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Efficiency Wages, Increasing Returns and Endogenous Fluctuations

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Abstract

This work studies the implications of efficiency wages, indivisible labour and increasing returns to endogenously create time series for real wages and employment exhibiting persistent fluctuations with a cyclical behavior similar to the one empirically observed. Our first result relates to the effects of introducing efficiency wages into a general equilibrium model. We show that this real wage rigidity can indeed explain involuntary unemployment but cannot affect the dynamical properties of an economic system. Hence, the remaining of the paper concerns basically to the role of indivisible labour and increasing returns to the occurrence of endogenous fluctuations. Our main results are: (*i*) when both constant returns and a standard elasticity of labour supply are considered, a small substitutability between factors is needed to generate endogenous fluctuations; this is just a replication of Reichlin (1986); (*ii*) the introduction of increasing returns is able to establish the possibility of endogenous fluctuations, even when the elasticity of inputs substitution is relatively large; but as in Cazzavilan et al (1998), the amount of increasing returns necessary to get this result is still large; (*iii*) the introduction of the indivisible labour hypothesis can generate endogenous fluctuations with both an elasticity of inputs substitution and an amount of increasing returns in consonance with empirical evidence. Moreover, in this latter case we replicate two labour market regularities not usually captured by general equilibrium models: 1) employment is more volatile than real wages, and 2) real wages are acyclical.

1. Introduction

In this work we analyse the cyclical properties of real wages and employment exhibited by trajectories displaying endogenous fluctuations.

From a theoretical point of view, two different types of explanations for the existence of persistent fluctuations may arise. Persistent fluctuations may be due to the existence of repeated exogenous shocks on technologies, preferences and policy variables, or they can emerge endogenously as an equilibrium result of dynamic market interactions. While the literature of business cycles has traditionally emphasised the exogenous shocks approach, since the mid eighties the endogenous fluctuations approach has experienced a revival.

The latter approach has shown that simple deterministic dynamic models can generate very complex behavior if nonlinearities, are strong enough to keep within bounded intervals trajectories that do not converge to a fixed point.

This work is an extension of some earlier works on endogenous fluctuations in overlapping generations (OLG) models. In the 1985's seminal paper Grandmont has employed bifurcation theory to analyse the dynamic properties of a particular simple version of an OLG model. He has shown that a sucessions of flip bifurcations may occur, as a parameter of the model is varied, that leads to the existence of sucessively more complicated cycles. However, complicated dynamic behaviour occurs in Grandmont's model only if saving is a decreasing function of the interest rate. In fact no cycles can exist unless the elasticity of savings with respect to the interest rate is below $-\frac{1}{2}$. This may make endogenous cycles empirically unlikely.

Reichlin (1986) shows that Grandmont's result depends on his consideration of a very simple technology, i.e., one in which current labour is the only input required to produce current output. In fact, Reichlin considers that a model with a more general production function, i.e., with capital and labour can displays endogenous fluctuations through the occurrence of an Hopf bifurcation even when the saving rate has a positive elasticity with respect to the interest rate. However this only occurs for a sufficiently low elasticity between capital and labour, which again make endogenous cycles empirically unlikely.

This paper is, in many aspects, an extension to Reichlin (1986).

In Reichlin's paper there are three assumptions that can be relaxed in order to increase the likelihood of creating endogenous cycles. First, as he considers that returns to scale are constant, a natural extension is to consider increasing returns to scale. We show that increasing returns to scale (a realistic small amount) cannot create endogenous cycles. Reichlin also considers a standard elasticity of supply; however, since Hansen (1985), the indivisible labour assumption has been increasingly taken seriously in the profession. When this assumption is considered alone, endogenous fluctuations cannot emerge as well. However, the joint assumption of indivisible labour and increasing returns to scale makes endogenous fluctuations occur rather easily, through an Hopf bifurcation, without any "unlikely" requirement for the elasticity of savings related to the interest rate and the technical elasticity of substitution.

A final extension that can be made in the Reichlin's model is to introduce some mechanism that produces real wage rigidity. For this purpose, we consider the payment of efficiency wages[1].

The study of the conditions to the emergence of endogenous fluctuations throughout these different hypothesis is the first goal of this work.

Moreover, the second goal of this work is to see whether our model can reproduce some labour market regularities that usually are not captured by the models from the exogenous approach. The first one is known as "the employment variability puzzle". In fact because RBC models usually assume that most shocks hitting the economy shift labour demand, and since micro studies suggest that the labour supply is highly positive sloped, much of the adjustments to a productivity shock should be borne by wages, rather than employment. The consequence of this is that in these models employment is less variable and wage more variable than in reality[2].

Another consequence of this is that employment and the real wage would be highly correlated. However a number of authors[3] have argued that the correlation of employment and real wages is near zero, which was first identified by Dunlop (1938). This constitutes an equally significant puzzle usually known as "the productivity puzzle".

It seems that to solve the first puzzle, that is, to obtain large movements in employment accompanied by only small wage changes, we need a fairly flat labour supply function. One way to get this result is to assume that workers are constrained to work for a fixed amount of time or not at all (Hansen 1985). In this work we will consider this indivisible labour assumption as well.

In respect to the productivity puzzle, one way of reducing the model correlation

[1]Dynamic general equilibrium models with efficiency wages have been studied before, notably by Danthine and Donaldson (1990) and Uhlig and Xu (1996).

[2]The higher volatility exhibited by the labour input relatively to the volatility of real wages can be confirmed for example in Stadler (1994).

[3]See for example Christiano and Eichenbaum (1992).

between employment and the real wage is to introduce some mechanism that produce real wage rigidity. In this work, as we said before, we consider the existence of efficiency wages.

Briefly, the purpose of this work is to build a general equilibrium model with efficiency wages, indivisible labour and increasing returns and to find whether it is able to endogenously create time series for real wages and employment exhibiting persistent fluctuations with a cyclical behavior similar, from a qualitatively point of view, to the one empirically observed.

The structure of this paper is as follows: we present the model in Section 2 and Section 3 derives the necessary conditions to have endogenous deterministic fluctuations. Section 4 simulates the model and Section 5 discusses the main results.

2. The model

2.1. Assumptions

In this work, we consider a general equilibrium overlapping generations (OLG) model with perfect foresight.

In each period t, (with $t = 0...\infty$), there is a single final produced good, either used as a consumption good or as a capital good. Population is constant over time and is composed by a finite number of households living two periods: they are young in the first period and old in the second.

In each period t, there are L young households, born at t, and L old households, born at $t-1$. All households possess identical preferences, described by a welfare function additively separable between consumption (current and future) and effort. Households consume both in youth and in the old age, but they only work when young.

The utility function is homogeneous of degree one over current and future consumption. Young households can be employed or unemployed. If employed, the portion of their wages income not used for immediate consumption is saved in productive capital goods. When they get old, the capital goods are rented to the firms, that use them as a productive input, and all resulting rents are spent in consumption. If unemployed, they do not receive any compensation for that state and, also, they do not have the possibility of insuring themselves against unemployment risk.

Preferences of young consumers, identical for all individuals, are represented by the compound function

$$W = U(c_{1,t}, c_{2,t+1}) - \gamma V(e_t)$$

where c_1 and c_2 denote, respectively, present and future consumption of the only good produced in the economy. The function U is twice-differentiable, increasing and homogeneous of degree one. All variables are expressed in real terms. Workers do not dislike working[4], but they dislike effort and γ is a disutility scale parameter[5].

Technology is also identical for every firm and it is a function of capital and labour. We assume full depreciation of capital after one period of utilization in production[6]. We also consider that the social technology may differ from the technology faced by the representative agent, due to external effects in the production process. We use externalities to combine a social technology that displays increasing returns to scale with a competitive behavior by individual producers.

According to this view, the private total factor productivity is positively affected by an increase in the aggregate capital and labour stocks. At least two informal arguments exist to legitimate such a formulation. The first one accounts for the contribution of aggregate capital to the externalities and is now standard: positive externalities come from learning spillovers created through the production activities (see Arrow, 1962). The accumulation of productive capital increases public knowledge and, hence, factor productivities. The same sort of argument may be used to justify the positive externalities coming from the aggregate labour stock through learning by doing, associated to the level of aggregate production. A second justification for labour externalities is related to the exchange process in a market economy and, in particular, to the thick market externalities modeled by Diamond (1982)[7]. Following Cazzavilan et al (1998), when dealing with short run fluctuations, thick market externalities may appear more relevant than those originated through learning by doing, so we should expect externalities coming

[4] This simplification is just to avoid another parameter in the model; of course, it has no consequences.

[5] We consider that γ is known to the firm and that its value is equal to all workers.

[6] It has been argued that, given a relatively long (necessary in the OLG setting) lenght of the time period, the rate of depreciation should be high, and we set it equal to one, as in many other OLG models.

[7] The higher the level of employment, the higher the probability of a fast matching between sellers and buyers in the labour market. In this case, as search costs decrease, total factor productivity increases.

from labour to be more important than capital externalities. This is the kind of assumption we use in this paper.

Briefly, in each period t a consumption good y is produced combining labour in efficiency units en (supplied by the young generation) and capital k, rented from the old generation. The gross production function F exhibits constant returns to scale (homogeneous of degree one) and allows for input substitution. We assume that each producer benefits from positive productive externalities: the input services of labour and capital are, respectively, $E\left(\overline{k}, \overline{en}\right) k$ and $E\left(\overline{k}, \overline{en}\right) en$, where $\overline{k}$ and $\overline{en}$ are, respectively, the average capital stock and the average employment in efficiency units of the economy and where e is the effort per worker. Moreover, when the average capital or labour goes up, the productivity of each input also moves up.

Because we assume an external effect that is compatible with perfect competition in the product market, each (small enough) identical producer takes $\overline{k}$ and $\overline{en}$ as given when optimizing. Of course, $\overline{k} = k$ and $\overline{en} = en$ in equilibrium. Therefore, the quantity of y produced is

$$\begin{aligned} y_t &= F\left(E\left(\overline{k_t}, \overline{e_t n_t}\right) k_t, E\left(\overline{k_t}, \overline{e_t n_t}\right) e_t n_t\right) \\ &= E\left(\overline{k_t}, \overline{e_t n_t}\right) F\left(k_t, e_t n_t\right) = E\left(\overline{k_t}, \overline{e_t n_t}\right) e_t n_t f\left(x_t\right) \end{aligned}$$

The second and third equalities are deduced from the homogeneity of F, where $x = \frac{k}{en}$ is the capital labour ratio.

The externality function E is homogeneous of degree v (which should be small according to the estimates of Caballero and Lyons (1992)) and can be written, therefore, as $E\left(\overline{k}, \overline{en}\right) = A\left(\overline{en}\right)^v \Psi\left(\overline{x}\right)$, where A is a scaling factor. The parameter v measures the degree of increasing returns to scale, which is zero if no externalities are present. Social returns to scale are equal to $1 + v$ and are increasing, while private returns to scale are constant. The assumption of perfectly competitive product market can be maintained. Moreover, E is increasing in $\overline{k}$, i.e., the contribution of capital to the externalities is $\Psi_k = E'k/E \geq 0$, and increasing in $\overline{en}$. This means that the contribution of labour to the externalities is $\Psi_n = v - \Psi_k \geq 0$.

2.2. Employment contracts

At the beginning of each period, firms choose, from the pool of workers (L - labour supply), the number of employees they want to hire (n - labour demand) for the current period. To each one of its employees, each firm offers employment contracts specifying that the agreed wage w_t will be paid only if a specified productivity (e - effort) level is achieved, i.e., w and e are specified in an enforceable contract. Firms also choose the stock of capital.

The firm then chooses k_t, n_t, w_t and e_t to maximize real profits subject to the restriction that households are willing to work only if their welfare $W = U(c_{1,t}, c_{2,t+1}) - \gamma V(e_t)$ is at least equal to some level W^*, which is the utility the firm must offer in order to hire employees - the participation constraint. W^* is the utility level of being unemployed and, since we consider no unemployment benefits, W^* must be zero.

Assume for now that the indirect utility from consumption is a function of the current real wage and the future real interest factor: $U(c_{1,t}, c_{2,t+1}) = U^*(w_t, \theta_{t+1}) = w_t u(\theta_{t+1})$ (we derive this result in the following section).

$$\begin{aligned}
\max_{k_t, n_t, e_t, w_t} \pi_t &= y_t - w_t n_t - \theta_t k_t \\
s.t.\ y_t &= F(E_t k_t, E_t e_t n_t) = E_t e_t n_t f(x_t) \\
W &= w_t u(\theta_{t+1}) - \gamma V(e_t) \geq 0
\end{aligned}$$

The first order conditions are:

$$E_t f'(x_t) = \theta_t \tag{2.1}$$

$$E_t [f(x_t) - f'(x_t) x_t] = \frac{w_t}{e_t} \tag{2.2}$$

$$E_t [f(x_t) - f'(x_t) x_t] = \lambda \frac{\gamma V'(e_t)}{n_t} \tag{2.3}$$

$$n_t = \lambda u(\theta_{t+1}) \tag{2.4}$$

$$w_t u(\theta_{t+1}) - \gamma V(e_t) \geq 0, \ \lambda \geq 0 \tag{2.5}$$

From (2.3) or (2.4), the Lagrangian multiplier λ is positive and hence, from (2.5), the participation constraint is binding: $w_t u(\theta_{t+1}) = \gamma V(e_t)$. Total welfare is then zero and workers will be indifferent between being employed or unemployed, and the reservation wage is given by $w_t = \frac{\gamma V(e_t)}{u(\theta_{t+1})}$.

From the above expression we know that

$$\frac{\partial w_t}{\partial e_t} = \frac{\gamma V'(e_t)}{u(\theta_{t+1})}$$

Substituting the above expression into (2.3) and (2.4), we get

$$E_t\left[f(x_t) - f'(x_t)x_t\right] = \frac{\partial w_t}{\partial e_t}$$

and combining with (2.2), we obtain the condition that the elasticity of effort to wages equals one (the Solow's 1979 condition):

$$\varepsilon_{e,w} = \frac{\partial e_t}{\partial w_t}\frac{w_t}{e_t} = 1 \tag{2.6}$$

From the zero profit condition, we define the capital share $b(x)$ [8] $= \frac{f'(x)x}{f(x)}$ and the labour share $a(x) = \frac{[f(x)-f'(x)x]}{f(x)} = 1 - b(x)$.

If e and w are specified in an enforceable employment contract, the firm will never offer a wage above the minimum required to hire employees at that effort level - the reservation wage and no involuntary unemployment can result from the payment of an efficiency wage.

In this approach, we implicitly assumed perfect information or perfect monitoring of workers' effort. Alternatively, one can consider an employment contract that specifies a wage contingent on an observed signal of workers' effort. We do this exercise in Appendix A. There we show that the only difference between the perfect and the imperfect information cases is a scale effect on the wage equation, sufficient to create involuntary unemployment. It will not, however, affect the dynamics of the model. Since, we are basically interested in the study of the dynamics of the model, and just to avoid another constant in the model, from now on we continue to assume the perfect information case.

2.3. Savings decision

Given that wage and effort were already decided, households only have to decide c_1 and c_2 to maximize welfare subject to budget constraint.

[8] $\pi = y - wn - \theta k = 0 \Leftrightarrow y = wn - \theta k \Leftrightarrow 1 = \frac{wn}{Eenf(x)} + \frac{\theta xen}{Eenf(x)}$

The capital share $b = \frac{\theta xen}{Eenf(x)} = \frac{\theta x}{Ef(x)} = \frac{Ef'(x)x}{Ef(x)} = \frac{f'(x)x}{f(x)}$

$$\max_{c_{1,t},c_{2,t+1}} W = U(c_{1,t}, c_{2,t+1}) - \gamma V(e_t)$$
$$s.t.\ w_t = c_{1,t} + \frac{c_{2,t+1}}{\theta_{t+1}}$$

The first order condition of the above problem is

$$\frac{U_{c_{1,t}}}{U_{c_{2,t+1}}} = \theta_{t+1} \tag{2.7}$$

From (2.7), and by homogeneity of the function U, we have

$$\frac{U_{c_{1,t}}}{U_{c_{2,t+1}}} = g\left(\frac{c_{2,t+1}}{c_{1,t}}\right) = \theta_{t+1} \tag{2.8}$$

Using the above expression the intertemporal consumption structure is

$$g^{-1}(\theta_{t+1}) = \frac{c_{2,t+1}}{c_{1,t}} \tag{2.9}$$

We denote the elasticity of intertemporal substitution between c_2 and c_1 by $\frac{1}{\sigma}$

$$\frac{1}{\sigma(\theta)} = \frac{\left[g^{-1}(\theta)\right]' \theta}{g^{-1}(\theta)} > 0$$

Using (2.9) and the budget constraint, present and future consumption are, respectively, given by:

$$c_{1,t} = \frac{\theta_{t+1}}{\theta_{t+1} + g^{-1}(\theta_{t+1})} w_t = \left[1 - s(\theta_{t+1})\right] w_t \tag{2.10}$$

$$c_{2,t+1} = s(\theta_{t+1})\,\theta_{t+1} w_t \tag{2.11}$$

where $[1 - s(\theta_{t+1})]$ is the marginal propensity to consume of the young and $s(\theta_{t+1})$ is the savings rate

$$s(\theta_{t+1}) = \frac{g^{-1}(\theta_{t+1})}{\theta_{t+1} + g^{-1}(\theta_{t+1})} \tag{2.12}$$

From (2.9), (2.10) and (2.11) the optimum intertemporal consumption structure is

$$\frac{c_{2,t+1}}{c_{1,t}} = \frac{s(\theta_{t+1})\,\theta_{t+1}}{1 - s(\theta_{t+1})} = g^{-1}(\theta_{t+1}) \tag{2.13}$$

For future reference, the elasticity of the savings rate relative to the real interest factor is given by

$$\varepsilon_{s,\theta} = \frac{s^{'}(\theta)\,\theta}{s(\theta)} = \left[\frac{1}{\sigma(\theta)} - 1\right]\left[1 - s(\theta)\right] \tag{2.14}$$

The savings rate of the young is an increasing function of the real interest rate if $\sigma(\theta) < 1$ and a decreasing function if $\sigma(\theta) > 1$ and it is independent of the real interest rate if $\sigma(\theta) = 1$.

We now define the total utility derived from the optimal bundle U, which, given homogeneity of degree one, can be written as

$$U(c_{1,t}, c_{2,t+1}) = U^*(w_t, \theta_{t+1}) = w_t u(\theta_{t+1})$$

which is the utility function used in Section 2.2.

U is linear in the wage and increasing in the real interest factor θ, where the elasticity of U in respect to θ is[9]

$$\varepsilon_{U^*,\theta} = \varepsilon_{u,\theta} = s(\theta) \tag{2.15}$$

To obtain the effort function, we solve the bidding participation constraint in order to e

$$e_t = V^{-1}\left(\frac{U^*(w_t, \theta_{t+1})}{\gamma}\right) = e(w_t, \theta_{t+1}) \tag{2.16}$$

To find the wage set by firms, we use the Solow condition

$$\varepsilon_{e,w} = \frac{\partial V^{-1}(w,\theta)}{\partial w}\frac{w}{V^{-1}(w,\theta)} = \frac{\partial V^{-1}(w,\theta)}{\partial U^*(w,\theta)}\frac{U^*(w,\theta)}{V^{-1}(w,\theta)}\frac{\partial U^*(w,\theta)}{\partial w}\frac{w}{U^*(w,\theta)} = 1$$

Using (2.15), the Solow condition implies that

$$\frac{\partial V^{-1}(w,\theta)}{\partial U^*(w,\theta)}\frac{U^*(w,\theta)}{V^{-1}(w,\theta)} = 1 \tag{2.17}$$

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[9] Differentiating U^* we have that: $U^{*\prime} = U_{c_1}\frac{\partial c_1}{\partial \theta} + U_{c_2}\frac{\partial c_2}{\partial \theta}$. $U^{*\prime} = w\left[-U_{c_1}s'(\theta) + U_{c_2}s(\theta) + U_{c_2}s'(\theta)\,\theta\right]$ which, using (2.8), simplifies to $U^{*\prime} = wU_{c_2}s(\theta)$. From Euler's identity, we know that $U^* = U_{c_1}c_1 + U_{c_2}c_2 = w\left[U_{c_1}\left[1 - s(\theta)\right] + U_{c_2}s(\theta)\,\theta\right]$, which, again using (2.8), simplifies to $U^* = wU_{c_2}\theta$. Hence, $U^{*\prime} = \frac{U^*}{\theta}s(\theta)$ and $\frac{U^{*\prime}\theta}{U^*} = \varepsilon_{U^*,\theta} = s(\theta)$.

which, implicitly, defines the optimal wage as a function of the real interest rate

$$U^* (w, \theta) \frac{\partial V^{-1} (w, \theta)}{\partial U^* (w, \theta)} - V^{-1} (w, \theta) = 0 \Leftrightarrow h (w, \theta) = 0 \tag{2.18}$$

or

$$\theta_{t+1} = \theta (w_t)$$

From (2.18), the elasticity of the wage related to the real interest rate is given by[10]

$$\varepsilon_{w,\theta} = \frac{\partial w}{\partial \theta} \frac{\theta}{w} = -s (\theta)$$

and, as a consequence

Proposition 2.1: Constant effort

The effort will be constant in equilibrium.

Proof: From (2.16) and (2.18), $e_t = e (w_t, \theta (w_t))$*; then, the total effect of wages on effort is* $\varepsilon_{e,w} + \varepsilon_{e,\theta} \varepsilon_{\theta,w}$*. From (2.15), (2.16) and (2.17), we have that:* $\varepsilon_{e,w} = \frac{\partial e}{\partial w} \frac{w}{e} = \frac{\partial V^{-1}(w,\theta)}{\partial U^*(w,\theta)} \frac{U^*(w,\theta)}{V^{-1}(w,\theta)} \frac{\partial U^*(w,\theta)}{\partial w} \frac{w}{U^*(w,\theta)} = \frac{\partial V^{-1}(w,\theta)}{\partial U^*(w,\theta)} \frac{U^*(w,\theta)}{V^{-1}(w,\theta)} = \Phi$ *and* $\varepsilon_{e,\theta} = \frac{\partial e}{\partial \theta} \frac{\theta}{e} = \frac{\partial V^{-1}(w,\theta)}{\partial U^*(w,\theta)} \frac{U^*(w,\theta)}{V^{-1}(w,\theta)} \frac{\partial U^*(w,\theta)}{\partial \theta} \frac{\theta}{U^*(w,\theta)} = \Phi \varepsilon_{U^*,\theta} = \Phi s (\theta)$*. Hence,* $\varepsilon_{e,w} + \varepsilon_{e,\theta} \varepsilon_{\theta,w} = \Phi + \Phi s \left(-\frac{1}{s}\right) = 0.$

We can aggregate individuals' behavior to characterize the dynamics of the economy. The capital stock in period $t+1$ is the amount saved by young individuals in period t. Thus[11],

$$k_{t+1} = n_t w_t s (\theta_{t+1}) \tag{2.19}$$

[10] From (2.18)
$\frac{\partial w}{\partial \theta} \frac{\theta}{w} = -\frac{h_\theta}{h_w} \frac{\theta}{w} = -\frac{\left[\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)/\partial \theta\right] U^*(w,\theta) + \frac{\partial U^*(w,\theta)}{\partial \theta} \frac{\partial e_t}{\partial U^*(w,\theta)} - \frac{\partial e_t}{\partial \theta}}{\left[\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)/\partial w\right] U^*(w,\theta) + \frac{\partial U^*(w,\theta)}{\partial w} \frac{\partial e_t}{\partial U^*(w,\theta)} - \frac{\partial e_t}{\partial w}} \frac{\theta}{w} = -\frac{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)/\partial \theta}{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)/\partial w} \frac{\theta}{w}$. Since $\frac{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)}{\partial \theta} = \frac{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)}{\partial U^*(w,\theta)} \frac{\partial U^*(w,\theta)}{\partial \theta}$ and $\frac{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)}{\partial w} = \frac{\partial \left(\frac{\partial e_t}{\partial U^*(w,\theta)}\right)}{\partial U^*(w,\theta)} \frac{\partial U^*(w,\theta)}{\partial w}$, then the above expression simplifies to $\frac{\partial w}{\partial \theta} \frac{\theta}{w} = -\frac{\frac{\partial U^*(w,\theta)}{\partial \theta}}{\frac{\partial U^*(w,\theta)}{\partial w}} \frac{\theta}{w} = -\frac{\varepsilon_{U^*,\theta}}{\varepsilon_{U^*,w}} = -s (\theta)$.

[11] To use the same notation k (n) for the aggragate economy and for the individual firms, we consider, without any loss of generality, that there is a continuum of identical firms of total mass 1. Hence, $K = \int_0^1 k \partial \varphi = k$ and $N = \int_0^1 n \partial \varphi = n$, where φ is the index of firms, K (N) is the aggregate capital (employment) and k (n) is the capital (employment) in each firm.

2.4. The equilibrium dynamic system

To solve the model, we consider a symmetric equilibrium $\overline{k} = k$ and $\overline{en} = en$. Hence, the first order conditions from the producers' problem are:

$$A\left(en_t\right)^v \Psi\left(x_t\right) f'\left(x_t\right) = \theta_t \tag{2.20}$$

$$A\left(en_t\right)^v \Psi\left(x_t\right) e\left[f\left(x_t\right) - f'\left(x_t\right) x_t\right] = w_t \tag{2.21}$$

The other equations necessary to solve the model are the demand for capital (2.19), which yields

$$x_t = \frac{k_t}{en_t} = \frac{n_{t-1} w_{t-1} s\left(\theta_t\right)}{en_t} \tag{2.22}$$

and the relation between the rate of return and the wage (2.18)

$$\theta_t = \theta\left(w_{t-1}\right) \tag{2.23}$$

The above system is composed by four equations and six variables (θ_t, k_t, k_{t+1}, n_t, n_{t-1}, w_t, w_{t-1}) which means that the system can be reduced to a two dimensional nonlinear system in k and n, where k is a predetermined variable and n a non-predetermined variable. However, since we are interested in studying the cyclical properties of wages and employment, we express the system (2.20-2.23) in terms of wages and employment. Of course, the dynamical properties and the number of predetermined variables are the same in both systems.

Hence, we substitute equations (2.22) and (2.23) into (2.20) and (2.21). The result is a two dimensional nonlinear system in wage and employment, which means that, given some initial conditions for the two variables, we determine the optimal trajectories for the future.

Definition 2.1:

An intertemporal equilibrium with perfect foresight is a sequence of (w_{t-1}, n_{t-1}) *for* $t = 1, 2, ...$ *such that*

$$w_t = g\left(w_{t-1}, n_{t-1}\right) \tag{2.24}$$

$$n_t = h\left(w_{t-1}, n_{t-1}\right) \tag{2.25}$$

3. Steady state, local dynamics and bifurcation analysis

In this section, we analyse the emergence of endogenous fluctuations, and we start by finding the steady state of the system (2.24-2.25).

First of all, note that a steady state ($n_t = n_{t-1} = n$, $w_t = w_{t-1} = w$), for all t, is a solution of (2.24-2.25) if and only if

$$\begin{aligned} w &= g(w, n) \\ n &= h(w, n) \end{aligned}$$

In view of (2.20) - (2.23), finding a steady state amounts to finding a value for w that satisfies

$$\frac{e\left[f(x) - f'(x)x\right]}{f'(x)} = \frac{w}{\theta(w)} \text{ where } x = \frac{ws\left[\theta(w)\right]}{e} \tag{3.1}$$

For future reference, using (3.1), we derive the elasticity of substitution between factors ρ

$$\frac{1}{\rho(x)} = -\frac{f(x)f''(x)x}{f'(x)\left[f(x) - f'(x)x\right]} \tag{3.2}$$

From (2.16-2.18), we know that the relation between real interest rate and wage is parameterised on the scale parameter γ, hence we ensure the existence of a steady state, namely with $w = 1$, by choosing an appropriate value of the scaling parameter γ, so that:

$$\frac{e\left[f(x) - f'(x)x\right]}{f'(x)} = \frac{1}{\theta} \tag{3.3}$$

where $x = \frac{s(\theta)}{e}$, $e = e(1, \theta)$ and $\theta = \theta(1, \gamma)$.

We now use the employment equation (2.21) and the steady state value for w and we ensure the existence of a steady state for employment, namely $n = 1$, by choosing an appropriate value of the scaling parameter A, so that:

$$Ae^{v+1}\Psi(x)\left[f(x) - f'(x)x\right] = 1 \tag{3.4}$$

Proposition 3.1: Existence of the steady state

Let γ and A be the unique solutions of (3.3) and (3.4); then $(w, n) = (1, 1)$ is a stationary solution of the dynamical system (2.24-2.25).

Assuming that a unique steady state exists[12], we proceed to local stability and bifurcation analysis by the use of the geometrical method developed by Grandmont et al (1998).

To study the stability proprieties of our model, we first compute the Jacobian matrix of the system (2.24-2.25) evaluated at the steady state point. Following Cazzavillan et al (1998), we should expect externalities coming from labour to be more important than capital externalities. Therefore we consider the extreme case of having no external effects coming from capital, that is: $\Psi_k = 0$ and $v = \Psi_n \geq 0$[13].

$$J = \begin{bmatrix} g_w(w,n) & g_n(w,n) \\ h_w(w,n) & h_n(w,n) \end{bmatrix}$$

where

$$g_w(w,n) = \frac{\partial w_t}{\partial w_{t-1}} = \frac{(1+\varepsilon_{s,\theta}\varepsilon_{\theta,w})\,v}{\rho v + a} + \frac{\rho v + a - 1}{\rho v + a}\varepsilon_{\theta,w}$$

$$g_n(w,n) = \frac{\partial w_t}{\partial n_{t-1}} = \frac{v}{\rho v + a}$$

$$h_w(w,n) = \frac{\partial n_t}{\partial w_{t-1}} = \frac{(1+\varepsilon_{s,\theta}\varepsilon_{\theta,w})\,a + \varepsilon_{\theta,w}\rho}{\rho v + a}$$

$$h_n(w,n) = \frac{\partial n_t}{\partial n_{t-1}} = \frac{a}{\rho v + a}$$

The trace and determinant are, then, given by

$$T = \frac{a + v/\sigma}{\rho v + a}(1+\varepsilon_{\theta,w}) - T_1\varepsilon_{\theta,w}$$

where[14] $1 < T_1 = \frac{1+v(1-\rho)}{\rho v + a} < 2$;

$$D = -D_1\varepsilon_{\theta,w}$$

where[15] $0 < D_1 = \frac{1-a}{\rho v + a} < 1$.

D_1 and T_1 are, respectively, the values of D and T, when $\varepsilon_{\theta,w} = -1$.

[12] Appendix B addresses the issue of uniqueness versus multiplicity of steady states.

[13] Other cases, however, do not change the stability properties of the model.

[14] For v small $T_1 > 1$ since $1 - a > v(2\rho - 1)$ and $T_1 < 2$ since $2a - 1 > -v(3\rho - 1)$.

[15] $D_1 > 0$ because $a < 1$; $D_1 < 1$ because $a > 0.5$

From now on we assume[16]:

Assumption 3.1: $\sigma = 1$

The elasticity of intertemporal substitution between current and future consumption is one.

From the above expressions, we easily conclude that $D > 0$ (since $\varepsilon_{\theta,w} < 0$ and $D_1 > 0$) and that for a small amount of increasing returns[17] $T > 0$.

The geometrical method of Grandmont et al (1998) proceeds by dividing the plane (T, D) into several regions defined by the values assumed by the local eigenvalues of the dynamic system. There is a local eigenvalue equal to 1, when $1 - T + D = 0$, which is represented in the plane (T, D) by the line AC in Figure 3.1. Similarly, one eigenvalue is equal to -1 when (T, D) belongs to AB $(1 + T + D = 0)$. The two local eigenvalues are complex conjugate, of modulus 1, on the segment BC $(D = 1$ and $|T| < 2)$. The deterministic dynamics is locally stable - a sink (the two local eigenvalues have modulus less than 1) if and only if the point (T, D) lies in the interior of the triangle ABC, defined by $|T| < |1 + D|$, $|D| < 1$. The stationary state is a saddle in the region of the plane on the left of AB and on the right of AC, defined by $|T| > |1 + D|$. It is a source in all other cases, i.e., when $|T| < |1 + D|$, $|D| > 1$.

The basic idea of the method is to choose one parameter from the set of parameters - the bifurcation parameter, and to analyse how the trace and determinant evolves when the bifurcation parameter changes.

We consider $\varepsilon_{\theta,w}$ as the bifurcation parameter. Then, from the above expressions for the trace and the determinant, the locus $(T(\varepsilon_{\theta,w}), D(\varepsilon_{\theta,w}))$ is defined through the following relation (Δ line):

$$D = \Delta(T) = D_1 \left[\frac{(\rho v + a) T - (v + a)}{(\rho v + a) T_1 - (v + a)} \right]$$

[16] Obviously, we could have continued the analysis considering a general utility function. We know that the elasticity of intertemporal substitution could affect the stability of the system, since the trace depends on it, however, this is the standard case consider in the literature, and the analysis of its importance is not the focus of this papaer. For these reasons and also because this assumption permits us to make more conclusive statements we assume $\sigma = 1$

[17] An alternative expression for the trace is given by $T = \frac{a+v}{\rho v + a} - \frac{1}{\rho v + a} [(1 - a) - v\rho] \varepsilon_{\theta,w}$, hence, a sufficient condition for $T > 0$ is $v < \frac{1-a}{\rho}$.

where its slope (positive[18]) is equal to

$$\Delta' = \frac{\partial D}{\partial T} = \frac{D_1 (\rho v + a)}{(\rho v + a) T_1 - (v + a)}$$

Note that, the fact of $s \in [0, 1]$ made $\varepsilon_{\theta,w} = -\frac{1}{s} \in [-\infty, -1]$, hence only part of the Δ line is relevant in our model. That is the part beginning at point (T_1, D_1), where T_1 and D_1 define, respectively, the values taken by T and D when $\varepsilon_{\theta,w} = -1$.

We can now also define the ending point of the Δ line, which is, of course, given by the point where $s = 0$ or $\varepsilon_{\theta,w} = -\infty$, i.e., $(T_2, D_2) = (+\infty, +\infty)$.

Before continuing the analysis of the Δ line, lets us consider the case of constant returns to scale ($v = 0$). This is a particular case where the Δ line coincides with the line AC: $D = T - 1$, ($\Delta' = 1$, see Figure 3.1).

Proposition 3.2: The CRS economy

The CRS model degenerates into a non-hyperbolic steady state, where one of the eigenvalues is exactly one and the other is higher than one if $s < \frac{1-a}{a}$ *(thus unstable) and lower than one (thus stable) otherwise. Hence, endogenous fluctuations cannot occur.*

Proof: $D = -D_1 \varepsilon_{\theta,w} = \frac{1}{s}\frac{1-a}{a}$ *and* $D > 1$ iff $s < \frac{1-a}{a}$.

Back to our increasing returns model, the next proposition states the sufficient condition to the emergence of endogenous fluctuations through the occurrence of Hopf bifurcations (see Figure 3.1).

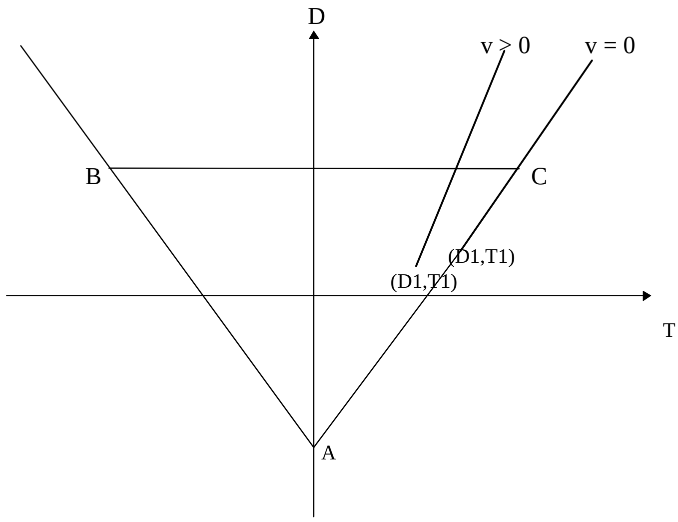
Proposition 3.3: Endogenous fluctuations

If the point (T, D) *crosses the interior of the segment* BC, *one gets, typically, a Hopf bifurcation with an invariant closed curve around the stationary state. To get this result, it is, in this model, sufficient to have the slope of the* Δ *line higher than one.*

Proof: Given that (T_1, D_1) *lies in the interior of the triangle* ABC, $(T_2, D_2) = (+\infty, +\infty)$ *and that the* Δ *line is positively sloped, a sufficient condition for having endogenous fluctuations through the occurrence of Hopf bifurcations is to have the slope of the* Δ *line higher than one* $(\Delta' > 1)$. *Given the above expressions, this will happen iff* $\sigma(\rho - 1) > -1$. *And the bifurcation point* $(D = 1)$ *occurs at* $|\varepsilon_{\theta,w}| = \frac{a+\rho v}{1-a} \Leftrightarrow s^0 = \frac{1-a}{a+\rho v}$. *Since* $D = -D_1 \varepsilon_{\theta,w} = \frac{D_1}{s}$, *we have* $\frac{\partial D}{\partial s} < 0$

[18] In fact, the Δ line is positively sloped if $T_1 > \frac{v+a}{\rho v+a}$, given the above definition for T_1, $\Delta' > 0$ if $T > 0$.

Figure 3.1: The Δ line



in $s \in (0,1)$ *containing* $s^0 = \frac{1-a}{a+\rho v}$. *The steady is locally stable when* $s > s^0$ *and locally unstable when* $s < s^0$. *The steady state undergoes an Hopf bifurcation when* s *goes through* s^0. *The bifurcation can be either subcritical or supercritical. In the first case, an unstable (repelling) invariant curve exists near the steady state on the stable side, but near* s^0 *and coalesces to it at* $s = s^0$. *In the second case, a stable invariant closed curve emerges from the steady state on the unstable side near* s^0, *the size of which shrinks to zero as* $|s - s^0| \to 0$.

Figure 3.1 shows the Δ lines for the cases of constant and increasing returns to scale. This picture is conceived in the neighbourhood of $\rho = 1$, the standard case analised in the literature. In Coimbra (1999) we study the importance of the elasticity of inputs substitution and we show that the analysis around $\rho = 1$ is sufficient, in the sense that it is quite general.

4. Simulation

In this section we obtain equilibrium trajectories of wages and employment displaying endogenous fluctuations and we analyse their co-movements and their relative volatility.

To simulate the model, we consider a CES production function and a Cobb-Douglas utility function $(\sigma = 1)$. In this case, as we showed in Appendix B, the steady state is unique.

We consider the following functional forms:

The production function is given by

$$y = \left[a (en)^{\frac{\rho-1}{\rho}} + (1-a) k^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} E,\ E = A (en)^{v}$$

and the welfare function is[19]

$$W = c_1^{\mu} c_2^{1-\mu} - \gamma \exp(e)$$

The four dynamic equations are given by

$$\begin{aligned}
w_t &= aAn_t^{v-\frac{1}{\rho}} \left[an_t^{\frac{\rho-1}{\rho}} + (1-a) k_t^{\frac{\rho-1}{\rho}} \right]^{\frac{1}{\rho-1}} \\
n_t &= \left(\frac{1-a}{a} \right)^{-\rho} \left(\frac{w_t}{\theta_t} \right)^{-\rho} k_t \\
k_t &= s n_{t-1} w_{t-1} \\
\theta_t &= \left[\frac{\left(\mu^{-\mu} (1-\mu)^{\mu-1} \exp(1) \gamma \right)}{w_{t-1}} \right]^{\frac{1}{1-\mu}}
\end{aligned}$$

With a Cobb-Douglas utility function, real interest rates have no effect on the savings rate, which is 0.5 (see Table 4.1 where we set $\mu = 0.5$), and the elasticity of the wage related to the real interest rate is -0.5. We assume the traditional values (see, for example, Uhlig and Xu 1996) for the share of wn and θk on the real product, respectively 0.64 and 0.36.

The remaining values in Table 4.1 are steady state values for variables and elasticities. We set w and n equal to one and use the two scale parameters A and γ to get this result. As a consequence, the steady state value for k is equal to 0.5 and for θ is equal to 1.125.

[19] We choose this functional form for the disutility of effort because, in this case, we obtain a concave relation between e and w, as is traditional in partial equilibrium efficiency wage models, and also a value for e equal to one in equilibrium.

$\sigma = 1$	$w = 1$
$\mu = 0.5$	$n = 1$
$\varepsilon_{s,\theta} = 0$	$e = 1$
$s = 0.5$	$k = 0.5$
$\varepsilon_{w,\theta} = -0.5$	$\theta = 1.125$
$\varepsilon_{e,w} = 1$	$\gamma = 0.195$
$wn/y = 0.64$	$\theta k/y = 0.36$

Table 4.1: Calibration I

ρ	0.5	2/3	1	2
a	0.78	0.72	0.64	0.56
A	1.91	1.95	2.03	2.06
v $(D = 1)$	0.16	0.12	0.08	0.04

Table 4.2: Calibration II

Using the steady state equations from the first order conditions of the firm we calibrate a using the values for the shares of labour and capital on real product. Hence,

$$a = \frac{0.5^{\frac{\rho-1}{\rho}}}{0.5625 + 0.5^{\frac{\rho-1}{\rho}}}$$

which means that, for different values of the elasticity of input substitution we get, different values for a and different values for A, thus ensuring the same steady state value for employment (see Table 4.2).

We now compute the trace and the determinant[20]

$$T = 1 + \frac{0.72 - v(3\rho - 1)}{0.64 + \rho v}$$

$$D = \frac{0.72}{0.64 + \rho v}$$

From the above expression for the determinant, we easily show that the Hopf

[20] In the general analysis of Section 3, we considered $\varepsilon_{\theta,w} = -\frac{1}{s}$ as the bifurcation parameter. Here, since we calibrated s to be 0.5, it does not make sense to consider a fixed parameter to bifurcate, hence we choose v for that purpose.

bifurcation ($D = 1$ and $|T| < 2$) occurs at[21] $v = 0.08/\rho$. Hence, for different elasticities of substitution we have different values for the increasing returns to scale causing the Hopf bifurcation (see Table 4.2); moreover, there is a substitution between ρ and v to scale to generate that bifurcation. This means that, for larger elasticity of capital and labour substitution, a small amount of increasing returns to scale is needed to generate an Hopf. As we have seen before, when constant returns to scale are considered, endogenous fluctuations cannot emerge.

We first simulate the case of $\rho = 1$, where, at the bifurcation point, social returns to scale are 1.08. The steady state is therefore unstable (a source) for returns to scale lower than 1.08 and stable (a sink) for returns to scale higher than 1.08. This means that we only require a little amount of increasing returns to have endogenous fluctuations.

Figures 4.1 to 4.2 display the simulated results[22] for the starting values $(w_0, n_0) = (1.01, 1)$. We simulate the model for a value of v slightly below 0.08. The steady state is then unstable and by starting from a point close to the steady state, the equilibrium dynamic sequences of both variables converge to an invariant curve[23]. Figure 4.1, depicts the wage and employment against time, for the last fifty periods. Also it shows that the duration of the cycle is fourteen periods and that this cycle repeats itself forever. From this picture, we easily conclude that employment varies much more than wages. Figure 4.2 describes the wage-employment relationship, where the circle implies a zero correlation between wage and employment. These two results are in accordance with the labour market regularities referred in the introduction.

Of course, the explanation for this has already been given: the efficiency wage theory and the indivisible labour assumption greatly improve the results, because the employment variability is generated exclusively by movements in and out of employment - with a fixed labour supply - rather than along the labour supply curve, as in Walrasian models.

[21] Substituting $v = 0.08/\rho$ into the expression for the trace we conclude that $T < 2$ if $v < 0.21$ which we consider to hold, since we want to analise the dynamics of the system for a small amount of increasing returns.

[22] The study of the cyclical behaviour of a macroeconomic variable is usually done by taking percentage deviations of the variable from its trend. Here, we consider that the equilibrium trajectories already represent detrended values of employment and real wages. Indeed, our simulated time series endogenously fluctuates around a steady state solution. Sustained growth is not accounted in our analysis and the need for detrend therefore disappears.

[23] The Hopf bifurcation is supercritical, i.e., the closed invariant curve appears on the unstable side of the system.

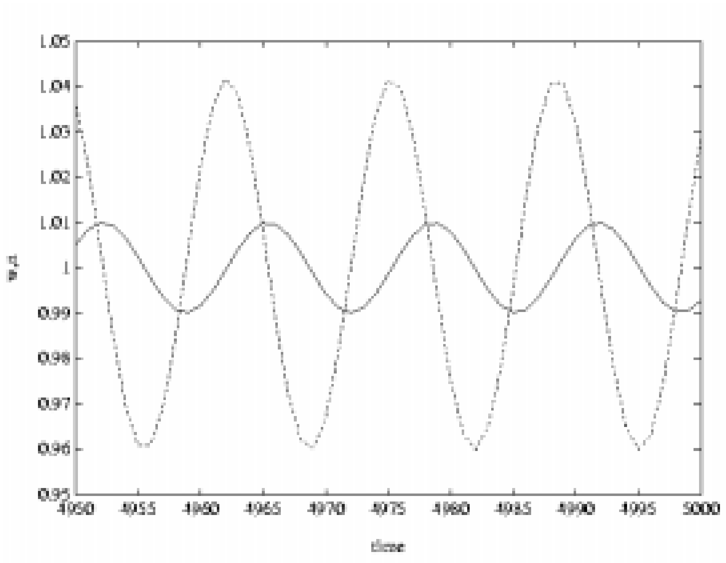
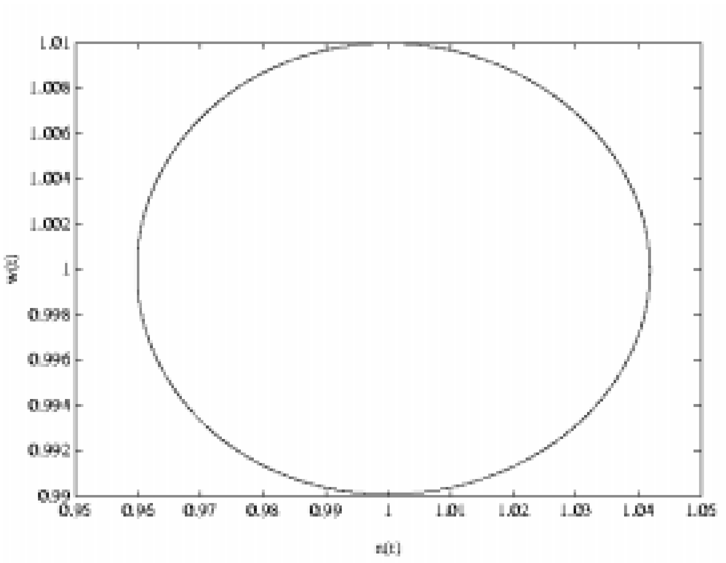


Figure 4.1:

Figure 4.2: Wage and employment



ρ	$STDn/STDw$	$Corrw, n$
0.5	1.83	-0.15
2/3	2.41	-0.07
1	4.12	0
2	9.17	0.05

Table 4.3: Cyclical behaviour

Table 4.3 summarises the cyclical behaviour of wages and employment for the case of $\rho = 1$ and others cases of interest.

The main findings from Table 4.3 are:

i) There is a trade-off between the two empirical results we want to replicate. When the input substitution increases, both the relative standard deviation and the correlation coefficient increases. We start from a low value of ρ (0.5), which shows a relative standard deviation in accordance with the data, but also shows a negative correlation which is not in consonance with the US data. When the elasticity of input substitution is increased (for example, to 1), we get a zero correlation between the two variables, but also a higher relative standard deviation;

ii) For $\rho < 1$, the correlation between the two variables is negative and for $\rho > 1$ the correlation is positive.

5. Discussion

To understand exactly the effect of the indivisible labour and increasing returns on the possibility of the occurrence of endogenous fluctuations, we must compare this model with a more standard one, where households choose hours of work. We do this exercise in Appendix C. There we show that the model is similar to (2.24-2.25). There is, however, a little difference, due to the endogenous hours assumption, which is sufficient to modify the stability properties of the system. In fact, the only difference in the expressions for the trace and determinant is the value for the elasticity of the labour supply.

In fact, in Appendix C Proposition C.1 we show that in a more standard model and for standard parameters the steady state is a saddle. The sufficient condition for this to occur is for the Cobb-Douglas economy:

$$\delta a - v > 0$$

where δ stands for the wage elasticity of labour supply; i.e., $\delta = \frac{\partial w}{\partial h}\frac{h}{w}$.

Since we consider throughout all this paper a small v the above condition will always be satisfied.

Note that the above condition is useful to understand the effect of the indivisible labour hypothesis. In fact, when labour is considered to be indivisible, the above condition simplifies to $v < 0$ which, since we are considering the presence of increasing returns to scale, will never occur. This proves that the Δ line in the model of Sections 2 to 4 lies on the left of the AC line[24], however, it can be placed above or below the line $D = 1$. Thus, also for the Cobb-Douglas economy, we derive the condition for a determinant to be less than one.

$$\delta < \frac{v + a}{1 - a}s - 1$$

With these two conditions we can divide the space (δ, v) into three regions, where each one of them yields a different stability result[25].

Region I in Figure 5.1, corresponds to a saddle point equilibrium, and this is the typical result one gets from the model of Appendix C. Region II and III, respectively the source and the sink regions, can easily describe the model of Section 2 to 4 also shows that the Hopf bifurcation to occur requires a positive amount of increasing returns and that this amount is increasing with the wage elasticity of labour supply.

Just to give an example, consider $a = 0.64$ and $s = 0.5$. As we have already shown, when labour is indivisible the value of v required to get the Hopf bifurcation is just 0.08, increasing to 0.8 when labour is divisible (with $\delta = 1$).

We can now, based on Figure 5.1 state that:

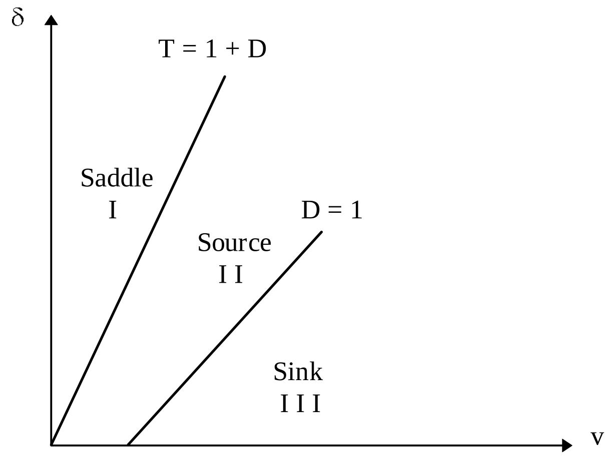
Statement 5.1:
In a Cobb-Douglas economy increasing returns to scale (a small amount) is a necessary but not sufficient condition to the emergence of endogenous fluctuations through the occurrence of Hopf bifurcations; it is also necessary for the wage elasticity of labour supply to be small.

Of course, in the more standard model, endogenous fluctuations are possible, even with constant returns to scale, but, as in Reichlin (1986) a small substitutability between factors is needed. As in Cazzavilan et al (1998) the introduction of a productive externality on the technological side is able to establish the possibility

[24]The same could occur in the standard model developed in Appendix C but, it requires a huge v; in particular higher than δa (for the case of $a = 0.64$ and $\delta = 1$ the condition is $v > 0.64$).

[25]Recall that both D and T are positive.

Figure 5.1: Stability analysis



of endogenous fluctuations, even when the elasticity of inputs substitution is relatively large. But the amount of increasing returns necessary to get this result is still large. The introduction of the indivisible labour hypothesis is what we needed to get endogenous fluctuations with both an elasticity of inputs substitution and an amount of increasing returns in consonance with empirical evidence.

To understand the emergence of endogenous fluctuations it is common practice (see for example Farmer and Guo (1994)) the use of the behaviour of the labour market, in particular the slopes of both labour demand and labour supply.

In terms of the elasticities of demand and supply of labour, again for the Cobb-Douglas case, the expression for a determinant equals to one is given by:

$$\left(1+\varepsilon^{d}\right) = \left(1+\varepsilon^{s}\right)\chi, \text{ where } \chi = \frac{1-a}{s} < 1 \tag{5.1}$$

where $\varepsilon^{d} = a + v - 1$ and $\varepsilon^{s} = \delta$.

The next proposition considers three different cases:

Proposition 5.1:

1) If $\varepsilon^{s} = 0$, *then (5.1) is verified for* $\varepsilon^{d} < 0$;

2) if $\varepsilon^{s} > 0$, *but a low value such that* $\left(1+\varepsilon^{s}\right)\chi - 1 < 0$, *then (5.1) is also verified for* $\varepsilon^{d} < 0$. *However, the condition* $\left(1+\varepsilon^{s}\right)\chi - 1 < 0$ *only occurs for small (near zero) elasticities of labour supply.*

3) if $\varepsilon^s > 0$, such that $(1+\varepsilon^s)\chi - 1 > 0$, which is the most probable case to occur, then (5.1) is verified for $\varepsilon^d > 0$.

Therefore, with indivisible labour, we can get endogenous fluctuations, in consonance with real data, with a negatively sloped labour demand (and a low amount of increasing returns). When labour is divisible, a positively sloped labour demand is necessary to fulfill condition (5.1). Since the elasticity of labour demand is given by $a + v - 1$, this will be positive for large amounts of increasing returns.

References

[1] Arrow, K. (1962), "The Economic Implications of Learning by Doing ", *Review of Economic Studies,* 29, pp 155-173.

[2] Caballero, R. and R. Lyons (1992), "External Effects in US Procyclical Productivity", *Journal of Monetary Economics*, 29, pp 209-225.

[3] Christiano, L. J. and M. Eichenbaum (1992), "Current Real Business-Cycle Theories and Aggregate Labor-Market Fluctuations", *American Economic Review*, 82(3), pp 430-50.

[4] Cazzavillan, G, Lloyd Braga, T. and Pintus, P. (1998), "Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production", *Journal of Economic Theory*, 80, pp 60-107.

[5] Coimbra, R. (1999), "Expectations-Driven Business Cycles Under the Efficiency Wages Hypothesis", *Ph.D dissertation,* Chap 6, University of York, UK.

[6] Danthine, J. and Donaldson, J. (1990), "Efficiency Wages and the Business Cycle Puzzle", *European Economic Review*, 34, pp 1275-1301.

[7] Diamond, P. (1982), "Aggregate Demand Management in Search Equilibrium", *Journal of Political Economy*, 90, pp 881-894.

[8] Dunlop, J. (1938), "The movement of Real Wages and Money Wages", *Economic Journal*, 48, pp 413-34.

[9] Farmer, R and Guo, J. (1994), "Real Business Cycles and the Animal Spirits Hypothesis", *Journal of Economic Theory*, 63, pp 42-72.

[10] Grandmont, J. M. (1985), "On endogenous Competitive Business Cycle", *Econometrica*, 53, pp 995-1046.

[11] Grandmont, J. M., Pintus, P. and R. de Vilder (1998), "Capital-Labour Substitution and Competitive Nonlinear Endogenous Business Cycles", *Journal of Economic Theory*, 80, pp 14-59.

[12] Hansen, G. (1985), "Indivisible Labor and the Business Cycle", *Journal of Monetary Economics*, 16, pp 309-25.

[13] Reichlin, P. (1986), "Equilibrium Cycles in an Overlapping Generations Economy with Production", *Journal of Economic Theory*, 40(1), pp 89-102.

[14] Solow, R. (1979), "Another possible source of wage stickiness", *Journal of Macroeconomics*, 1, pp 79-82.

[15] Stadler, G. (1994), "Real Business Cycles", *Journal of Economic Literature*, 32, pp 1750-83.

[16] Uhlig H. and Y. Xu (1996), "Effort and the cycle: Cyclical Implications of Efficiency Wages", *mimeo.*

A. Appendix: Incomplete contracts

Usually a firm has some problems in actually measuring the productivity of individual employees in a way that is sufficiently objective for it to form an enforceable contract. In this sense, now we consider that firms face a certain probability of having their workers shirking and not providing any effort at all. In this case, firms do not observe the effort of a particular worker ($e > 0$ or $e = 0$) but, for each worker, firms observe some signal s of his effort. This signal can take two values - s_1 (high effort) and s_2 (low effort) - and its law depends on the level of effort, according to:

$$\begin{aligned} pr\left(s = s_1 \left| e > 0\right.\right) &= 1,\ pr\left(s = s_2 \left| e > 0\right.\right) = 0 \\ pr\left(s = s_1 \left| e = 0\right.\right) &= \varphi,\ pr\left(s = s_2 \left| e = 0\right.\right) = 1 - \varphi \end{aligned}$$

where $0 < \varphi < 1$ is the probability of shirking with success.

Since the firm can observe the signal, the most general contract that a firm can offer is a contract contingent on s. A firm designs a contract that offers:

$$\begin{cases} w > 0 & if \quad s = s_1 \\ w = 0 & if \quad s = s_2 \end{cases}$$

Hence, the utility from the three alternatives that the households face are:

$$W = \begin{cases} U^*(w_t, \theta_{t+1}) - \gamma V(e) & if \quad not\ shirking \\ \varphi U^*(w_t, \theta_{t+1}) & if \quad shirking \\ 0 & if \quad not\ working \end{cases}$$

The problem faced by the firm is, then, the same as before. However, W^* is now the welfare level of the shirking worker, since the firm wants to offer an incentive compatible contract, i.e., one where the non shirking strategy is optimal[26]

$$(1-\varphi) U^*(w_t, \theta_{t+1}) \geq \gamma V(e)$$

Thus, firms' problem is now given by

$$\begin{aligned} \max_{k_t, n_t, e_t, w_t} \pi_t &= y_t - w_t n_t - \theta_t k_t \\ s.t.\ y_t &= F(E_t k_t, E_t e_t n_t) = E_t e_t n_t f(x_t) \\ W - W^* &= (1-\varphi) U^*(w_t, \theta_{t+1}) - \gamma V(e) \geq 0 \end{aligned}$$

The first order conditions are again (2.1) to (2.3) and also:

$$n_t = \lambda (1-\varphi) U^{*\prime}(w_t, \theta_{t+1}) \tag{2.4'}$$

$$(1-\varphi) U^*(w_t, \theta_{t+1}) - \gamma V(e_t) \geq 0, \ \lambda \geq 0 \tag{2.5'}$$

Again, from (2.3) or (2.4'), we have $\lambda > 0$, hence, from (2.5') the constraint is binding: $(1-\varphi) U^*(w_t, \theta_{t+1}) = \gamma V(e_t)$.

Notice that the only difference between this relation and the correspondent in the perfect information case is φ, which is zero in the latter case. This means that, for a given effort and future real interest factor, the wage will be higher in the

[26]This contract is also individually rational, in the sense that workers prefer working to not working - $U^*(w_t, \theta_{t+1}) \geq \gamma V(e)$.

imperfect information case. Employed workers will be better off than unemployed workers and involuntary unemployment emerges, caused by imperfect information. In fact, employed workers' welfare will be given by $\varphi U^* (w_t, \theta_{t+1})$.

Since $(1-\varphi) U^* (w_t, \theta_{t+1}) = (1-\varphi) w_t u (\theta_{t+1}) = \gamma V (e_t)$, then

$$\frac{\partial w_t}{\partial e_t} = \frac{\gamma V' (e_t)}{(1-\varphi) u (\theta_{t+1})}$$

Substituting the above expression into (2.3) and (2.4'), we get

$$E_t [f (x_t) - f' (x_t) x_t] = \frac{\partial w_t}{\partial e_t}$$

and combining with (2.2), we obtain again the Solow condition (2.6).

Note that this alternative only represents a scale effect, in the sense that it only increases wages, and does not affect in anything the dynamics of the model.

It carries however welfare implications. In Section 2 we referred that individual welfare is zero, independent on the values for wage and interest rate along the cycle; the same occurs with aggregate welfare. Here we show that with imperfect information individual workers will get a positive welfare; however, constant - $\frac{\partial \varphi w_t u[\theta(w_t)]}{\partial w} = 0$. But the aggregate welfare, which is simply employment times individual welfare, will obviously varies positively along the cycle, i.e., higher levels of employment means that there will be more people getting the same positive welfare.

B. Appendix: Uniqueness versus multiplicity

In this appendix, we study the uniqueness or multiplicity of stationary states for the dynamical system defined by (2.24-2.25).

In Section 3 we showed that finding a stationary solution for the dynamics of the system requires finding a value that satisfies (3.1) or, more simply, that satisfies:

$$F (w) = \frac{e [f (x) - f' (x) x]}{f' (x)} \frac{\theta (w)}{w} = 1 \text{ where } x = \frac{ws [\theta (w)]}{e} \tag{B.1}$$

Studying the existence of multiple steady states involves studying the number of solutions in w for (B.1). In particular, if $F (w)$ is monotonic (i.e. either $F' (w) > 0$ for all w or $F' (w) < 0$ for all w), there is, at most, one steady state, the one defined in Proposition 3.1 for the case where $F'(w) < 0$). On the other hand, if $F'(w)$ changes its sign only once, then, at most, two steady-state solutions generically exist. Finally, if $F(w)$ is constant, then $F(w) = F(1) = 1$ for all values of w and there is a continuum of steady states.

To check whether $F(w)$ is monotonic, we now analyse the sign of $F'(w)$. Differentiating $F(w)$ and using the expressions for ρ, $\varepsilon_{\theta,w}$ and $\varepsilon_{s,\theta}$ we obtain:

$$\frac{F'(w)\,w}{F(w)} = Z(w) = \frac{1}{s(w)}\left(\frac{1}{\rho(w)} - 1 - \frac{1}{\rho(w)}\frac{1}{\sigma(w)}\right) + \left(\frac{1}{\rho(w)}\frac{1}{\sigma(w)} - 1\right)$$

The following proposition presents the sufficient condition for uniqueness.

Proposition B.1: Uniqueness of the steady state

There is, at most, one steady state if one of the following conditions is satisfied:
(i) $s(w) < \frac{\sigma(w)(1-\rho(w))-1}{\sigma(w)\rho(w)-1} \Rightarrow Z(w) > 0$ *for all* w
(ii) $s(w) > \frac{\sigma(w)(1-\rho(w))-1}{\sigma(w)\rho(w)-1} \Rightarrow Z(w) < 0$ *for all* w

Note that the case of $\rho = 1$ falls into configuration (ii) because, in that case, $Z(w) = -\frac{1}{\sigma(w)}\left(\frac{1-s(w)}{s(w)}\right) - 1$, which is negative for all w. Moreover, when $\sigma = 1$, the savings rate is constant (c.f. (2.14)) and Z is equal to $-\frac{1}{s} - 1 + \frac{1}{\rho(w)}$. Z in this case could be either positive or negative; in fact, it is positive for low values of ρ $\left(< \frac{s}{1+s}\right)$ and negative for high values of ρ $\left(> \frac{s}{1+s}\right)$. Hence Z can change its sign. However, for a constant elasticity of substitution it will always be positive or negative, and the unicity of the steady state is guaranteed.

We now address the issue of multiplicity versus unicity by analysing the circumstances in which the conditions stated in Proposition B.1 are violated. We focus our attention on the case in which $Z(w)$ changes sign, at most, only once. In this case, $F(w)$ is either single-caved or single peaked, so that there are, at most, two steady states.

Proposition B.2: Multiplicity of steady states

There are, at most, two steady states if either (i) $Z'(w) > 0$ or (ii) $Z'(w) < 0$.

Proof: Under case (i), Z *is an increasing function of* w. *Hence,* Z *changes its sign once if* $Z(w_1) < 0$ *and* $Z(w_2) > 0$, $F(w)$ *is single caved and two steady states exist. Under case* (ii), Z *is a decreasing function of* w. *Hence,* Z *changes its sign once if* $Z(w_1) > 0$ *and* $Z(w_2) < 0$, $F(w)$ *is single peaked and also two steady states exist.*

C. Appendix: A more standard model (divisible labour)

To understand exactly the effect that efficiency wages and the indivisible labour assumption have on the possibility of the occurrence of endogenous fluctuations, we must compare the model in the text with a more standard one where there are no efficiency wages and households choose hours of work. We do this exercise in this appendix.

The firm only chooses[27] capital and hours of work - h. There will be no Solow condition and the wage clears the market.

The households problem is now given by

$$\begin{aligned} \max_{c_{1,t},c_{2,t+1},h_t} W &= U\left(c_{1,t},c_{2,t+1}\right)-\gamma V\left(h_t\right) \\ s.t.w_t h_t &= c_{1,t}+\frac{c_{2,t+1}}{\theta_{t+1}} \end{aligned}$$

The model is basically the same as before. However, now households also decide the number of hours they wish to work:

$$\gamma V'\left(h_t\right)=w_t U_{c_{1,t}}\left(\theta_{t+1}\right) \tag{C.1}$$

which can be written as

$$\theta_{t+1}=\theta\left(w_t,h_t\right) \tag{C.2}$$

This equation is the only difference between this standard model and the equivalent one from the text (c.f. (2.18)).

From (C.1) and (C.2), we define the following elasticities[28]

$$\varepsilon_{\theta,w}=-\frac{1}{s\left(\theta\right)} \text{ and } \varepsilon_{\theta,n}=\frac{\delta}{s\left(\theta\right)}$$

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[27] We set e equal to one, as usual, in a non-efficiency wage model.

[28] Since U is homogenous of degree one, U_{c_1} is homogeneous of degree zero and, thus $U_{c_1}=U_{c_1}\left(\frac{c_2}{c_1}\right)=U_{c_1}\left[\frac{s(\theta)\theta}{1-s(\theta)}\right]$.

From (C.1), we have $w_t=\frac{\gamma V'(n)}{U_{c1}(\theta)}$, hence $\varepsilon_{w,\theta}=-\frac{U'_{c1}\theta}{U_{c1}}=-U_{c_1,c_2}\frac{\partial\left(\frac{c_2}{c_1}\right)}{\partial\theta}\frac{\theta}{U_{c1}}$

Differentiating (2.8), we have: $\frac{\partial\left(\frac{c_2}{c_1}\right)}{\partial\theta}\left[U_{c_1,c_2}-U_{c_2,c_2}\theta\right]=\frac{U_{c_1}}{\theta}$. Applying Euler's identity to $U_{c_2}=U_{c_2}\left(c_1,c_2\right)$, we get $U_{c_2,c_2}=-\frac{c_1}{c_2}U_{c_2,c_1}$ and noting that $U_{c_2,c_1}=U_{c_1,c_2}$, the above expression becomes $\frac{\partial\left(\frac{c_2}{c_1}\right)}{\partial\theta}U_{c_1,c_2}\left[1+\frac{c_1\theta}{c_2}\right]=\frac{\partial\left(\frac{c_2}{c_1}\right)}{\partial\theta}U_{c_1,c_2}\frac{1}{s(\theta)}=\frac{U_{c_1}}{\theta}$, thus $\varepsilon_{w,\theta}=-U_{c_1,c_2}\frac{\partial\left(\frac{c_2}{c_1}\right)}{\partial\theta}\frac{\theta}{U_{c1}}=-s\left(\theta\right)$ and $\varepsilon_{\theta,w}=-\frac{1}{s(\theta)}$.

To find the effect of the real interest rate on the number of hours, note that $\varepsilon_{h,\theta}=\frac{\partial h}{\partial\theta}\frac{\theta}{h}=$

where δ is the wage elasticity of labour supply, which in the model considered in the text was zero.

Of course, as the equilibrium condition in the goods market is the same, the system becomes very similar to the one in (2.24-2.25).

For the same steady state than before, the trace and the determinant are given by

$$\begin{aligned} T &= \frac{\left[a + v/\sigma - (a - \rho v)\left(\frac{1}{\sigma} - 1\right)\delta\right]}{\rho v + a}(1 + \varepsilon_{\theta,w}) - T_1 \varepsilon_{\theta,w} \\ \text{where } T_1 &= \frac{1 + v(1 - \rho) + \delta\rho}{\rho v + a} > 0 \end{aligned}$$

$$D = -D_1 \varepsilon_{\theta,w}, \text{ where } D_1 = \frac{(1+\delta)(1-a)}{\rho v + a} > 0$$

where T_1 and D_1 have the same meaning as before, and from now on we assume $\sigma = 1$ (Assumption 4.1). From the above expressions, we easily conclude that $D > 0$ (Since $\varepsilon_{\theta,w} < 0$ and $D_1 > 0$) and that for a small amount of increasing returns[29] $T > 0$.

We can easily check that the above expressions for the trace and determinant are the same as in the text, for the case $\delta = 0$ (the indivisible labour).

Consider $\varepsilon_{\theta,w}$ as the bifurcation parameter. Then the Δ line is given by

$$D = \Delta(T) = D_1 \left[\frac{(\rho v + a)T - (v + a)}{(\rho v + a)T_1 - (v + a)}\right]$$

where its slope is

$$\Delta' = \frac{\partial D}{\partial T} = \frac{D_1(\rho v + a)}{(\rho v + a)T_1 - (v + a)}$$

and we conclude that the Δ line is positively sloped if $T_1 > \frac{v+a}{\rho v + a}$. Given the above definition for T_1, $\Delta' > 0$ if $T > 0$.

$\frac{1}{V''}\frac{\partial V'}{\partial \theta}\frac{\theta}{h} = \frac{1}{V''}\frac{w}{\gamma}U'_{c_1}\frac{\theta}{h} = \frac{1}{V''}\frac{V'}{h}\frac{U'_{c_1}\theta}{U_{c_1}}$

Since $\frac{U'_{c_1}\theta}{U_{c_1}} = s$ the above expresion becomes $\varepsilon_{h,\theta} = \frac{1}{V''}\frac{V'}{h}s(\theta) = \frac{1}{\delta}s(\theta)$, or, equivalently, $\varepsilon_{\theta,h} = \frac{\delta}{s(\theta)}$.

[29] An alternative expression for the trace $T = \frac{a+v}{\rho v+a} - \frac{1}{\rho v + a}[(1-a) - v\rho + \delta\rho]\varepsilon_{\theta,w}$, hence, $T > 0$ if $v < \frac{1-a}{\rho} + \delta$ which is true for low values of v and standard values of the elasticities.

We will assume that this condition holds, which happens, for example, in the Cobb-Douglas economy.

The next proposition shows the sufficient condition to rule out endogenous fluctuations from the standard model.

Proposition C.1: Endogenous fluctuations in the standard model

Endogenous fluctuations cannot emerge because, with standard parameters, the steady state is a saddle.

Proof: The steady state is always a saddle when the system lies on the right of the AC line ($T > 1 + D$). Hence, using the expressions for the determinant and the trace, we have $T > 1 + D$ if $v < \frac{\delta(a+\rho-1)}{\rho-s(1-\rho)}$. For low values of v and standard values (around one) for ρ the above condition is verified.