)]$, $\theta_2 = (g_Z - (pz/pc)f_C)$, and $\theta_3 = U_{ZZ} + \psi g_{ZZ} - 2(pz/pc)U_{ZC} + (pz/pc)^2(U_{CC} + \psi f_{CC})$. $\dot{H}$, $\dot{\psi}$ and C may all be expressed in terms of Z and H using Eqs. (2) to (9) and ψ is obtained from rearranging Eq. (10). This is a highly complex expression which is difficult to analyse without assuming specific functional forms. The numerical simulation of section 4 is therefore used again to suggest a possible shape for the locus, and the movement of Z above and below it. The locus is shown by the simulation to reach a maximum between $Z_{\min}$ and $Z_{\max}$, with consumption of Z falling above it and rising below it. It is marked in figure 1 as $\dot{Z} = 0$ with the directions of movement of Z above and below it marked by the horizontal arrows.

Inherited health H_0 and the 'death' stock $H_{\min}$. The 'death' stock of health is marked as $H_{\min}$ and lies below the inherited stock of health H_0.

In figure 1, the feasible region for the problem is bounded by the $Z_{\min}$ and $Z_{\max}$ loci and the levels chosen for H_0 and $H_{\min}$. The $\dot{Z} = 0$ and $\dot{H} = 0$ loci form the boundaries of the four *isosectors* of the phase diagram, marked I-IV. Within each isosector, the movement of H and Z at any point is in the direction indicated by the arrows. For example, in isosector I, health stock and consumption of Z are falling, and paths move in a 'south-westerly' direction.

There are three *equilibrium points* in the phase diagram, where consumption of H and Z do not change with time. These are at the origin, point e and point f. Above point f on the $Z_{\max}$ locus, health stock falls because depreciation of health stock outweighs gross investment in health. Below point f on the $Z_{\max}$ locus health stock rises because gross investment in health outweighs depreciation. The nature of these equilibria is investigated further in the simulations presented in section 4.

3.4 Comment

The phase diagram of figure 1 is central to illustrating the models in section 4 and two points are worthy of note at this stage. Firstly, it is drawn in two-dimensional space. Had a capital market been included, a state and costate variable for assets would have entered the model and the phase diagram could not be drawn in two dimensions.

Secondly, the system of equations used to draw the phase diagram (Eqs. (2) and (15)) is *autonomous*, that is, time does not feature as an independent term in either equation. Introducing a time-varying depreciation rate in a similar way to Grossman (1972a, 1972b, 1998), Ried (1998) and Ehrlich and Chuma (1990) means that t enters the system of equations as an independent term and the phase diagram changes as t changes, making it impossible to draw.

As will be seen in section 4, a constant depreciation rate is consistent with finite and optimal terminal values for life-span because of the equation of motion for the marginal value of health stock ψ, (Eq. (9)). This is different to Grossman's model, in which the marginal cost of health investment is time invariant (see Grossman's 1972a, page 14 equation (2-3) and the accompanying reasoning for an age-dependent rate of health capital depreciation). Such a scenario does not arise in this paper, since in Eq. (9) the time-invariant δ affects the rate of change of ψ, which in turn, via Eqs. (7), (8) and (2) influences the time paths of Z, C and H and results in finite life-spans.

4 Numerical simulations

Numerical simulations are carried out using Maple V Release 5.1 by simulating the differential equations (15) and (2).[3] The utility function is of the form $U(C, Z, H) = C^a Z^b H^{(1-a-b)}$, where $0 < a < 1$, $0 < b < 1$ and $a+b < 1$. The individual's income production function is of the form $w(H) = \zeta H^\omega$, where $\zeta > 0$ and $0 < \omega < 1$. Consumption of C reduces the stock of health via the relationship $f(C) = \eta C^\beta$ where $\eta < 0$ and $\beta > 1$ and consumption of Z increases the stock of health via the relationship $f(Z) = \gamma Z^\alpha$ where $\gamma > 0$ and $0 < \alpha < 1$. These functional forms satisfy the restrictions imposed in section 3.

To run the simulations, parameter values are chosen to yield phase diagrams that are suitable for the purposes of illustration, rather than to yield

[3] A sample Maple worksheet is available from the author on request.

Table 2: Parameter values for numerical simulation of the 'benchmark' model

parameter	definition	value
T	terminal time	6.000
$H_{\min}$	terminal stock of health	0.400
H_0	inherited level of health	1.000
a	weight on C in $U(\cdot)$	0.400
b	weight on Z in $U(\cdot)$	0.300
ρ	rate of time preference	0.050
ζ	'efficiency parameter' on production of income	1.000
ω	index for production of income function	0.250
η	'efficiency parameter' for effect of C on H	-1.000
β	index for effect of C on H	4.000
γ	'efficiency parameter' for effect of Z on H	0.500
α	index for effect of Z on H	0.600
δ	rate of depreciation on health stock	0.300
pc	price of C	1.000
pz	price of Z	10.000

'realistic' life-spans. For all versions of the model in this section, the individual inherits a stock of health set at $H_0 = 1.000$. For the 'benchmark' model, the individual dies when health falls to the 'death' stock set at $H_{\min} = 0.400$. The parameter values for the 'benchmark' model are presented in table 2. Subsequent versions of the model differ only in the parameter values that apply to $H_{\min}$ and T, depending on whether these become choice variables.

The phase diagram for the numerical simulation of the benchmark model is presented in figure 2. The initial and terminal stocks of health are marked as $H_0 = 1.000$ and $H_{\min} = 0.400$. The loci denoting maximum and minimum levels of consumption of Z are marked as $Zmax$ and $Zmin$ respectively. The $\dot{H} = 0$ locus is marked as $Hdot = 0$ and the $\dot{Z} = 0$ locus as $Zdot = 0$. The arrows show the direction of movement of H and Z, measured by $\dot{H}/\dot{Z}$, at each arrow's centre.

The three equilibrium points in the phase diagram are (0,0), (0.046,0.251) (point e) and (0.077, 0.359) (point f). Paths approaching the origin are ruled out by the lower bound on Z, $Z_{\min}$. The nature of points e and f can be investigated by taking a first-order approximation of the system given by Eqs. (15) and (2) and establishing the sign of the determinant and eigenvalues of the Jacobian matrix J at each equilibrium point (Z^*, H^*):

$$J = \begin{bmatrix} \partial\dot{Z}/\partial Z & \partial\dot{Z}/\partial H \\ \partial\dot{H}/\partial Z & \partial\dot{H}/\partial H \end{bmatrix}_{(Z^*,H^*)}. \tag{16}$$

- *Point e.* $|J| \approx -.080$ and its eigenvalues are real and of opposite sign. Hence point e is a *saddle point* and is *conditionally stable.* There are

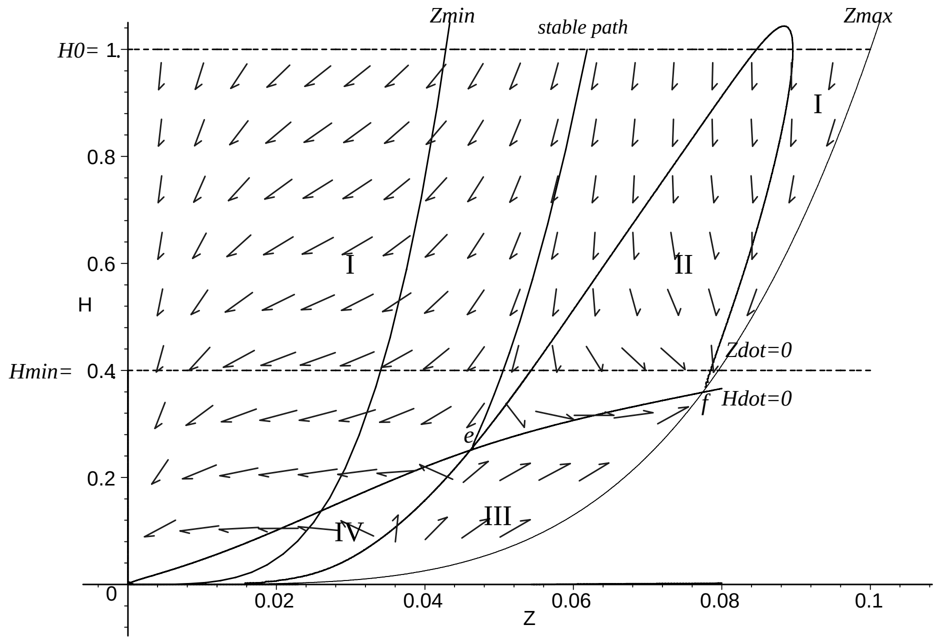

Figure 2: Phase diagram for the numerical simulation with path to saddle-point stable equilibrium point e.

two *stable paths* in regions I and III that lead towards point e and two *unstable paths* in regions II and IV that lead away from point e. The stable path in region I is marked in figure 2 as *stable path*. It will take an infinite amount of time to travel from H_0 to point e along this path, since changes in H and Z become infinitely small in the vicinity of e. To the left of the stable path, paths move in the direction indicated by the arrows towards the Z_{min} locus; to the right they move in the direction indicated by the arrows towards point f.

- *Point f.* $|J| \approx +0.123$ and the eigenvalues are both negative. Hence this point is an *improper node* and is *asymptotically stable*, that is, there exists a neighbourhood of point f such that, for any point within the neighbourhood, the paths of Z and H approach point f as t tends to infinity.

Finally, consider the upper and lower bounds on T. The lower bound on T can be calculated by considering the path from H_0 that reaches the intersection of the locus Z_{min} and the terminal level of health $H_{\text{min}} = 0.400$. This is the point (0.035,0.400) and it defines a path for which $T_{\text{lower}} \approx 4.312$. Any path to the left of this one will take less than 4.312 units to link H_0 with H_{min} and will violate the restriction that $\psi \geq 0$, since it will cross the

$Z_{\min}$ locus before reaching $H_{\min}$. The upper limit on T is defined by the path that moves down $Z_{\max}$ from $H_0 = 1.000$ to $H_{\min} = 0.400$. This will not be a feasible path for the individual, given the form of the utility function. Nevertheless, it can be used to determine the upper bound on T by solving Eqs. (2) and (15) with $C = 0$ for all t, yielding $T_{\text{upper}} \approx 10.467$.

4.1 Simulations of the 'benchmark' model and its variants

We now have a full description of the phase diagram, and can consider the 'benchmark' model and its variants in more detail. In doing so, it is crucial to note the role played by the terminal condition for health stock and the terminal time in determining the optimal path: the necessary conditions in Eqs. (7) to (9) have been used to construct the phase diagram, but it is impossible to isolate which of these paths is optimal except by considering the terminal conditions.

Model 1. *The 'benchmark' model - fixed terminal stock of health and a fixed terminal time.* In the 'benchmark' model, the individual knows that he/she has a fixed life-span, $T = 6$, and will die when health falls to the fixed 'death' stock, $H_{\min} = 0.400$. The optimisation problem consists of selecting the path for Z and C such that, in travelling from H_0 to $H_{\min}$ in $T = 6$ units, V is maximised.

Determining the path that takes exactly $T = 6$ units to travel from H_0 to $H_{\min}$ is a *two point boundary value problem* (see Judd 1998, pages 336 and 350). The path is established by setting $H_0 = 1.000$ and varying choices of $Z(0)$ to 'shoot' a path defined by Eqs. (2) and (15) such that it takes exactly 6 units of time to reach the terminal stock $H_{\min}$. This path is defined by the initial conditions $Z(0) \approx 0.064, H(0) = 1.000$, and is marked in figure 3 as $T = 6$. Along the path, consumption of Z falls initially and then rises. Paths of the variables H, Z, C and ψ are plotted against time for this path in figure 4.[4] ψ increases and consumption of C falls throughout the individual's life. The simulation calculates discounted lifetime utility to be 1.175 units along this path.

What would happen if the individual's planning horizon is lower than 6? The individual must reach $H_{\min}$ more quickly, and must consume less of Z and more of C. An example for a life-span of $T = 5$ is shown in figure 3: consumption of Z is lower for all levels of health, and falls through time until $H_{\min}$ is reached. Consumption of C is higher. The simulation calculates discounted lifetime utility for this planning horizon to be approximately 1.060 units.

If the individual has longer to live, the opposite occurs: with more time to reach the terminal stock of health the individual consumes more Z and

[4] To ensure a suitable scale for the variables in figure 4, levels of Z are scaled up by a factor of ten, and those for ψ are scaled down by a factor of ten.

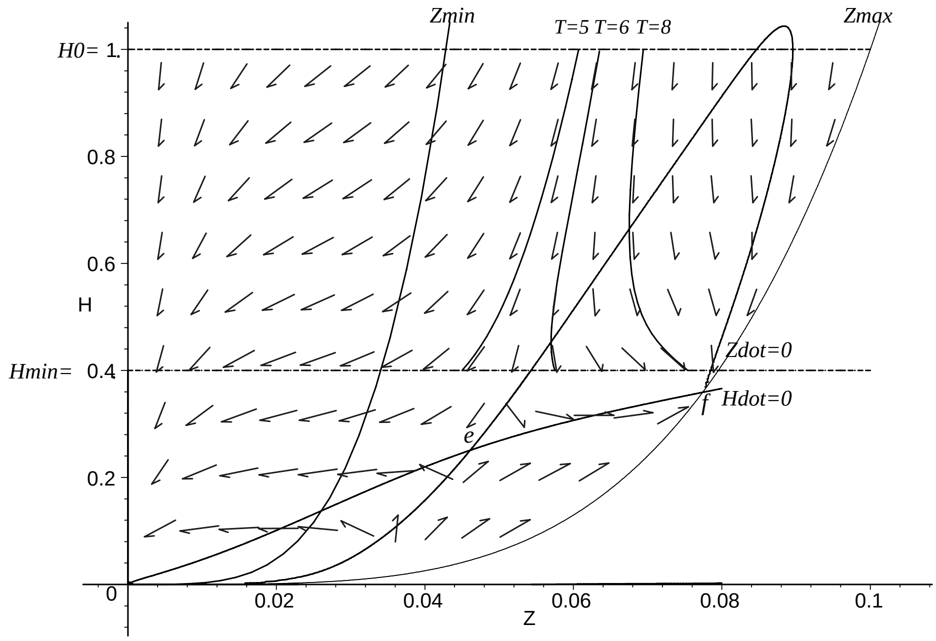

Figure 3: Optimal paths for the 'benchmark' fixed terminal time, fixed terminal stock problem with $T = 6$. Paths for $T = 5$ and $T = 8$ also drawn.

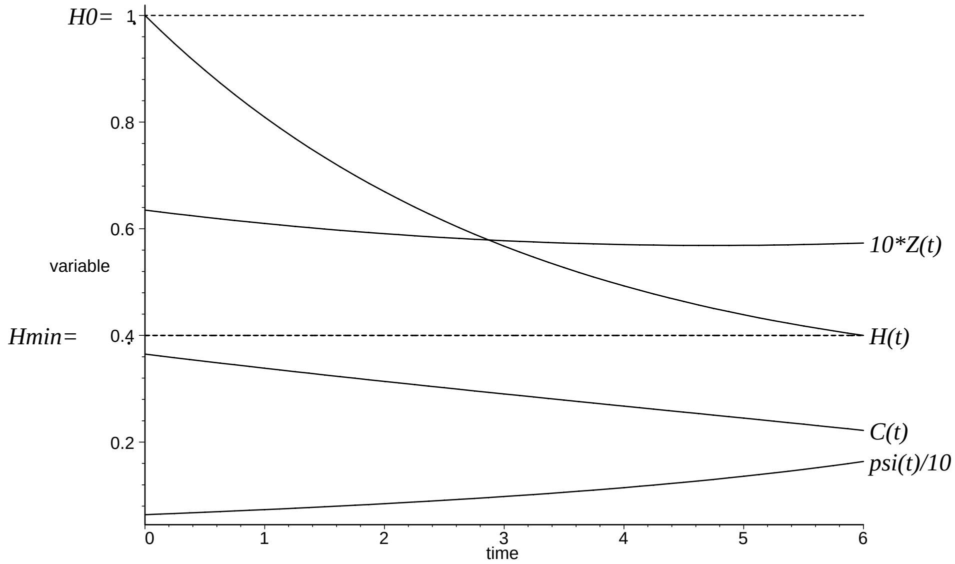

Figure 4: Time paths of health, consumption and the costate variable for health for the 'benchmark' fixed terminal time, fixed terminal stock problem.

less C. Such a path is marked for a life-span of $T = 8$ in figure 3, and discounted lifetime utility is calculated at 1.240 units.

Model 2. *Free terminal stock of health and fixed terminal time.* Consider now the problem in which the terminal time is again fixed at $T = 6$, but the terminal conditions in Eq. (4) are replaced by:

$$H(0) = H_0,\ H(t) > H_{\min}\ \forall t \in [0, T),\ H(T) = H_{\min} \text{ free.}$$

In this scenario, the individual knows that he/she has exactly 6 units of time to live, but is allowed to choose the 'death' stock of health optimally. This is the terminal condition imposed by B. Forster (1989) and Eisenring (1999). The individual will 'use up' the stock of health until its marginal value equals zero, that is, until the transversality condition:

$$\psi(T) = 0 \tag{17}$$

holds (Léonard and van Long, 1992, page 222).

The individual now chooses a path of consumption that is defined by the initial condition H_0, the *fixed* life-span T and the transversality condition (17). This is the path linking H_0 to a (freely chosen) terminal stock of health in exactly $T = 6$ units of time. The locus $Z_{\min}$ is the locus of all points for which $\psi = 0$. The individual will therefore choose the path of consumption that links inherited health H_0 to a point on $Z_{\min}$ in exactly 6 units of time.

The problem is again of a two point boundary value nature, and is solved numerically using *reverse shooting* methods described in Judd (1998, page 355). Firstly, the right hand sides of Eqs. (2) and (15) are multiplied by -1 to allow paths of consumption and health to be traced back from the terminal values of Z and H. Then ψ is set to zero in Eq. (10) and values for Z are varied to define *terminal* pairs of values $(Z(T), H(T))$ on the $Z_{\min}$ locus which can be used to 'shoot' paths *back* towards the initial stock of health $H_0 = 1.000$. The one taking $T = 6$ units is the optimal path.

Figure 5 shows the optimal path $Hfree$ for this problem, and figure 6 plots the paths of health, consumption of C and Z, and ψ against time along this path. The terminal stock of health is now $H_{\min} \approx 0.326$. ψ falls over time and reaches zero, satisfying the transversality condition, when $T = 6$. Consumption of Z falls over time in response to the diminishing value the individual places on health (measured by ψ). Consumption of C at first falls and then rises. The simulation calculates discounted lifetime utility to be 1.207 units along this path.

Model 3. *Fixed terminal stock of health and free terminal time.* In this scenario, the constraints on health stock in Eq. (4) apply, but T is chosen optimally. This is the terminal condition imposed by Ehrlich and Chuma (1990). The individual now knows that death occurs at the fixed $H_{\min}$,

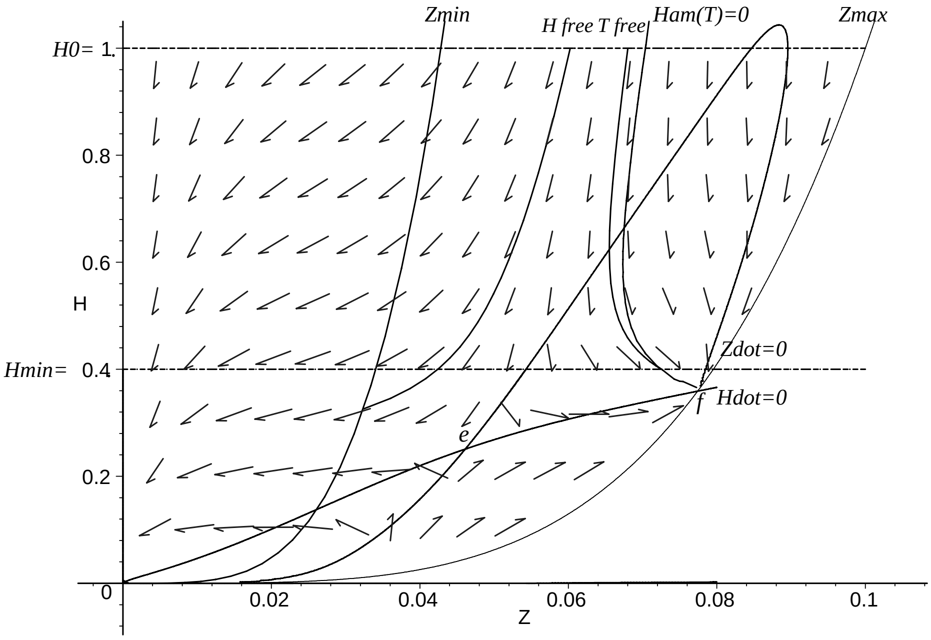

Figure 5: Optimal paths for the free $H_{\min}$, fixed T and fixed $H_{\min}$, free T problems.

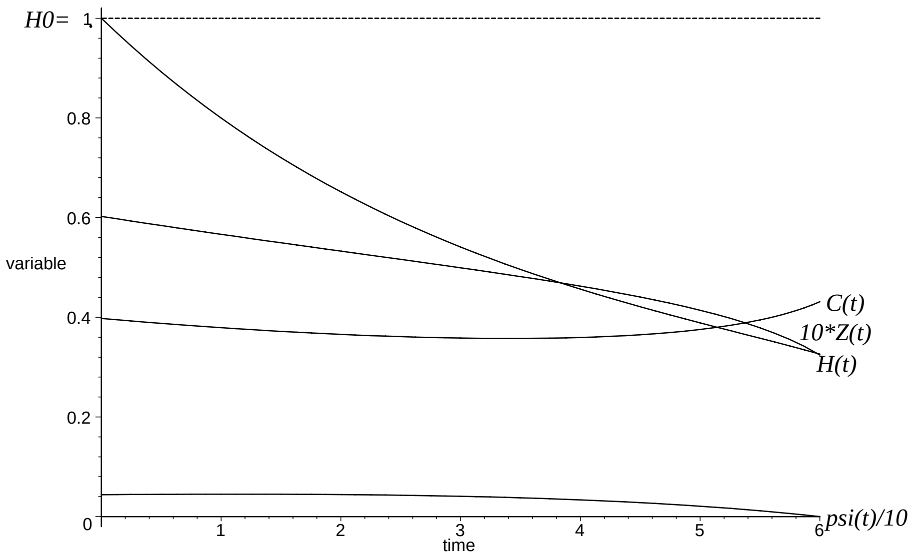

Figure 6: Time paths of health, consumption and the costate variable for health for the fixed terminal time, free terminal stock problem.

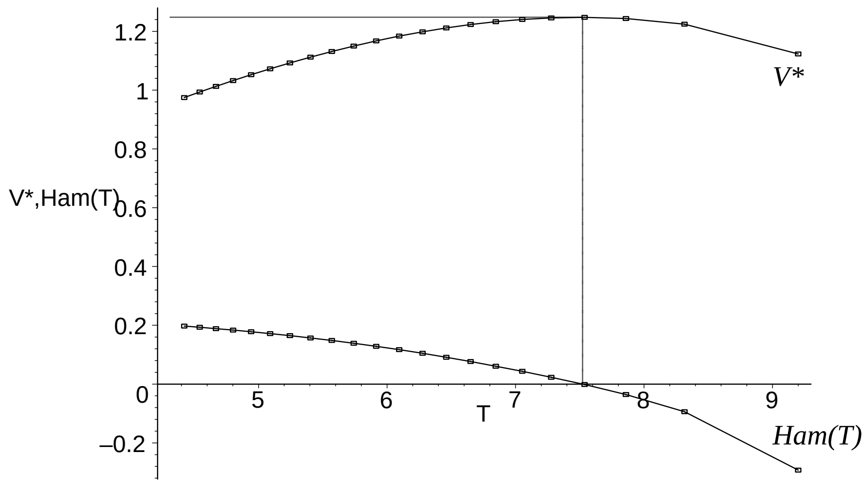

Figure 7: Plot of concave maximum value function V^* and decreasing $\mathcal{H}(T)$ against T for a series of fixed terminal stock, fixed terminal time problems that differ in their choice of T.

but can choose the optimal T to reach it. Such a problem requires the transversality condition that determines the optimal choice of T (Léonard and van Long, 1992, page 240):

$$\mathcal{H}(T) = 0. \tag{18}$$

Further, the maximum value function for the problem V^* must be concave in T, implying that the Hamiltonian for the problem is a decreasing function of T (Léonard and van Long, 1992, page 253). This is confirmed in a plot of V^* and $\mathcal{H}$ for a series of fixed terminal time, fixed terminal stock problems which differ only in their choice of T in figure 7.

The transversality condition in Eq. (18) can be expressed as an implicit function of Z and H, and is plotted as $Ham(T) = 0$ in figure 5. This is the locus of all optimal terminal points in model 3. The optimal path is found by substituting $H_{\min} = 0.400$ into (18) and solving for Z. This yields $(Z(T), H_{\min})$, the optimal terminal pair, which lies on $Ham(T) = 0$. Using the system of differential equations from model 2, the optimal T is established by calculating how long it takes to link the optimal terminal pair to the initial condition $H_0 = 1.000$. The optimal path is marked in figure 5 as $T\ Free$ and optimal life-span is calculated at $T \approx 7.522$. Paths taken by the variables along the optimal path are plotted in figure 8. Health and consumption of C fall with time; ψ rises with time, and consumption of Z falls at first and then rises. Discounted lifetime utility is 1.248 units.[5]

[5] $\rho > 0$ in the simulations to represent an individual's 'impatience' for consuming now

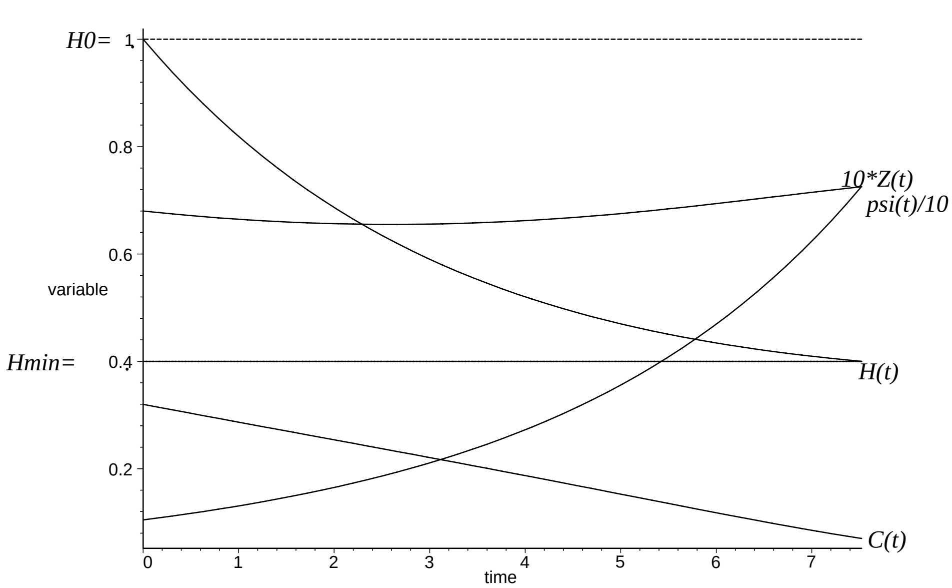

Figure 8: Time paths of health, consumption and the costate variable for health for the free terminal time, fixed terminal stock problem.

Model 4. *Free terminal stock of health and free terminal time.* The individual can now choose the optimal time T to reach the optimally chosen 'death' stock, $H_{\min}$. Both transversality conditions in Eqs. (17) and (18) must hold at the point of death, which, by substituting Eq. (17) into Eq. (18), implies:

$$U(C(T), Z(T), H(T)) = 0. \tag{19}$$

In terms of the phase diagram, the optimal point will occur at the intersection of the two loci $Z_{\min}$ and $Ham(T) = 0$. For Eq. (19) to hold when $U(C(T), Z(T), H(T)) = C^a Z^b H^{(1-a-b)}$, one or more of C, Z and H must equal zero at the optimally chosen T. This only occurs at the origin in figure 1, where $H = 0$ and $Z = 0$, or point f, where $C = 0$. Both points have been ruled out of the set of feasible solutions for a finite t, hence neither scenario can hold, and there is no feasible solution to this problem in a finite time. This point has already been noted by Eisenring (1998) when trying to solve a free terminal time version of B. Forster's (1989) model.

rather than later. Nevertheless, the primary role of a discount factor in dynamic optimisation models is either to (i) ensure that, for problems with an infinite terminal time, the improper integral of the objective function converges as $t \to \infty$ or, (ii) ensure that an individual has an incentive to consume in the present when there is a capital market and the rate of interest is positive. Since this paper deals with neither an infinite terminal time nor a capital market, model 3 was re-run with $\rho = 0$. Results do not change significantly: optimal life-span is $T \approx 7.71$ and $V \approx 1.463$ for this simulation.

Table 3: Summary of the simulation results for model 2 (fixed terminal time, free terminal stock)

parameter (τ)	V_τ	T_τ	$H_{\min\tau}$	$Z(0)_\tau$	$Z(T)_\tau$	$C(0)_\tau$	$C(T)_\tau$	$\psi(0)_\tau$	$\psi(T)_\tau$
H_0	>0	-	>0	>0	>0	>0	>0	<0	0
ρ	<0	-	<0	<0	<0	>0	<0	<0	0
ζ	>0	-	>0	>0	>0	>0	>0	>0	0
η	>0	-	>0	<0	>0	>0	>0	>0	0
γ	>0	-	>0	>0	>0	<0	>0	<0	0
δ	<0	-	<0	<0	<0	>0	<0	<0	0
pz	<0	-	<0	<0	<0	>0	<0	<0	0
pc	<0	-	>0	<0	>0	<0	<0	<0	0

4.2 Comparative static results

Comparative static results for models 2 and 3 are reported in tables 3 and 4. Results are established by varying the parameter of interest in the vicinity of the value reported in table 2 and, using the methods for establishing the optimal paths described in section 4.1, noting the changes in the variables of interest. Such an exercise can only be used as a guide to comparative static effects within the models, since it applies only to a small change in a parameter value on either side of the value used in the simulations.

The effect of parameter changes on discounted lifetime utility V are the same in both models: V increases with increases in the initial stock of health, the efficiency parameter for the production of income ζ and the efficiency parameters for the effect of Z and C on health (γ and η respectively). It decreases with increases in the rate of time preference, the rate of depreciation of health capital and the prices of Z and C.

Table 3 shows that, for model 2, T is, as expected, invariant to changes in any of the parameters. Also, since $\psi(T) = 0$ is the transversality condition for this model, it is also invariant to changes. The optimal value of $H_{\min}$ increases with increases in the initial stock of health, the efficiency parameters for the production of income, the efficiency parameter for the effect of Z and C on health and the price of C. It decreases with increases in the rate of time preference, the rate of depreciation of health capital and the price of Z.

Table 4 shows that, for model 3, the terminal stock of health is, as expected, invariant to changes in the parameter values. Additionally, the terminal values of Z, C and ψ are invariant with respect to changes in H_0 and ρ because these parameters do not affect the optimal terminal point of the problem (that is, they do not appear as arguments of Eq. (18) which used to define the optimal terminal point). Optimal life-span, or T, is increasing in the initial stock of health, the efficiency parameter for the production of income and the efficiency parameter for effect of Z on health. It is decreasing

Table 4: Summary of the simulation results for model 3 (free terminal time, fixed terminal stock)

parameter (τ)	V_τ	T_τ	$H_{\min\tau}$	$Z(0)_\tau$	$Z(T)_\tau$	$C(0)_\tau$	$C(T)_\tau$	$\psi(0)_\tau$	$\psi(T)_\tau$
H_0	>0	>0	-	>0	0	>0	0	<0	0
ρ	<0	<0	-	<0	0	>0	0	<0	0
ζ	>0	>0	-	>0	>0	>0	<0	>0	>0
η	>0	<0	-	<0	<0	>0	>0	<0	<0
γ	>0	>0	-	>0	>0	<0	<0	>0	>0
δ	<0	<0	-	<0	<0	>0	>0	<0	<0
pz	<0	<0	-	<0	<0	>0	>0	<0	<0
pc	<0	<0	-	<0	<0	<0	<0	<0	<0

in the rate of time preference, the rate of depreciation of health stock, the price of Z, the price of C and the efficiency parameter on the unhealthy good.

5 Discussion

Table 5 summarises the results of models 1 to 3. The choice of the terminal conditions for $H_{\min}$ and T has a marked effect on optimal plans. To explain the intuition behind these solutions we start with the 'benchmark' model.

In the benchmark model the individual chooses a path for Z (and, as a result, one for C), that moves him/her from $H_0 = 1.000$ to $H_{\min} = 0.400$ in exactly $T = 6$ units of time. The higher the value of T, the higher the level of consumption of Z along the optimal path. This makes sense, since with longer to live, the individual must 'slow down' the rate of decay of health stock by reducing unhealthy consumption and increasing healthy consumption.

Now consider an individual seeking the optimal T (which we shall denote T^*) by evaluating a series of these fixed terminal stock, fixed terminal time problems which differ only in the choice of the terminal time, T. A plot of the maximum value function V^* and the Hamiltonian at the terminal time for a series of these problems is shown in figure 7. The individual will not wish to reach the 'death' stock $H_{\min}$ at a time $T^* - dT$, when the overall utility prospect as measured by the Hamiltonian is still positive, since figure 7 shows that V^* will be rising at $T^* - dT$. Nor will the individual wish to arrive there at a time $T^* + dT$, when the Hamiltonian is negative, since figure 7 shows that V^* is falling. Hence the optimal T for the problem occurs at the time when V^* is stationary, or $\mathcal{H}(T) = 0$. Eq. (18) is therefore no more than the necessary condition for the choice of T given by:

$$V_T^* = \mathcal{H}(T) = 0. \tag{20}$$

Léonard and van Long (1992, page 241) and Caputo (1995, page 356) provide

Table 5: Summary of models 1 to 3

	'Benchmark', fixed $H_{\min}$, T	free $H_{\min}$, fixed T	fixed $H_{\min}$, free T
$H_{\min}$ free/fixed?	fixed	free	fixed
Value	0.400	0.326	0.400
T free/fixed?	fixed	fixed	free
Value	6.000	6.000	7.522
$\psi(0)$	0.640	0.439	1.046
$\psi(T)$	1.637	0.000	7.257
$\dot{\psi}$	> 0	< 0	> 0
$Z(0)$	0.064	0.060	0.068
$Z(T)$	0.057	0.032	0.073
$\dot{Z}$	$< 0, > 0$	< 0	> 0
$C(0)$	0.365	0.398	0.320
$C(T)$	0.222	0.432	0.070
$\dot{C}$	< 0	$< 0, > 0$	< 0
V	1.175	1.207	1.248

further details . Eq. (20) is a sufficient condition for an optimal T given the concavity of V^* with respect to T.

This illustrates how an iterative process similar to those described by Grossman and Ried might be used to determine the optimal terminal time in a continuous time model. The individual solves a series of fixed terminal stock, fixed terminal time problems by varying the choice of T and establishes which choice of T maximises V^*. However, it is clear that application of the transversality condition in Eq. (18) yields exactly the same result in the continuous time model, obviating the need for an iterative approach.[6]

Figure 7 also illustrates the tradeoff that occurs between discounted lifetime utility V^* and life-span T in model 3. By choosing the optimal 'mix' of healthy and unhealthy consumption the individual does not live as long as possible. Maximising life-span could only occur if the individual consumed Z throughout time, but there comes a point at which the individual is living 'too healthily' and gains increased life-span at the 'cost' of reduced lifetime utility.

Such a scenario is completely different in model 2. Here, the individual knows that he/she has only T units of time to live, and that all health capital of value should be used up within this time. Hence the 'death' stock of health is chosen to ensure that the marginal value of health capital, measured by ψ, equals zero. This is transversality condition (17), and it explains why consumption of Z falls and consumption of C rises in the latter periods of life: there is no in point consuming Z for its beneficial effect on health when $\psi = 0$. Hence Z and C are consumed solely for their utility benefits at the point of death.

[6] It is not clear whether a discrete time equivalent of Eq. (18) exists.

5.1 The role of ψ and the 'optimal mix' of healthy and unhealthy consumption

The results illustrate the crucial role played by ψ in determining the optimal 'mix' of healthy and unhealthy consumption at the point of death. In model 2, health stock will always have zero marginal value at T because of the transversality condition Eq. (17). Hence consumption of Z will always fall to $Z_{\min}$. However, in model 3, Eq. (18) implies that:

$$\psi(T) = \frac{-U(C(T), Z(T), H(T))}{f(C(T)) + g(Z(T)) - \delta H(T)}. \tag{21}$$

In the simulations $\psi(T) > 0$ since health stock is still yielding income, utility (the numerator of Eq. (21)) is positive and the rate of change of health stock (the denominator of Eq. (21)) is negative. This implies that, from Eq. (10), $Z(T) > Z_{\min}$. Furthermore, since in the simulation the optimal value of $Z(T)$ is located in isosector II, healthy consumption is *rising* at the terminal time. If H_0 had been chosen to intersect $Ham(T) = 0$ in isosector I, healthy consumption would be falling at the terminal time. We consider this result in more detail in section 5.2.

As well as determining the optimal terminal levels of healthy and unhealthy consumption, ψ also determines the optimal mix through time: the more valuable is health, the greater the individual's emphasis on healthy consumption. However, as figures 4 and 8 show, $\dot{\psi} > 0$ does not necessarily imply that the absolute level of healthy consumption is rising. This somewhat counterintuitive result arises because *absolute* levels of healthy and unhealthy consumption crucially depend on the amount of income yielded by the current stock of health. In periods when health capital is falling, falling income can reduce opportunities for *both* healthy and unhealthy consumption.

The emphasis on healthy consumption as a function of ψ is better viewed using a ratio of Z to C. This is shown in figure 9 for model 3. The higher the marginal value of health stock, the greater the individual's emphasis on healthy consumption relative to unhealthy consumption even if, as shown in figure 8, absolute levels of both are falling and ψ is rising in the initial stages of life.

5.2 Additional terminal conditions

So far, we have considered the transversality conditions for models 2 and 3. In model 4, when health stock is freely chosen (or, equivalently, when its marginal value equals zero at the point of death) and life-span is freely chosen, *both* transversality conditions, Eqs. (17) and (18) apply. However, because the utility function in model 4 could not be set equal to zero to satisfy Eq. (19), there was no solution in a finite time.

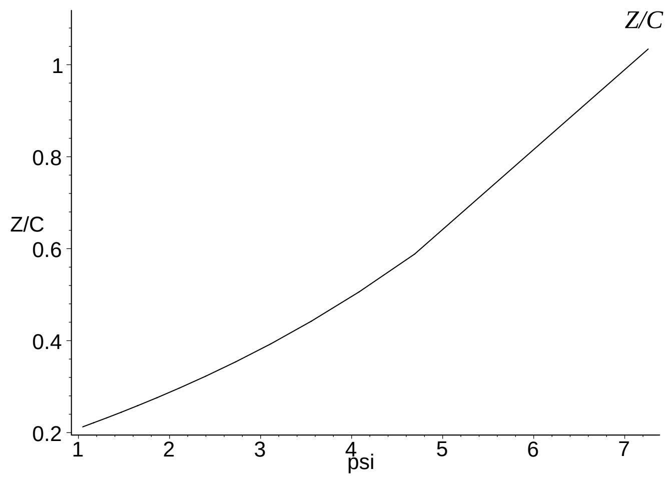

Figure 9: Ratio of Z to C plotted against ψ for model 3.

Nevertheless, it is clear from section 2 that in some models, namely Grossman's (1972a, 1972b, 1998) and Ried's (1998), (i) the marginal value of health stock equals zero at the point of death *and* (ii) life-span is freely chosen. Rather than explicitly modelling (i) as a freely chosen terminal stock of health, these models assume that the 'exogenously' set death stock is defined as that which has a zero marginal value (see the discussion on page 4). Simulating a model which applies both of these terminal conditions involves choosing a utility function that can take on a zero value at the point of death and so satisfy Eq. (19).

A simple function that can be used to illustrate the intuition behind the result is $U(\cdot) = a\ln(C) + b\ln(Z) + (1 - a - b)\ln H$ and so, as a final exercise, a simulation of the 'benchmark' model using this functional form is considered. The two endpoint restrictions are: (i) Eq (4), where $H_{\min}$ is defined as having a zero marginal value and (ii) T is freely chosen. Both transversality conditions Eqs. (17) and (18) now hold at the point of death, implying from Eq. (19) that utility must equal zero. This set of terminal conditions most closely resembles those of Grossman (1972a, 1972b, 1998) and Ried (1998): life-span is freely chosen, health stock is used up until its marginal value equals zero and utility equals zero at the point of death. Although the individual's income does not equal zero at the point of death, residual income is of no value to the individual either in current utility terms (since $U = 0$) or in future utility terms (since $\psi = 0$).

The new form of the utility function means that new parameter values, including one for H_0, must be chosen so as to ensure feasible solutions are obtained from the simulation.[7] The phase diagram for the simulation is

[7]The parameter values selected are as reported in table 2 except that $pz = pc = 0.300$, $\rho = 0.05$, $\delta = 0.01$, $H(0) = 0.550$, and , of course, $H_{\min}$ and T are freely chosen.

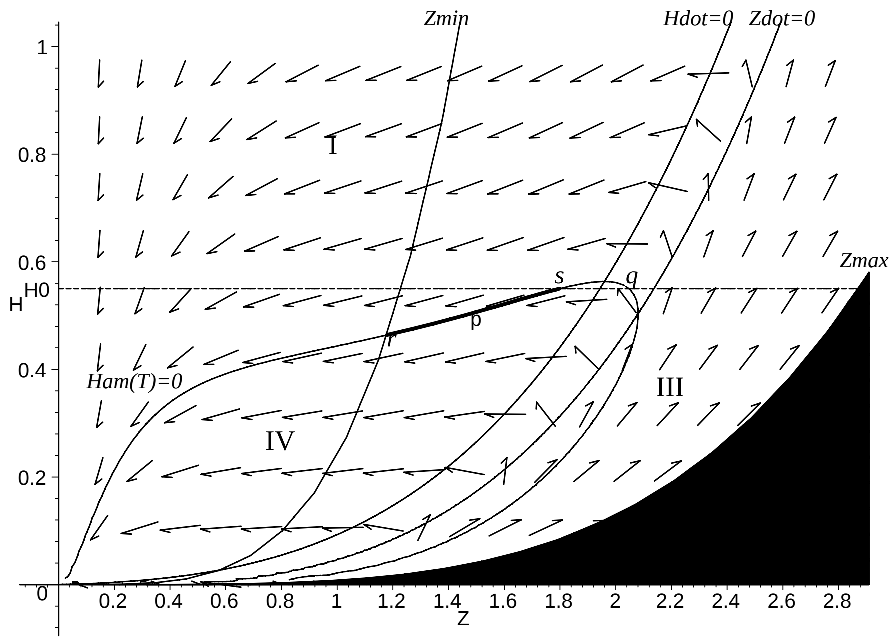

Figure 10: Intersection of $Z_{\min}$ and $Ham(T) = 0$ loci determining terminal point r in the final simulation.

plotted in figure 10. The Z axis is scaled differently to reflect the new utility function and choice of parameters, and infeasible points to the right of the $Z_{\max}$ locus are erased to delete distracting, non-feasible arrows and loci. The $Z_{\min}$ locus is an upward-sloping convex function as before and $Hdot = 0$ and $Zdot = 0$ loci are upward-sloping functions that do not intersect in the region of interest. Isosectors I, III and IV are marked for comparison to figure 1. The locus of points satisfying Eq. (18) is marked as $Ham(T) = 0$.

The terminal point satisfying both transversality conditions is at the intersection of the $Z_{\min}$ and the $Ham(T) = 0$ loci and is marked r. The path defined by this terminal point follows the $Ham(T) = 0$ locus[8] and intersects

[8]To see why it coincides with the $Ham(T) = 0$ locus, consider the derivative of $\mathcal{H}(t)$ with respect to time:

$$\frac{d\mathcal{H}}{dt} = \mathcal{H}_C\dot{C} + \mathcal{H}_Z\dot{Z} + \mathcal{H}_H\dot{H} + \mathcal{H}_\psi\dot{\psi}. \tag{22}$$

From the necessary conditions, $\mathcal{H}_C = \mathcal{H}_Z = 0$, $\mathcal{H}_\psi = \dot{H}$ and $\mathcal{H}_H = -\dot{\psi} + \rho\psi$ which, upon substituting into Eq. (22) gives $d\mathcal{H}/dt = \rho\dot{H}\psi$, the rate of change of $\mathcal{H}$ with time along any path (Chiang, 2000, page 212). Since on path p, $\psi(T) = 0$, it follows that $d\mathcal{H}/dt = 0$ and since $\mathcal{H}(T) = 0$ by Eq. (18), the Hamiltonian must be constant and equal to zero along the optimal path.

Table 6: Summary of the simulation results for free terminal time, free terminal stock model

parameter (τ)	V_τ	T_τ	$H_{\min\tau}$	$Z(0)_\tau$	$Z(T)_\tau$	$C(0)_\tau$	$C(T)_\tau$	$\psi(0)_\tau$	$\psi(T)_\tau$
H_0	>0	>0	0	>0	0	<0	0	>0	0
ρ	<0	<0	0	<0	0	>0	0	<0	0
ζ	>0	>0	<0	>0	>0	<0	>0	>0	0
η	>0	>0	0	>0	0	<0	0	>0	0
γ	>0	>0	0	>0	0	<0	0	>0	0
δ	<0	<0	0	<0	0	>0	0	<0	0
pz	<0	<0	>0	<0	<0	>0	>0	<0	0
pc	<0	<0	>0	<0	>0	>0	<0	<0	0

initial health $H_0 = 0.55$ at two points, s and q. A check of the second-order conditions shows that path p linking r to s maximises V, since consuming to point q means that the individual lives longer but accumulates more 'negative' utility through the term $(1-a-b)\ln(H) < 0$ in U. Along path p, the individual reduces consumption of Z throughout time, and health falls to the level $H_{\min}$ at which point its marginal value equals zero. $T \approx 0.047$ units along this path.

A set of comparative static results for this model are listed in table 6, and combine many of the features of tables 3 and 4: results for V are identical to those in these previous tables, $\psi(T)$ is, as expected, invariant to changes in any of the parameter values and $H_{\min}$, $Z(T)$ and $C(T)$ are invariant to changes in H_0, ρ, η, γ and δ, none of which feature in Eq. (19). Optimal life-span is decreasing for increases in the rate of time preference, the rate of health capital depreciation and the prices of Z and C and is increasing in initial health and the productivity parameters for the production of income, and the healthy and unhealthy goods.

The result of this final model highlights the need to choose a utility function that satisfies both transversality conditions Eqs. (17) and (18) and that can therefore equal zero within a finite time. As such, it attaches 'meaning' to a specific level of utility that occurs at the point of death. However, the functional form chosen for this final simulation also introduces a problem of 'negative' utility occurring in periods prior to death and stresses the importance of checking the second-order conditions for the optimal choice of T: although consuming to point q means that the individual lives longer and consumes more, the accumulation of additional health valued at $(1 - a - b)\ln(H) < 0$ means that consuming to this point minimises discounted lifetime utility. Consuming only to point s maximises it. It is therefore not clear from this simulations whether imposing both transversality conditions can yield 'meaningful' results for optimising behaviour.

6 Conclusion

This paper has presented numerical simulations of a model of healthy and unhealthy consumption which differ in the way they model the terminal conditions for health stock, the terminal time and the value of utility at the point of death. This reflects the range of approaches to modelling the point of death in existing models of the demand for health and also generalises the 'medical-care' focus of such models. Results have shown how time paths for consumption and health differ significantly depending on the terminal conditions that are used. The same is true for the comparative static results. Further, the continuous time approach has shown how an iterative process might be used to establish an optimal life-span, although this is not necessary in continuous time models because they utilise transversality conditions. The role of the marginal value of health stock, ψ, has been further highlighted, building upon previous work in this area by Ehrlich and Chuma (1990). The value of ψ at the terminal time of the problem has been shown to be crucial in determining whether any health investment occurs in the final instant of life. Results have also shown that time invariant depreciation rates are consistent with finite life-spans.

What way forward? There are two main directions. Firstly, for the model presented here, there is scope for testing the sensitivity of results to the inclusion of time varying depreciation rates, a capital market and perhaps a stock of addiction to accompany consumption of the unhealthy good C (which would parallel Becker and Murphy's (1988) model with the benefit that health stock would also be included). Results could be generalised so that they might become more amenable to empirical testing. Sadly, such additions are likely to be accompanied by increased complexity and the loss of the benefit of two dimensional phase diagrams to inform the numerical simulations.

Secondly, there is major scope for adopting numerical simulations to *complement* the analytical material that is already established in the literature, whether it be in a discrete time or a continuous time setting. This would certainly benefit some of the debates about iterative methods to establish an optimal life-span in the discrete time models that were discussed in the introduction: a sure way to test whether they work as described is to simulate them. Comparative static/dynamic analyses can then be carried out to check arguments about the sensitivity of optimal life-spans to changes in parameters of interest.

Finally, there is a need to pursue further the meaning of 'death' and its implications for optimal plans in all such models, since the current literature does not appear to have converged on a standard definition. Such research must consider carefully the marginal value of health at the point of death, the optimal choice of life-span, and values attached to utility and earnings at the point of death. In doing so it must also check carefully the necessary and sufficient conditions for optimising behaviour.

7 Appendix

The features of the phase diagram presented in figure 1 and not derived in the main body of the text are derived below.

Upper bound on consumption of Z. The upper bound on Z is given by considering the locus of points where the individual spends all available income on consuming Z. This implies that consumption of C equals zero and therefore, from Eq. (3):

$$Z \equiv \phi(H) = w(H)/pz.$$

Define $\varrho(Z)$ as the inverse function of $\phi(H)$. Then, by the inverse function theorem, $\varrho_Z = 1/\phi_H > 0$ and $\varrho_{ZZ} = -(\varrho_Z \phi_{HH})/\phi_H^2 > 0$. Hence the locus $Z_{\max}$ is increasing and convex.

Lower bound on consumption of Z. Differentiating Eq. (12) with respect to time yields:

$$\begin{aligned} \left.\frac{dH}{dZ}\right|_{\Psi=0} &= -\frac{\partial\Psi/\partial Z}{\partial\Psi/\partial H} \\ &= -\frac{(U_{CC}C_Z + U_{CZ})pz - (U_{ZC}C_Z + U_{ZZ})pc}{(U_{CC}C_H + U_{CH})pz - (U_{ZC}C_H + U_{ZH})pc}, \end{aligned} \tag{23}$$

the sign of which is ambiguous. The locus $Z_{\min}$ is drawn as an upward-sloping convex function in figure 1, since this is the shape of the locus that results from the numerical simulations.

The $\dot{H} = 0$ locus. From Eq. (13), when $Z = 0$, $\Omega_1(H, 0) = f\left(\frac{1}{pc}[w(H)]\right) - \delta H$ equals zero only when $H = 0$. Hence this locus passes through the origin. When $w(H) = pzZ$, that is, when the individual spends all available income on Z, the locus intersects the $Z_{\max}$ locus at the point defined by the solution to the equations $H = g(Z)/\delta$ (from Eq. (13)) and $pzZ = w(H)$.

The slope of the $\dot{H} = 0$ locus is given by considering the expression:

$$\left.\frac{dH}{dZ}\right|_{\Omega_1=0} = -\frac{\partial\Omega_1/\partial Z}{\partial\Omega_1/\partial H}, \tag{24}$$

where: $\partial\Omega_1/\partial Z = f_C(-pz/pc) + g_Z > 0$, $\partial\Omega_1/\partial H = f_C(w_H/pc) - \delta < 0$ so this locus slopes upwards.

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