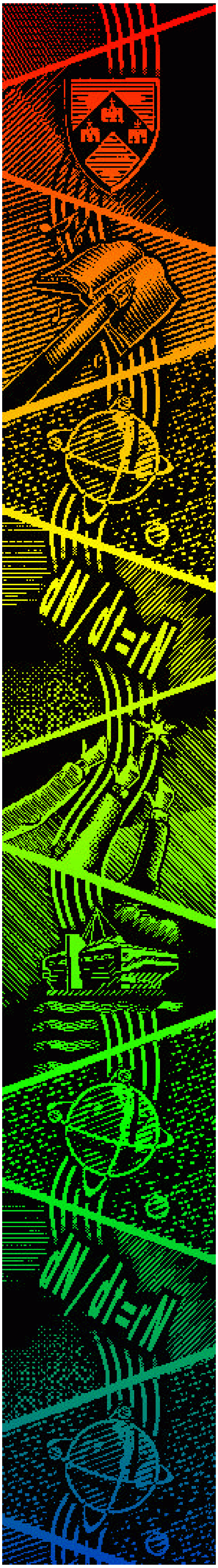

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Some Exact Formulae for the Constant Correlation and Diagonal M - Garch Models

by

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SOME EXACT FORMULAE FOR THE CONSTANT CORRELATION AND DIAGONAL M-GARCH MODELS

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Abstract

The purpose of this paper is to examine the covariance structure of multivariate GARCH (M-GARCH) models that have been introduced in the literature the last fifteen years, and have been greatly favoured by time series analysts and econometricians. In particular, we analyze the second moments of the constant conditional correlation M-GARCH model introduced by Bollerslev (1990) and the diagonal M-GARCH model introduced by Bollerslev, Engle and Wooldridge (1988).

Key Words: Autocovariance Generating Function, ARMA representations, Diagonal Multivariate GARCH.

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1 INTRODUCTION

One of the most popular class of non-linear processes are the generalized autoregressive conditional heteroscedasticity (GARCH) models[1]. The existence of the huge literature which uses these processes in modelling conditional volatility in high frequency financial assets demonstrates the popularity of the various GARCH models (see, for example, the surveys by Bollerslev, Chou and Kroner, 1992, Bera and Higgins, 1993, Bollerslev, Engle and Nelson, 1994, Diebold and Lopez, 1995, Palm, 1996, Shephard, 1996 and Pagan, 1996; see also the book by Engle, 1995, and the book by Gourieroux, 1997 for a detail discussion of the GARCH models and financial applications).

As economic variables are inter-related, generalisation of univariate models to the multivariate set-up is quite natural-this is more so for the GARCH models[2]. From the many different multivariate functional forms the diagonal Multivariate GARCH (M-GARCH) model originally suggested by Bollerslev, Engle and Wooldridge (1988), hereafter BEW, and the constant conditional correlation (ccc) M-GARCH model put forward in B (1990), have become perhaps the most common.

In particular, various cases of the diagonal representation of the M-GARCH model (with various mean specifications) have been applied by many researchers. For example, it has been used by BEW(1988) for their analysis of returns on bills, bonds and stocks, by Engel and Rodrigues (1989) to test the international CAPM, by Kaminsky and Peruga (1990) to examine the risk premium in the forward market for foreign exchange, by McCurby and Morgan (1991) to test the uncovered interest rate parity, and by Baillie and Myers (1991) to estimate optimal hedge ratios in commodity markets.

B (1990) illustrated the validity of the ccc M-GARCH model for a set of five nominal European U.S. dollar exchange rates following the inception of the European Monetary System. The ccc M-GARCH model has also been used by Cecchetti, Cumby and Figlewski (1988) to estimate the optimal future hedge, McCurby and Morgan (1989) to examine risk premia in foreign currency futures market, Schwert and Seguin (1990) and Karolyi (1995) to analyze stock returns, Baillie and Bollerslev (1990) to model risk premia in forward foreign exchange rate markets, Kroner and Claessens (1991) to analyze the optimal currency composition of external debt, Ng (1991) to test the CAPM, and Kroner and Sultan (1993) to estimate futures hedge ratios.

Although the M-GARCH models were introduced almost fifteen years ago and have been widely used in empirical applications, their statistical properties have only recently been examined by researchers[3]. However, the analysis of the covariance structure of the various M-GARCH models has not been considered yet. This article attempts to fill in this gap in the GARCH literature. The focus will be on the fourth moment of the errors and on the theoretical auto/cross covariance functions of the conditional variances/covariances. In this context, the paper generalizes the results for the univariate GARCH model given in Karanasos[4] (2000a), hereafter K, to M-GARCH models.

In Section 2 we contribute to the theoretical developments in the M-GARCH literature by presenting the covariance structure of the conditional variances/covariances for the ccc M-GARCH model. In Section 3 we present the theoretical auto/cross covariance functions of the conditional variances/covariances for the diagonal M-GARCH model[5]. The goal of this article is to provide a comprehensive methodology for the analysis of the covariance structure in M-GARCH models. First, it provides the univariate

[1] The ARCH model was originally proposed by Engle (1982), whereas Taylor (1986) and Bollerslev (1986), hereafter B, independently of each other, presented the generalised ARCH model.

[2] The first paper on M- GARCH models was written by Engle, Granger and Kraft (1984). They used a bivariate ARCH(1) process to combine forecasts in two models of US inflation.

[3] Engle and Kroner (1995) examined the identification and maximum likelihood estimation of the vech, diagonal and BEKK representations of the M-GARCH model. Lin (1997), hereafter L, provided a comprehensive analytical tool for the impulse response analysis for all the aforementioned representations of the M-GARCH model. Tse (1998) developed the Lagrange multiplier test for the hypothesis of constant correlation in Bollerslev's representation, whereas Jeantheau (1998) and Ling and McAleer (1999) investigated the asymptotic theory of the quasi maximum likelihood estimator for an extension of the ccc M-GARCH model.

[4] Karanasos(2000a) obtained the autocovariances of the conditional variance for the GARCH- in-mean model. Fountas, Karanasos and Karanassou (2000) did that for the GARCH-in-mean-level model. Karanasos(1999) derived the autocovariances of the squared errors for the simple GARCH and the N Component GARCH(1,1) model (for the simple GARCH model, see also He and Terasvirta, 1999). Finally, Karanasos(2000b) obtained the auto/cross covariances of the conditional variances and the squared errors for the N Component GARCH(n,n) model.

[5] Karanasos, Psaradakis and Sola (1999), hereafter KPS, examined a sum of GARCH processes which follow a diagonal M-GARCH (S-GARCH) model and applied it to option pricing (see also Zaffaroni, 1999).

ARMA representations of the conditional variances/covariances and it gives general conditions for stationarity, and irreducibility of these representations. Second, it uses the canonical factorization (cf) of the auto/cross covariance generating functions (agf/cgf) of these ARMA representations to obtain the auto/cross covariances of the conditional variances/covariances. Finally, it gives the conditions for the existence of the second moments of the conditional variances/covariances. It should be noted that we only examine the case of distinct roots in the AR polynomials of the ARMA representations and we express the auto/cross covariances in terms of the roots of the AR polynomials and the parameters of the MA polynomials of the ARMA representations.

2 Constant Correlation M-GARCH Model

In what follows we will examine the p-th order ccc M-GARCH(r_i^*, s_i) [M_cGARCH(r_i^*,s_i,p)] model ($i = 1, \ldots, p$).

We have p error terms and each of these terms follows a GARCH(r_i^*,s_i) process

$$\varepsilon_{it} = h_{it}^{\frac{1}{2}} e_{it},\ \varepsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{pt} \end{bmatrix},\ H_t = \begin{bmatrix} h_{1,t} & \cdots & h_{1p,t} \\ \cdots & \cdots & \cdots \\ h_{1p,t} & \cdots & h_{p,t} \end{bmatrix} \tag{2.1}$$

$$\widehat{B}_i(L) h_{it} = \omega_i + A_i(L)\varepsilon_{it}^2,\ \ \widehat{B}_i(L) = 1 - \sum_{l=1}^{r_i^*} \widehat{\beta}_{il} L^l,\ \ A_i(L) = \sum_{l=1}^{s_i} a_{il} L^l \tag{2.2}$$

where H_t denotes the conditional variance-covariance matrix of the errors ($\sigma_\varepsilon^2 | F_{t-1} = H_t$) and $h_{ij,t}$ denotes the conditional covariance between the ε_{it} and the ε_{jt} error terms [$h_{ij,t} = cov_{t-1}(\varepsilon_{it}, \varepsilon_{jt})$].

In addition, the e_{it} ($i = 1, \ldots, p$) terms follow the multivariate normal distribution

$$e_t = \begin{bmatrix} e_{1t} \\ \vdots \\ e_{pt} \end{bmatrix},\ \ e_t \sim IIN(0, \sigma_e^2),\ \ \sigma_e^2 = \begin{bmatrix} \sigma_{1,e} & \cdots & \sigma_{1p,e} \\ \cdots & \cdots & \cdots \\ \sigma_{p1,e} & \cdots & \sigma_{p,e} \end{bmatrix},\ \ \sigma_{ij,e} = cov(e_{it}, e_{jt}) = p_{ij} \tag{2.3}$$

Finally, the conditional correlation between the errors is constant

$$\frac{cov_{t-1}(\varepsilon_{it}, \varepsilon_{jt})}{\sqrt{var_{t-1}(\varepsilon_{it})(\varepsilon_{jt})}} = \frac{h_{ij,t}}{\sqrt{h_{it} h_{jt}}} = p_{ij} \tag{2.4}$$

Note that the conditional variance-covariance matrix (H_t) is positive semidefinite when $\omega_i,\ a_{ik},\ \widehat{\beta}_{il} \geq 0$ for $i = 1, \ldots, p$, $k = 1, \ldots s_i$, $l = 1, \ldots r_i^*$.

Corollary 2.1 *The ARMA representations of the conditional variances (h_{it}) are*

$$B_i(L) h_{it} = \omega_i + A_i(L) v_{it},\ \ v_{it} = \varepsilon_{it}^2 - h_{it},\ i = 1, \ldots, p \tag{2.5a}$$

$$B_i(L) = \widehat{B}_i(L) - A_i(L) = 1 - \sum_{l=1}^{r_i} \beta_{il} L^l = \prod_{l=1}^{r_i} (1 - \lambda_i L), \tag{2.5b}$$

$$r_i = \max(r_i^*, s_i),\ \ \beta_{il} = \begin{cases} \widehat{\beta}_{il} + a_{il} & \text{if } r_i^*, s_i > l \\ \widehat{\beta}_{il} & \text{if } r_i^*, l > s_i \\ a_{il} & \text{if } s_i, l > r_i^* \end{cases} \tag{2.5c}$$

In addition, the covariances between the v_{it} and the v_{jt} terms are

$$E(v_{it}) = 0,\ \ cov(v_{it}, v_{j,t-k}) = 0,\ \ \sigma_{ij,v} = cov(v_{it}, v_{jt}) = 2p_{ij}^2 E(h_{it} h_{jt}),\ \ i, j = 1, \ldots, p \tag{2.6}$$

where the second moments of the conditional variances [$E(h_{it} h_{jt})$] are given in Proposition 2.1.

Since the v_{it} terms are uncorrelated, equation (2.5a) gives the ARMA(r_i, s_i) representations of the conditional variances (h_{it}).

Proof. In the right hand side of (2.2) we add and subtract $A_i(L)h_t$ and we get (2.5a). The proof of (2.6) is given in the Appendix. ■

Assumption 1. The roots of the autoregressive polynomials $B_i(L)$ lie outside the unit circle (Covariance-Stationary conditions).

Assumption 2. The polynomials $B_i(L)$ and $A_i(L)$ are left coprime. In other words the representation $\frac{A_i(L)}{B_i(L)}$ is irreducible.

In what follows we only examine the case where the roots of the autoregressive polynomials $B_i(L)$ are distinct ($\lambda_{il} \neq \lambda_{ik}$, for $l \neq k$).

Theorem 2.1 *Under assumptions 1 and 2 the auto/cross covariance generating functions of the conditional variances are*

$$g_{ij,z} = \frac{A_i(z)A_j(z^{-1})\sigma_{ij,v}}{B_i(z)B_j(z^{-1})} = \sum_{m=-\infty}^{\infty} cov(h_{it}, h_{j,t-m})z^m, \quad cov(h_{it}, h_{j,t-m}) = \begin{cases} \gamma_{ij}^m \sigma_{ij,v} & m \geq 0 \\ \gamma_{ji}^m \sigma_{ij,v} & m \leq 0 \end{cases}, \tag{2.7a}$$

$$\gamma_{ij}^m = \sum_{l=1}^{r_i} \varsigma_{il,j}^m \Phi_{l,m}^{ij} + \sum_{k=1}^{r_j} \varsigma_{jk,i}^m \Phi_{ij}^{km}, \quad \varsigma_{il}^m = \frac{\lambda_{il}^{r_i-1+m}}{\prod_{\substack{f=1 \\ f\neq l}}^{r_i} (\lambda_{il} - \lambda_{if})}, \quad \varsigma_{il,j}^m = \frac{\varsigma_{il}^m}{\prod_{f=1}^{r_j} (1 - \lambda_{il}\lambda_{jf})}, \tag{2.7b}$$

$$\Phi_{l,m}^{ij} = \sum_{c=0}^{s_j} \sum_{d=1}^{s_i'} a_{id}a_{j,d+c}\lambda_{il}^c + \sum_{c=1}^{m^*} \sum_{d=1}^{s_j'} a_{jd}a_{i,d+c}\lambda_{il}^{-c}, \quad \Phi_{ij}^{km} = \sum_{c=m^*+1}^{s_i} \sum_{d=1}^{s_j'} a_{jd}a_{i,d+c}\lambda_{jk}^{c-2m}, \quad m \geq 0, \tag{2.7c}$$

$$s_i' = \min(\mathrm{s}_i, s_j - c), \quad s_j' = \min(\mathrm{s}_j, s_i - c), \quad m^* = \min(\mathrm{m},\mathrm{s}_i), \tag{2.7d}$$

The proof of Theorem 2.1 is given in the Appendix.

Proposition 2.1 *The first and second moments of the conditional variances are given by*

$$E(h_{it}) = \frac{\omega_i}{B_i(1)}, \quad i = 1, \ldots, p, \tag{2.8a}$$

$$E(h_{it}h_{jt}) = \frac{E(h_{it})E(h_{jt})}{1 - 2p_{ij}^2\gamma_{ij}^0}, i, j = 1, \ldots, p \tag{2.8b}$$

where γ_{ij}^0 is defined by (2.7b). Note that (2.8b), when i=j, gives the second moment of the i-th conditional variance. The condition for the existence of the second moments [E($\mathrm{h}_{it}h_{jt}$)] is $2p_{ij}^2\gamma_{ij}^0 < 1$.

The proof of Proposition 2.1 is given in the Appendix.

Example 1. Consider the p-th order ccc M-GARCH model where the i-th ($1 \leq i \leq p$) conditional variance (h_{it}) follows a GARCH(2,2) process and the j-th ($1 \leq j \leq p, \quad j \neq i$) conditional variance ($h_{jt}$) follows a GARCH(2,1) process. Let the ARMA representations of the two GARCH processes be

$$\begin{aligned} (1 - \lambda_{i1}L)(1 - \lambda_{i2}L)h_{it} &= \omega_i + a_{i1}v_{i,t-1} + a_{i2}v_{i,t-2}, \quad \lambda_{i1}, \lambda_{i2} < 1, \quad \lambda_{i1} \neq \lambda_{i2}, \\ (1 - \lambda_{j1}L)(1 - \lambda_{j2}L)h_{jt} &= \omega_j + a_{j1}v_{j,t-1}, \quad \lambda_{j1}, \lambda_{j2} < 1, \quad \lambda_{j1} \neq \lambda_{j2} \end{aligned}$$

The cross covariances between h_{it} and h_{jt} are

$$cov(h_{it}, h_{j,t-m}) = \begin{cases} [\frac{\lambda_{i1}^{m+1}[a_{i1}a_{j1}+a_{j1}a_{i2}\lambda_{i1}^{-1}]}{(\lambda_{i1}-\lambda_{i2})(1-\lambda_{i1}\lambda_{j1})(1-\lambda_{i1}\lambda_{j2})} + \frac{\lambda_{i1}^{m+1}[a_{i1}a_{j1}+a_{j1}a_{i2}\lambda_{i2}^{-1}]}{(\lambda_{i2}-\lambda_{i1})(1-\lambda_{i2}\lambda_{j1})(1-\lambda_{i2}\lambda_{j2})}]2p_{ij}^2 E(h_{it}h_{jt}), & m > 0 \\ [\frac{\lambda_{j1}^{|m|+1}[a_{i1}a_{j1}+a_{j1}a_{i2}\lambda_{j1}^{1}]}{(\lambda_{j1}-\lambda_{j2})(1-\lambda_{i1}\lambda_{j1})(1-\lambda_{j1}\lambda_{i2})} + \frac{\lambda_{j2}^{|m|+1}[a_{i1}a_{j1}+a_{j1}a_{i2}\lambda_{j2}^{1}]}{(\lambda_{j2}-\lambda_{j1})(1-\lambda_{i1}\lambda_{j2})(1-\lambda_{j2}\lambda_{i2})}]2p_{ij}^2 E(h_{it}h_{jt}), & m \leq 0 \end{cases},$$

$$E(h_{it}h_{jt}) = \frac{E(h_{it})E(h_{jt})}{1 - 2p_{ij}^2\gamma_{ij}^0}, \quad E(h_{it}) = \frac{\omega_i}{(1 - \lambda_{i1})(1 - \lambda_{i2})}, \quad E(h_{jt}) = \frac{\omega_i}{(1 - \lambda_{j1})(1 - \lambda_{j2})},$$

$$\gamma_{ij}^0 = \frac{1 + (\lambda_{j1} + \lambda_{j2}) + \lambda_{j1}\lambda_{j2}(\lambda_{i1} + \lambda_{i2} - \lambda_{i1}\lambda_{i2})}{(1 - \lambda_{j1}\lambda_{i1})(1 - \lambda_{j1}\lambda_{i2})(1 - \lambda_{j2}\lambda_{i1})(1 - \lambda_{j2}\lambda_{i2})}$$

3 Diagonal M-GARCH Model

In what follows we will examine the p-th order diagonal M-GARCH(r^*_{ij}, s_{ij}) [MGARCH(r^*_{ij}, s_{ij}, p)] model (i,j= $1, \ldots, p$).

We have p error terms

$$\varepsilon_{it} = h_{it}^{\frac{1}{2}} e_{it}, \ \varepsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{pt} \end{bmatrix}, \ H_t = \begin{bmatrix} h_{11,t} & \cdots & h_{1p,t} \\ \cdots & \cdots & \cdots \\ h_{1p,t} & \cdots & h_{pp,t} \end{bmatrix}, \tag{3.1}$$

where H_t denotes the conditional variance-covariance matrix of the errors ($\sigma^2_\varepsilon | F_{t-1} = H_t$) and $h_{ij,t}$ denotes the conditional covariance between the ε_{it} and the ε_{jt} error terms [$h_{ij,t} = cov_{t-1}(\varepsilon_{it}, \varepsilon_{jt})$]. Each conditional variance/covariance follows a GARCH(r^*_{ij}, s_{ij}), process ($i, j = 1, \ldots, p$)

$$\widehat{B}_{ij}(L) h_{ij,t} = \omega_{ij} + A_{ij}(L) \varepsilon_{it} \varepsilon_{jt}, \ \ \widehat{B}_{ij}(L) = 1 - \sum_{l=1}^{r^*_{ij}} \widehat{\beta}^l_{ij} L^l, \ \ A_{ij}(L) = \sum_{l=1}^{s_{ij}} a^l_{ij} L^l \tag{3.2}$$

In addition, the e_{it} ($i = 1, \ldots, p$) terms follow the multivariate normal distribution

$$e_t = \begin{bmatrix} e_{1t} \\ \vdots \\ e_{pt} \end{bmatrix}, \quad e_t \sim IIN(0, \sigma^2_e), \quad \sigma^2_e = \begin{bmatrix} \sigma_{11,e} & \cdots & \sigma_{1p,e} \\ \cdots & \cdots & \cdots \\ \sigma_{p1,e} & \cdots & \sigma_{pp,e} \end{bmatrix}, \ \ \sigma_{ij,e} = cov(e_{it}, e_{jt}) = p_{ij} \tag{3.3}$$

The conditions on the parameters to ensure that the variance-covariance matrix (H_t) is positive semidefinite are given in Attanasio (1991).

Corollary 3.1 *The ARMA representations of the conditional variances/covariances ($h_{ij,t}$) are*

$$\begin{aligned} B_{ij}(L) h_{ij,t} &= \omega_{ij} + A_{ij}(L) v_{ij,t}, \ \ v_{ij,t} = \varepsilon_{it}\varepsilon_{jt} - h_{ij,t}, \ i, j = 1, \ldots, p && (3.4a) \\ B_{ij}(L) &= \widehat{B}_{ij}(L) - A_{ij}(L) = 1 - \sum_{l=1}^{r_{ij}} \beta^l_{ij} L^l = \prod_{l=1}^{r_{ij}} (1 - \lambda^l_{ij} L) && (3.4b) \\ r_{ij} &= \max(r^*_{ij}, s_{ij}), \ \ \beta^l_{ij} = \begin{cases} \widehat{\beta}^l_{ij} + a^l_{ij} & \text{if } \ r^*_{ij}, s_{ij} > l \\ \widehat{\beta}^l_{ij} & \text{if } \ r^*_{ij}, l > s_{ij} \\ a^l_{ij} & \text{if } \ s_{ij}, l > r^*_{ij} \end{cases} && (3.4c) \end{aligned}$$

In addition, the covariances between the $v_{ij,t}$ and the $v_{nk,t}$ terms are

$$\begin{aligned} E(v_{ij,t}) &= 0, \ \ cov(v_{ij,t}, v_{nk,t-m}) = 0, \ \ \ i, j, n, k = 1, \ldots, p && (3.5a) \\ \sigma^v_{ij,nk} &= cov(v_{ij,t}, v_{nk,t}) = E(h_{ik,t} h_{jn,t}) + E(h_{in,t} h_{jk,t}), && (3.5b) \end{aligned}$$

where the second moments of the conditional variances/covariances [$E(h_{ik,t}h_{jn,t})$] are given in Proposition 3.1. Since the $v_{ij,t}$ terms are uncorrelated, equation (3.4a) gives the ARMA(r_{ij}, s_{ij}) representations of *the conditional variances/covariances.*

Proof. The proof of Corollary 3.1 is similar to that of Corollary 2.1. ∎

Assumption 1. The roots of the autoregressive polynomials $B_{ij}(L)$ lie outside the unit circle (Covariance-Stationary conditions).

Assumption 2. The polynomials $B_{ij}(L)$ and $A_{ij}(L)$ are left coprime. In other words the representation $\frac{A_{ij}(L)}{B_{ij}(L)}$ is irreducible.

In what follows we only examine the case where the roots of the autoregressive polynomials $B_{ij}(L)$ are distinct ($\lambda_{ij,l} \neq \lambda_{ij,k}$, for $l \neq k$).

Theorem 3.1 *Under assumptions 1 and 2 the auto/cross covariance generating functions of the conditional variances/covariances are*

$$g^z_{ij,nk} = \frac{A_{ij}(z)A_{nk}(z^{-1})\sigma^v_{ij,nk}}{B_{ij}(z)B_{nk}(z^{-1})} = \sum_{m=-\infty}^{\infty} cov(h_{ij,t}, h_{nk,t-m})z^m, \quad (3.6a)$$

$$cov(h_{ij,t}, h_{nk,t-m}) = \begin{cases} \gamma^{ij,nk}_m \sigma^v_{ij,nk} & m \geq 0 \\ \gamma^{nk,ij}_m \sigma^v_{ij,nk} & m \leq 0 \end{cases}, \quad (3.6b)$$

$$\gamma^{ij,nk}_m = \sum_{l=1}^{r_{ij}} \varsigma^{lm}_{ij,nk}\Phi^{ij,nk}_{l,m} + \sum_{f=1}^{r_{nk}} \varsigma^{fm}_{nk,ij}\Phi^{f,m}_{ij,nk}, \quad \varsigma^{lm}_{ij,nk} = \frac{\varsigma^m_{ij,l}}{\prod_{f=1}^{r_{nk}}(1-\lambda^l_{ij}\lambda^f_{nk})}, \quad \varsigma^m_{ij,l} = \frac{(\lambda^l_{ij})^{r_{ij}-1+m}}{\prod_{f=1, f\neq l}^{r_{ij}}(\lambda^l_{ij}-\lambda^f_{ij})},$$

$$\Phi^{ij,nk}_{l,m} = \sum_{c=0}^{s_{nk}}\sum_{d=1}^{s'_{ij}} a^d_{ij}a^{d+c}_{nk}(\lambda^l_{ij})^c + \sum_{c=1}^{m^*}\sum_{d=1}^{s'_{nk}} a^d_{nk}a^{d+c}_{ij}(\lambda^l_{ij})^{-c}, \quad m \geq 0,$$

$$\Phi^{f,m}_{ij,nk} = \sum_{c=m^*+1}^{s_{ij}}\sum_{d=1}^{s'_{nk}} a^d_{nk}a^{d+c}_{ij}(\lambda^f_{nk})^{c-2m}, \quad m \geq 0,$$

$$s'_{ij} = \min(\mathrm{s}_{ij}, s_{nk}-c), \quad s'_{nk} = \min(\mathrm{s}_{nk}, s_{ij}-c), \quad m^* = \min(\mathrm{m}, \mathrm{s}_{ij}), \quad i,j,n,k = 1,\ldots,p$$

Proof. The proof of Theorem 3.1 is similar to that of Theorem 2.1. ■

Proposition 3.1. *The first and second moments of the conditional variances/covariances are*

$$E(h_{ij,t}) = \frac{\omega_{ij}}{B_{ij}(1)}, \quad i,j,n,k = 1,\ldots,p, \quad (3.7a)$$

$$E(h_{ij,t}h_{nk,t}) = \frac{\gamma_{ij,nk}E(h_{ij,t})E(h_{nk,t}) + \gamma_{in,jk}E(h_{in,t})E(h_{jk,t}) + \gamma_{ik,jn}E(h_{ik,t})E(h_{jn,t})}{1-\gamma_{ij-nk}}, \quad (3.7b)$$

$$\gamma_{ij-nk} = \gamma^{ij,nk}_0\gamma^{in,jk}_0 + \gamma^{ij,nk}_0\gamma^{ik,jn}_0 + \gamma^{in,jk}_0\gamma^{ik,jn}_0 - 2\gamma^{ij,nk}_0\gamma^{in,jk}_0\gamma^{ik,jn}_0, \quad (3.7c)$$

$$\gamma_{ij,nk} = 1-\gamma^{in,jk}_0\gamma^{ik,jn}_0, \quad \gamma_{in,jk} = \gamma^{ij,nk}_0(1+\gamma^{in,jk}_0), \quad \gamma_{ik,jn} = \gamma^{ij,nk}_0(1+\gamma^{ik,jn}_0) \quad (3.7d)$$

Note that, when $k = n = j = i$, equations (3.7b)-(3.7d) give

$$E(h^2_{ii,t}) = \frac{[E(h_{ii,t})^2]}{1-2\gamma^{ii,ii}_0}$$

and, when $n = i, k = j$, equations (3.7b)-(3.7d give

$$E(h^2_{ij,t}) = \frac{(1+\gamma^{ij,ij}_0)\{[E(h_{ij,t})]^2 + E(h_{it})E(h_{jt})\gamma^{ij,ij}_0\}}{1-\gamma^{ij,ij}_0[2\gamma^{ii,jj}_0 + \gamma^{ij,ij}_0 + 2\gamma^{ij,ij}_0\gamma^{ii,jj}_0]}$$

The proof of Proposition 3.1 is given in the Appendix.

Example 2. Consider the p-th order diagonal M-GARCH(1,1) model where the ij-th (i,j=1,… ,p) conditional variances/covariances follow GARCH(1,1) processes. Let the ARMA representations of these GARCH(1,1) processes be

$$(1-\lambda_{ij}L)h_{ij,t} = \omega_{ij} + a_{ij}v_{ij,t-1}, \quad \lambda_{ij} < 1, \quad i,j = 1,\ldots,p,$$

The cross covariances between $\mathrm{h}_{ij,t}$ and $\mathrm{h}_{nk,t}$ (i,j,n,k=1,… ,p) are

$$cov(h_{ij,t}, h_{nk,t-m}) = \begin{cases} \frac{(\lambda_{ij})^m a_{ij}a_{nk}}{1-\lambda_{ij}\lambda_{nk}}[E(h_{ik,t}h_{jn,t}) + E(h_{in,t}h_{jk,t})] & m \geq 0 \\ \frac{(\lambda_{nk})^{|m|} a_{ij}a_{nk}}{1-\lambda_{ij}\lambda_{nk}}[E(h_{ik,t}h_{jn,t}) + E(h_{in,t}h_{jk,t})] & m \leq 0 \end{cases},$$

where the second moments of the conditional variances/covariances are given by (3.7b) and the first moments of the conditional variances/covariances and the γ_0's are given by

$$
\begin{aligned}
E(h_{ij,t}) &= \frac{\omega_{ij}}{1-\lambda_{ij}}, \quad E(h_{in,t}) = \frac{\omega_{in}}{1-\lambda_{in}}, \quad E(h_{ik,t}) = \frac{\omega_{ik}}{1-\lambda_{ik}}, \\
E(h_{nk,t}) &= \frac{\omega_{nk}}{1-\lambda_{nk}}, \quad E(h_{jn,t}) = \frac{\omega_{jn}}{1-\lambda_{jn}}, \quad E(h_{jk,t}) = \frac{\omega_{jk}}{1-\lambda_{jk}}, \\
\gamma_0^{ij,nk} &= \frac{a_{ij}a_{nk}}{1-\lambda_{ij}\lambda_{nk}}, \quad \gamma_0^{in,jk} = \frac{a_{in}a_{jk}}{1-\lambda_{in}\lambda_{jk}}, \quad \gamma_0^{ik,jn} = \frac{a_{ik}a_{jn}}{1-\lambda_{ik}\lambda_{jn}}
\end{aligned}
$$

4 Conclusions

This paper has contributed to the theoretical developments in the multivariate GARCH literature. In Section 2 we presented the auto/cross covariances of the conditional variances for the constant correlation Multivariate GARCH model. In Section 3 we presented the auto/cross covariance generating function of the conditional variances/covariances for the diagonal Multivariate GARCH model. For both these Multivariate GARCH models we also gave the conditions for the existence of the second moments of the conditional variances/covariances. The technique used in this paper (i.e. the autocovariance generating function of the ARMA representations of the conditional variances/covariances) can be applied to obtain the covariance structure of more complex multivariate GARCH models like the BEKK one.

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A APPENDIX

PROOF OF COROLLARY 2.1

The covariances between the v_{it} and the v_{jt} terms are

$$cov(v_{it}, v_{jt}) = E(v_{it}v_{jt}) = E(\varepsilon_{it}^2 - h_{it})(\varepsilon_{jt}^2 - h_{jt}),\ i,j = 1,\ldots,p$$

In the above equation using

$$\varepsilon_{it}^2 = h_{it}^{\frac{1}{2}} e_{it},\quad E(e_{it}^2 e_{jt}^2) = 1 + 2p_{ij}^2,\quad E(e_{it}^2) = E(e_{jt}^2) = 1$$

we get

$$\sigma_{ij,v} = cov(v_{it}, v_{jt}) = E(h_{it}h_{jt})E[e_{it}^2 e_{jt}^2 - e_{it}^2 - e_{jt}^2 + 1] = E(h_{it}h_{jt})2p_{ij}^2.$$

When i=j the above equation gives $var(v_{it}) = 2E(h_{it}^2)$. ■

PROOF OF THEOREM 2.1

>From (2.5b) we get

$$\frac{1}{B_i(z)B_j(z^{-1})} = \frac{1}{\prod_{l=1}^{r_i}(1-\lambda_{il}z)\prod_{l=1}^{r_j}(1-\lambda_{jl}z^{-1})} = \sum_{k=1}^{r_j}\sum_{l=1}^{r_i}\frac{\varsigma_{il}^0\varsigma_{jk}^0}{(1-\lambda_{il}z)(1-\lambda_{jk}z^{-1})} \tag{A.1}$$

>From (2.2) we get

$$\begin{aligned} A_i(z)A_j(z^{-1}) &= (\sum_{f=1}^{s_i} a_{if}z^f)(\sum_{f=1}^{s_j} a_{jf}z^{-f}) = \sum_{l=0}^{s_i}\sum_{k=1}^{s_j'} a_{jk}a_{i,k+l}z^l + \sum_{l=1}^{s_j}\sum_{k=1}^{s_i'} a_{ik}a_{j,k+l}z^{-l}, \\ s_j' &= \min(s_j, s_i - l),\ s_i' = \min(s_i, s_j - l) \end{aligned} \tag{A.2}$$

where ς_{il}^0 is defined by (2.7b). From the preceding equations we have

$$\frac{A_i(z)A_j(z^{-1})}{(1-\lambda_{il}z)(1-\lambda_{jk}z^{-1})} = \frac{\varsigma_{il}^0\varsigma_{jk}^0}{(1-\lambda_{il}\lambda_{jk})}\sum_{m=0}^{\infty} f_m[(\Phi_{l,m}^{ij}\lambda_{il}^m + \Phi_{ij}^{km}\lambda_{jk}^m)z^m + (\Phi_{km}^{ji}\lambda_{jk}^m + \Phi_{ji}^{lm}\lambda_{il}^m)z^{-m}]$$

where $\Phi_{l,m}^{ij}$, and Φ_{ij}^{km} are given by (2.7c). Next, using

$$\sum_{k=1}^{r_j}\frac{\varsigma_{jk}^0}{1-\lambda_{il}\lambda_{jk}} = \frac{1}{\prod_{k=1}^{r_j}(1-\lambda_{il}\lambda_{jk})}$$

and (A.1)-(A.2) we obtain

$$\begin{aligned} \frac{A_i(z)A_j(z^{-1})}{B_i(z)B_j(z^{-1})} &= \sum_{k=1}^{r_j}\sum_{l=1}^{r_i}\sum_{m=0}^{\infty}\frac{\varsigma_{il}^0\varsigma_{jk}^0}{(1-\lambda_{il}\lambda_{jk})}[(\Phi_{l,m}^{ij}\lambda_{il}^m + \Phi_{ij}^{km}\lambda_{jk}^m)z^m + (\Phi_{km}^{ji}\lambda_{jk}^m + \Phi_{ji}^{lm}\lambda_{il}^m)z^{-m}] = \\ &= \sum_{m=0}^{\infty} f_m[(\sum_{l=1}^{r_i}\Phi_{l,m}^{ij}\varsigma_{il,j}^m + \sum_{k=1}^{r_j}\Phi_{ij}^{km}\varsigma_{jk,i}^m)z^m + (\sum_{k=1}^{r_j}\Phi_{k,m}^{ji}\varsigma_{jk,i}^m + \sum_{l=1}^{r_i}\Phi_{ji}^{lm}\varsigma_{il,j}^m)z^{-m}] = \\ &= \sum_{m=0}^{\infty} f_m[\gamma_{ij}^m z^m + \gamma_{ji}^m z^{-m}] \end{aligned}$$

■

PROOF OF PROPOSITION 2.1

>From (2.7a) we get

$$cov(h_{it}, h_{jt}) = E(h_{it}h_{jt}) - E(h_{it})E(h_{jt}) = \gamma_{ij}^0 \sigma_{ij,v}$$

Using (2.6) the above equation gives

$$E(h_{it}h_{jt}) - E(h_{it})E(h_{jt}) = 2p_{ij}^2 \gamma_{ij}^0 E(h_{it}h_{jt}) \Rightarrow E(h_{it}h_{jt}) = \frac{E(h_{it})E(h_{jt})}{1 - 2p_{ij}^2 \gamma_{ij}^0}$$

■

PROOF OF PROPOSITION 3.1

>From (3.6b) we get

$$cov(h_{ij,t}h_{nk,t}) = \gamma_0^{ij,nk} \sigma_{ij,nk}^v$$

Using (3.5b) the above equation gives

$$E(h_{ij,t}h_{nk,t}) - E(h_{ij,t})E(h_{nk,t}) = \gamma_0^{ij,nk}[E(h_{ik,t}h_{jn,t}) + E(h_{in,t}h_{jk,t})] \tag{A.4}$$

or

$$\begin{aligned} E(h_{ik,t}h_{jn,t}) - E(h_{ik,t})E(h_{jn,t}) &= \gamma_0^{ik,jn}[E(h_{ij,t}h_{kn,t}) + E(h_{in,t}h_{jk,t})], & \text{(A.5)} \\ E(h_{in,t}h_{jk,t}) - E(h_{in,t})E(h_{jk,t}) &= \gamma_0^{in,jk}[E(h_{ij,t}h_{kn,t}) + E(h_{ik,t}h_{jn,t})] & \text{(A.6)} \end{aligned}$$

Rewriting equations (A.4)-(A.6) in a matrix form we have

$$\begin{bmatrix} 1 & -\gamma_0^{ij,nk} & -\gamma_0^{ij,nk} \\ -\gamma_0^{ik,jn} & 1 & -\gamma_0^{ik,jn} \\ -\gamma_0^{in,jk} & -\gamma_0^{in,jk} & 1 \end{bmatrix} \begin{bmatrix} E(h_{ij,t}h_{nk,t}) \\ E(h_{ik,t}h_{jn,t}) \\ E(h_{in,t}h_{jk,t}) \end{bmatrix} = \begin{bmatrix} E(h_{ij,t})E(h_{nk,t}) \\ E(h_{ik,t})E(h_{jn,t}) \\ E(h_{in,t})E(h_{jk,t}) \end{bmatrix}$$

Solving the above system of equations we get (3.7b). ■