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On the density of generalised noncentral quadratic forms with applications to asymptotic expansions for test statistics

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Abstract

In this note we derive a general formula useful to express the density of generalised noncentral quadratic forms (i.e. of a scalar random variable obtained by contracting non zero mean multivariate normal vectors over multidimensional arrays) in terms of linear combinations of noncentral chi square random variables.

The formula can be used to obtain explicit expressions for the terms appearing in the asymptotic expansions for test statistics under a local alternative.

Keywords and Phrases: Edgeworth expansions, Generalised noncentral quadratic forms, Local alternatives

JEL subject classification: C12

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1 Introduction

In their seminal papers on asymptotic expansions of asymptotically equivalent χ^2 test statistics under a sequence of local alternatives, Peers (1971) and Hayakawa (1975) propose a method which is based on inverting the approximate characteristic function of the test under consideration. In order to do so, one has to find a stochastic expansion for the statistic, and then calculates its characteristic function via a multivariate (type A) Edgeworth expansion. This procedure delivers valid (in the sense of Chandra & Ghosh (1979)) asymptotic expansions, but it becomes extremely complicated to apply especially when one is considering third order local analysis in a multiparameter setting. An alternative approach is to consider the signed square root of the stochastic expansion of the test statistic and evaluate the approximate multivariate cumulants (usually up to the fourth). An Edgeworth expansion argument can then be used to obtain a valid asymptotic expansion for the original test statistic. We believe that this second approach highlights in a neater way the relevant features of the higher order asymptotic behaviour of the test statistic under investigation, both under the null (for example the eventual Bartlett correctability, see Bickel & Ghosh (1990)), and under the alternative (for example deficiency analysis, see Chandra & Joshi (1983)). As far as we know, there are no papers which follow the second approach in a multiparameter setting under a local alternative: the main result of the note will, hopefully, fill this gap.

In this note we describe a simple method to derive the density of noncentral generalised quadratic forms, such as cubic, quartic forms and more generally the density of any scalar obtained by repeated contractions between nonzero mean (asymptotically) normal random vectors with multidimensional arrays of constants (i.e. a noncentral ν form). The method is based on a partial differential operator representation of the product of the generalised noncentral quadratic form with the a nonzero mean multivariate standard normal density in terms of a linear combination of partial derivatives of an auxiliary function. The resulting formula can be used directly to obtain explicit expression for the various scalar terms appearing in the asymptotic expansions for test statistics under a local alternative in terms of linear combinations of noncentral chi squares random variates with coefficients given by appropriate contractions between the generalised arrays and the mean vectors. As an application of the formula, we obtain a third order asymptotic expansion for the local power function of the general class of test statistic introduced by Chandra & Joshi (1983) which includes the likelihood ratio, Rao's efficient score and Wald test.

Notice that throughout the rest of the note we use (unless otherwise stated) tensor notation and the summation convention (i.e. for any two repeated indices, their sum is understood), as described for example in McCullagh (1987). Also, each index $r, s, ...$in the set R_ν of indices runs from 1 to q.

2 Main result

Let us introduce some notation. Let $w^r \sim \phi_q(\gamma^r, \delta^{rs})$ where $\phi_q(\cdot,\cdot)$ is the multivariate q dimensional normal density with mean vector γ^r and identity covariance matrix δ^{rs} (the kronecker delta); let $g_{q,\tau}(x)$ be the density of a noncentral chi square random variate with q degrees of freedom and noncentrality parameter $\tau = \gamma^r\gamma^r < \infty$, b^{R_ν} be a q^ν dimensional array of constants not depending on n (i.e. b^r is a vector, b^{rs} is a matrix and so on). Finally, let $\partial^\nu(\cdot) = \partial^\nu(\cdot)/\partial t^{r_1}\partial t^{r_2}...\partial t^{r_\nu}$ be the (componentwise) partial derivative operator.

We can now state the following theorem.

Theorem 1 *Let $w^{R_\nu} = w^{r_1}w^{r_2}...w^{r_\nu}$, and t^r be a $q \times 1$ vector of auxiliary real variables. Assume that the function $a(x,t) : R^q \to R$*

(1) belongs to $\mathcal{C}_0^k(\mathcal{N})$, the space of k times continuously differentiable functions on an open set $\mathcal{N}$ of $t = 0$, almost surely dx,

(2) satisfies the following dominance condition

$$\int \sup_{t\in\mathcal{N}} |\partial^{\nu} a(x,t)|\, dx < \infty \quad \textit{almost surely } dx.$$

Then for any arbitrary noncentral ν form $w^{R_\nu} b^{R_\nu}$, the following holds:

$$w^{R_\nu} b^{R_\nu} \phi_q(\gamma^r, \delta^{rs}) = \sum\nolimits_{\Upsilon} b^{R\nu} \gamma^{R_\nu} g_{q+2\nu,\tau}(x) \tag{1}$$

where the sum is over $\Upsilon = \{p, \nu_1, ..., \nu_p\}$ - the number of ways of partitioning a set of $p = \nu_1 + 2\nu_2 + ... + p\nu_p$ different indices into ν_k subsets containing k indices for $k = 1, 2, ..., p$, such that the resulting homogeneous polynomial in γ^{R_ν} is even or odd according to the number of indices in the set R_ν.

Proof. We use the transformation from R^q to R^{q+1}

$$T : w^r \to (x, v^r) \tag{2}$$

where $x = w^r w^r$, $v^r = w^r / (w^s w^s)^{1/2}$ and the following identity

$$w^{R_\nu} b^{R_\nu} \phi_q(\gamma^r, \delta^{rs}) \equiv \sum\nolimits_{R_\nu} b^{R_\nu} \partial^{\nu} \left(\phi_q(\gamma^r, \delta^{rs}) \exp\{w^r \delta^{rs} t^s\}\right) |_{t^{r_i}=0} \qquad i = 1, 2, ..., \nu$$

where $\sum_{R_\nu}$ indicates summation over the superscripts of b^{R_ν} (i.e. the components of the b array) with the correspondent components of the various t^{r_i}). Using T, the density for x is obtained by integrating out the vector $v^r \in V_{1,q}$ (i.e. over the unit sphere $v^r v^r = 1$ in R^q), that is:

$$(2\pi)^{-q/2} \sum\nolimits_{R_\nu} \int_{v^r v^r=1} b^{R_\nu} \exp\{-(x+\tau)/2\} |J| \, \partial^{\nu} \left(\exp\left\{x^{1/2} v^r \delta^{rs} t^s\right\}\right) |_{t^{r_i}=0} \, (dv^r) \tag{3}$$

for all the $i = 1, 2, ..., \nu$, where $|J| = x^{q/2-1}/2$ is the Jacobian of the transformation T and (dv^r) denotes the unnormalised Haar measure on the Stiefel manifold $V_{1,q}$. Upon normalising the measure on $V_{1,q}$ by the constant $2\pi^{q/2}/\Gamma(q/2)$ and interchanging differentiation and integration which is permissible under (1) for

$a(x,t) := \exp\left\{x^{1/2} v^r \delta^{rs} t^s\right\}$ (note also that the transformation T is essentially a polar coordinate type transformation), we can then use Theorem 7.4.1 in Muirhead (1982), to get:

$$K(x,\tau) \sum_{R_\nu} b^{R_\nu} \partial^\nu {}_0F_1\left(;q/2; x\left(\tau + t^r t^r + 2\gamma^r t^r\right)/4\right)|_{t^{r_i}=0}$$

with

$$\begin{aligned} K(x,\tau) &= x^{q/2-1} \exp\left\{-\left(x+\tau\right)/2\right\}/2^{q/2}\Gamma\left(q/2\right), \quad {}_0F_1\left(;c;z\right) = \sum_{k=0}^{\infty} z^k/\left(c\right)_k k!, \\ \left(c\right)_k &= \Gamma\left(c+k\right)/\Gamma\left(c\right). \end{aligned}$$

Let $[k]$ denote the k different ways a contraction between indices belonging to a given set can be performed. Differentiating now ${}_0F_1(;\cdot;\cdot)$ and evaluating the resulting derivatives at $t^r = 0$, we obtain:

$$\sum_{r=1}^{q} b^r \partial\, {}_0F_1\left(;\cdot;\cdot\right)|_{t=0} = b^r \gamma^r x\, {}_0F_1\left(;q/2+1; x\tau/4\right)/2\left(q/2\right),$$

$$\begin{aligned} \sum_{r,s=1}^{q} b^{rs} \partial^2\, {}_0F_1\left(;\cdot;\cdot\right) \quad | \quad {}_{t=0} &= b^{rr} x\, {}_0F_1\left(;q/2+1; x\tau/4\right)/2\left(q/2\right) + \\ & b^{rs}\gamma^r\gamma^s x^2\, {}_0F_1\left(;q/2+2; x\tau/4\right)/4\left(q/2\right)_2, \end{aligned}$$

$$\begin{aligned} \sum_{r,s,t=1}^{q} b^{rst} \partial^3\, {}_0F_1\left(;\cdot;\cdot\right) \quad | \quad {}_{t=0} &= [3]\, b^{rss}\gamma^r x^2\, {}_0F_1\left(;q/2+2; x\tau/4\right)/4\left(q/2\right)_2 + \\ & b^{rst}\gamma^r\gamma^s\gamma^t x^3\, {}_0F_1\left(;q/2+3; x\tau/4\right)/8\left(q/2\right)_3, \end{aligned}$$

$$\begin{aligned} \sum_{r,s,t,u=1}^{q} b^{rstu} \partial^4\, {}_0F_1\left(;\cdot;\cdot\right) \quad | \quad {}_{t=0} &= [3]\, b^{rrss} x^2\, {}_0F_1\left(;q/2+2; x\tau/4\right)/4\left(q/2\right)_2 + \\ & [6]\, b^{rstt}\gamma^r\gamma^s x^3\, {}_0F_1\left(;q/2+3; x\tau/4\right)/8\left(q/2\right)_3 + \\ & b^{rstu}\gamma^r\gamma^s\gamma^t\gamma^u x^4\, {}_0F_1\left(;q/2+4; x\tau/4\right)/16\left(q/2\right)_4, \end{aligned}$$

where for example $[3]\, b^{rss}\gamma^r = b^{rss}\gamma^r + b^{rsr}\gamma^s + b^{rrs}\gamma^s$.

On inspecting these first four terms, a clear pattern for the number $[k]$ of contractions between the components of the b^{R_ν} and γ^{R_ν} arrays emerges, and it can be expressed using standard results in combinatorial analysis. Specifically, Let $\#_\nu$ denote the number of different indices in the set R_ν (strictly) less than the dimension of ν (i.e. $\#_\nu << \nu$), and let

$$\begin{aligned} \pi &= 1^{\nu_1}, 2^{\nu_k}, ..., \nu^{v_p}, \quad \nu = \nu_1 + 2\nu_2 + ... + p\nu_p \\ k_{\nu_p} &= \nu!/\left[(1!)^{\nu_1} \nu_1! \, (2!)^{\nu_2} \nu_2! ... (\nu!)^{\nu_{k_p}} \nu_{\nu_{kp}}!\right]. \end{aligned}$$

Then we can conclude that for ν odd, say ν^o,

$$\begin{aligned} b^{R_\nu} \partial^{\nu^o} {}_0F_1(;\cdot;\cdot) \quad | \quad {}_{t=0} &= \left[k_{\nu_{p_1}}\right] b^{rS_{\nu_{p1}}} \gamma^r x^{\#_{\nu_{p1}}} {}_0F_1\left(; q/2 + \#_{v_{p_1}}; x\tau/4\right) / 2^{\#_{\nu_{p1}}} (q/2)_{\#_{\nu_{p_1}}} + \\ & \left[k_{\nu_{p_3}}\right] b^{rstU_{\nu_{p3}}} \gamma^r \gamma^s \gamma^t x^{\#_{\nu_{p3}}+1} {}_0F_1\left(; q/2 + \#_{v_{p_3}} + 1; x\tau/4\right) / 2^{\#_{\nu_{p3}}+1} (q/2)_{\#_{\nu_{p_3}}+1} \\ & \left[k_{\nu_{p_5}}\right] b^{rstuvW_{\nu_{p5}}} \gamma^r ... \gamma^v x^{\#_{\nu_{p5}}+2} {}_0F_1\left(; q/2 + \#_{v_{p_5}} + 2; x\tau/4\right) / 2^{\#_{\nu_{p5}}+2} (q/2)_{\#_{\nu_{p_5}}} \\ & .... + b^{R_\nu} \gamma^{R_\nu} x^\nu {}_0F_1(; q/2 + \nu; x\tau/4) / 2^\nu (q/2)_\nu, \end{aligned}$$

and for ν even, say ν^e

$$\begin{aligned} b^{R_\nu} \partial^{\nu^e} {}_0F_1(;\cdot;\cdot) \quad | \quad {}_{t=0} &= \left[k_{\nu_{p_o}}\right] b^{R_{\nu_{p0}} S_{\nu_{p0}}} x^{\#_{\nu_{p0}}} {}_0F_1\left(; q/2 + \#_{v_{p_0}}; x\tau/4\right) / 2^{\#_{\nu_{p0}}} (q/2)_{\#_{\nu_{p0}}} + \\ & \left[k_{\nu_{p2}}\right] b^{rsT_{\nu_{p2}}} \gamma^r \gamma^s x^{\#_{\nu_{p2}}+1} {}_0F_1\left(; q/2 + \#_{v_{p_2}} + 1; x\tau/4\right) / 2^{\#_{\nu_{p2}}+1} (q/2)_{\#_{\nu_{p_2}}+1} + \\ & \left[k_{\nu_{p_4}}\right] b^{rstuV_{\nu_{p4}}} \gamma^r ... \gamma^u x^{\#_{\nu_{p4}}+1} {}_0F_1\left(; q/2 + \#_{v_{p_4}} + 2; x\tau/4\right) / 2^{\#_{\nu_{p4}}+1} (q/2)_{\#_{\nu_{p_4}}+1} \\ & .... + b^{R_\nu} \gamma^{R_\nu} x^\nu {}_0F_1(; q/2 + \nu; x\tau/4) / 2^\nu (q/2)_\nu, \end{aligned}$$

where each free index in the b arrays is contracted with each others. Expression (1) follows immediately. ■

As an illustration of (1), consider a 5 and a 6 noncentral form. For $R_5 = \{r, s, t, u, v\}$ we have

$$\begin{aligned} b^{rstuv} \partial^5 {}_0F_1(;\cdot;\cdot) / \partial t^{r_1} ... \partial t^{r_5} \quad | \quad {}_{t=0} &= \left[k_{\nu_{5_1}}\right] b^{rsstt} \gamma^r x^3 {}_0F_1(; q/2 + 3; x\tau/4) / 2^3 (q/2)_3 + \\ & \left[k_{\nu_{5_3}}\right] b^{rstuu} \gamma^r \gamma^s \gamma^t x^4 {}_0F_1(; q/2 + 4; x\tau/4) / 2^4 (q/2)_4 + \\ & b^{rstuv} \gamma^r \gamma^s \gamma^t \gamma^u \gamma^v x^5 {}_0F_1(; q/2 + 5; x\tau/4) / 2^5 (q/2)_5, \end{aligned}$$

with $\left[k_{\nu_{5_1}}\right] = [15]$ because we have one contraction with the vector γ^r and 4 indices ss and tt, $\left[k_{\nu_{5_3}}\right] = [10]$ as we have 3 contractions over $\gamma^r, \gamma^s, \gamma^t$ and 2 indices uu.

For $R_6 = \{r, s, t, u, v, w\}$ we have:

$$\begin{aligned} b^{rstuvw} \partial^6 {}_0F_1 (; \cdot; \cdot) / \partial t^{r_1} ... \partial t^{r_6} \quad | \quad_{t=0} = & \left[k_{\nu_{6_0}} \right] b^{rrsstt} {}_0F_1 (; q/2 + 3; x\tau/4) / 2^3 (q/2)_3 + \\ & \left[k_{\nu_{6_2}} \right] b^{rsttuu} \gamma^r \gamma^s x^4 {}_0F_1 (; q/2 + 4; x\tau/4) / 2^4 (q/2)_4 + \\ \left[k_{\nu_{6_4}} \right] \quad & b^{rstuvv} \gamma^r \gamma^s \gamma^t \gamma^u x^5 {}_0F_1 (; q/2 + 5; x\tau/4) / 2^5 (q/2)_5 + \\ & b^{rstuvw} \gamma^r \gamma^s \gamma^t \gamma^u \gamma^v \gamma^w {}_0F_1 (; q/2 + 6; x\tau/4) / 2^6 (q/2)_6 , \end{aligned}$$

where $\left[k_{\nu_{6_0}} \right] = [15]$ because we have 2^3 repeated indices r, s, t, $\left[k_{\nu_{6_2}} \right] = [45]$ because we have 2 contractions with the vectors γ^r, γ^s and 4 indices tt and uu, and finally $\left[k_{\nu_{6_2}} \right] = [15]$ as we have 4 contractions over the indices $\gamma^r, \gamma^s, \gamma^t, \gamma^u$ and 2 indices vv.

As $g_{q,\tau}(x) = \exp \{ - (x + \tau) / 2 \} x^{q/2-1} {}_0F_1 (; q/2; x\tau/4) / 2^{q/2} \Gamma (q/2)$, we easily obtain

$$\begin{aligned} b^r w^r \phi_q (\gamma^r, \delta^{rs}) &= b^r \gamma^r g_{q+2,\tau}(x), \qquad (4) \\ b^{rs} w^r w^s \phi_q (\gamma^r, \delta^{rs}) &= b^{rs} \gamma^r \gamma^s g_{q+4,\tau}(x) + b^{rr} g_{q+2,\tau}(x), \\ b^{rst} w^r ... w^t \phi_q (\gamma^r, \delta^{rs}) &= b^{rst} \gamma^r \gamma^s \gamma^t g_{q+6,\tau}(x) + [3] b^{rrs} \gamma^s g_{q+4,\tau}(x), \\ b^{rstu} w^r ... w^u \phi_q (\gamma^r, \delta^{rs}) &= b^{rstu} \gamma^r ... \gamma^u g_{q+8,\tau}(x) + [6] b^{rstt} \gamma^r \gamma^s g_{q+6,\tau}(x) + [3] b^{rrss} g_{q+4,\tau}(x), \\ b^{rstuv} w^r ... w^v \phi_q (\gamma^r, \delta^{rs}) &= b^{rstuv} \gamma^r ... \gamma^v g_{q+10,\tau}(x) + [10] b^{rstuu} \gamma^r \gamma^s \gamma^t g_{q+8,\tau}(x) + \\ & \quad [15] b^{rsstt} \gamma^r g_{q+6,\tau}(x), \\ b^{rstuvw} w^r ... w^w \phi_q (\gamma^r, \delta^{rs}) &= b^{rstuvw} \gamma^r ... \gamma^w g_{q+12,\tau}(x) + [15] b^{rstuvv} \gamma^r \gamma^s \gamma^t \gamma^u g_{q+10,\tau}(x) + \\ & \quad [45] b^{rsttuu} \gamma^r \gamma^s g_{q+8,\tau}(x) + [15] b^{rrsstt} g_{q+6,\tau}(x). \end{aligned}$$

For third order asymptotic expansions based on the Edgeworth series the first six generalised noncentral forms are enough, however we report in the Appendix the densities for generalised noncentral quadratic forms up to $\nu = 10$.

We consider now the very general class of test statistics T_n as in Chandra & Joshi (1983). Let $f(\cdot, \theta^r)$ denote the common density of a sequence of IID vector valued random variables $X_1, X_2, ..., X_n$, with a q dimensional vector parameter $\theta^r \in \Theta \subseteq R^q$. Chandra & Joshi (1983) consider the problem of testing $H_0 : \theta^r = \theta_0^r$ against a sequence of local alternatives of the form $\theta_n^r = \theta_0^r + n^{-1/2}\gamma^r$ where $0 < \gamma^r\gamma^r < \infty$ and n is the sample size. Let

$$\begin{aligned} l_{r_1 \ldots r_v} &= E_{\theta_0}\left(\partial^{r_1\ldots r_v}\log f\left(X_1, \theta_0^r\right)/\partial\theta^{r_1}\ldots\partial\theta^{r_v}\right), \\ Z_{r_1\ldots r_v} &= n^{-1/2}\sum_{i=1}^{n}\left[\partial^{r_1\ldots r_v}\log f\left(X_i, \theta_0^r\right)/\partial\theta^{r_1}\ldots\partial\theta^{r_v} - l_{r_1\ldots r_v}\right] \end{aligned} \tag{5}$$

denote $\bigotimes_{k=1}^{v} q^{r_k}$ dimensional arrays of expectations and standardised random varables, respectively. Under the general assumptions of Chandra & Ghosh (1980), there exists a set $\mathcal{K}_n$ with $\Pr_{\theta_n^r}$ -probability $1 + o\left(n^{-1}\right)$ uniformly over compact subsets of γ^r, such that over $\mathcal{K}_n$ the any test statistic belonging to W_n admits a stochastic expansion of the form $W_n = W_n^r W_n^r + o\left(n^{-1}\right)$, where

$$\begin{aligned} W_n^r &= \left(I^{rs}\right)^{1/2} Z_s + n^{-1/2}\left(A_{st}^1 Z_{rs} Z_t + A_{rst}^2 Z_s Z_t\right) + \\ & n^{-1}\left(B_{stu}^1 Z_{rs} Z_t Z_u + B_{stuv}^2 Z_{rs} Z_{tu} Z_v + B_{rstu}^3 Z_s Z_t Z_u + B_{stuv}^4 Z_{rst} Z_u Z_v\right) \end{aligned} \tag{6}$$

is the signed square root of W_n, and $I^{rs} = -\left(l_{rs}\right)^{-1}$ is the inverse of the Fisher information per observations. The constants A and B are free from n and depend on the particular test statistic under consideration; for example for Rao's score test we have $A_{st}^1 = A_{rst}^2 = B_{stu}^1 = B_{stuv}^2 = B_{rstu}^3 = B_{stuv}^4 = 0$.

Using the formal delta method as in Bhattacharya & Ghosh (1978), it is then possible to evaluate the cumulants of (6) under the local alternative θ_n^r. Let $k^{r_1, r_2, \ldots, r_k}$ denote the approximate joint kth order cumulant of W_n^r; under IID sampling, it is possible to show that, in general, the asymptotic order of the cumulants is

$$\begin{aligned} k^r &= \gamma^r + n^{-1/2} k_{11}^r + n^{-1} k_{12}^r + O\left(n^{-3/2}\right), \\ k^{r,s} &= \delta^{rs} + n^{-1/2} k_{21}^{r,s} + n^{-1} k_{22}^{r,s} + O\left(n^{-3/2}\right), \\ k^{r,s,t} &= n^{-1/2} k_{31}^{r,s,t} + n^{-1} k_{32}^{r,s,t} + O\left(n^{-3/2}\right), \\ k^{r,s,t,u} &= n^{-1} k_{41}^{r,s,t,u} + O\left(n^{-3/2}\right), \\ k^{r_1,\ldots,r_k} &= O\left(n^{-3/2}\right) \quad k \geq 5. \end{aligned}$$

Let h^{R_ν} be the R_νth multivariate Hermite tensor (see for example McCullagh (1987, Ch. 5))

$$h^{R_\nu} := h\left(w^{R_\nu}\right) = (-1)^\nu \, \partial^{R_\nu} \phi_q\left(\gamma^r, \delta^{rs}\right) / \partial w^{r_1} \partial w^{r_2} ... \partial w^{r_\nu},$$

and $\nabla^k G_{q,\tau}(\cdot)$ be the kth (double) forward difference operator applied to the distribution function of a noncentral chi square distribution $G_{q,\tau}(\cdot)$ with noncentrality parameter τ (i.e. $\nabla^k G_{q,\tau}(\cdot) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} G_{q+2(k-j),\tau}(\cdot)$). We can then prove the following theorem:

Theorem 2 *The power function of T_n under a sequence of local alternatives $\theta_n^r = \theta_0 + n^{-1/2}\gamma^r$ has (third order) asymptotic expansion*

$$Pr_{\theta_n}(T_n \geq z) = 1 - G_{q,\tau}(z) + n^{-1/2} P_1(z, \gamma^r) + n^{-1} P_2(z, \gamma^r), \tag{7}$$

where $P_i(z,\gamma)$ $i = 1, 2$ are linear combinations of noncentral chi square distributions with coefficients given by odd or even, respectively, polynomials in γ obtained by appropriate contractions between the approximate cumulants (6) and γ^r, whose expressions are:

$$\begin{aligned}
P_1(z,\gamma^r) \;=\; & \left(k_{21}^{r,r}/2 + k_{11}^{r}\gamma^r - [3]\, k_{31}^{r,s,s}\gamma^r/6 - k_{21}^{r,s}\gamma^r\gamma^s/2 + k_{31}^{r,s,t}\gamma^r\gamma^s\gamma^t/6\right) G_{q,\tau}(z) + \\
& \left(-k_{21}^{r,r}/2 - k_{11}^{r}\gamma^r + [3]\, k_{31}^{r,s,s}\gamma^r/3 + k_{21}^{r,s}\gamma^r\gamma^s - k_{31}^{r,s,t}\gamma^r\gamma^s\gamma^t/2\right) G_{q+2,\tau}(z) - \\
& \left([3]\, k_{31}^{r,s,s}\gamma^r/6 + k_{21}^{r,s}\gamma^r\gamma^s/2 - k_{31}^{r,s,t}\gamma^r\gamma^s\gamma^t/2\right) G_{q+4,\tau}(z) - k_{31}^{r,s,t}\gamma^r\gamma^s\gamma^t G_{q+6,\tau}(z)/6,
\end{aligned}$$

$$\begin{aligned}
P_2(z,\gamma^r) \;=\; & \Big[\left(k_{11}^{r}k_{11}^{r} + k_{22}^{r,r}\right)/2 - [3]\left(k_{11}^{r}k_{31}^{r,s,s}/6 + k_{12}^{r,r}k_{12}^{s,s}/8 + k_{41}^{r,r,s,s}/24\right) - [15]\, k_{31}^{r,r,s}k_{31}^{s,t,t}/72 + \\
& k_{12}^{r}\gamma^r - [3]\left(k_{11}^{r}k_{21}^{s,s}/2 + k_{32}^{r,s,s}/6\right)\gamma^r + [15]\, k_{21}^{r,s}k_{31}^{s,t,t}\gamma^r/12 + \\
& [6]\left(k_{11}^{r}k_{31}^{s,t,t}/6 + k_{12}^{r,s}k_{12}^{t,t}/8 + k_{41}^{r,s,t,t}/24\right)\gamma^r\gamma^s - \left(k_{11}^{r}k_{11}^{s} + k_{22}^{r,s}\right)\gamma^r\gamma^s/2 + \\
& \left(k_{11}^{r}k_{21}^{s,t}/2 + k_{32}^{r,s,t}/6\right)\gamma^r\gamma^s\gamma^t - [10]\, k_{21}^{r,s}k_{31}^{t,u,u}\gamma^r\gamma^s\gamma^t/12 - \\
& \left(k_{11}^{r}k_{31}^{s,t,u}/6 + k_{12}^{r,s}k_{12}^{t,u}/8 + k_{41}^{r,s,t,u}/24\right)\gamma^r\gamma^s\gamma^t\gamma^u + [15]\, k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^r\gamma^s\gamma^t\gamma^u/72 + \\
& k_{21}^{r,s}k_{31}^{t,u,v}\gamma^r\gamma^s\gamma^t\gamma^u\gamma^v/12 - k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^r\gamma^s\gamma^t\gamma^u\gamma^v\gamma^w/72\Big] G_{q,\tau}(z) +
\end{aligned}$$

$$
\begin{aligned}
&\Big[-\left(k_{11}^{r}k_{11}^{r}+k_{22}^{r,r}\right)/2+2\,[3]\left(k_{11}^{r}k_{31}^{r,s,s}/6+k_{12}^{r,r}k_{12}^{s,s}/8+k_{41}^{r,r,s,s}/24\right)-[15]\,k_{31}^{r,r,s}k_{31}^{s,t,t}/24\\
&k_{12}^{r}\gamma^{r}+2\,[3]\left(k_{11}^{r}k_{21}^{s,s}/2+k_{32}^{r,s,s}/6\right)\gamma^{r}-[15]\,k_{21}^{r,s}k_{31}^{s,t,t}\gamma^{r}/4+\\
&\left(k_{11}^{r}k_{11}^{s}+k_{22}^{r,s}\right)\gamma^{r}\gamma^{s}-3\,[6]\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^{r}\gamma^{s}+\\
&[45]\,k_{31}^{r,s,t}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}/18-3\left(k_{11}^{r}k_{21}^{s,t}/2+k_{32}^{r,s,t}/6\right)\gamma^{r}\gamma^{s}\gamma^{t}+[10]\,k_{21}^{r,s}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}\gamma^{t}/3+\\
&[4]\left(k_{11}^{r}k_{31}^{s,t,u}/6+k_{12}^{r,s}k_{12}^{t,u}/8+k_{41}^{r,s,t,u}/24\right)\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}-5\,[15]\,k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}/72-\\
&5k_{21}^{r,s}k_{31}^{t,u,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}/12+k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}/18\Big]\,G_{q+2,\tau}\left(z\right)+\\
&\Big[[15]\,k_{31}^{r,r,s}k_{31}^{s,t,t}/24-[3]\left(k_{11}^{r}k_{31}^{r,s,s}/6+k_{12}^{r,r}k_{12}^{s,s}/8+k_{41}^{r,r,s,s}/24\right)+\\
&[15]\,k_{21}^{r,s}k_{31}^{s,t,t}\gamma^{r}/4-[3]\left(k_{11}^{r}k_{21}^{s,s}/2+k_{32}^{r,s,s}/6\right)\gamma^{r}-\left(k_{11}^{r}k_{11}^{s}+k_{21}^{r,s}\right)\gamma^{r}\gamma^{s}/2+\\
&3\,[6]\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^{r}\gamma^{s}-[45]\,k_{31}^{r,s,t}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}/18+\\
&3\left(k_{11}^{r}k_{21}^{s,t}/2+k_{32}^{r,s,t}/6\right)\gamma^{r}\gamma^{s}\gamma^{t}-[10]\,k_{21}^{r,s}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}\gamma^{t}/2-\\
&[6]\left(k_{11}^{r}k_{31}^{s,t,u}/6+k_{12}^{r,s}k_{12}^{t,u}/8+k_{41}^{r,s,t,u}/24\right)\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}+5\,[15]\,k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}/36+\\
&5k_{21}^{r,s}k_{31}^{t,u,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}/6-5k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}/24\Big]\,G_{q+4,\tau}\left(z\right)+\\
&\Big[-[15]\,k_{31}^{r,r,s}k_{31}^{s,t,t}/72-[15]\,k_{21}^{r,s}k_{31}^{s,t,t}\gamma^{r}/12+[45]\,k_{31}^{r,s,t}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}/18-\\
&[6]\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^{r}\gamma^{s}-\left(k_{11}^{r}k_{21}^{s,t}/2+k_{32}^{r,s,t}/6\right)\gamma^{r}\gamma^{s}\gamma^{t}+\\
&[10]\,k_{21}^{r,s}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}\gamma^{t}/3+4\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}-\\
&5\,[15]\,k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}/36-5k_{21}^{r,s}k_{31}^{t,u,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}/6+\\
&5k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}/18\Big]\,G_{q+6,\tau}\left(z\right)+\Big[-[45]\,k_{31}^{r,s,t}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}/72-\\
&[10]\,k_{21}^{r,s}k_{31}^{t,u,u}\gamma^{r}\gamma^{s}\gamma^{t}/12-\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}+\\
&+5\,[15]\,k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}/72+5k_{21}^{r,s}k_{31}^{t,u,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}/12-\\
&5k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}/24\Big]\,G_{q+8,\tau}\left(z\right)+\Big[-[15]\,k_{31}^{r,s,t}k_{31}^{u,v,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}/6-\\
&k_{21}^{r,s}k_{31}^{t,u,v}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}+k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}\Big]\,G_{q+10,\tau}\left(z\right)/12-\\
&k_{31}^{r,s,t}k_{31}^{u,v,w}\gamma^{r}\gamma^{s}\gamma^{t}\gamma^{u}\gamma^{v}\gamma^{w}G_{q+12,\tau}\left(z\right)/72.
\end{aligned}
$$

Proof. Let

$$
E_{h} = 1+\left(k_{11}^{r}h^{r}+k_{21}^{r,s}h^{rs}/2+k_{31}^{r,s,t}h^{rst}/3!\right)/n^{1/2}+\left[k_{12}^{r}h^{r}+\left(k_{11}^{r}k_{11}^{s}+k_{22}^{r,s}\right)h^{rs}/2+\right.
$$

$$\left(k_{11}^{r}k_{21}^{s,t}/2+k_{32}^{r,s,t}/6\right)h^{rst}+\left(k_{11}^{r}k_{31}^{s,t,u}/6+k_{12}^{r,s}k_{12}^{t,u}/8+k_{41}^{r,s,t,u}/24\right)h^{rstu}+$$
$$k_{21}^{r,s}k_{31}^{t,u,v}h^{rstuv}/12+k_{31}^{r,s,t}k_{31}^{u,v,w}h^{rstuvw}/72\Big]/n, \tag{8}$$

where the various k's are the approximate cumulants defined in (6). A formal Edgeworth expansion for the local power for the family W_n of tests as given in (5) can be obtained by exponentiating and successive inversion of the approximate cumulant generating function implied by (6), which gives:

$$\Pr\left(W_n \leq z\right)=\int_{-\infty}^{z} E_h\phi_q\left(\gamma^r,\delta^{rs}\right)+O\left(n^{-3/2}\right).$$

We can now use Theorem 1 to evaluate the above integral, by considering each of the terms appearing in (8) as generalised noncentral quadratic forms, i.e.

$$\Pr\left(W_n \geq z\right)=\int_{w^r w^r \geq z}\left[\sum\nolimits_{R_6} k^{R_6}\partial^6\left(\phi_q\left(\gamma^r,\delta^{rs}\right)\exp\left\{w^r\delta^{rs}t^s\right\}\right)|_{t^r=\mathbf{0}}\right]dw^r,$$

where k^{R_6} denotes all the different cumulants appearing in (6); after some lengthy but straightforward algebra, using repeatedly (4), we get

$$\begin{aligned}P_1(z,\tau) \;=\; & -\Big(k_{11}^{r}\gamma^r\nabla G_{q,\tau}(z)+k_{21}^{r,r}\nabla G_{q,\tau}(z)/2+k_{21}^{r,s}\gamma^r\gamma^s\nabla^2 G_{q,\tau}(z)/2+\\ & [3]\,k_{31}^{r,s,s}\gamma^r\nabla^2 G_{q,\tau}(z)/6+k_{31}^{r,s,t}\gamma^r\gamma^s\gamma^t\nabla^3 G_{q,\tau}(z)/6\Big),\end{aligned}$$

$$\begin{aligned}P_2(z,\tau) \;=\; & -\left(k_{12}^{r}\gamma^r\nabla G_{q,\tau}(z)+\left(k_{11}^{r}k_{11}^{r}+k_{22}^{r,r}\right)\nabla G_{q,\tau}(z)/2+\right.\\ & \left(k_{11}^{r}k_{11}^{s}+k_{22}^{r,s}\right)\gamma^r\gamma^s\nabla^2 G_{q,\tau}(z)/2+[3]\left(k_{11}^{r}k_{21}^{s,s}/2+k_{32}^{r,s,s}/6\right)\gamma^r\nabla^2 G_{q,\tau}(z)+\\ & \left(k_{11}^{r}k_{21}^{s,t}/2+k_{32}^{r,s,t}/6\right)\gamma^r\gamma^s\gamma^t\nabla^3 G_{q,\tau}(z)+\\ & [3]\left(k_{11}^{r}k_{31}^{r,s,s}/6+k_{12}^{r,r}k_{12}^{s,s}/8+k_{41}^{r,r,s,s}/24\right)\nabla^2 G_{q,\tau}(z)+\\ & [6]\left(k_{11}^{r}k_{31}^{s,t,t}/6+k_{12}^{r,s}k_{12}^{t,t}/8+k_{41}^{r,s,t,t}/24\right)\gamma^r\gamma^s\nabla^3 G_{q,\tau}(z)+\\ & \left(k_{11}^{r}k_{31}^{s,t,u}/6+k_{12}^{r,s}k_{12}^{t,u}/8+k_{41}^{r,s,t,u}/24\right)\gamma^r\gamma^s\gamma^t\gamma^u\nabla^4 G_{q,\tau}(z)+\\ & [15]\,k_{21}^{r,s}k_{31}^{s,t,t}\gamma^r\nabla^3 G_{q,\tau}(z)/12+\\ & \Big([10]\,k_{21}^{r,s}k_{31}^{t,u,u}\gamma^r\gamma^s\gamma^t\nabla^4 G_{q,\tau}(z)+k_{21}^{r,s}k_{31}^{t,u,v}\gamma^r\gamma^s\gamma^t\gamma^u\gamma^v\nabla^5 G_{q,\tau}(z)\Big)/12+\end{aligned}$$

$$\Big([15]\, k_{31}^{r,r,s} k_{31}^{s,t,t} \nabla^3 G_{q,\tau}(z) + [45]\, k_{31}^{r,s,t} k_{31}^{t,u,u} \gamma^r \gamma^s \nabla^4 G_{q,\tau}(z) +$$
$$[15]\, k_{31}^{r,s,t} k_{31}^{u,v,v} \gamma^r \gamma^s \gamma^t \gamma^u \nabla^5 G_{q,\tau}(z) +$$
$$k_{31}^{r,s,t} k_{31}^{u,v,w} \gamma^r \gamma^s \gamma^t \gamma^u \gamma^v \gamma^w \nabla^6 G_{q,\tau}(z)\Big) /72\Big),$$

and (7.1) and (7.2) follows after some simplifications and noting that $\int_{w^r w^r \geq z} \prod_{r=1}^{q} \phi(\gamma^r, 1)\, dw^r = 1 - G_{q,\tau}(z)$. ■

References

Bhattacharya, R. & Ghosh, J. (1978), 'On the validity of formal Edgeworth expansion', *Annals of Statistics* **6**, 434–451.

Bickel, P. & Ghosh, J. (1990), 'A decomposition for the likelihood ratio statistic and the Bartlett correction - A Bayesian argument', *Annals of Statistics* **18**, 1070–1090.

Chandra, T. & Ghosh, J. (1979), 'Valid asymptotic expansions for the likelihood ratio and other perturbed chi-square variables', *Sankhyā A* **41**, 22–47.

Chandra, T. & Ghosh, J. (1980), 'Valid asymptotic expansions for the likelihood ratio and other statistics under contiguous alternatives', *Sankhyā A* **42**, 170–184.

Chandra, T. S. & Joshi, S. (1983), 'Comparison of the likelihood ratio, Rao's and Wald's tests and a conjecture of C.R.Rao', *Sankhyā A* **45**, 226–246.

Hayakawa, T. (1975), 'The likelihood ratio criterion for a composite hypothesis under a local alternative', *Biometrika* **62**, 451–460.

McCullagh, P. (1987), *Tensor Methods in Statistics,*, London: Chapman and Hall.

Muirhead, R. (1982), *Aspects of Multivariate Statistical Theory*, New York: Wiley.

Peers, H. (1971), 'Likelihood ratio and associated test criteria', *Biometrika* **58**, 577–587.

APPENDIX

Densities for noncentral quadratic forms

Let $g_{q+.,\tau}(x) := g_{q+.,\tau}$. Then,

$$
\begin{aligned}
b^{r_1...r_7} w^{r_1}...w^{r_7} \phi_q(\gamma^r, \delta^{rs}) &= b^{r_1...r_7}\gamma^{r_1}...\gamma^{r_7} g_{q+12,\tau} + [21]\, b^{r_1 r_2 r_3 r_4 r_5 ss}\gamma^{r_1}\gamma^{r_2}\gamma^{r_3}\gamma^{r_4}\gamma^{r_5} g_{q+10,\tau} + \\
&\quad [105]\, b^{r_1 r_2 r_3 sstt}\gamma^{r_1}\gamma^{r_2}\gamma^{r_3} g_{q+8,\tau} + [105]\, b^{rssttuu}\gamma^{r} g_{q+6,\tau}, \\
b^{r_1...r_8} w^{r_1}...w^{r_8} \phi_q(\gamma^r, \delta^{rs}) &= b^{r_1...r_8}\gamma^{r_1}...\gamma^{r_8} g_{q+14,\tau} + [28]\, b^{r_1...r_6 ss}\gamma^{r_1}...\gamma^{r_6} g_{q+12,\tau} + \\
&\quad [210]\, b^{r_1...r_4 sstt}\gamma^{r_1}...\gamma^{r_4} g_{q+10,\tau} + [420]\, b^{r_1 r_2 ssttuu}\gamma^{r_1}\gamma^{r_2} g_{q+8,\tau} + \\
&\quad [105]\, b^{rrssttuu} g_{q+6,\tau}, \\
b^{r_1...r_9} w^{r_1}...w^{r_9} \phi_q(\gamma^r, \delta^{rs}) &= b^{r_1...r_9}\gamma^{r_1}...\gamma^{r_9} g_{q+16,\tau} + [36]\, b^{r_1...r_7 ss}\gamma^{r_1}...\gamma^{r_7} g_{q+14,\tau} + \\
&\quad [378]\, b^{r_1...r_5 sstt}\gamma^{r_1}...\gamma^{r_5} g_{q+12,\tau} + [1260]\, b^{r_1 r_2 r_3 ssttuu}\gamma^{r_1}\gamma^{r_2}\gamma^{r_3} g_{q+10,\tau} + \\
&\quad [945]\, b^{rssttuuvv}\gamma^{r} g_{q+8,\tau}, \\
b^{r_1...r_{10}} w^{r_1}...w^{r_{10}} \phi_q(\gamma^r, \delta^{rs}) &= b^{r_1...r_{10}}\gamma^{r_1}...\gamma^{r_{10}} g_{q+18,\tau} + [45]\, b^{r_1...r_8 ss}\gamma^{r_1}...\gamma^{r_8} g_{q+16,\tau} + \\
&\quad [630]\, b^{r_1...r_6 sstt}\gamma^{r_1}...\gamma^{r_5} g_{q+14,\tau} + [3150]\, b^{r_1...r_4 ssttuu}\gamma^{r_1}...\gamma^{r_4} g_{q+12,\tau} + \\
&\quad [4725]\, b^{rsttuuvvww}\gamma^{r}\gamma^{s} g_{q+10,\tau} + [945]\, b^{rrssttuuvv} g_{q+8,\tau}.
\end{aligned}
$$