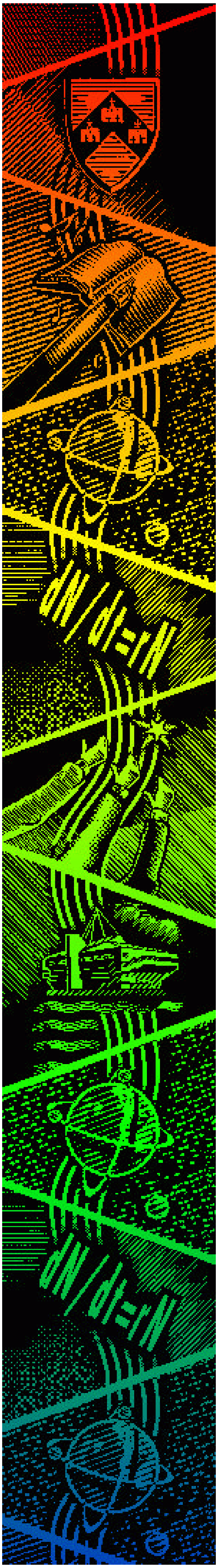

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Cross-Sectional Aggregation and Persistence in Conditional Variance

by

Menelaos Karanasos, Zacharias Psaradakis and Martin Sola

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

cross-sectional aggregation and persistence in conditional variance[¤]

Menelaos Karanasos
Department of Economics, University of York , U.K.
Zacharias Psaradakis
School of Economics, Mathematics and Statistics, Birkbeck College, U.K.
Martin Sola
School of Economics, Mathematics and Statistics, Birkbeck College, U.K.
Departamento de Economía, Universidad Torcuato di Tella, Argentina

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Abstract

This paper explores the interactions between cross-sectional aggregation and persistence of volatility shocks. We derive the ARMA-GARCH representation that linear aggregates of ARMA processes with GARCH errors admit, and establish conditions under which persistence in volatility of the aggregate series is higher than persistence in the volatility of the individual series. The practical implications of the results are illustrated empirically in the context of an option pricing exercise.

Keywords: ARMA process; Cross-sectional aggregation; GARCH process; Volatility persistence.

[¤]Correspondence to: Zacharias Psaradakis, School of Economics, Mathematics and Statistics, Birkbeck College, 7-15 Greese Street, London W1P 2LL, U.K.; fax: +44 20 76316416; e-mail: zpsaradakis@econ.bbk.ac.uk.

1 Introduction

In the last fifteen years or so, the family of Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models has been used extensively in the modelling of the conditional second moments of financial time series. Within this class of models, it is almost a 'stylized fact' that the estimated coefficients in the conditional variance function sum to very close to one, especially for high-frequency financial data. In such so-called integrated GARCH models (Engle and Bollerslev, 1986), shocks to the conditional variance are persistent in the sense that current information remains important for volatility forecasts of all horizons. To the extent that this apparent nonstationary behaviour of volatility is not the result of misspecification of the conditional variance function (cf. Diebold, 1986; Lamourex and Lastrapes, 1990; Hamilton and Susmel, 1994), it has broad implications for the construction of long-term volatility forecasts which are essential in many asset-pricing models and is also important from a theoretical point of view (see Poterba and Summers, 1986).

The purpose of this paper is to explore the interaction between cross-sectional aggregation and persistence of volatility shocks. The motivation for our work is the common empirical finding that the conditional variance of aggregate series, such as weighted market indices, typically exhibits higher degree of persistence than the conditional variance of the constituent series. To investigate this issue, we first establish the properties that simple linear aggregates of ARMA processes with GARCH errors have by deriving the ARMA-GARCH representation of the aggregate series. These results are then used to establish conditions under which the conditional variance of the aggregate process is more persistent than the conditional variance of the individual processes.

The plan of the paper is as follows. Section 2 sets out the assumptions on which our analysis is based and introduces the necessary notation. Section 3 establishes the aggregation properties of ARMA processes with GARCH errors. Section 4 gives results concerning the degree of persistence in the conditional variance of linear aggregates of ARMA-GARCH processes. Section 5 illustrates the empirical implications of our results in the context of a simple option pricing exercise. Section 6 summarizes and concludes.

2 Notation and Assumptions

Throughout the paper we consider the situation where the stochastic process $\{y_t\}$ is a simple linear aggregate of k processes $\{y_{it}\}$, i.e.

$$y_t = \sum_{i=1}^{k} w_i y_{it} \qquad (t = 0, \pm 1, \pm 2, \ldots), \tag{1}$$

where $(w_1, \ldots, w_k)$ are real constants. Each individual process $\{y_{it}\}$ is assumed to be a causal ARMA(r_i, s_i) process satisfying

$$\Phi_i(L)y_{it} = \phi_{ic} + \Theta_i(L)\varepsilon_{it} \qquad (i = 1, \ldots, k), \tag{2}$$

where

$$\Phi_i(L) = -\sum_{j=0}^{r_i} \phi_{ij} L^j = \prod_{j=1}^{r_i} (1 - \phi'_{ij} L) \qquad (\phi_{i0} = -1,\ \phi_{i,r_i} \neq 0), \tag{3}$$

$$\Theta_i(L) = -\sum_{j=0}^{s_i} \theta_{ij} L^j \qquad (\theta_{i0} = -1,\ \theta_{i,s_i} \neq 0), \tag{4}$$

ϕ'_{ij} $(j = 1, \ldots, r_i)$ are the inverse zeros of $\Phi_i(z)$ and L is the conventional lag operator. Further, the polynomials $\Phi_i(z)$ and $\Theta_i(z)$ are assumed to have no common zeros.

Regarding the innovations $\{\varepsilon_{it}\}$, we assume that they follow a diagonal multivariate GARCH process (cf. Bollerslev et al., 1988). More specifically, letting F_{t-1} denote the σ-field of events generated by $\{\boldsymbol{\varepsilon}_s = [\varepsilon_{1s}, \ldots, \varepsilon_{ks}]^\top, s \leq t-1\}$, we have

$$(\boldsymbol{\varepsilon}_t | F_{t-1}) \sim N(0, H_t), \qquad H_t = \begin{bmatrix} h_{1t} & h_{12,t} & \cdots & h_{1k,t} \\ h_{21,t} & h_{2t} & \cdots & h_{2k,t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k1,t} & h_{k2,t} & \cdots & h_{kt} \end{bmatrix}, \tag{5}$$

where each individual variance h_{it} satisfies a GARCH(p_i, q_i) model,

$$B_i(L)h_{it} = \alpha_{ic} + A_i(L)\varepsilon^2_{it} \qquad (i = 1, \ldots, k), \tag{6}$$

with

$$B_i(L) = -\sum_{j=0}^{q_i} \beta_{ij} L^j \quad (\beta_{i0} = -1) \qquad \text{and} \qquad A_i(L) = \sum_{j=1}^{p_i} \alpha_{ij} L^j. \tag{7}$$

Moreover, each covariance $h_{uv,t}$ satisfies a GARCH(p_{uv}, q_{uv}) model,

$$B_{uv}(L)h_{uv,t} = \alpha_{uv,c} + A_{uv}(L)\varepsilon_{ut}\varepsilon_{vt} \qquad (u = 1, \ldots, k-1,\ v = u+1, \ldots, k), \tag{8}$$

where

$$B_{uv}(L) = -\sum_{j=0}^{q_{uv}} \beta_{uv,j} L^j \quad (\beta_{uv,0} = -1) \qquad \text{and} \qquad A_{uv}(L) = \sum_{j=1}^{p_{uv}} \alpha_{uv,j} L^j.$$

The parameters in (6) and (8) are appropriately restricted to ensure that $\{\boldsymbol{\varepsilon}_t\}$ is stationary up to order 2 and that H_t is positive definite with probability one for all t (see Engle and Kroner, 1995).

For future development, it is helpful to note that (6) may be expressed as an ARMA(p_i^*, q_i) model,

$$B_i^*(L)h_{it} = \alpha_{ic} + A_i(L)\eta_{it}, \qquad \eta_{it} = \varepsilon^2_{it} - h_{it} \tag{9}$$

with

$$B_i^*(L) = -\sum_{j=0}^{p_i^*} \beta_{ij}^* L^j = \prod_{j=1}^{p_i^*} (1 - \beta_{ij}^0 L);$$

where β_{ij}^0 $(j = 1, \dots, p_i^*)$ are the inverse zeros of $B_i^*(z)$, $p_i^* = \max\{p_i, q_i\}$, and

$$\beta_{ij}^* = \begin{cases} \alpha_{ij} & \text{if } p_i > j > q_i \\ \beta_{ij} & \text{if } q_i > j > p_i \\ \alpha_{ij} + \beta_{ij} & \text{if } p_i, q_i > j. \end{cases}$$

In a similar fashion, (8) may be rewritten as an ARMA(p_{uv}^*, p_{uv}) model,

$$B_{uv}^*(L) h_{uv,t} = \alpha_{uv,c} + A_{uv}(L)\eta_{uv,t}; \qquad \eta_{uv,t} = \varepsilon_{ut}\varepsilon_{vt} - h_{uv,t}; \tag{10}$$

with

$$B_{uv}^*(L) = -\sum_{j=0}^{p_{uv}^*} \beta_{uv,j}^* L^j = \prod_{j=1}^{p_{uv}^*} (1 - \beta_{uv,j}^0 L);$$

where $\beta_{uv,j}^0$ $(j = 1, \dots, p_{uv}^*)$ are the inverse zeros of $B_{uv}^*(z)$, $p_{uv}^* = \max\{p_{uv}, q_{uv}\}$, and

$$\beta_{uv,j}^* = \begin{cases} \beta_{uv,j} & \text{if } q_{uv} > j > p_{uv} \\ \alpha_{uv,j} & \text{if } p_{uv} > j > q_{uv} \\ \beta_{uv} + \alpha_{uv} & \text{if } p_{uv}, q_{uv} > j \end{cases} \qquad \left(\begin{array}{l} u = 1, \dots, k-1 \\ v = u+1, \dots, k \end{array} \right).$$

It is also worth noting that the framework described by (1)-(8) includes as a special case the component model proposed in Ding and Granger (1996) for modelling persistence in volatility. Specifically, since (1)–(8) imply that

$$\text{var}(y_t | F_{t-1}) = \sum_{i=1}^{k} w_i^2 h_{it} + 2 \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_u w_v h_{uv,t};$$

it is easily seen that $\{y_t\}$ satisfies a $\frac{1}{2}k(k+1)$–component GARCH(p, q) model when $\sum_{i=1}^{k} w_i = 1$, the errors $\{\varepsilon_{it}\}$ in (2) are the same for all $i = 1, \dots, k$, $p_i = p_{uv} = p$, and $q_i = q_{uv} = q$. Hence, our framework may be seen as providing an alternative way of analyzing the long-memory characteristics of the volatility of aggregate time series.

3 Main Results

In this section we derive the ARMA-GARCH representation that the aggregate process $\{y_t\}$ admits. We also give results concerning the moments of the conditional variance $h_t = \text{var}(y_t | F_{t-1})$. All proofs are deferred to the Appendix.

Theorem 1 Under the assumptions of Section 2, $\{y_t\}$ is an ARMA($\tilde{r}, \tilde{l}$) process,

$$\Phi(L)y_t = \imath + \varepsilon_t + \Theta(L)(\varepsilon^*_t + \varepsilon_t); \tag{11}$$

$$\Phi(L) = -\sum_{i=0}^{\tilde{r}} \phi_i L^i = \prod_{i=1}^{\tilde{r}} (1 - \phi'_i L); \qquad \Theta(L) = -\sum_{i=1}^{\tilde{l}} \theta_i L^i \tag{12}$$

where $\imath$ is a constant, $\{\varepsilon_t\}$ is a GARCH($\tilde{p}, \hat{p}$) process such that $(\varepsilon_t | F_{t-1}) \sim N(0, h_t)$,

$$B^*(L)h_t = \alpha^*_0 + A^*(L)\varepsilon^2_t; \tag{13}$$

$$B^*(L) = -\sum_{i=0}^{\hat{p}} \beta^*_i L^i; \qquad A^*(L) = \sum_{i=1}^{\tilde{p}} \alpha^*_i L^i; \tag{14}$$

and $\{\varepsilon^*_t\} \overset{iid}{\sim} N(0, \sigma^2_{\varepsilon^*})$, independent of $\{\varepsilon_t\}$. In (12), $\tilde{r}$ is the number of different zeros of the k $\Phi_i(z)$ polynomials and $\tilde{l} = \max\{\tilde{l}_1, \ldots, \tilde{l}_k\}$, where $\tilde{l}_i = \tilde{r} - r_i + s_i$. In (14), $\tilde{p} = \max\{p'_i, p'_{uv}\}$ and $\hat{p} = \max\{\tilde{p}, \bar{p}\}$, where $\bar{p}$ is the number of different inverse zeros of the k $B^*_i(z)$ polynomials and of the $\frac{1}{2}k(k-1)$ $B^*_{uv}(z)$ polynomials, $p'_i = \bar{p} - p^*_i + p_i$, and $p'_{uv} = \bar{p} - p^*_{uv} + p_{uv}$. The $\sigma^2_{\varepsilon^*}$, ϕ'_i, θ_i, β^*_i and α^*_i can be found in the proof of the theorem.

The moments of the volatility process $\{h_t\}$ are given in the next theorem.

Theorem 2 The first two moments of the volatility process $\{h_t\}$ in Theorem 1 are:

$$E(h_t) \equiv f_1 = \frac{\sum_{m=1}^{k} w_m^2 \alpha_{mo} \prod_{i=1}^{\bar{p}-p^*_m} (1-\beta'_i) + 2 \sum_{u=1}^{k-1} \sum_{v=u+1}^{k} w_u w_v \alpha_{uv;c} \prod_{l=1}^{\bar{p}-p^*_{uv}} (1-\beta'_l)}{\prod_{i=1}^{\bar{p}} (1-\beta'_i)}; \tag{15}$$

$$\begin{aligned} E(h_t^2) \equiv f_2 = {} & \sum_{l=1}^{k} \sum_{i=1}^{k} w_i^2 w_l^2 f^{2;il} + 4 \sum_{i=1}^{k} \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_i^2 w_v w_u f^{2;i;uv} \\ & + 4 \sum_{s=m+1}^{k} \sum_{m=1}^{k-1} \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_s w_m w_v w_u f^{2;uv;ms}, \end{aligned} \tag{16}$$

where β'_i are the $\bar{p}$ different inverse zeros of the k $B^*_i(z)$ polynomials and of the $\frac{1}{2}k(k-1)$ $B^*_{uv}(z)$ polynomials, $\beta'_i \neq \beta'_{mj}$ $(j = 1, \ldots, p^*_m)$, $\beta'_l \neq \beta_{uv;j}$ $(j = 1, \ldots, p^*_{uv})$, $f^{2;il} \equiv E(h_{it}h_{lt})$, $f^{2;i;uv} \equiv E(h_{it}h_{uv;t})$, and $f^{2;uv;ms} \equiv E(h_{uv;t}h_{ms;t})$. Further, the unconditional kurtosis of ε_t is $\kappa_\varepsilon \equiv E(\varepsilon_t^4)/[E(\varepsilon_t^2)]^2 = 3f_2/f_1^2$.

As an illustration of how the results in Theorem 1 simplify in specific cases, we conclude this section by giving two relatively simple examples. The first example considers a linear aggregate of two ARMA processes with GARCH innovations.

Proposition 1 Let $y_t = y_{1t} + y_{2t}$ where $\{y_{it}\}$ $(i = 1, 2)$ are ARMA(r_i, s_i) processes which satisfy (2)–(4). Suppose further that $\epsilon_t = [\varepsilon_{1t}, \varepsilon_{2t}]^\top$ follows a bivariate GARCH$(1,1)$ process like (5) $-$ (7) with $h_{12,t} = \lambda \in (0, \infty)$ for all t. Then, $\{y_t\}$ admits the ARMA$(\tilde r, \tilde s)$ representation (11) $-$ (12), where $\{\varepsilon_t\}$ is a GARCH$(2,2)$ process with

$$h_t = \alpha^*_c + \alpha^*_1 \varepsilon^2_{t-1} + \alpha^*_2 \varepsilon^2_{t-2} + (\alpha'_{11} + \alpha'_{21} - \alpha^*_1) h_{t-1} - (\alpha^*_2 + \alpha'_{11}\alpha'_{21}) h_{t-2};$$

$\tilde r$ is the number of different zeros of $\Phi_1(z)$ and $\Phi_2(z)$ and $\tilde s = \max\{\tilde r - r_1 + s_1, \tilde r - r_2 + s_2\}$. The coefficients α^*_1 and α^*_2 are given by:

$$\begin{aligned}
\alpha^*_1 = \; & \pm\{w_1^4 \alpha_{11}^2 (1 + \alpha'^2_{21}) f_{2,1} + w_2^4 \alpha_{21}^2 (1 + \alpha'^2_{11}) f_{2,2} + 2 w_1^2 w_2^2 \alpha_{11}\alpha_{21}(1 + \alpha'_{21}\alpha'_{11})\lambda^2 \\
& \pm (2f_2)^{-1}\{w_1^8 \alpha_{11}^4 (1 - \alpha'^2_{21})^2 f^2_{2,1} + w_2^8 \alpha_{21}^4 (1 - \alpha'^2_{11})^2 f^2_{2,2} \\
& + w_1^4 w_2^4 \alpha_{11}^2 \alpha_{21}^2 [(1 - \alpha'_{21})^2 (1 + \alpha'_{11})^2 + (1 - \alpha'_{11})^2 (1 + \alpha'_{21})^2] f_{2,1} f_{2,2} \\
& + 4 w_1^4 w_2^4 \alpha_{11}^2 \alpha_{21}^2 [(1 + \alpha'_{21}\alpha'_{11})^2 - (\alpha'_{11} + \alpha'_{21})^2]\lambda^4 \\
& + w_1^6 w_2^2 \alpha_{11}^3 \alpha_{21} [(1 + \alpha'_{21})^2 (1 + \alpha'_{21}\alpha'_{11} - \alpha'_{11} - \alpha'_{21}) \\
& + 2(1 - \alpha'_{21})^2 (1 + \alpha'_{21}\alpha'_{11} + \alpha'_{11} + \alpha'_{21})]\lambda^2 f_{2,1} + w_2^6 w_1^2 \alpha_{21}^3 \alpha_{11} [(1 + \alpha'_{11})^2 \\
& (1 + \alpha'_{21}\alpha'_{11} - \alpha'_{11} - \alpha'_{21}) + 2(1 - \alpha'_{11})^2 (1 + \alpha'_{21}\alpha'_{11} + \alpha'_{11} + \alpha'_{21})]\lambda^2 f_{2,2}\}^{1/2}\}^{1/2}; \qquad (17)
\end{aligned}$$

and

$$\alpha^*_2 = -[w_1^4 \alpha_{11}^2 \alpha'_{21} f_{2,1} + w_2^4 \alpha_{21}^2 \alpha'_{11} f_{2,2} + w_1^2 w_2^2 \alpha_{11}\alpha_{21}(\alpha'_{11} + \alpha'_{21})\lambda^2]/\alpha^*_1 f_2; \qquad (18)$$

where $\alpha'_{i1} = \alpha_{i1} + \beta_{i1}$ $(i = 1, 2)$. Expressions for $f_{2,i} \equiv E(h^2_{it})$ are given in He and Teräsvirta (1997) and Karanasos (1999), and $f_2 \equiv E(h_t^2)$ is given in (16).

Our second example considers the case of a linear aggregate of MA(1) processes with GARCH innovations. This is an interesting case from a practical point of view since many stock-price series appear to be adequately described by low order MA models.

Proposition 2 Let $\{y_t\}$ be a linear aggregate of k MA(1) processes $\{y_{it}\}$ which satisfy (2) $-$ (4). Suppose further that $\epsilon_t = [\varepsilon_{1t}, \ldots, \varepsilon_{kt}]^\top$ follows a GARCH process like (5) $-$ (7). Then, $\{y_t\}$ admits the MA(1) representation

$$y_t = \mu + \varepsilon_t - \theta(\varepsilon_{t-1} + \varepsilon^*_{t-1}); \qquad (19)$$

where $\{\varepsilon_t\}$ is a GARCH process like (13) $-$ (14), $\varepsilon^*_t \overset{iid}{\sim} N(0, \sigma^2_{\varepsilon^*})$, independently of $\{\varepsilon_t\}$,

$$\theta = \sum_{i=1}^{k} w_i^2 \theta_i^2 (f_{1i}/f_1) + \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_u w_v (\theta_u + \theta_v)(f_{1,uv}/f_1);$$

and

$$\sigma^2_{\varepsilon^*} = \frac{1}{\theta^2 f_1} \left\{ \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_u^2 w_v^2 (\theta_u - \theta_v)^2 (f_{1,u} f_{1,v} - f^2_{1,uv}) \right.$$

$$+ 2 \sum_{\substack{m=v+1;\\ m\neq u}}^{k} \sum_{\substack{v=1;\\ v\neq u}}^{k-1} \sum_{u=1}^{k} w_u^2 w_v w_m (\theta_u - \theta_m)(\theta_v - \theta_m)(f_{1;u} f_{1;vm} - f_{1;uv} f_{1;um})$$

$$+ 2 \sum_{s=m+1}^{k} \sum_{m=v+1}^{k-1} \sum_{v=u+1}^{k-2} \sum_{u=1}^{k-3} w_u w_v w_m w_s$$

$$\times \sum_{i=v;m;s} f_{1;ui} f_{1;i_1;i_2} [(\theta_u - \theta_{i_1})(\theta_i - \theta_{i_2}) + (\theta_u - \theta_{i_2})(\theta_i - \theta_{i_1})] \Bigg\} ;$$

with

$$(i_1; i_2) = \begin{cases} (m; s) & \text{if } i = v; \\ (v; s) & \text{if } i = m; \\ (v; m) & \text{if } i = s; \end{cases}$$

$f_{1i} \equiv E(h_{it})$ and $f_{1;uv} \equiv E(h_{uv;t})$.

4 Persistence of Volatility Shocks

Using the results in the previous section, we can now examine how the persistence of a shock to the aggregate conditional variance h_t is related to the persistence of shocks to the k individual variances h_{it}. As in Engle and Mustafa (1992), the persistence of a volatility shock is thought of here in terms of the coefficients of the MA representation of the relevant volatility process. Thus, in the case of $\{h_t\}$, for instance, persistence depends primarily on $\gamma = 1 + A^*(1) - B^*(1)$.

We shall distinguish between two cases of interest, depending on whether the conditional covariance matrix H_t is diagonal or not. Henceforth, we let $h_t^k = \sum_{i=1}^{k} w_i^2 h_{it}$, $h_t^{k_i} = h_t^k - w_i^2 h_{it}$, and $h_t^{k*} = 2 \sum_{v=u+1}^{k} \sum_{u=1}^{k-1} w_u w_v h_{uv;t}$. The sum of the coefficients of the lag polynomials in the GARCH equations h_t^k, $h_t^{k_i}$ and h_t^{k*} is respectively denoted by γ^k, γ^{k_i} and γ^{k*}.

CASE I: The polynomials $B_1^*(z), \ldots, B_k^*(z)$ have no common zeros and H_t is diagonal. In this case, the denominator in (15) is equal to

$$\prod_{i=1}^{k} (1 - \gamma_i) = 1 - \gamma^k;$$

where

$$\gamma_i = \sum_{j=1}^{p^*} \beta^*_{ij} \qquad (i = 1; \ldots; k);$$

and

$$\gamma_k = \gamma_1 + (1 - \gamma_1)[\gamma_2 + (1 - \gamma_2)[\gamma_3 + (1 - \gamma_3)[\cdots[\gamma_{k-1} + (1 - \gamma_{k-1})\gamma_k]]\cdots]:$$

We have, therefore,

$$\gamma^k = \gamma_i + (1-\gamma_i)\gamma^{k_i} > \gamma_i,$$

and

$$\gamma^k = \gamma^{k_i} + (1-\gamma^{k_i})\gamma_i > \gamma^{k_i}.$$

Hence, the sum of the coefficients of the GARCH equation $h_t^k = h_t\ (\gamma^k)$ will be greater than the sum of the coefficients of each of the k GARCH equations $h_{it}\ (\gamma_i)$, and it will also be greater than the sum of the coefficients of $h_t^{k_i}\ (\gamma^{k_i})$.

CASE II: The polynomials $B_i^*(z)$ and $B_{uv}^*(z)$ $(i; u; v = 1; \ldots; k)$ have no common zeros and H_t is non-diagonal. In this case, the denominator in (15) is equal to

$$\prod_{i=1}(1-\gamma_i)\prod_{v=u+1}\prod_{u=1}^{k-1}(1-\gamma_{uv}) = (1-\gamma^k)(1-\gamma^{k*}) = 1-\gamma,$$

where

$$\gamma_{uv} = \sum_{j=1}^{p^*_{uv}} \beta^*_{uv,j},$$

and

$$\begin{aligned}\gamma^{k*} = {} & \gamma_{12} + (1-\gamma_{12})[\gamma_{13} + (1-\gamma_{13})[\cdots \\ & [\cdots\gamma_{1k} + (1-\gamma_{1k})[\gamma_{23} + (1-\gamma_{23})[\cdots[\gamma_{k-2,k} + (1-\gamma_{k-2,k})\gamma_{k-1,k}]\cdots].\end{aligned}$$

From the last equation we have

$$\gamma = \gamma^k + (1-\gamma^k)\gamma^{k*}.$$

Hence, if the sum of the coefficients of all the $B_{uv}^*(z)$ polynomials (γ^{k*}) is positive (negative), then the sum of the coefficients of the GARCH equation $h_t\ (\gamma)$ will be greater (smaller) than the sum of the coefficients of $h_t^k\ (\gamma^k)$.

Furthermore, from (15) we obtain

$$\gamma = \gamma_i + (1-\gamma_i)[\gamma^{k_i} + (1-\gamma^{k_i})\gamma^{k*}],$$

$$\gamma = \gamma^{k_i} + (1-\gamma^{k_i})[\gamma_i + (1-\gamma_i)\gamma^{k*}].$$

Hence, when the sum of the coefficients of all the $B_{uv}^*(z)$ polynomials (γ^{k*}) is negative, the sum of the coefficients of the GARCH equation $h_t\ (\gamma)$ will be greater than the sum of the coefficients of $h_t^{k_i}\ (\gamma^{k_i})$ if $\gamma_i > (1-\gamma_i)\gamma^{k*}$, and it will be greater than the sum of the coefficients in the GARCH equation $h_{it}\ (\gamma_i)$ if $\gamma^{k_i} > (1-\gamma^{k_i})\gamma^{k*}$.

Also, if $\gamma^{k*,r}$ and $\gamma^{r,k*}$ denote the sums of coefficients in the GARCH equation for respectively r out of the $\frac{1}{2}k(k-1)$ terms in h_t^{k*} and $\frac{1}{2}k(k-1)-r$ terms in h_t^{k*}, we have from (15) that

$$\gamma = \gamma^{k} + (1 - \gamma^{k})[\gamma^{k*;r} + (1 - \gamma^{k*;r})\gamma^{r;k*}]:$$

Hence, when the sum of the coefficients of the r out of the $\frac{1}{2}k(k - 1)$ $B^{*}_{uv}(z)$ polynomials is positive, the sum of the coefficients of the GARCH equation h_t (γ) will be greater (smaller) than the sum of the coefficients in h^k_t if $\gamma^{k*;r} > (1 - \gamma^{k*;r})\gamma^{r;k*}$ ($\gamma^{k*;r} < (1 - \gamma^{k*;r})\gamma^{r;k*}$).

Finally, in the extreme case where all the polynomials $B^{*}_{i}(z)$ and $B^{*}_{uv}(z)$ are identical, the sum of the coefficients of the GARCH equation h_t will be equal to the sum of the coefficients of each GARCH equation h_{it}.

5 Aggregation and Option Pricing

As an illustration of some of the practical implications of the results given in the previous two sections of the paper, we consider the effects of cross-sectional aggregation in the context of GARCH option pricing. More specifically, we price options on individual stocks and on an equally weighted index and compare the price of a call option on the index to the average cost of the calls on the individual stocks. Since the volatility of the index typically exhibits more persistence than the volatility of the individual stocks, a forecast of the volatility of the index would take longer to revert to the unconditional variance. Hence, whenever forecasting from a period of high volatility, the forecast values will be above the unconditional variance, and whenever forecasting from a state with low volatility, the forecast will be below the unconditional variance. The effects on the price of the option pricing would be more dramatic for the index than for the individual stocks.

Our analysis here is based on daily data for the price of stocks of seven U.K. companies, namely Allied-Lyons (ALLD), ASDA, Blue Circle Industries (BCI), Cadbury Schweppes (CBRY), Courtaulds (CTLD), National Westminster Bank (NWB), and Royal Insurance (ROYL), as well as on a simple linear aggregate of the seven stocks with equal weights (referred to hereafter as the 'index'). The sample covers the period **** (920 observations in total), and is chosen so as to avoid the possibility of structural breaks which would spuriously increase volatility persistence. A simple specification search revealed that all individual price series can be characterized as GARCH(1; 1); the fitted models show little or no signs of residual serial correlation in the residuals, and no signs of serial correlation in the squared residuals. Table 1 reports quasi-maximum likelihood estimates of our persistence measure (i.e. the sum of GARCH coefficients) for the individual stocks and the index, along with their asymptotic standard errors (computed using the usual sandwich covariance matrix estimator). Clearly, the estimates for the individual stocks are smaller than the estimate for the index.

To assess the effects of aggregation on the persistence of volatility shocks (and on option pricing), we must distinguish between what we shall call diversification effect and increased

Table 1. Estimates of Persistence[a]

ALLD	0.7301	(0.1476)	CTLD	0.6617	(0.2307)
ASDA	0.5432	(0.1780)	NWB	0.6706	(0.1626)
BCI	0.2401	(0.1473)	ROYL	0.7036	(0.1498)
CBRY	0.5729	(0.4038)	Index	0.8736	(0.0978)

[a]Figures in parentheses are asymptotic standard errors.

persistence e¤ect. Clearly, taking a weighted average of the individual stocks would reduce the unconditional variance of the index. Therefore, we are interested in assessing how much of the di¤erence between the value of the call on the index and the average of the calls on the individual stocks using GARCH option pricing comes from the reduction of the variance associated with averaging (diversifying the portfolio) and how much comes from the e¤ects of the increased persistence. In order to do so, we have also created a synthetic option using constant variances.

Before analyzing the e¤ects of aggregation on GARCH pricing, it is worth examining the plot in Figure 1 which shows the unconditional variance, the …tted conditional variance for the index, and 30 forecasts of the conditional variance. It is evident that the end of the sample coincides with a period of low volatility and that the forecast values are all below the unconditional variance. This will have implications for option pricing since, when compared with the option prices computed using historical volatility, GARCH pricing will give lower or higher values for the relevant forecast period depending on whether the economy is in a period of high or low volatility at the forecast origin and on whether the option is in-the-money or out-of-the-money.

In our pricing exercise we follow Bollerslev and Mikkelsen (1996) in using the Black and Scholes (1973) option pricing formula to calculate the price of a European call option written at date T as a function of the volatility of stock prices, the maturity time of the option (¿), the exercise price (K), the stock price at date T (P_T), and the risk-free interest rate over the life of the option (r). This formula is evaluated using both historical volatility and the average (over the life of the option) of forecasts from the …tted GARCH models for the relevant stock price or index.[1] Our exercise consists of evaluating option prices for maturity times ¿ = 1; : : : ; 30. We consider options that are deep-in-the-money ($K = 0{:}8P_T$), in-the-money ($K = 0{:}9P_T$), and at-the-money ($K = P_T$), and take $r = 0{:}08$ per year. Under these scenarios, we compare the option price of the index with the average of the option prices of the individual stocks.

As is evident from Table 1, the individual stocks are characterized by relatively small persistence, so they tend to revert to the unconditional variance in few time periods after a volatility

[1]We also priced the option by computing the average of the calls evaluated using instantaneous variances (in our context, GARCH forecasts). Hull and White (1987) have shown that this is equivalent to Black-Scholes pricing whenever the continuous-time volatility process is uncorrelated with the aggregate consumption in the economy. The results obtained with this alternative pricing scheme are qualitatively similar to those reported here and do not change our conclusions (detailed results are available upon request).

shock. This implies that, for our sample, GARCH pricing and historical volatility pricing would yield very similar results for the individual stocks. For the purpose of our exercise, this result is very informative since it allows us to distinguish between diversification effects and increased persistence effects.

Figure 2 shows the results of our simulations for both GARCH and historical volatility pricing. The values of the average of the calls of the individual stocks using either of the two pricing methods are indistinguishable for the reason explained before. The Black-Scholes value of the call on the index is higher since the volatility of the index is smaller (because of aggregation) and the stock is deep in the money (and therefore the prospect of the price falling bellow the strike price is smaller). Nevertheless, the value of the GARCH option is even higher since the forecast origin was a low variance state.[2] Figure 3 shows qualitatively similar results for in-the-money options. Finally, the results shown in Figure 4 for options that are at-the-money reveal once again that option prices for the individual stocks using either pricing method are very similar and are higher than the values of the options on the index (since the lower is the variance the less likely it is that the an option at-the-money has any value). As before, the difference between prices obtained by the two alternative pricing schemes reveals how much of the differences in option prices is due to the increased persistence that characterizes the index.

6 Summary

This paper has investigated the properties of linear aggregates of ARMA processes with errors that follow a diagonal multivariate GARCH process. We have derived the ARMA-GARCH representations that such linear aggregates admit. We have also shown that, under conditions that are typically satisfied in practice, persistence in the volatility of the aggregate series is higher than persistence in the volatility of the individual series. As an empirical illustration of the importance of the issues analyzed, we have discussed the results of a simple option pricing exercise involving seven U.K. individual stocks and an equally weighted index.

7 Appendix: Proofs

Proof of Theorem 1. First note that from (1) and (2) we have

$$h_t = E_{t-1}\left[\left\{\sum_{i=1}^{k} w_i \varepsilon_{it} - E_{t-1}\left(\sum_{i=1}^{k} w_i \varepsilon_{it}\right)\right\}^2\right] = \sum_{i=1}^{k} w_i^2 h_{it} + 2\sum_{v=u+1}^{k}\sum_{u=1}^{k-1} w_u w_v h_{uv,t}; \quad \text{(A.1)}$$

[2] In this example, the increased persistence has the effect of producing slowly declining forecasts with lower than average variance. In such a case, both the persistence effect and the diversification effect reduce the variance. Had this exercise been conducted at observation 740 (associated with high conditional variance), the persistence effect would have had opposite sign form the diversifiaction effect since the forecast of the conditional variance at that date would produce values considerably higher than the unconditional variance.

where $E_{t-1}(\cdot)$ denotes conditional expectation with respect to F_{t-1}. Next, consider the polynomial

$$B(L) = \prod_{i=1}^{\bar{p}}(1-\beta_i' L) = -\sum_{i=0}^{\bar{p}}\beta_i L^i \qquad (\beta_0 = -1); \tag{A.2}$$

where β_i' are all the $\bar{p}$ different inverse zeros of the k $B_i^*(z)$ polynomials and of the $\frac{1}{2}k(k-1)$ $B_{uv}^*(z)$ polynomials. Since each $B_i^*(z)$ has p_i^* zeroes and each $B_{uv}^*(z)$ has p_{uv}^* zeroes, the maximum value of $\bar{p}$ is $\sum_{i=1}^{k} p_i^* + \sum_{v=u+1}^{k}\sum_{u=1}^{k-1} p_{uv}^*$ (assuming that the zeroes of each polynomial are different); the minimum value of $\bar{p}$ is the $\max\{p_i^*; p_{uv}^*\}$.

Now, multiplying (A.1) by (A.2) and using (6) and (8) we obtain

$$B(L)h_t = \alpha_0^* + \sum_{i=1}^{k} A_i'(L) w_i^2 \eta_{it} + 2\sum_{v=u+1}^{k}\sum_{u=1}^{k-1} A_{uv}'(L) w_u w_v \eta_{uv;t} \tag{A.3}$$

where

$$A_i'(L) = \prod_{l=1}^{\bar{p}-p_i^*}(1-\beta_l' L)A_i(L) = \sum_{j=1}^{p_i'}\alpha_{ij}' L^j; \quad \beta_l' \neq \beta_{ij}' \ (j = 1; \ldots; p_i^*);$$

$$A_{uv}'(L) = \prod_{l=1}^{\bar{p}-p_{uv}^*}(1-\beta_l' L)A_{uv}(L) = \sum_{j=1}^{p_{uv}'}\alpha_{uv;j}' L^j; \quad \beta_l' \neq \beta_{uv;j}' \ (j = 1; \ldots; p_{uv}^*);$$

$$\alpha_0^* = \sum_{i=1}^{k} w_i^2 a_{i0}\prod_{l=1}^{\bar{p}-p_i^\star}(1-\beta_l') + 2\sum_{v=u+1}^{k}\sum_{u=1}^{k-1} w_u w_v \alpha_{uv;0}\prod_{s=1}^{\bar{p}-p_{uv}^\star}(1-\beta_s'); \quad \beta_l' \neq \beta_{ij}'; \quad \beta_s' \neq \beta_{uv;j}'.$$

In the right-hand side of (A.3), we have k $A_i'(L)$ polynomials and $\frac{1}{2}k(k-1)$ $A_{uv}'(L)$ polynomials. Each $A_i'(L)$ is of order $p_i' = \bar{p} - p_i^* + p_i$ and each $A_{uv}'(L)$ is of order $p_{uv}' = \bar{p} - p_{uv}^* + p_{uv}$. In other words, the right-hand side of (A.3) is equal to the sum of k MA(p_i') parts and $\frac{1}{2}k(k-1)$ MA(p_{uv}') parts. Hence, it can be expressed as an MA of order $\hat{p} = \max\{p_i'; p_{uv}'\}$;

$$B(L)h_t = \alpha_0^* + A^*(L)\eta_t; \tag{A.4}$$

where $\eta_t = \varepsilon_t^2 - h_t$ and $A^*(L) = \sum_{i=1}^{\hat{p}}\alpha_i^* L^i$. Denoting the right-hand side expressions in (A.3) and (A.4) by ρ_t and μ_t, respectively, we have

$$\begin{aligned}
\mathrm{cov}(\rho_t;\rho_{t-j}) \;=\; & \sum_{l=1}^{p_i'-j}\sum_{r=1}^{k}\sum_{i=1}^{k}\alpha_{i;l}'\alpha_{r;l+j}' w_i^2 w_r^2 \mathrm{cov}(\eta_{it};\eta_{rt}) \\
& + 2\sum_{l=1}^{p_u'-j}\sum_{v=u+1}^{k}\sum_{u=1}^{k-1}\sum_{i=1}^{k}\alpha_{i;l}'\alpha_{uv;l+j}' w_i^2 w_u w_v \mathrm{cov}(\eta_{it};\eta_{uv;t}) \\
& + 4\sum_{l=1}^{p_m'-j}\sum_{s=m+1}^{k}\sum_{m=1}^{k-1}\sum_{v=u+1}^{k}\sum_{u=1}^{k-1}\alpha_{uv;l}'\alpha_{ms;l+j}' w_u w_v w_m w_s \mathrm{cov}(\eta_{uv;t};\eta_{ms;t});
\end{aligned}$$

$$\mathrm{cov}(\mu_t;\mu_{t-j}) \;=\; 2\sum_{i=1}^{\hat{p}-j}\alpha_i^*\alpha_{i+j}^* f_2; \qquad f_2 \equiv E(h_t^2).$$

But since $\mathrm{cov}(\rho_t; \rho_{t-j}) = \mathrm{cov}(\mu_t; \mu_{t-j})$, the α_i^*'s can be obtained by equating the right-hand sides of the above two equations for $j = 0, \ldots, \tilde p - 1$ and solving the resulting system of $\tilde p$ equations.

Next, from (A.4) we have

$$B^*(L)h_t = \alpha_0^* + A^*(L)\varepsilon_t^2; \qquad B^*(L) = -\sum_{i=0}^{\hat p} \beta_i^* L^i \quad (\beta_0^* = -1);$$

where $\hat p = \max\{\tilde p, \bar p\}$ and

$$\beta_i^* = \begin{cases} \alpha_i^* + \beta_i & \text{if } \tilde p, \bar p > i \\ \alpha_i^* & \text{if } i, \tilde p > \bar p \\ \beta_i & \text{if } i, \bar p > \tilde p. \end{cases}$$

Now, consider the polynomial

$$\Phi(L) = \prod_{i=1}^{\bar r}(1 - \phi_i' L) = -\sum_{i=0}^{\bar r} \phi_i L^i \quad (\phi_0 = -1); \tag{A.5}$$

where ϕ_i' are all the $\bar r$ different inverse zeros of the k $\Phi_i(z)$ polynomials (each of which has r_i zeros). Clearly, $\max\{r_1, \ldots, r_k\} \leq \bar r \leq \sum_{i=1}^k r_i$. Multiplying (1) by (A.5), and using (2) and (6), we obtain

$$\Phi(L)y_t = \omega + \sum_{i=1}^{k} \Theta_i'(L) w_i \varepsilon_{it}; \tag{A.6}$$

where

$$\Theta_i'(L) = \sum_{j=0}^{\tilde r_i} \theta_{ij}' L^j = \prod_{l=1}^{\bar r - r_i}(1 - \phi_l' L)\Theta_i(L).$$

In the right-hand side of (A.6), we have k $\Theta_i'(L)$ polynomials, each of which is of order $\tilde r_i = \bar r - r_i + s_i$. In other words, the right-hand side of (A.6) is equal to the sum of k MA($\tilde r_i$) parts. Hence, it can be expressed as an MA term of order $\tilde r = \max\{\tilde r_1, \ldots, \tilde r_k\}$;

$$\Phi(L)y_t = \varepsilon_t + \Theta(L)(\varepsilon_t^* + \varepsilon_t); \tag{A.7}$$

where

$$\Theta(L) = -\sum_{i=1}^{\tilde r} \theta_i L^i; \qquad (\varepsilon_t | F_{t-1}) \sim N(0, h_t); \qquad \varepsilon_t^* \sim N(0, \sigma_{\varepsilon^*}^2).$$

Denoting the right-hand sides of (A.6) and (A.7) by $\tilde\rho_t$ and μ_t, respectively, we have

$$\mathrm{cov}(\tilde\rho_t; \tilde\rho_{t-j}) = \sum_{l=0}^{\tilde r_v - j} \sum_{v=1}^{k} \sum_{u=1}^{k} w_u w_v \theta_{u,l}' \theta_{v,l+j}' f_{1,uv^*} \qquad (j = 0, \cdots, \tilde r);$$

$$\mathrm{cov}(\mu_t; \mu_{t-j}) = \sum_{l=1}^{\tilde r - j} \theta_l \theta_{l+j} f_1 + \sum_{l=1}^{\tilde r - j} \theta_l \theta_{l+j} \sigma_{\varepsilon^*}^2 \qquad (j = 0, \cdots, \tilde r);$$

where $f_1 \equiv E(h_t^2)$ and

$$f_{1;uv*} = \begin{cases} E(h_{uv;t}) & \text{if } v \neq u \\ E(h_{ut}) & \text{if } v = u. \end{cases}$$

Setting $\mathrm{cov}(\rho_t; \rho_{t-j}) = \mathrm{cov}(\mu_t; \mu_{t-j})$ we obtain a system of $\ell + 1$ equations which can be solved for the θ_i's and $\sigma^2_{\varepsilon*}$. ∎

Proof of Theorem 2. Observe first that, under the assumption of conditional normality, we may write

$$\varepsilon_{ut} = e_{ut}\sqrt{h_{ut}} \qquad (u = 1; \ldots; k);$$

where $\{e_{ut}\} \overset{iid}{\sim} N(0; 1)$. It follows, therefore, that

$$E_{t-1}(e_{ut}e_{vt}) \equiv \tau_{uv} = h_{uv;t}/\sqrt{h_{ut}h_{vt}}; \qquad E_{t-1}(e_{ut}^2 e_{vt}^2) = 1 + 2\tau_{uv}^2; \qquad E_{t-1}(e_{ut}^3 e_{vt}) = 3\tau_{uv};$$

$$E_{t-1}(e_{ut}^2 e_{vt} e_{mt}) = \tau_{vm} + 2\tau_{um}\tau_{uv}; \qquad E_{t-1}(e_{ut}e_{vt}e_{mt}e_{st}) = \tau_{uv}\tau_{ms} + \tau_{um}\tau_{vs} + \tau_{us}\tau_{vm}.$$

Also note that, from the definition of η_{it}, $\eta_{uv;t}$ and ε_{ut}, we have

$$\mathrm{cov}(\eta_{ut}; \eta_{vt}) = 2E(h_{uv;t}^2) \equiv f_{2;uv}; \qquad \tfrac{1}{2}\mathrm{var}(\eta_{ut}) = E(h_{ut}^2) \equiv f_{2;u}; \tag{A.8}$$

$$\mathrm{var}(\eta_{uv;t}) = E(h_{ut}h_{vt}) + E(h_{uv;t}^2) \equiv f^{2;uv} + f_{2;uv}; \tag{A.9}$$

$$\tfrac{1}{2}\mathrm{cov}(\eta_{ut}; \eta_{uv;t}) = E(h_{ut}h_{uv;t}) \equiv f^{2;u;uv}; \tag{A.10}$$

$$\mathrm{cov}(\eta_{ut}; \eta_{vm;t}) = E(h_{uv;t}h_{um;t}) \equiv f^{2;uv;um}; \tag{A.11}$$

$$\mathrm{cov}(\eta_{uv;t}; \eta_{um;t}) = f^{2;u;vm} + f^{2;uv;um}; \tag{A.12}$$

$$\mathrm{cov}(\eta_{uv;t}; \eta_{ms;t}) = f^{2;um;vs} + f^{2;us;vm}. \tag{A.13}$$

Furthermore, there exist constants γ_{u0}, γ_{v0}, γ_{uv0}, $\gamma^{uv;0}$ and $\gamma^{u;uv;0}$ such that

$$\mathrm{var}(h_{ut}) = \gamma_{u0}\mathrm{var}(\eta_{ut}); \qquad \mathrm{var}(h_{vt}) = \gamma_{v0}\mathrm{var}(\eta_{vt}); \qquad \mathrm{var}(h_{uv;t}) = \gamma_{uv0}\mathrm{var}(\eta_{uv;t});$$

$$\mathrm{cov}(h_{ut}; h_{vt}) = \gamma^{uv;0}\mathrm{cov}(\eta_{ut}; \eta_{vt}); \qquad \mathrm{cov}(h_{ut}; h_{uv;t}) = \gamma^{u;uv;0}\mathrm{cov}(\eta_{ut}; \eta_{uv;t});$$

so we may write

$$f_{2;uv} = \frac{[E(h_{uv;t})]^2 + \gamma_{uv;0}E(h_{ut})E(h_{vt})}{1 - \gamma_{uv;0}(1 + 2\gamma^{uv;0})}; \qquad f_{2;u} = \frac{[E(h_{ut})]^2}{1 - 2\gamma_{u;0}}; \tag{A.14}$$

$$f^{2;u;uv} = \frac{E(h_{ut})E(h_{uv;t})}{1 - 2\gamma^{u;uv;0}}; \qquad f^{2;u;vm} = \frac{E(h_{ut})E(h_{vm;t})}{1 - \gamma^{u;vm;0}}; \tag{A.15}$$

$$f^{2;uv;um} = \frac{E(h_{uv;t})E(h_{um;t}) + \gamma^{uv;um;0}E(h_{ut}h_{vm;t})}{1 - \gamma^{uv;um;0}}; \tag{A.16}$$

and

$$\begin{bmatrix} E(h_{uv,t}h_{ms,t}) \\ E(h_{um,t}h_{vs,t}) \\ E(h_{us,t}h_{vm,t}) \end{bmatrix} = \begin{bmatrix} 1 & -\gamma_{uv,ms,0} & -\gamma_{uv,ms,0} \\ -\gamma_{um,vs,0} & 1 & -\gamma_{um,vs,0} \\ -\gamma_{us,vm,0} & -\gamma_{us,vm,0} & 1 \end{bmatrix} \begin{bmatrix} E(h_{uv,t})E(h_{ms,t}) \\ E(h_{um,t})E(h_{vs,t}) \\ E(h_{us,t})E(h_{vm,t}) \end{bmatrix}. \quad (A.17)$$

Finally, notice that

$$f_{1i} \equiv E(h_{it}) = \frac{\alpha_{ic}}{B_i(1) - A_i(1)}, \quad (A.18)$$

and

$$f_{1,uv} \equiv E(h_{uv,t}) = \frac{\alpha_{uv,c}}{B_{uv}(1) - A_{uv}(1)}. \quad (A.19)$$

Now, from (A.1), using (A.18)-(A.19) and taking into account all the common zeros of the autoregressive polynomials, we obtain (15). Moreover, squaring (A.1), taking expectations and using (A.8)-(A.13),we get (16), where the f_2's and f^2's are given by (A.14)-(A.17). $\blacksquare$

Proof of Proposition 1. From (A.1) it follows that

$$h_t = w_1^2 h_{1t} + w_2^2 h_{2t} + 2\lambda w_1 w_2.$$

Multiplying the above equation by $(1-\alpha'_{11}L)(1-\alpha'_{21}L)$ and noticing that

$$(1-\alpha'_{i1}L)h_{it} = \alpha_{ic} + \alpha_{i1}\eta_{i,t-1}, \qquad \eta_{it} = \varepsilon_{it}^2 - h_{it},$$

we obtain

$$(1-\alpha'_{11}L)(1-\alpha'_{21}L)h_t = \alpha_0^* + w_1^2\alpha_{11}(1-\alpha'_{21}L)\eta_{1t-1} + w_2^2\alpha_{21}(1-\alpha'_{11}L)\eta_{2t-1}.$$

Therefore, writing $\psi_t = (1-\alpha'_{11}L)(1-\alpha'_{21}L)h_t - \alpha_0^*$, we have

$$\mathrm{var}(\psi_t) = 2w_1^4\alpha_{11}^2(1+\alpha_{21}'^2)f_{2,1} + 2w_2^4\alpha_{21}^2(1+\alpha_{11}'^2)f_{2,2} + 4w_1^2w_2^2\alpha_{11}\alpha_{21}(1+\alpha'_{21}\alpha'_{11})\lambda^2,$$

$$\mathrm{cov}(\psi_t,\psi_{t-1}) = -2w_1^4\alpha_{11}^2\alpha'_{21}f_{2,1} - 2w_2^4\alpha_{21}^2\alpha'_{11}f_{2,2} - 2w_1^2w_2^2\alpha_{11}\alpha_{21}(\alpha'_{11}+\alpha'_{21})\lambda^2,$$

where $\alpha_0^* = 2w_1w_2\lambda(1-\alpha_{11})(1-\alpha_{21}) + w_1^2\alpha_{10}(1-\alpha_{21}) + w_2^2\alpha_{20}(1-\alpha_{11})$. Also, since h_t may be expressed as

$$(1-\alpha'_{11}L)(1-\alpha'_{21}L)h_t = \alpha_0^* + \alpha_1^*\eta_{t-1} + \alpha_2^*\eta_{t-2},$$

where $\eta_t = \varepsilon_t^2 - h_t$, we have

$$\mathrm{var}(\psi_t^*) = 2(\alpha_1^{*2} + \alpha_2^{*2})f_2, \qquad \text{and} \qquad \mathrm{cov}(\psi_t^*,\psi_{t-1}^*) = 2a_1^*\alpha_2^*f_2,$$

where $\psi_t^* = \alpha_1^*\eta_{t-1} + \alpha_2^*\eta_{t-2}$. Setting $\mathrm{var}(\psi_t) = \mathrm{var}(\psi_t^*)$ and $\mathrm{cov}(\psi_t,\psi_{t-1}) = \mathrm{cov}(\psi_t^*,\psi_{t-1}^*)$ and solving for α_1^* and α_2^* yields the results in (17)-(18). $\blacksquare$

Proof of Proposition 2. From (1), it follows that

$$\mathrm{var}(y_t) = \sum_{i=1}^{k} w_i^2(1+\theta_i^2)f_{1i} + 2\sum_{v=u+1}^{k}\sum_{u=1}^{k-1} w_u w_v (1+\theta_u\theta_v) f_{1;uv}; \tag{A.20}$$

$$\mathrm{cov}(y_t; y_{t-1}) = -\sum_{i=1}^{k} w_i^2 \theta_i f_{1i} - \sum_{v=u+1}^{k}\sum_{u=1}^{k-1} w_u w_v (\theta_u + \theta_v) f_{1;uv}. \tag{A.21}$$

Moreover, (19) implies that

$$\mathrm{var}(y_t) = (1+\theta^2)f_1 + \theta^2\sigma^2_{\varepsilon}; \tag{A.22}$$

$$\mathrm{cov}(y_t; y_{t-1}) = -\theta f_1. \tag{A.23}$$

The derided results are obtained by equating the right-hand sides of (A.20) and (A.22) and (A.21) and (A.23) and solving for σ^2_{ε} and θ. ∎

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