THE UNIVERSITY of York

Discussion Papers in Economics

No. 2000/22

Growth Constrained by Exhaustible Resources: A Creative Destruction Approach

by

Farhad Nili

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Growth Constrained by Exhaustible Resources: A Creative Destruction Approach

Farhad Nili[1]

Draft: 13 Apr 2000

Abstract. This paper is about how the constraints imposed by natural resource scarcity affects economic growth and its sustainability. We extend the creative destruction model of Aghion and Howitt(1998, ch.5) to study its transitional dynamics. This extension allows us to describe the dynamics of the economy on the stable saddle path and its rate of convergence toward the steady state. Moreover we show situation in which even in the presence of sustainability condition, suggested by Aghion and Howitt, it is optimal for consumption to display negative growth rate on the transition path for a finite period. The conditions for uniqueness of the steady state are also defined. Under plausible assumptions the closed forms of the fundamentals of the model in the steady state is determined. We study the stability of the model around the steady state and analyse the nonlinear system of differential equations, describing the dynamics of the economy, by numerical methods.

Keywords : endogenous growth, exhaustible resources, optimal growth, steady state, transitional dynamics
JEL classification: C62 , O13 , O41, Q32.

1 Introduction

Oil shock of 1973/74 had intellectual effects on the research activities as well as its economic effects on the industrial economies[2]. It called economists' attention to the question of whether it is possible to deal with the finite stock of exhaustible resources as a binding constraint on the worldwide economic growth[3]. Available technology at that time was suggesting fossil energy as an essential input of

---

[1] PhD student, Department of Economics, University of York. Email: fn101@york.ac.uk

[2] See for example Nordhaus(1980), Liewelwyn(1983), Marion and Svensson(1984), Sachs(1981) and Foreman-Peck(1995, pp:335-38) for the effects of oil shocks on the world economy.

[3] Apart from initiation of *growth constrained* as a new branch of growth theory, oil shocks raised other new issues

aggregate production. OPEC agreement about a significant reduction in the oil production and its effect on the economic performance of industrial economies encouraged economists to think about *growth constrained by natural resources* as a way to analyse the effect of natural resource scarcity on the economic growth.

The other stimulus for study of the problem was the studies have been carried out, in a multi-disciplinary approach, by Forrester and the Club of Rome. Their pessimistic approach was indicating that worldwide economic growth is constrained by natural resource scarcity. The majority of economists however despite the above mentioned Malthusian view found that technical progress, capital substitution and increasing returns to scale are three ways to compensate the declining trend of rate of utilization of an essential and nonrenewable resource in the aggregate production function.

The purpose of this paper is to study the transitional dynamics of the creative destruction model[4] of Aghion and Howitt in the presence of exhaustible resource introduced in chapter 5 of their book, *Endogenous Growth Theory*. They touch environmental issues and nonrenewable resources to address the advantage of the Shumpeterian approach to AK model to analyse the main questions related to sustainability. They focus on the steady state and show that a permanent process of innovation ensures that an economy overcomes natural resource scarcity and provides a permanent level of positive consumption. In Aghion-Howitt model the level of initial physical capital has been considered as a choice variable implying that economy can jump to the steady state instantaneously. In this paper instead the physical capital is taken given at the time of planning and so having the transitional dynamics is unavoidable. We thus extend Aghion-Howitt model to study its transitional dynamics. Moreover we derive the conditions for the uniqueness of the steady state and for its feasibility and compute the rate of convergence of the economy toward it. A closed form for the fundamental variables of the economy in the long run is derived too. In the case of nonuniqueness of the steady state, the characteristics of multiple equilibria is also found. We furthermore provide the stability analysis of the steady state and consider the effect of return to scale on the economy as a whole and in particular balanced rate of growth and rate of resource depletion.

After considering the link of the paper to the existing literature in next section, we explain the micro foundations of the model in section 3. In section 4 and 5 the optimal growth and steady state are studied respectively. We analyse the dynamics of the model including the stability analysis and the study of transitional dynamics in section 6. We conclude then in section 7.

2 Literature review

The problem of the effect of natural resource scarcity (in particular nonrenewable one) on the economic growth has been explored during 1970's by Stiglitz(1974a,b), Solow(1974a), Dasgupta(1974), and Dasgupta and Heal(1974, 1979). They consider in a neo-classical growth framework the rate of utilization of natural resources as a factor of production. The results was optimistic and dramatically

in macroeconomics, see e.g. Sachs(1983) and Tornell and Lane(1998).

[4] For comparison of the two main approaches to endogenous growth theory, namely *expanding product variety* and *quality ladder* (or creative destruction) and some of shortcomings of the former respect to the latter see Helpman(1992).

in contrast with the Malthusian view presented by Forrester(1971) and the Club of Rome[5]. They found that the world economy can overcome the finite supply of exhaustible resources which are essential for production if there is a continuous flow of technical progress or if the share of physical capital in aggregate production function is larger than the share of natural resources. So the world in their view would survive and technical progress or capital substitution can compensate the natural resource scarcity[6].

Although during 1980s many research carried out on the macroeconomic effects of natural resource scarcity (see for example Chichilnisky 1985; Chichilinsky and Heal 1983; Marion and Svensson 1984 ; Hellinell et al. 1982 ; Sachs 1981,1983 ; Krugman 1983 and Buiter and Purvis 1983), but by 1990s the Stiglitz' findings was still the state of the art in the area of growth constrained by natural resource. In his model the main source for overcoming the natural resource scarcity was the exogenous - and hence unexplained- technical progress factor.

One of the first attempts to endogens technical progress in a growth model in the presence of exhaustible resource is Suzuki(1976) and Kamien and Schwartz(1978). In the former technical progress is the result of R&D activities that absorbs part of output. Chiarella(1980) extends Suzuki's analysis by endogenizing the aggregate saving rate. However these studies investigate only efficient solutions, namely in the case of market economy neither externalities nor monopolistic competition. They demonstrate that growing per capita consumption is possible in a world which exhaustible resources are essential for production.

The wave of endogenous growth theory arose by Romer(1986, 1990), Lucas(1988), Aghion and Howitt(1992) and Helpman and Grossman(1991) affected the Stiglitz' model too, though until recently the endogenous growth literature has not been concerned about the contribution of natural resources.

Recently Barbier(1999) has reconciled the Stiglitz' model of constrained growth and Romer's model of endogenous growth to study the role of innovation in overcoming natural resource scarcity. He also considers the possibility that in low income, natural resources abundant economies the supply of innovation may be adversely affected by the rate of resource utilization. Scholz and Ziemes(1999) demonstrate that in the decentralized version of the expanding varieties growth model, indeterminacy of equilibrium trajectories arises when the Romer's model is extended to incorporate exhaustible resources. In their paper, two types of inefficiencies are responsible for this result: inefficiencies owing to monopolistic competition and information spillover. Schou, as reported by Scholz and Ziemes(1999) shows that compared to the social optimum, in the decentralized version of the model the resource extraction rate could be too low or too high.

Farzin(1999) in a *AK-type* model claims that growth models that incorporate the flow of

---

[5] Meadows et. al. (1972,1974) based on the World Dynamics' model of Forrester investigate five major trends of global concern at 1970s: industrialization, population growth, food-consumption, nonrenewable resources depletion, and pollution; their interconnection, and their implications on growth and the quality of life.

[6] The contribution of neoclassical growth constrained by exhaustible resource has been summarized by Solow (1999) and in more detail by Toman et. al. (1995) and Krautkraemer(1998).

nonrenewable resources as an essential input of production are not the suitable framework for the exhaustible resource-exporting developing economies (EREs), where among all of their features production linkages between the resource sector and the rest of the economy are weak and the gross national income is generated primarily from liquidation and the direct export of the exhaustible asset. He produces a *bang-bang* characteristic for the optimal extraction policy which is totally different from the policies that have been derived so far. Characterization of optimal saving policy for ERE's is the main contribution of Farzin(1999) who shows that, both in magnitude and time profile, the optimal saving policy for an ERE sharply differs from that of a non extractive economy and that derived from the conventional models of growth with exhaustible resources. For a wide range of plausible parameter values, he also shows that a set of selected EREs have been substantially under saving.

There is similarities between incorporating exhaustible resources and the environmental considerations(e.g. the accumulation of pollution) in the growth models. From methodological point of view, growth models dealing with pollution usually consider environmental quality as a renewable resource and enter it as an argument in the utility function. Growth models constrained by exhaustible resources instead consider rate of natural resource use as a factor of production. Aghion and Howitt(1998, ch.5) for example consider environmental quality as a renewable resource and regard to environmental considerations and exhaustible resource scarcity as two different aspects of *sustainability*. They show that sustainability in the latter version can be attained with weaker assumptions. They also show that AK model, due to lack of distinguishing between physical and intellectual capital, can not deal with the sustainability issue well .They propose then *Shumpeterian approach of creative destruction* to endogenous growth as an appropriate way to analyse both environmental considerations and natural resource scarcity.

Grimaud(1998) illustrates how the optimal growth paths in Aghion-Howitt model can be implemented in a decentralized economy. He also studies the suboptimality of the market equilibrium respect to central planner's version of the model and moreover analyses the effects of government intervention on the characteristics of the economy, in particular the rate of growth and the rate of resource use in the long run.

3 The model[7]

Consider an economy populated by a fixed continuous mass of households, each endowed with one unit of skilled labour and supplies it inelastically. The population size is normalized to unity and so, one is also equal to the aggregate flow of labour supply. This means that we abstract from the size effect and variables in the model are in per capita terms .

There are two type of activities for labour force: working in the final product market or doing research. We denote the number of people producing final product and doing research with L and n respectively and hence in each period we will have

[7] This section is heavily relied on the model of Aghion and Howitt (1998, ch.3) in the presence of exhaustible resource.

$$n_t + L_t = 1 \tag{1}$$

Following Aghion-Howitt we extend the conventional growth model of creative destruction to incorporate use of natural resources by including *R*, the rate of extraction of an exhaustible resource, S, as an additional input in the production of final good. According to the theory of exhaustible resource(See Dasgupta and Heal 1979, ch.6) the relation between *R* and *S* is described as

$$S_t = S_0 - \int_0^t R_\tau d\tau \tag{2}$$

where S_t (for t $\geq$0) is the stock of resource at t. We assume that S_0 is given and its amount is known with certainty. Moreover for convenience we assume that there is one pool of resource and we do not consider the problem concerning common utilization of the resource and its externality effect.

The economy produces one (numeraire) final product, denoted by *Y*, and a continuum of intermediate (capital) goods indexed on the unit interval [0,1] and denoted by x_i where 0 $\leq$ i $\leq$1. Each intermediate good is produced from physical capital and each can be used to produce the final good independently of other intermediate goods, with no complementaries between them.

More specifically, the flow of final good that can be produced using intermediate good i depends only on the flow x_i of intermediate good i according to the production function

$$Y_i = B_i x_i^\alpha L^\beta R^\nu \qquad\qquad 0 \leq i \leq 1$$

where *L* and *R* as introduced before are the total amount of labour working in final-product section and rate of extraction of the natural resource respectively; B_i is the highest level of technology in section i, which represents the productivity of the latest generation of intermediate good i. Aggregate output of the final good is therefore the sum

$$Y = L^\beta R^\nu \int_0^1 B_i x_i^\alpha di \tag{3}$$

where α, β, ν ∈ (0,1) are technology parameters which represent the importance of intermediate good (or physical capital as will be described later), labour and natural resource in the aggregate product respectively.

The engine of growth in the model is technical progress through innovation in the intermediate sector. Since the variety of intermediate goods is assumed to be constant, innovation leads to improvement in the quality of the existing goods which has been interpreted as *vertical innovation*. Research activities are sector specific and lead to innovation in a stochastic fashion. More research effort in a sector more likely is the innovation and improvement in the quality of the product of that sector along the quality ladder. Furthermore innovation has a *spillover effect* through knowledge accumulation in the whole economy. Innovation in each sector has a positive externality on the other sectors by increasing the level of public knowledge accessible for all researchers.

Beside this positive externality, quality improvement of the intermediate goods makes the old products less attractive. In the extreme case where we suppose perfect substitution among the products of each sector, new products make the last products *obsolete*. This negative externality of new innovations on the incumbent producers will be explained in the following in terms of monopolistic competition among researchers.

Each intermediate sector is monopolized by the holder of a patent to the last generation of that good. The local monopolist sells its output to the competitive final-good sector in which its marginal product, and thus its price measured in final good, is given by

$$P_i = \alpha L^{\beta} R^{\nu} B_i x_i^{\alpha-1} \tag{4}$$

which gives the demand for i-th intermediate as

$$x_i = (\alpha L^{\beta} R^{\nu} B_i / P_i)^{1/(1-\alpha)} \tag{4'}$$

Suppose now that there is a stock of physical capital *K*, embodied in durable machines and belongs to the households. Capital is produced, along with consumption goods *C*, according to the production function (3), where the factors of production are employed in two activities of producing consumption goods and physical capital

$$Y = C + \dot{K} \tag{5}$$

where there is no depreciation.

The only input into the production of intermediate goods is capital. The monopolist in sector i to produce x_i requires $B_i x_i$ units of capital and the i-th intermediate is produced according to the (linear) constant returns production function $x_i = K_i / B_i$ for $0 \leq i \leq 1$, where K_i is the amount of capital used to produce good i. Thus more advanced technologies (larger B_i) are more capital intensive.

Assume that each monopolist rents its capital from households in an imperfectly competitive market, where the rental rate at each date t is r_t. Then its average cost will be $B_{it} r_t$. Thus the monopolist's profit will be

$$\pi_i = (P_i - rB_i) x_i$$

The first order condition of the profit-maximization problem yields the monopolist's output in section i as

$$x_i = x := (\alpha^2 L^{\beta} R^{\nu} / r)^{1/(1-\alpha)} \qquad \text{for all } i \in [0,1] \tag{6}$$

which is constant across sectors. Thus the optimal supply of intermediate goods is proportional to supply of labour in product market and the rate of utilization of resource and is inversely related to the rental rate of capital.

Replacing the optimal (symmetric) supply of intermediate good in the inverse demand function (4) leads the (markup) monopoly pricing as $P_{it} = B_{it} r_t / \alpha$ which is not constant neither over time, nor across sectors. The monopoly's flow of profit will be then

$$\pi_i = [(1-\alpha)/\alpha]\, r B_i x_i \tag{7}$$
$$= (1-\alpha) P_i x_i \tag{7'}$$

which the latter shows that the revenue from each innovation is distributed in the fraction of (1- α) and α to the profit and cost of renting capital which are received by innovators and households respectively.

Now let

$$B = \int_0^1 B_i \, di \tag{8}$$

denote the average productivity parameter across all sectors. Since each sector i uses $B_i x_i$ units of capital and there is a total capital stock of *K*, capital market equilibrium requires

$$K = \int_0^1 B_i x_i \, di \tag{9}$$

According to (6), all sectors produce the same amount of intermediate goods at any given time[8]. This with two above equations imply that

$$x_i = x = k := K/B \qquad \text{for all } i \in [0,1] \tag{10}$$

That is, the equilibrium flow of intermediate output from each sector at date t must be equal to the *capital intensity k* .

Substituting x_i from (10) into (3) yields the familiar Cobb-Douglas aggregate production function

$$Y = B x^{\alpha} L^{\beta} R^{\nu} \tag{11}$$
$$= B^{1-\alpha} K^{\alpha} L^{\beta} R^{\nu} \tag{11'}$$

This functional form is consistent with the theory of exhaustible resources where it is quite natural to consider the flow of utilization of the resource as a factor of production in the aggregate production function. Moreover in a CES world where the elasticities of substitution between natural resource and the aggregate capital is constant, only the Cobb-Douglas form exhibits realistic characteristics which are consistent with the feasibility conditions implied by the scarcity of resources. [See Dasgupta and Heal(1974) for more details.]

What we have just described in (11') is a three sectors growth model where the level of output is determined by three stocks: the stock of capital K, the level of knowledge which is proportional to B

---

[8] Maximizing Y subject to (9) yields the optimality condition $x_i = x = L(\psi/\alpha)^{1/(\alpha-1)}$, where ψ=r/α is the lagrange multiplier of the constraint expressed in (9). This is another way to find that the supply of intermediate goods is symmetric across sectors.

and the rate of utilization of resources R, which depends on S. We refer through this paper to these three types of capitals as physical (or tangible), intellectual and natural capital.

The above functional form indicates that for a given state of knowledge, α + β + ν is a measure of returns to scale. In spite of studies that carried out so far and are reviewed in the previous section we do not impose the constant returns to scale(CRTS) on the production function because in this case one can not distinguish the effect of change of α and ν.

Combining (6) and (10) gives us rental rate of capital as a function of capital intensity

$$r = \alpha^2 k^{\alpha-1} L^{\beta} R^{\nu} \tag{12}$$

According to (12), an increase in the historically predetermined capital intensity will reduce the equilibrium rental rate that a monopolist must pay for capital. This can be interpreted as a consequence of decreasing returns at the accumulation of capital intensity. Comparing (12) and (11') we find that the rental rate of capital is α (< 1) times of its marginal product, i.e

$$r = \alpha\, \partial Y/\partial K \tag{12'}$$

$$= \alpha^2 Y/K \tag{12''}$$

which means higher the degree of monopoly in the intermediate section, less households will receive from lending capital. This is another expression for what we found in (7') that capital expenditure is distributed between monopolists and households according to parameter α .

Research in intermediate section is sector specific and the probability of success of research in each sector proportionately depends on the research effort in that sector which we measure it with the amount of researchers in the sector. Following Aghion- Howitt we assume that the probability of innovation per researcher is constant over time and across sectors. Thus if we denote by n_i, the research employment in sector i then the arrival rate of innovation in sector i would be ηn_i where η>0 is a parameter indicating the productivity of the research activities. Although the arrival rates in different sectors are independent of each other, the innovations themselves all contribute to raise the level of public knowledge. The state of this knowledge is represented by a *leading-edge technology*, whose amount at date t is denoted with

$$B_t^{\max} = \max\{B_{it}:\ 0 \le i \le 1\}$$

This parameter grows gradually at a rate proportional to the aggregate flow of innovations

$$n_t = \int_0^1 n_{it}\, di$$

with a factor of proportionality equal to σ>0. Thus in the economy as a whole there will be a continuous flow of ηn innovation per unit of time. This implies

$$\dot{B}^{\max} = \sigma \eta n B^{\max} \tag{13}$$

Equation (13) is the law of motion governing the evolution of public knowledge. At any point in time there will be a distribution of productivity parameters B_{it} across the sectors with values ranging from 0 to $B_t^{\max}$. Over time the distribution will be displaced upward as innovating sectors move up to $B_t^{\max}$ and rightward as technological progress raises $B_t^{\max}$ itself. Fortunately, the shape of the distribution does not change, even if the order of the sectors occupying the different places in the distribution are continually changing. More specifically, in the long run the cross-sectoral distribution of the relative productivity parameters $b_{it} = B_{it}/B_t^{\max}$ will be given by the distribution function[9]

$$H(b) = b^{1/\sigma}, \qquad 0 \le b \le 1$$

According to the definition of relative productivity parameter b, equation (8) gives an expression for its average which is equal to $E(b_i) = B / B^{\max}$.On the other hand by definition we have

$$E(b_i) = \int_0^1 b_i h(b_i)\, d b_i = \frac{1}{1+\sigma} \qquad\qquad \text{for all } \ i \in (0,1)$$

where $h(b) = b^{(1-\sigma)/\sigma}/\sigma$ is the probability density function of relative productivity. Equating these two expressions obtained for E(b) gives

$$B_t = B_t^{\max}/(1+\sigma)$$

From the above equation and considering (13), we obtain

$$\frac{\dot{B}}{B} = \frac{\dot{B}^{\max}}{B^{\max}} = \sigma\eta n \tag{14}$$

which shows that research effort, n has a positive growth effect on the accumulation of knowledge and hence on the economy as a whole. The overall structure of the model is now complete and depicted in figure 1.

Figure 1, in here

4 Optimal growth

Now suppose that households have identical lifetime utility function as

$$W = \int_0^\infty e^{-\rho t} U(C_t)\, d t \tag{15}$$

where U is the instant utility of consumption, and $\rho > 0$ is the rate of time preference. The problem of optimal growth is that of choosing the rates of consumption C, research employment n, and extraction of resource R, at each date so as to maximise W subject to

[9] See section 3.1.2 of Aghion-Howitt for details.

$$\dot{S} = -R \tag{2'}$$

$$\dot{K} = B^{1-\alpha} K^{\alpha} (1-n)^{\beta} R^{\nu} - C \tag{5'}$$

$$\dot{B} = \sigma \eta n B \tag{14}$$

where B_0, K_0 and S_0 are given. The coefficients α , β and ν are all in [0,1) and α + β + ν is the measure of returns to scale which otherwise stated we do not impose any constraint on it.

The Hamiltonian of the optimization problem is

$$H = U(C) + \lambda[B^{1-\alpha} K^{\alpha} (1-n)^{\beta} R^{\nu} - C] + \mu \sigma \eta n B - \xi R$$

There are three state variables (*K*, *B* and *S*), three costate variables (λ, μ and ξ), and three control variables (*C*, *n* and *R*). Assuming an interior solution, the first order (static efficiency) conditions are

$$H_C = 0 \Rightarrow U^{/}(C) = \lambda \tag{16}$$

$$H_n = 0 \Rightarrow \lambda \beta Y/(1-n) = \mu \eta \sigma B \tag{17}$$

$$H_R = 0 \Rightarrow \lambda \nu Y/R = \xi \tag{18}$$

The Euler equations (dynamic efficiency conditions) are

$$H_K = -\dot{\lambda} + \rho \lambda \Rightarrow g_{\lambda} = \rho - \alpha Y/K \tag{19}$$

$$H_B = -\dot{\mu} + \rho \mu \Rightarrow g_{\mu} = \rho - (\lambda/\mu)(1-\alpha)(Y/B) - \eta \sigma n \tag{20}$$

$$H_S = -\dot{\xi} + \rho \xi \Rightarrow g_{\xi} = \rho \tag{21}$$

where for each variable like z, we denote its exponential rate of growth with $g_z = \frac{dz/dt}{z}$. The transversality conditions are

$$\lim_{t \uparrow \infty} e^{-\rho t} \lambda K = \lim_{t \uparrow \infty} e^{-\rho t} \mu B = \lim_{t \uparrow \infty} e^{-\rho t} \xi S = 0$$

(22)

Before returning to the algebra, we follow Barbier(1999) to build intuition about these conditions. Equation (16) describes the optimality rule for consumption. It indicates that, along the optimal path, the marginal utility of consumption must equal the shadow price of accumulated capital. Condition (17) determines the optimal amount of research effort. It shows that the marginal productivity of labour force in final product relative to marginal contribution of research in innovation, must equal the relative price of intellectual capital to tangible one. Condition (18) indicates that the marginal productivity of resource inputs must equal the relative price of the resource stock to capital. Condition (19) shows that the percentage change in shadow price of physical capital must equal the discount factor less the marginal productivity of capital. Similarly, condition (20) shows that the percentage rate of change in shadow value of intellectual capital should equal the discount factor less the rate of growth of social knowledge and the relative price of physical capital to intellectual one times the marginal productivity of social knowledge. Finally, condition (21) indicates that the capital gains of holding on the resource stock on the optimal path must equal to its opportunity costs.

We assume households have isoelastic preferences which means U in (15) is of the form $U(C) = (C^{1-\varepsilon})/(1-\varepsilon)$ and $\varepsilon = -U^{//}(C)C/U^{/}(C)$, the inverse of elasticity of substitution, is a positive constant parameter. Considering this functional form for the preferences, from (16) and (19) we obtain the familiar *Ramsey equation for consumption*[10]

$$g_C = \frac{\alpha Y/K - \rho}{\varepsilon} \tag{23}$$

Taking into account (11'), Eq. (23) gives the optimal rate of growth of the consumption as

$$g_C(t) = [\alpha(B_t/K_t)^{1-\alpha}L_t^{\beta}R_t^{\nu} - \rho]/\varepsilon \tag{23'}$$

which means that *in the absence of population growth, an economy with exhaustible resources is sustainable if increases in the ratio of intellectual capital to the tangible capital can compensate the finiteness of the essential resources*.

Plugging (17) into (20) yields

$$g_\mu = \rho - \sigma\eta n - \frac{(1-\alpha)(1-n)\sigma\eta}{\beta} \tag{24}$$

Log-differentiating from (11'),(17) and (18) respectively yields

$$g_Y = (1-\alpha)g_B + \alpha g_K - \beta g_n + \nu g_R \tag{25}$$

$$g_\lambda + g_Y + g_n = g_\mu + g_B \tag{26}$$

$$g_\lambda + g_Y - g_R = \rho \tag{27}$$

where we used (1) and (21) in (25) and (27) respectively. Replacing (19) in (27) gives

$$g_R = g_Y - \alpha Y/K \tag{28}$$

---

[10] Under *CRRA*, for W in (15) to be well-defined and considering (23) we should have

$$\alpha\, Y/K\,(1-\varepsilon) < \rho$$

which is trivial for $\varepsilon \geq 1$.

Following Barbier(1999) we call this condition, the *basic Hotelling rule for resource flows* which indicates growth in resource use is determined by the growth rate of output less the marginal productivity of capital.

We can rewrite (5) as

$$g_K = \frac{Y-C}{K} \tag{5'}$$

substitution of g_K and g_R from (5') and (28) in (25) gives

$$g_Y = \frac{1}{1-\nu}[(1-\alpha)\sigma\eta n - \beta g_n - \alpha\frac{C}{K}] + \alpha\frac{Y}{K} \tag{29}$$

where we have rewritten (14) as

$$g_B = \sigma\eta n \tag{14'}$$

Now consider (26). On the LHS we can substitute g_λ and g_Y from (19) and (29) respectively and on the RHS we can substitute g_μ and g_B from (24) and (14'). Assuming $\beta + \nu \neq 1$ and after manipulation we obtain

$$g_n = \frac{(1-\alpha)\sigma\eta}{\beta}(n + \frac{\nu-1}{1-\beta-\nu}) + \frac{\alpha}{1-\beta-\nu}\frac{C}{K} \tag{30}$$

Substituting from (30) into (29) gives the optimal rate of growth of final product as

$$g_Y = (\frac{1-\alpha}{1-\beta-\nu})\sigma\eta - (\frac{\alpha}{1-\beta-\nu})\frac{C}{K} + \alpha\frac{Y}{K} \tag{29'}$$

which after plugging in (28), leads the optimal rate of decline of resource use as

$$g_R = (\frac{1-\alpha}{1-\beta-\nu})\sigma\eta - (\frac{\alpha}{1-\beta-\nu})\frac{C}{K} \tag{28'}$$

This considering (2') for *R/S* , the rate of utilization of the resource[11] gives

$$g_{R/S} = (\frac{1-\alpha}{1-\beta-\nu})\sigma\eta - (\frac{\alpha}{1-\beta-\nu})\frac{C}{K} + \frac{R}{S} \tag{31}$$

Furthermore by definition we have

$$g_S = -R/S \tag{32}$$

[11] Similarly *S/R* indicates the life time of the resource.

Equations (19),(21),(23-24),(5'),(14'),(28'),(29'),(30),(31) and (32) define the law of motion of the economy on the optimal growth path. As we consider the optimal growth path of the all variables are linear combinations of *Y/K*, *C/K*, *n* and *R/S* which will be called *the fundamentals of the model.*

5 Steady state analysis

Definition of the steady state requires that *C, K* and *Y* grow at a constant common rate denoted by

$$g := g_Y = g_C = g_K \tag{33}$$

This implies that capital productivity remains constant at the steady state which from (12") results $g_r = 0$ where $r = \alpha^2 Y/K$ is the rental rate of physical capital. Moreover research effort and portion of labour force working in product market remains constant which requires

$$g_n = g_L = 0 \tag{34}$$

Furthermore following Barbier(1999), in a growth model constrained by a natural resource and to investigate whether natural resource scarcity operates as a binding constraint on the economic growth, it is worth exploring the condition under which the long run equilibrium is characterised by the rate of resource utilization *R/S,* converging to a steady state value (*R/S*)*. Thus an additional steady state condition imposed on the optimal path of the economy is

$$g_{R/S} = 0 \tag{35}$$

Beside transversality conditions defined by (22), equations (33-35) define constraints determining the balanced growth path in the steady state. In the following subsections, firstly we impose the steady state conditions on the optimal growth equations to derive the balanced optimal growth path. We derive then the conditions determining uniqueness of the steady state and then solve equations governing the balanced optimal growth path to derive the long run values of the fundamenatls.

5.1 Balanced optimal growth

In this subsection we impose the constraints determine the steady state, i.e conditions (33-37) on the equations define optimal growth path derived in section 4. Imposing (35) on (31) gives the steady state rate of utilization of the resource as

$$\frac{R}{S} = \frac{\alpha}{1-\beta-v}\left(\frac{C}{K}\right) - \left(\frac{1-\alpha}{1-\beta-v}\right)\sigma\eta \tag{36}$$

Now using (35) in (31)and considering (32) gives

$$g_R = g_S = \left(\frac{1-\alpha}{1-\beta-v}\right)\sigma\eta - \frac{\alpha}{1-\beta-v}\left(\frac{C}{K}\right) < 0 \tag{37}$$

which means that *in the steady state, the stock of resources and its rate of utilization both decline with a common rate which is inversely related to the consumption-capital ratio.* In addition the transversality condition corresponding to S, using (21) gives

$$\lim_{t\uparrow\infty}\int_0^t R_\tau d\tau = S_0$$

Due to finiteness of S and non negativity of R, this implies

$$\lim_{t\uparrow\infty} R_t = 0 \tag{38}$$

If we impose now (33) and (34) on (25) and substitute g_R from (37) we obtain

$$g = g_B - \frac{\nu}{1-\alpha}\left(\frac{R}{S}\right) \tag{39}$$

This, considering (R/S) ≥ 0, means that *in the presence of exhaustible resource the balanced rate of growth is lower than rate of knowledge accumulation. Higher the rate of utilization of the resource or higher the resource contribution in the final product, lower is the balanced rate of growth.* This is a justification of the argument that *natural resource intensity is harmful for growth*[12]. Using a Cobb-Douglas production function with a declining flow of natural resource as an essential factor of production, makes the growth slower respect to no-resource case, ie when $\nu = 0$. In the latter case the balanced rate of growth is equal to the rate of knowledge accumulation.

From (16), for *CRRA* case we have $\lambda = C^{-\varepsilon}$ which considering (33) gives $g_\lambda = -\varepsilon g$. This, so long as g > 0, indicates that *along the balanced growth path, the shadow value of physical capital is declining.*

Imposing (33) and (34) on (25) gives $g_R = (1-\alpha)(g - \sigma\eta n)/\nu$. Substituting these two expressions for g_λ and g_R into (27) gives the optimal balanced rate of growth of the economy as an increasing function of research effort

$$g(n_t) = \frac{(1-\alpha)\sigma\eta n_t - \rho\nu}{(1-\alpha)+(\varepsilon-1)\nu} \tag{40}$$

We have assumed $(1-\alpha)/\nu \neq 1-\varepsilon$ for derivation of (40) which in the cases of constant or decreasing returns to scale (ie when $1-\alpha-\nu \geq \beta > 0 > -\varepsilon$) or when $\varepsilon \geq 1$ is obviously valid.

What we have found in (40) is that research effort has a positive growth effect on the steady state rate of growth which is consistent with conventional creative destruction model. But in that model, the balanced growth rate is equal to the growth rate of intellectual capital which according to (14') is equal to $\sigma\ \eta\ n^*$,where n* is the steady state amount of research employment. In the conventional model of creative destruction, in the steady state output, consumption, physical and intellectual capital grow with the same constant rate $g = \sigma\ \eta\ n^*$ (see section 3.2.2 of Aghion and Howitt 1998). What we found here

[12] This argument is entirely different from the approach undertaken by Sachs and Warner(1995) and related studies about the effect of natural resource abundance on economic growth.

is that incorporating natural resource into the model not only causes the share of natural resource, ν comes into account but also in this case the degree of risk aversion of consumers does matter. The relation between balanced growth rate, *g* and the level of research effort, *n* is depicted in figure 2. As we consider in the presence of exhaustible resource a minimum level of research effort, namely

$$n_{\min} = \frac{\rho \nu}{(1-\alpha)\sigma\eta}$$

is required to compensate the declining trend of resource use and provide a positive rate of growth.

Figure 2, in here

Imposing (33) on (23),(5') and (29') gives two independent linear equations involving Y/K and C/K. Using algebra we derive the steady state level of output-capital ratio and consumption-capital ratio from these equations respectively as

$$\left(\frac{Y}{K}\right)^* = \frac{\varepsilon(1-\alpha)\sigma\eta - \rho\delta}{\alpha[(\varepsilon(\beta+\nu)-\delta]} \tag{41}$$

$$\left(\frac{C}{K}\right)^* = \frac{(\alpha-1)[(\alpha-\delta)\rho + (\alpha-\varepsilon)\sigma\eta]}{\alpha[\varepsilon(\beta+\nu)-\delta]} \tag{42}$$

where δ = 1 - α - β - ν is the degree of increasing return to scale, and we have assumed $(\alpha-\varepsilon)(\alpha-\delta) \neq 0$ for derivation of these expressions. In CRTS these expressions reduce to

$$\left(\frac{Y}{K}\right)^* = \frac{\sigma\eta}{\alpha} \tag{41'}$$

$$\left(\frac{C}{K}\right)^* = \frac{\sigma\eta}{\alpha} - \frac{\sigma\eta-\rho}{\varepsilon} \tag{42'}$$

Now we impose (34) on (30). This ,after substitution from (42), gives the steady state level of research effort as

$$n^* = \frac{1-\nu}{\alpha-\delta} + \frac{\beta}{\varepsilon(\beta+\nu)-\delta}\left(\frac{\alpha-\varepsilon}{\alpha-\delta} - \frac{\rho}{\sigma\eta}\right) \tag{43}$$

which in CRTS reduces to

$$n^* = \frac{\nu}{1-\alpha} + \frac{\beta}{(1-\alpha)\sigma\eta}\left(\frac{\sigma\eta-\rho}{\varepsilon}\right) \tag{43'}$$

By substituting the steady state level of consumption-capital ratio from (42) in (36) and assuming $\beta+\nu \neq 1$ we derive the rate of utilization of the resource at the steady state as

$$\left(\frac{R}{S}\right)^* = \frac{(1-\alpha)[\sigma\eta(\varepsilon-1)+\rho]}{\varepsilon(\beta+\nu)-\delta} \tag{44}$$

which in CRTS reduces to

$$\left(\frac{R}{S}\right)^{*} = \sigma\eta - \left(\frac{\sigma\eta - \rho}{\varepsilon}\right) \tag{44'}$$

Eq. (37) indicates that the negative of these expressions gives the rate of decline of resource stock and resource use at the steady state. In particular, according to (37), in CRTS we have

$$g_S = g_R = (\frac{1-\varepsilon}{\varepsilon})\sigma\eta - \frac{\rho}{\varepsilon}$$

and thus the optimal rate of depletion of resource will be

$$R_t = R_0 \exp\{[(\frac{\sigma\eta - \rho}{\varepsilon}) - \sigma\eta]t\}$$

In the simplest case where ε =1, rate of decline of resource use is equal to the discount rate, ρ. This means that our model , like those of Barbier(1999), can produce the Hotelling's rule as a special case. Moreover in the absence of R&D (i.e. when σ = η = 0) we will have $g_R = -\rho/\varepsilon$ which is the same as findings of the neoclassical models (see e.g Dasgupta and Heal 1974).

For deriving the balanced rate of growth, we can substitute (*C/K*)* from (46) into (27), or (*Y/K*)* and (*C/K*)* from (45) and (46) respectively into (5'), or n* from (47) into (44). After manipulation and rearrangement we obtain

$$g = \frac{(1-\alpha)\sigma\eta - \rho(\beta + \nu)}{\varepsilon(\beta + \nu) - \delta} \tag{49}$$

which in CRTS becomes

$$g = \frac{\sigma\eta - \rho}{\varepsilon} \tag{49'}$$

The effect of change in the parameters of the model on the long run growth of the economy is described in the following statement, which is the result of simple manipulations on equation (49):

Proposition 1 *The balanced rate of growth is an increasing function of the productivity of research activities, σ and η ; the degree of return to scale, δ and a decreasing function of the willingness to consume, ρ and ε. It also decreases when the contribution of each of the factors of production increases if and only if the long run rate of growth does not exceed the incremental value of public knowledge per researcher, ie*

$$\frac{\partial g}{\partial \theta} < 0 \qquad \text{iff} \qquad g \le \sigma\eta \qquad \text{where} \qquad \theta = \alpha, \beta, \nu$$

which is always true in case of CRTS.

By comparison of (47') and (49') one finds the following relationship between research effort in the steady state and balanced rate of growth

$$g = \frac{\sigma\eta}{\beta}[(1-\alpha)n^* - \nu]$$

which results

$$g = g(\underset{-}{\alpha}, \underset{-}{\beta}, \underset{-}{\nu}, \underset{+}{\sigma}, \underset{+}{\eta}, \underset{+}{n^*})$$

The rate of knowledge accumulation at the steady state can be obtained by substituting n* from (47) into (14') as

$$g_B = \sigma\eta n^* = \sigma\eta(\beta - \delta + \varepsilon\nu) - \frac{\beta\rho}{\varepsilon(\beta+\nu)-\delta} \tag{50}$$

which in case of CRTS reduces to

$$g_B = \sigma\eta(\beta + \varepsilon\nu) - \frac{\beta\rho}{\varepsilon(\beta+\nu)} \tag{50'}$$

5.2 The existence and uniqueness

As we considered on the balanced growth path *C*, *K* and *Y* grow with the same rate, *n* does not change and *R* and *S* decline with the same rate too.Thus the dynamical analysis can be done on the four variables *Y/K*, *C/K*, *n* and *R/S* whose rates of growth on the steady state must be zero.

According to optimal growth equations, from (29') and (5') we have

$$g_{Y/K} = \frac{1}{1-\beta-\nu}[(1-\alpha)\sigma\eta + (1-\alpha-\beta-\nu)\frac{C}{K}] + (\alpha-1)\frac{Y}{K} \tag{47}$$

Moreover from (23) and (5') we obtain

$$g_{C/K} = \frac{\alpha-\varepsilon}{\varepsilon}\frac{Y}{K} + \frac{C}{K} - \frac{\rho}{\varepsilon} \tag{48}$$

Now (47), (48), (30) and (31) define the growth rates of fundamental variables of the model in a system of equations, which due to Cobb-Douglas functional form of production function is linear respect to those variables itself:

$$\begin{bmatrix} g_{Y/K} \\ g_{C/K} \\ g_n \\ g_{R/S} \end{bmatrix} = \begin{bmatrix} \alpha-1 & 1-\dfrac{\alpha}{D_1} & 0 & 0 \\ \dfrac{\alpha-\varepsilon}{\varepsilon} & 1 & 0 & 0 \\ 0 & \dfrac{\alpha}{D_1} & \dfrac{-D_2}{\beta} & 0 \\ 0 & \dfrac{-\alpha}{D_1} & 0 & 1 \end{bmatrix} \begin{bmatrix} Y/K \\ C/K \\ n \\ R/S \end{bmatrix} - \begin{bmatrix} D_2/D_1 \\ \rho/\varepsilon \\ (\nu-1)D_2/\beta D_1 \\ D_2/D_1 \end{bmatrix} \tag{49}$$

where

$$D_1 = 1 - \beta - \nu \qquad \text{and} \qquad D_2 = (\alpha - 1)\sigma\eta$$

We call the RHS of the above system *Ax - b* , where *A* is the 4×4 matrix of coefficients,

$$x = (\frac{Y}{K}, \frac{C}{K}, n, \frac{R}{S})^T \tag{54}$$

is the vector of fundamental variables and *b* is the constant vector of parameters.

The nonzero entry in the last row of *A* indicates that *the rate of decline of resource utilization is affected by the consumption-capital ratio,* while the zero entries of the last column of *A* indicate that *the dynamics of the core of the economy is not affected by the rate of resource utilization.*

According to the previous section, the steady state will be defined as

$$SS = \{x \in \mathrm{R}_+^4 : Ax = b\}$$

So the existence and uniqueness of the steady state can be analysed by characterization of matrix of coefficient *A* for which we have

$$\det A = \frac{\alpha(1-\alpha)\sigma\eta}{\beta\varepsilon}[\frac{\alpha - \varepsilon(\beta+\nu)}{1-\beta-\nu} - 1]$$

Range of possible values of the parameters of the model indicates that A is nonsingular (ie det A $\neq$ 0) if and only if the following assumption holds

Assumption 1 $\quad \dfrac{\alpha - 1}{\beta + \nu} \neq \varepsilon - 1$

Thus we conclude the following statement

Proposition 2 *The necessary and sufficient condition for uniqueness of the steady state in the model is assumption 1. In this case the steady state is completely described by the parameters of the model by*

$$x^* = A^{-1}b \tag{55}$$

The existence of steady state requires that the RHS vector in (53), b can be generated by the columns of matrix of coefficient, A.

Obviously in the case of *CRTS*, *A* is lower triangular and we have

$$\det A = -\frac{(1-\alpha)^2 \sigma\eta}{\beta} < 0$$

Moreover if $\varepsilon \geq 1$, the *RHS* of assumption 1 becomes nonnegative while the *LHS* remains negative. Thus we conclude

Corollary 1 *The steady state is unique if the aggregate production function exhibits constant returns to scale or if the degree of risk aversion of consumers is higher than or equal to one.*

The linear form of growth equation (49) helps us to characterise the steady state in case of nonuniqueness as follows

Corollary 2 *There is a possibility for indeterminacy in the model by which we mean multiple equilibria. This possibility depends on the parameters of the model and occurs when assumption 1 violates. If A is singular, multiple equilibria are not isolated and there is a continuum of steady states. This happens because*

$$z, z' \in SS \Rightarrow \theta z + (1-\theta) z' \in SS \qquad \textit{for} \qquad 0 < \theta < 1$$

The linear dimension of the steady state is equal to the dimension of the kernel of A which defined as

$$\ker A = \{x \in \mathrm{R}^4 : Ax = 0\}$$

In addition SS and kerA are parallel. In other words

$$z \in SS \Rightarrow z + z' \in SS \qquad \textit{for all } z' \in \ker A$$

SS is different from ker A so long as b $\neq 0$.

We limit our attention in what follows to CRTS, so *A* is triangular and we can solve (49) analytically to obtain

$$\left(\frac{Y}{K}\right)^* = \frac{\sigma\eta}{\alpha} \tag{52a}$$

$$\left(\frac{C}{K}\right)^* = \frac{1}{\varepsilon}[\rho + (\varepsilon - \alpha)\frac{\sigma\eta}{\alpha}] \tag{52b}$$

$$n^* = 1 - \frac{\beta}{1-\alpha}[1 + \frac{1}{\varepsilon}(\frac{\rho}{\sigma\eta} - 1)] \tag{52c}$$

$$\left(\frac{R}{S}\right)^{*} = \frac{1}{\varepsilon}[\rho + (\varepsilon - 1)\sigma\eta] \tag{52d}$$

which are the same as what we derived before in equations (41'),(42'),(43') and (44').

Now it is the time to ask how central planner sets the control variables at initial position. The best set of control variables for him is the values that equate x_0 with x^* where the elements of x^* have been described by the RHS of equations 52(a-d). Thus given K_0, B_0 and S_0 and the parameters of the model, to avoid the social cost of being out of steady state, social planner should solve the Eq. (52a) at time of planning so as

$$\left(\frac{B_0}{K_0}\right)^{1-\alpha}\left(\frac{\beta}{1-\alpha}\right)^{\beta}\left(1-\frac{\sigma\eta-\rho}{\sigma\eta\varepsilon}\right)^{\beta}\left(\sigma\eta-\frac{\sigma\eta-\rho}{\varepsilon}\right)^{v} S_0{}^{v} = \frac{\sigma\eta}{\alpha}$$

or simply

$$\left(\frac{B_0}{K_0}\right)^{1-\alpha} S_0{}^{v} = cste$$

One degree of freedom is required to equate the LHS of the above equation with the RHS which is given by parameters of the model. Aghion and Howitt(1998, pp164), by taking K_0 as a function of B_0 and S_0, avoid the analysis of transitional dynamics in their model and conclude that the optimal balanced growth path can be reached instantaneously from the chosen initial position. We consider instead the initial value of physical capital, as well as intellectual and natural capital, as given to the central planner. According to this argument *in general being on the steady state is unlikely and the economy described by this model is almost always on the transitional path toward the steady state.*

According to (11') and (52a) now we can derive the ratio of intellectual to tangible capital in the steady state as

$$(\frac{K}{B})^{*} = [\frac{\alpha}{\sigma\eta}(1-n^{*})^{\beta} R^{*v}]^{\frac{1}{1-\alpha}}$$

Equation (52a) shows that the marginal productivity of physical capital at the steady state is equal to the percentage rate of increase of knowledge per researcher:

$$\left(\frac{\partial Y}{\partial K}\right)^{*} = \sigma\eta$$

Furthermore (52a) implies

$$r^{*} = \alpha^{2}\left(\frac{Y}{K}\right)^{*} = \alpha\sigma\eta$$

By plugging the steady state research effort n* from (52c) into (40) one can derive a closed form for the balanced rate of growth as follows

$$g = \frac{\sigma\eta - \rho}{\varepsilon} \tag{53}$$

which is the same as (45'). For g to be positive it is necessary to have $\rho < \sigma\eta$. In addition for g to be less than r* we should have[13] $\sigma\eta - \rho < \sigma\eta\alpha\varepsilon$. We combine these inequalities by assuming

Assumption 2 $0 < \sigma\eta - \rho < \sigma\eta\alpha\varepsilon$

We show in the following that assumption 2 is the exact condition required for the steady state to be feasible.

Proposition 3 *Assumption 2 is the necessary and sufficient condition for sustainability of the economy (ie for g > 0) in the presence of an exhaustible resource which is essential for production[14], and also for g to be less than r*. Moreover beside output-capital ratio which based on the range of the parameters of the model is positive, assumption 2 is the necessary and sufficient condition for research effort to be at the steady state in its acceptable range(ie 0 < n* < 1). In addition the second part of that assumption (ie σ η - ρ < σ η α ε) is sufficient for both the consumption-capital ratio and the rate of utilization of the resource to be positive.*

Eq. (52a) and (52b) imply an expression for consumption-output ratio and saving rate, $s = \dot{K}/Y$,whose characteristics are described below:

Corollary 3 *In the steady state, consumption-output ratio is constant and we will have*

$$\left(\frac{C}{Y}\right)^* = 1 - \frac{\alpha}{\sigma\eta}\left(\frac{\sigma\eta - \rho}{\varepsilon}\right)$$

This, considering (53), implies $C^* = (1 - \alpha g / \sigma\eta)Y^*$ *where g is the balanced rate of growth of the economy.*

The portion of output which is consumed in the steady state is a function of the parameter of the model as follows:

$$\left(\frac{C}{Y}\right)^* = \varphi(\underset{-}{\alpha}, \underset{+}{\varepsilon}, \underset{+}{\rho}, \underset{+}{\sigma}, \underset{+}{\eta})$$

Moreover assuming assumption 2, we will obtain $1 - \alpha^2$ *as a lower bound for consumption-output ratio. In addition, saving has a positive growth effect as* $g = (\sigma\eta / \alpha)s$ *. Thus higher the productivity of*

---

[13] When ε < 1, this is also the sufficient condition for W in (15) to be bounded (see footnote 10).

[14] Obviously the condition for sustained growth would be stronger in the presence of population growth or capital depreciation. See Helpman(1992) for details.

R&D, or lower the physical capital share in aggregate output, higher is the growth effect of saving in the model.

By comparison of g in (53) and g_B in (14') we find that g is less than g_B if we have

$$n* > \tilde{n} := \frac{1}{\sigma\eta}\left(\frac{\sigma\eta - \rho}{\varepsilon}\right)$$

But from (52c)

$$n* = 1 - \frac{\beta}{1-\alpha}(1 - \tilde{n})$$

which implies

$$\frac{1 - n*}{1 - \tilde{n}} = \frac{\beta}{1-\alpha}$$

Since $\nu > 0$, the RHS is less than one. This implies $\tilde{n} < n*$ which results

$$g < \sigma\eta n^* = \frac{\sigma\eta\nu}{1-\alpha} + \frac{\beta}{1-\alpha}\left(\frac{\sigma\eta - \rho}{\varepsilon}\right)$$

We conclude then

Corollary 4 *In the presence of exhaustible resource (ie when $\nu > 0$), and assuming assumption 2 , rate of balanced growth is less than rate of knowledge accumulation. This implies that in the steady state the supply of intermediate goods are declining without bound.*

From the Cobb-Douglas form of the aggregate production function in (11) we know that both intermediate goods *x*, and rate of extraction of the resources *R*, are essential for production. In (38) we conclude that in the steady state *R* is declining without bound. In the above we found the same result for *x*. Furthermore according to (10),(31),(33),(35) and (39) rate of decline of both variables are proportional in the steady state:

$$g_x = \frac{\nu}{1-\alpha}\, g_R$$

This indicates that *integrating exhaustible resource into the aggregate production is harmful for sustainability of the growth directly through its declining rate of extraction and also indirectly via the supply of intermediate goods.*

In next section we investigate the dynamical properties of the model around the steady state and its transitional path toward it.

6 Dynamical analysis

The purpose of this section is to study the dynamical properties of the model. We investigate firstly the stability of the model by linearization of the dynamical system describing the optimal growth paths around the steady state. We will study then the transitional path of the economy toward the steady state.

6.1 Stability

In this section we suppose either assumption 1 or its stronger version by assuming CRTS or $\varepsilon > 1$. Hence we assume that x^* ,described in (51), is unique. In this case the local dynamical properties of the model around x^* will be the same as the following system of linear differential equation

$$\frac{d}{dt}(x - x^*) = A^*(x - x^*) \tag{54}$$

where A^* is the Jacobian matrix of the dynamical system described in (49) evaluated at x^*. A simple manipulation shows that

$$\det A^* = \left(\frac{Y}{K}\right)^* \left(\frac{C}{K}\right)^* n^* \left(\frac{R}{S}\right)^* \det A$$

By assuming assumption 2 we conclude that both det A^* and det A are of the same sign. But only based on the assumption 1 we cannot determine the sign of det A and so to take the argument one step further in the remainder we assume CRTS. In this case A^* like A is lower triangular and we can explicitly determine its eigenvalues as follows

$$\lambda(A^*) = \{\ (\alpha - 1)\left(\frac{Y}{K}\right)^*,\ \left(\frac{C}{K}\right)^*\ ,\ \frac{(1-\alpha)\sigma\eta}{\beta} n^*\ ,\ 1\ \}$$

or after substitution from (52a-d)

$$\lambda(A^*) = \{\ \frac{(\alpha - 1)\sigma\eta}{\alpha}\ ,\ \frac{\sigma\eta}{\alpha} - \left(\frac{\sigma\eta - \rho}{\varepsilon}\right),\ \frac{\nu\sigma\eta}{\beta} + \frac{\sigma\eta - \rho}{\varepsilon}\ ,\ 1\ \}$$

based on assumption 2, A* has one negative eigenvalue $\lambda_1 = (\alpha - 1)\sigma\eta/\alpha$ and three positive eigenvalues: $\lambda_2, \lambda_3, \lambda_4$. The eigenvalues are real but the positive eigenvalues are not necessarily distinct. So there is a possibility for overshooting (see Hirsch and Smale 1974, ch.6 for details), but as will be described in the follows, we can set the initial conditions in such a way that the dynamics of the system becomes more tractable.

The solution of the linearized system (54) can be written in the general form as

$$x_t - x^* = \sum_{i=1}^{4} c_i \Gamma_i t^{\delta_i} e^{\lambda_i t} \tag{55}$$

where Γ_i is the eigenvector corresponds to eigenvalue λ_i, δ_i is a nonnegative integer less than multiplicity of λ_i, and c_i is the constant of integration. Considering initial conditions, although B_0, K_0 and S_0 are given we can freely determine C_0 , n_0 and R_0 and thus Y_0. Hence we can choose x_0 in such a way that $x_0 - x*$ has no component in the subspace generated by $\Gamma_i\ (i = 2, 3, 4)$. This implies $c_i = 0$ for i = 2 , 3 , 4. By replacing these constraints in (55), the stable solution can be characterised as

$$x_t - x^* = (x_0 - x^*) e^{\frac{(\alpha-1)\sigma\eta}{\alpha} t}$$

Thus the rate of convergence of the economy toward the steady state is determined solely by α, σ and η. As we considered, in $\mathbb{R}^4$ the only subspace generated by Γ_1 is stable, so we have a one-dimensional stable saddle path converging to the steady state x^*.

6.2 Transitional dynamics

To analyse the transitional dynamics of the model in the phase space, one should derive the locus $\dot{x}_i = 0$ of the components of the vector *x*. Assuming *CRTS*, from (49) the dynamical system describing the behaviour of *x* is as follows

$$\frac{d}{dt}\left(\frac{Y}{K}\right) = (\alpha - 1)\left(\frac{Y}{K}\right)\left[\left(\frac{Y}{K}\right) - \frac{\sigma\eta}{\alpha}\right] \tag{56a}$$

$$\frac{d}{dt}\left(\frac{C}{K}\right) = \left(\frac{C}{K}\right)\left[\frac{\alpha - \varepsilon}{\varepsilon}\left(\frac{Y}{K}\right) + \left(\frac{C}{K}\right) - \frac{\rho}{\varepsilon}\right] \tag{56b}$$

$$\frac{dn}{dt} = n\left[\frac{C}{K} + \frac{(1-\alpha)\sigma\eta}{\beta}\left(n + \frac{\nu - 1}{\alpha}\right)\right] \tag{56c}$$

$$\frac{d}{dt}\left(\frac{R}{S}\right) = \frac{R}{S}\left[\frac{R}{S} - \frac{C}{K} - \frac{(\alpha - 1)\sigma\eta}{\alpha}\right] \tag{56d}$$

As we see the dynamics of (*Y/K*) and (*C/K*) can be described respectively in one and two dimensions, but the dynamics of *n* and (*R/S*) cannot be analytically described in phase plane. The transitional dynamics of output-capital ratio is depicted in figure 3. As we see beside the origin which is a trivial steady state, $(Y/K)^* = \sigma\eta/\alpha$ is the stable equilibrium of the differential equation described in (56a). We see from (12") and (23) that $r = \alpha^2 Y/K$ and $g_C = (\alpha Y/K - \rho)/\varepsilon$ are linear transformations of (*Y/K*). So the transitional dynamics of r and g_C are the same as (*Y/K*) except that they converge to ασ
and (ση - ρ) / ε respectively.

Figure 3, in here

By assuming ρ < ση in assumption 2, we ensure that in the long run output, consumption and physical capital will grow with a positive (and constant) rate. Sustainability, interpreted as non-negative rate of growth of per capita consumption, requires that α Y/K ≥ρ. Now if $(Y/K)_0 < \rho/\alpha$, then for a finite time (0< t ≤T), g_C would be negative (see figure 3) and intergenerational equity a la Solow(1974b) violates, where

$$T = \min\{t : (Y / K)_t = \rho / \alpha\}$$

Note that to be on the stable saddle path, for every given value of K_0, there is a unique C_0. Although C_t is decreasing for 0< t ≤T, but (C/K) is increasing toward its long rum target (C/K)*.

The transitional dynamics of (*C/K*), based on the equation (56a-b) is depicted in figure 4. As we see $((Y/K)^*,(C/K)^*)$ is saddle stable. Beside this steady state, and the origin which is the trivial equilibrium, there are two attractors in the phase diagram: $(\sigma\eta/\alpha, 0)$ and $(\sigma\eta/\alpha, +\infty)$. The former is intertemporally inefficient and the latter is not feasible. The slope of the saddle path is positive (negative) if ε is greater (less) than α.

Figure 4, in here

6.3 Numerical illustration

To illustrate the working of the model, a handy example is given in this section. We should emphasis however that it is not a calibration exercise but to show how model works. For this reason we select a set of plausible values for technology parameters: $\alpha = 0.5$, $\beta = 0.25$, $\nu = 0.25$; preference parameters: $\varepsilon = 2$, $\rho = 0.005$; and R&D parameters: $\sigma = 0.1$, $\eta = 1$. So we have assumed *CRTS* to ensure the uniqueness of the steady state. Moreover we have set the degree of risk aversion higher than one. We call the above set of values *basic setting*.

variables	Basic Setting (BS)	BS but $\alpha = .6$, $\beta = \nu = .2$ (2)	BS but $\alpha = .65$, $\beta = .2$, $\nu = .15$	BS but $\rho = .075$	(2) but $\rho = .075$
$(Y/K)^*$	0.20	0.167	0.154	0.20	0.167
$(C/K)^*$	0.175	0.142	0.129	0.1875	0.154
$(C/Y)^*$	0.875	0.85	0.8375	0.9375	0.925
n^*	0.625	0.625	0.571	0.5625	0.5625
$(R/S)^*$ (%)	7.5	7.5	8.75	8.75	8.75
r^* (%)	5	6	4.2	5	6
$\alpha\ (Y/K)^*$	0.10	0.10	0.10	0.10	0.10
g (%)	2.5	2.5	2.5	1.25	1.25
g_B (%)	6.25	6.25	5.7	5.625	5.625
$(1-\alpha)\sigma\eta / \alpha$ (%)	10	6.67	5.38	10	6.67
$g_R = g_S$ (%)	- 7.5	-7.5	-8.75	- 8.75	- 8.75

The Table shows the size of the variables of the model in the basic setting and some of the other settings which are described there. We have also drawn the dynamics of the fundamental variables around their steady state values and based on the basic setting in figure (5-8). These figures represent the numerical solutions of the differential equations described in equations 56(a-d). The phase diagram of the output-capital ratio, and those of the output-capital and consumption-capital ratios are depicted in figures 5 and 6 respectively. Indeed, figures 5 and 6 are the numerical versions of figures 3 and 4 respectively. The dynamics of output-capital and consumption-capital ratios and research effort is depicted in figure 7. In addition the dynamics of output-capital and consumption-capital ratios and rate of utilization of the resource is depicted in figure 8. Figures 7 and 8 are three dimensional phase diagrams.

Figures 5-8 , in here

7 Conclusion and possible extensions

We conclude our findings as follows:

• Innovation is always profitable and the profit is distributed to innovators and households who lend capital for producing intermediate goods. The parameter that measures the contribution of capital in the aggregate production function α, indicates the share of researchers and households in the profit. Higher α, higher is the competition in the product market and higher households will benefit from lending capital.

• The share of natural resource in the production function has a positive and demanding effect on the steady state amount of research effort. Moreover in the presence of natural resource a minimum level of research effort is required to overcome the declining trend of the exhaustible resource use.

• In the absence of population growth, an economy with exhaustible resources is sustainable if increases in the ratio of intellectual capital to the tangible capital can compensate the finiteness of the essential resources.

• The necessary and sufficient condition for uniqueness of the nontrivial steady state in the model is derived. As a result the steady state is unique if the aggregate production function exhibits constant returns to scale or if the degree of risk aversion of consumers is higher than or equal to one. In either cases the steady state is completely described by the parameters of the model.

• There is a possibility for indeterminacy in the model by which we mean multiple equilibria. This possibility depends on the parameters of the model. Multiple equilibria are not however isolated and there is a continuum of steady states in case of nonuniqueness.

• Concerning Aghion and Howitt(1998, pp164)'s argument, we show that given the level of physical capital, public knowledge and natural asset at the initial stage of planning, jumping instantaneously to the steady state is unlikely and the economy described by this model is almost always on its transitional path toward the steady state.

• The steady state is one dimensional saddle stable. The number of given and free variables in the model ensures that by correct selection of initial values for rate of consumption , research effort and rate of extraction of the resource, localization of the economy on the stable saddle path is always possible. Rate of convergence of the economy on the stable saddle path is also determined by parameters of the model.

• Under plausible assumptions the steady state is feasible. The condition is also valid for sustainability and for balanced rate of growth to be less than rental rate of capital.

• In the presence of exhaustible resource, the balanced rate of growth is lower than rate of knowledge accumulation. This implies that in the steady state the supply of intermediate goods is declining without bound. Higher the rate of utilization of the resource or higher the resource contribution in the final product, lower is the balanced rate of growth. This may provide a justification for the argument that natural resource intensity is harmful for growth. Thus integrating exhaustible resource into the aggregate production adversely affects sustainability of the growth directly through its declining rate of extraction and indirectly via its effect on the supply of intermediate goods.

• In the steady state, the stock of natural resources and its rate of utilization both decline with a common rate which is inversely related with the consumption-capital ratio.

• The optimal rate of depletion of resource is derived which incorporates the Hotelling's rule as a special case.

• Although the dynamics of the rate of utilization of the resource is affected by the fundamentals of the model, it does not by itself affect the rest of the economy.

• The balanced rate of growth is an increasing function of the productivity of research activities, the degree of return to scale, and a decreasing function of the willingness to consume. It also decreases when the contribution of each of the factors of production increases if and only if the long run rate of growth does not exceed the incremental value of public knowledge per researcher which is true in case of constant returns to scale.

• Investment in physical capital has a positive growth effect. Higher the productivity of R&D, or lower the physical capital share in aggregate output, higher is the growth effect of saving in the model.

• Although by imposing sustainability condition , introduced by Aghion and Howitt , growth on the steady state is sustained ,but we provide conditions upon which it is optimal for consumption per capita to display negative growth rate on the transition for a finite period.

We now address some debates concerning the model described here. They can probably lead to some possibilities for its extensions.

1) In the exogenous growth framework, Hartwick(1977) establishes that investing the extracted resources in the form of building new physical capital, provides a sustainable growth path for an economy which constrained by finite essential resource. Is there a counterpart for this policy in the Shumpeterian framework? What is the Hartwick's rule of knowledge accumulation to compensate finiteness of the resources and to provide the sustainability of the economy?

2) A comprehensive measure of capital (including physical, intellectual, and natural) can be introduced and the time pattern of share of each type of capital in the comprehensive measure (in particular transformation of natural capital to physical and intellectual one) can be monitored through time. What is the optimum combination of three kinds of capital in the portfolio of social planner? And how the composition of national wealth, i.e $B + K + S$, evolves through time?

3) What is the result if we compare the findings of this model with those of the neo-classical and *AK* endogenous growth models in the presence of exhaustible resources? In particular, how we can address intergenerational justice, a la Rawls, in this context?

4) Another possible extension is to consider the step size between innovations as an endogenous variable, by allocating a portion of the extracted resource to education and by assuming that σ is a

positive function of the level of education and historically determined state of man power. This is slightly similar to one of the extension of the basic Shumpeterian model which takes the productivity of research activity, η as positive function of *n*.

5) Considering the possibility of population growth in the model, allow us to examine the effect of the size of the economy on the results, and analyse sustainability in per capita terms.

6) Does K and B are complementary or substitutable? What is the answer of the similar question about S , K and B? See Krautkraemer(1998) for a discussion about whether S and K are complementary or substitutable. Chichilnisky and Heal(1983) also investigate the effect of energy price on the substitutability of K and S.

7) One should also extend the model to a small open economy where r is constant and rate of return on accumulated capital does not tend to zero.

References

Aghion,P. and P.Howitt(1992)"A model of growth through creative destruction" , *Econometrica* , 60(2) ,323-51.

----------------------------(1998) *Endogenous Growth Theory* The MIT Press, Cambridge, Mass.

Barbier, E.B.(1999),"Endogenous growth and natural resource scarcity",*Environment and Resource Economcs*,14,51-74.

Bhandari,J.S. and B.H.Putnam(eds) *Economic Interdependence and Flexible Exchange Rates* 1983 The MIT Press, Cambridge, Mass.

Buiter,W.H. and D.D.Purvis(1983),"Oil, disinflation and export competitiveness: A model of the Dutch disease", in Bhandari and Putnam(eds),pp:221-47.

Chiarella,C.(1980),"Optimal depletion of a nonrenewable resource when technological progress is endogenous", in M.C.Kemp and N.v.Long(eds) *Exhaustible Resources, Optimality and Trade* Amsterdam et al, North Holland Publishing Company.

Chichilnisky,G.(1985),"International trade in resources: A general equilibrium analysis", reprinted in Heal(ed), pp:339-89.

----------------- and G.Heal(1983),"Energy-capital substitution: A general equilibrium analysis ", reprinted in Heal(ed), pp:390-401.

Dasgupta,P. (1974)"Some recent theoretical explorations in the Economics of Exhaustible Resources",in H.Gottinger(ed) *Systems Approaches and Environmental Problems* 193-214.Reprinted in Heal(ed).

-------------- and G.Heal (1974)"The optimal depletion of exhaustible resources",*Reviews of Economic Studies*, Symposium on the Economics of Exhaustible Resources,3-28.

-----------------------------(1979) *Economic Theory and Exhaustible Resources* Cambridge University Press, Cambridge.

Farzin,Y.H(1999),"Optimal saving policy for exhaustible resource economies", *Journal of Development Economics*,58,149-84.

Foreman-Peck,J.(1995) *A History of the World Economy* 2nd ed, Prentice Hall.

Forrester,J.W(1971) *World Dynamics* Wrigth-Allen Press, Inc. Cambridge, Mass.

Grimaud,A(1998),"Implementation de l'optimum dans un models Shumpeterian en presence d'une resource non renouvelable",*GREMAQ Working Paper* no.98.20.507,Toulose, France.

Grossman,G.M. and E.Helpman(1991) *Innovation and Growth in the Global Economy* The MIT Press, Cambridge, Mass.

Hartwick,J.M(1977),"Intergenerational equity and the investing of rents from exhaustible resources", *American Economic Review*, 66,972-4.

Heal,G. (ed) *The Economics of Exhaustible Resources* Edward Elgar Publishing Limited,1993.

-------- (1993)"The optimal use of Exhaustible Resources",in A.Kneese and J.Sweeney(eds) *Handbook of Natural Resource and Energy Economics* vol III,Elsevier Science Publishers, the Netherlands.

Hellinell,J.F. et al.(1982),"Stabilization, allocation and the 1970s oil price shocks", *Scandinavian Journal of Economics*, 84(2),259-88.

Helpman,E.(1992),"Endogenous macroeconomic growth theory", *European Economic Review*, 36, 237-67.

Hirsch,M.W and S.Smale(1974) *Differential Equations, Dynamical Systems, and Linear Algebra* Academic Press.

Kamien,M.I. and N.L.Schwartz(1978),"Optimal exhaustible resource depletion with endogenous technical change", *Reviews of Economic Studies*, 35, 179-96.

Krautkraemer,J.A(1998),"Nonrenewable resource scarcity", *Journal of Economic Literature*,36,2065-2107.

Krugman,P.(1983),"Oil and the Dollar", in Bhandari and Putnam(eds), pp:179-190.

Liewelwyn(1983),"Resource price and macroeconomic policies: Lessons from two oil price shocks", reprinted in Heal(ed), pp:461,478.

Lucas,R.(1988),"On the mechanics of economic development", *Journal of Monetary Economics*, 22, 3-22.

Marion,N.P. and L.E.O.Svensson(1984),"World equilibrium with oil price increases: An intertemporal analysis", *Oxford Economic Papers*, 36:86-102.

Meadows,D.H., D.L.Meadows, J.Randers, and W.W.Behrens III (1972) *The Limits to Growth* Earth Island, London.

Meadows,D.L, W.W.Behrens III, D.M.Meadows, R.F.Naill, J.Randers and E.K.O.Zahn(1974) *Dynamics of Growth in a Finite World* Wright-Allen Press,Inc. Cambridge, Mass.

Nordhaus,W.(1980),"Oil and economic performance in Industrial countries", reprinted in Heal(ed), pp:402-460.

Romer,P.(1986),"Increasing returns and long-run growth",*Journal of Political Economy*, 94,1002-37.

---------- (1990)"Endogenous technological change",*Journal of Political Economy*,98(5),S71-S102.

Sachs,J.D.(1981),"The current account and macroeconomic adjustment in the 1970s",*Brooking Papers on Economic Activity*,1,201-68.

------------(1983),"Energy and growth under flexible exchange rates: A simulation study", in Bhandari and Putnam(eds), pp:191-220.

------------ and A.M.Warner(1995),"Natural resource abundance and economic growth",*NBER Working Paper* 5398.

Scholz,C. and G.Ziemes(1999)"Exhaustible resources, monopolistic competition, and endogenous growth", *Environmental & Resource Economics*,13(2),169-85.

Solow,R.M.(1974a)"The economics of resources or the resources of economics", *American Economic Review,* Papers and Proceedings 64(2),1-14.

-------------- (1974b)"Intergenerational equity and exhaustible resources",*Review of Economic Studies* , Symposium on the Economics of Exhaustible Resources,29-45.

-------------- (1999),"Neoclassical growth theory", in Taylor,J.B. and M.Woodford (eds) *Handbook of Macroeconomics* , Vol 1, Elsevier.

Stiglitz,J(1974a)"Growth with exhaustible natural resources: efficient and optimal growth paths",*Review of Economic Studies,*Symposium,123-37.

--------- (1974b)"Growth with exhaustible natural resources: the competitive economy",*Review of Economic Studies* , Symposium,139-52.

Suzuki,H.(1976)"On the possibility of steadily growing per capita consumption in an economy with a wasting and non-replenishable resource", *Review of Economic Studies* ,33,527-35.

Toman,M.A, J.Pezzey and J.Krautkraemer(1995),“ Neoclassical economic growth theory and sustainability", in Bromley,D.W.(ed) *The Handbook of Environmental Economics* Blackwell Publishers.

Tornell,A. And P.R.Lane(1998),"Are windfalls a curse? A non-representative agent model of the current account", *Journal of International Economics*,44,83-112.

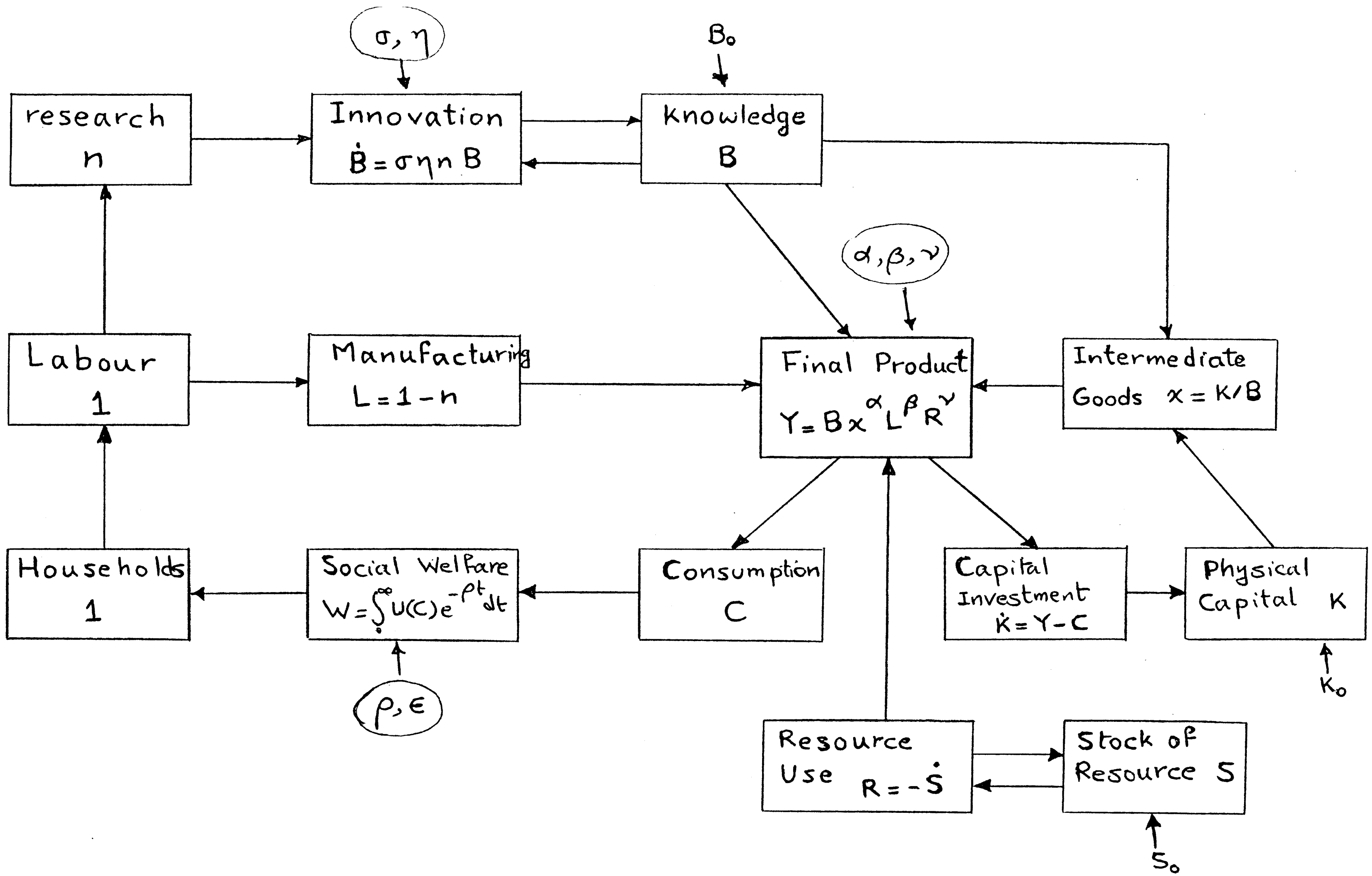

Figure 1. A schematic representation of economic activities in the model.

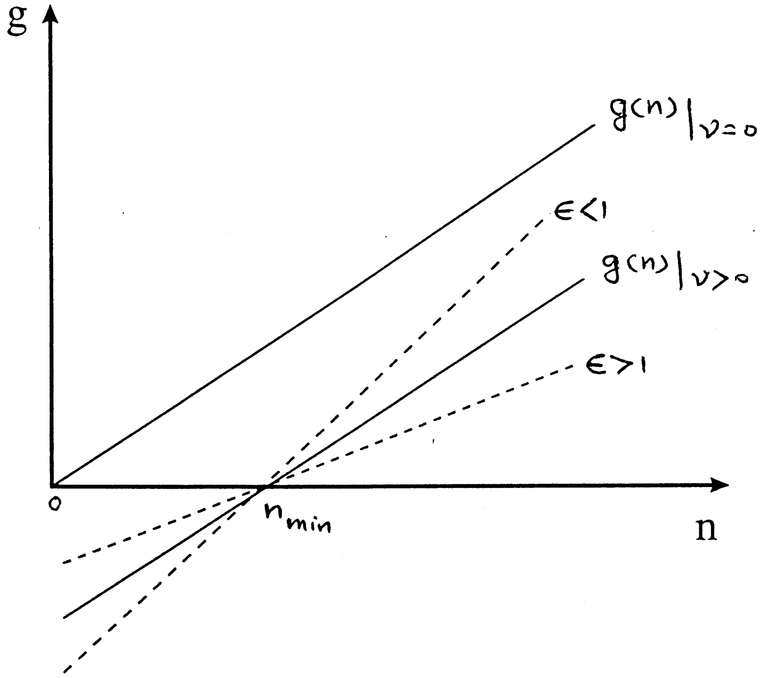

Figure 2. The optimal long run growth rate (g) as a function of research effort (n).

$$g = \frac{(1-\alpha)\sigma\eta n - \rho\nu}{(1-\alpha) + (\epsilon-1)\nu}$$

$$n_{min} = \frac{\rho\nu}{(1-\alpha)\sigma\eta} > 0$$

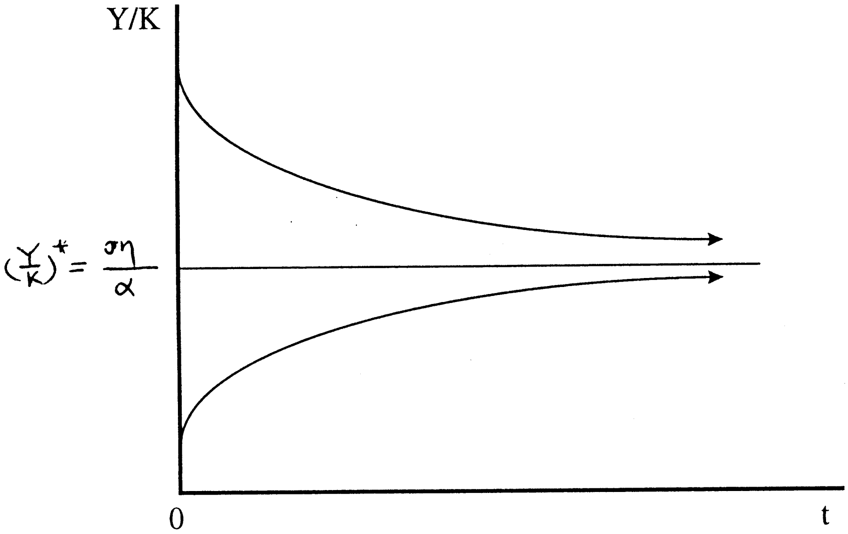

Figure 3. The dynamics of output-capital ratio.

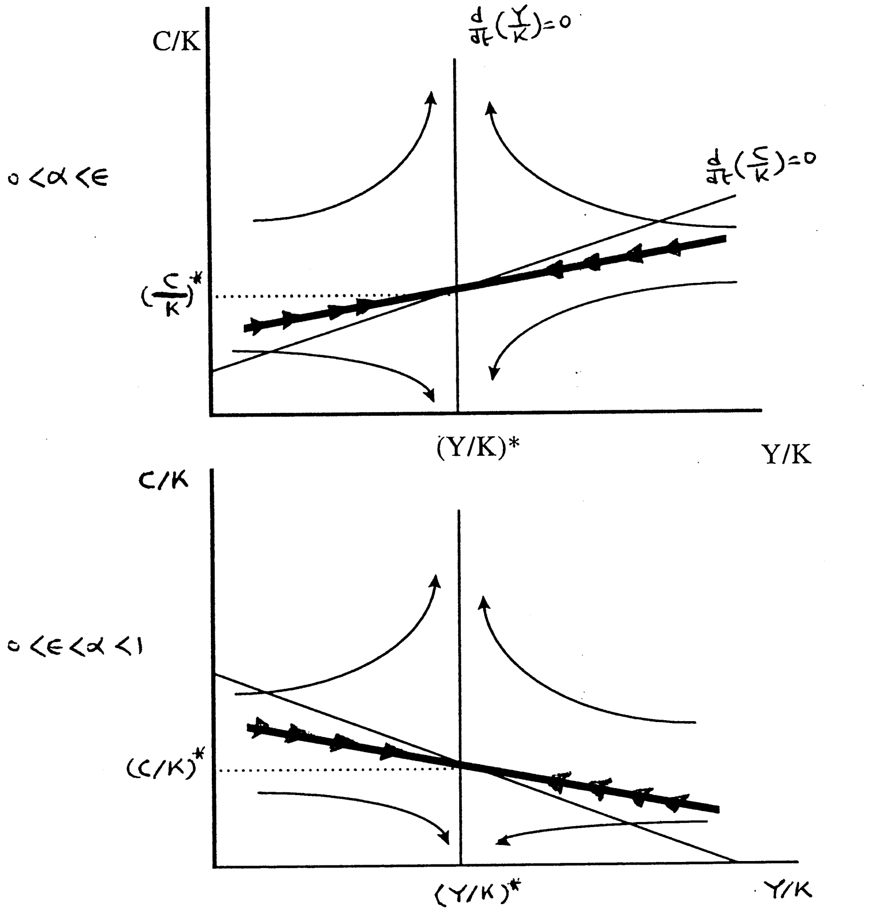

Figure 4. The dynamics of output-capital ration and Consumption-capital ratio. The slope of saddle path depends on whether $\alpha > \epsilon$ or not.

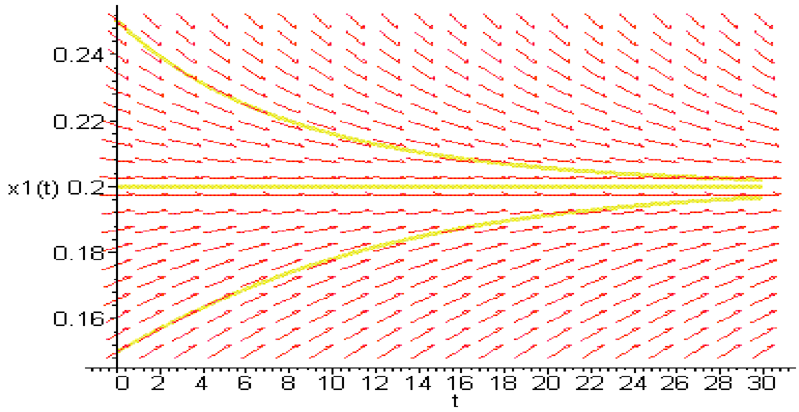

Figure 5. The dynamics of output-capital ratio, under basic setting.

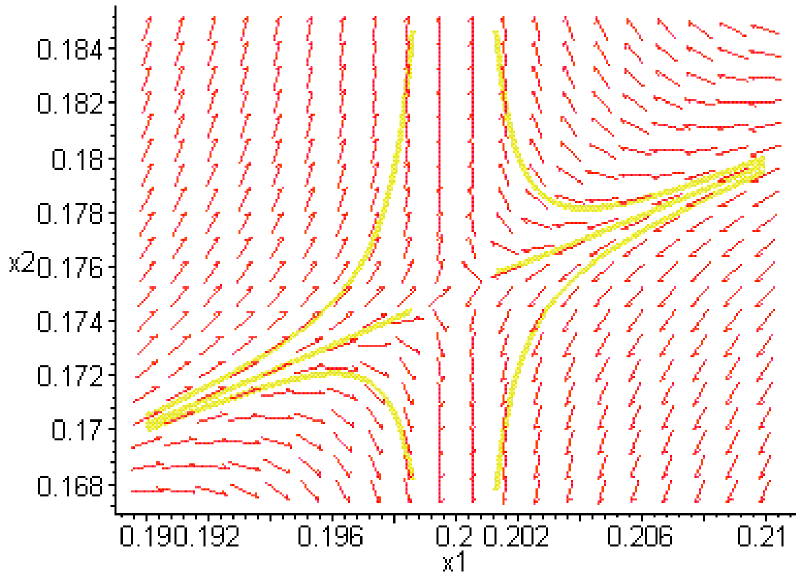

Figure 6. The dynamics of output-capital ratio(x1) and consumption-capital ratio(x2),under basic setting.

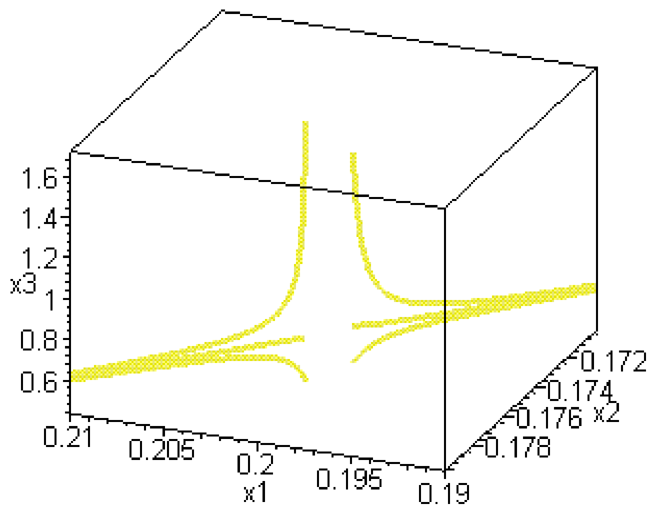

Figure 7. The dynamics of output-capital ratio(x1), consumption-capital ratio(x2), and research effort(x3), under basic setting.

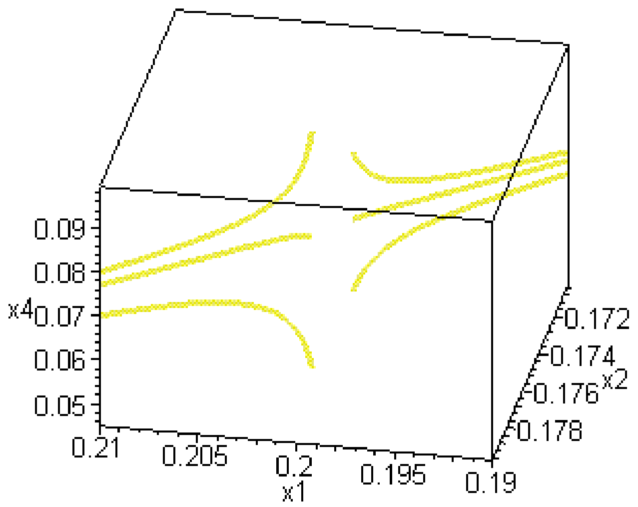

Figure 8. The dynamics of output-capital ratio(x1), consumption-capital ratio(x2), and rate of utilization of the resource(x4), under basic setting.