# Designing Realised Kernels to Measure the Ex-Post Variation of Equity Prices in the Presence of Noise

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# 1. Introduction

- Last 6 years: revolution in volatility modelling - High frequency data improved our understanding of financial volatility.
- Quadratic variation: ex-post variation of asset prices.
- Estimators of increments of $QV$ — forecasts of future volatility.

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$$RV_i = \sum_{j=1}^{n} \left( X_{i+\delta j} - X_{i+\delta(j-1)} \right)^2$$

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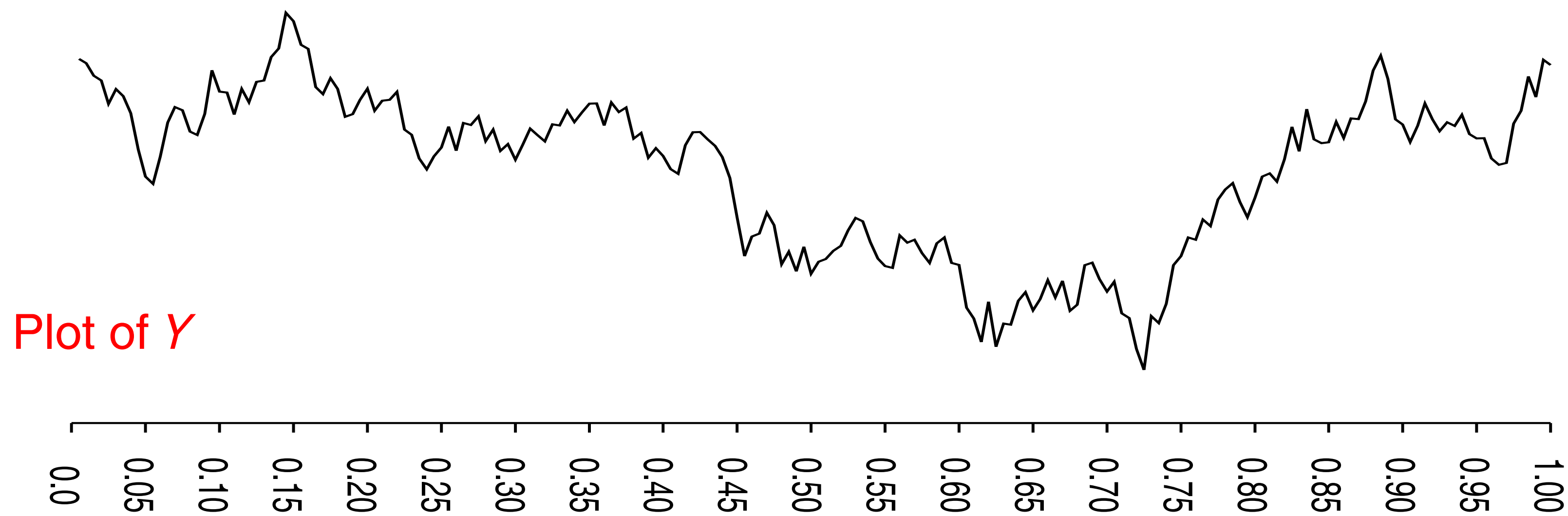
**Weakness:** $RV$ can be unacceptably sensitive to market frictions (NOISE) when applied to ultra High frequency returns such as 1 minute, or even more ambitiously 1 second.




- Log-Price, Brownian Semimartingale:

$$Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s,$$

where $a$ is predictable, $\sigma$ is cadlag, and $W$ is a standard Brownian Motion.

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and

$$n^{1/2}(RV^{(n)} - IV) \xrightarrow{d} MN(0, 2IQ), \quad \text{where} \quad IQ \equiv \int_0^1 \sigma^4(s) ds.$$

- Andersen & Bollerslev (1998), Barndorff-Nielsen & Shephard (2002*c*), Meddahi (2002).
- Jacod (1994), Jacod & Protter (1998), Barndorff-Nielsen & Shephard (2002*a*), Mykland & Zhang (2005) and Goncalves & Meddahi (2005).

# Objective:

- Want to estimate

$$[Y]_t = \int_0^t \sigma_u^2 \mathrm{d}u$$

  when we observe $X = Y + U$,

  → i.e. $Y$ is cloaked in noise.

- Study the class of *realised kernel* estimators of $QV$.

- Design these estimators to be robust to certain types of frictions and to be statistically efficient.

- We observe:

$$X_t = Y_t + U_t,$$

  - $X_t$ is the observed process.
  - $Y_t$ is the latent BSM.
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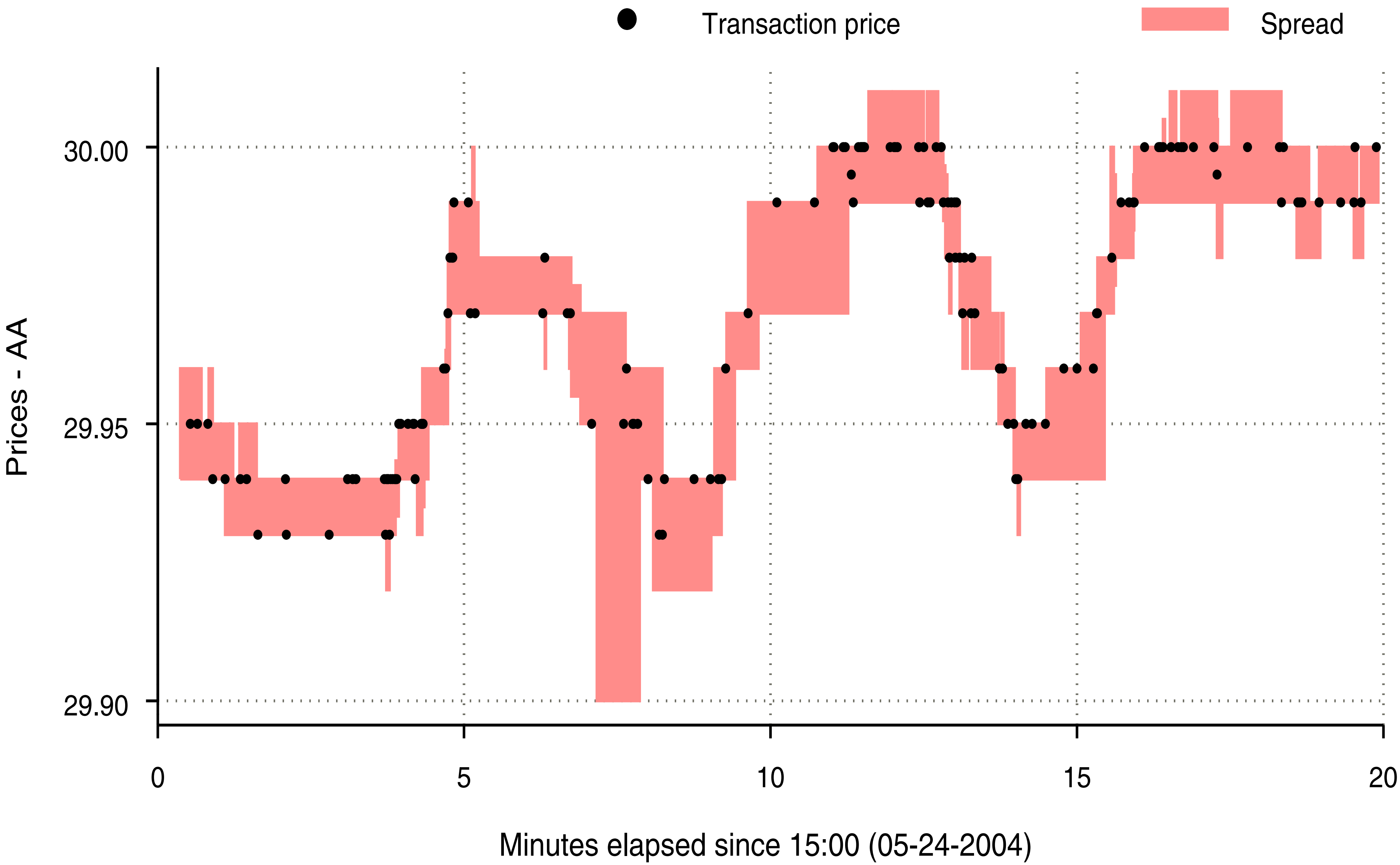
  - Temporary demand-supply imbalance (large trades).
  - Data errors (zeros, comma, time-stamp)

## ❑ A typical scenario ....



- Given

$$\underbrace{X_{\frac{i}{n}} - X_{\frac{i-1}{n}}}_{=x_i} = \underbrace{Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}}_{=y_i} + \underbrace{U_{\frac{i}{n}} - U_{\frac{i-1}{n}}}_{=u_i}.$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.7)*

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$$\begin{aligned} RV^{(n)} &\equiv \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (y_i + u_i)^2 \\ &= \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} u_i y_i + \sum_{i=1}^{n} u_i^2 \end{aligned}$$

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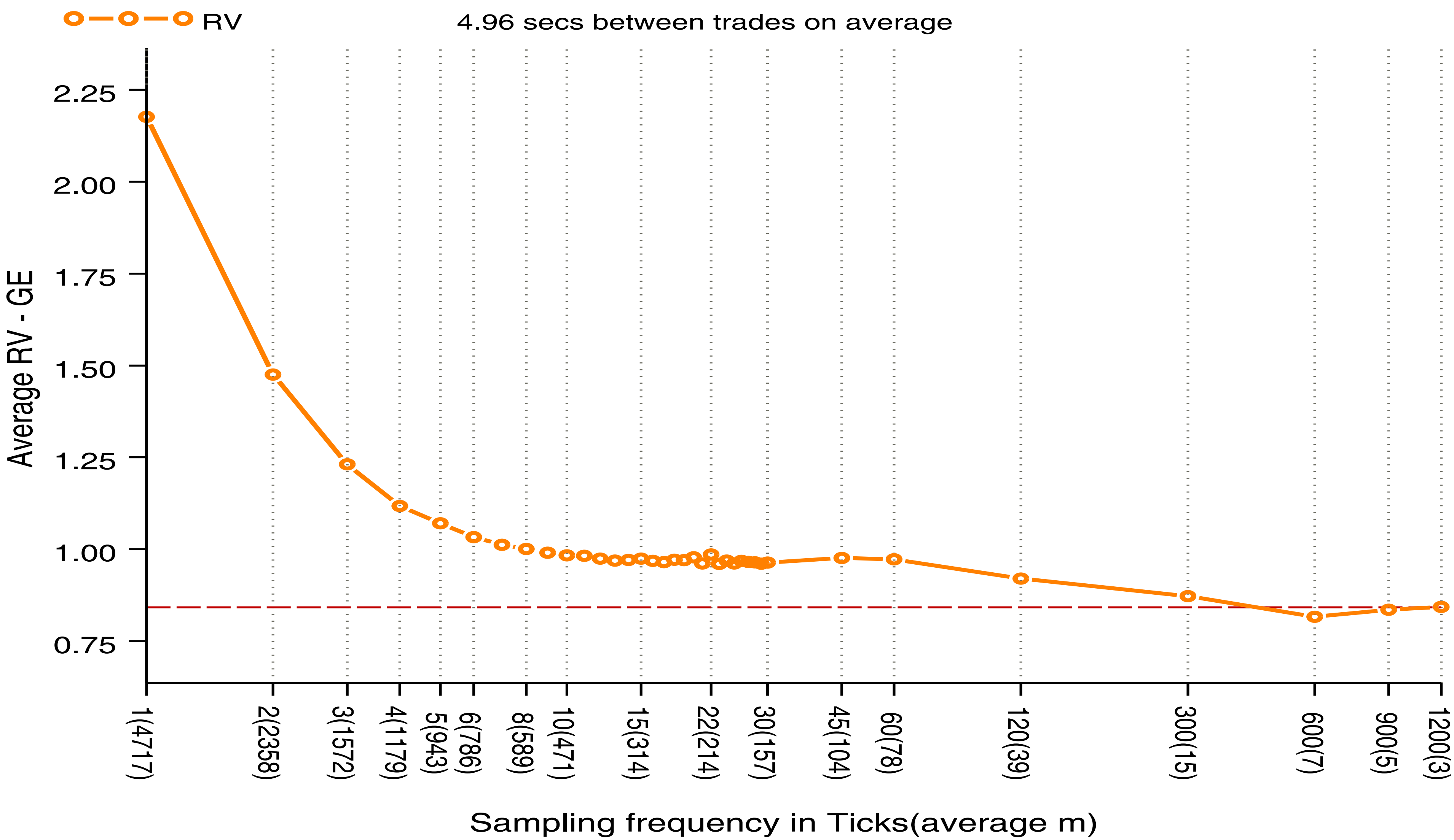


Realized variance for GE (x-axis in logs).

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- **Parametric estimators:**
  - Aït-Sahalia, Mykland & Zhang (2005*a*) and Oomen (2005)

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  - Zhou (1996), Zhang, Mykland & Aït-Sahalia (2005), Zhang (2004), Aït-Sahalia, Mykland & Zhang (2005*b*) and Li & Mykland (2006)

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- Delattre & Jacod (1997) studied **rounding** on realised variances where the size of the noise falls as the sample size increases.

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- General treatment of kernel estimators.
- Feasible inference... CLT.
  - Estimator of IQ
- General Noise...
  - time-dependent...
  - endogenous!
- Effect of Stochastically Spaced Data.
- Monte Carlo and Emprical Analysis.

## 1.1. Assumptions about noise

**Assumption 1** $U$ is a zero mean, weak white noise process:

$$\text{var}(U_t) = \omega^2, \quad \text{var}(U_t^2) = \lambda^2\omega^4, \quad U_s \perp\!\!\!\perp U_t \quad (s \neq t).$$

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- Resonable when sampling every 1-minute.
- Analytically tractable... estimators can be made robust to:
  - Dependent and Endogenous noise!

## 1.2. Realised autocovariation process

- Define the realised autocovariance process

$$\gamma_h(Z_\delta, X_\delta)_t = \sum_{j=1}^{n} \left(Z_{j\delta} - Z_{(j-1)\delta}\right)\left(X_{(j-h)\delta} - X_{(j-h-1)\delta}\right).$$

- Note

$$\gamma_h(X_\delta) = \gamma_h(Y_\delta) + \gamma_h(U_\delta) + \gamma_h(Y_\delta, U_\delta) + \gamma_h(U_\delta, Y_\delta).$$

- Throughout we write, for $h > 0$,

$$\widetilde{\gamma}_h(Z_\delta, X_\delta) = \gamma_h(Z_\delta, X_\delta) + \gamma_{-h}(Z_\delta, X_\delta).$$

## 1.3. Defining the realised kernel

- Realised kernel

$$\widetilde{K}(X_\delta) = \gamma_0(X_\delta) + \sum_{h=1}^{H} k(\tfrac{h-1}{H})\widetilde{\gamma}_h(X_\delta).$$

where the weights, $k(\frac{h-1}{H})$, are non-stochastic.

→ Considerable empirical literature on the use of realised kernels to deal with the effect of noise. There is no theory which justifies it.

*Special Cases:*

$RV = \gamma_0(X)$ *Realized Variance.*

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$$RV_{NW} = \gamma_0(X) + \sum_{h=1}^{H} \tfrac{h-1}{H}(\gamma_{-h}(X) + \gamma_h(X))$$ *Bartlett.*

- Recall

$$\gamma_h(X) \equiv \sum_{j=1}^{n} x_j x_{j-h},$$

where

$$x_i x_{i+h} = [y_i y_{i+h}] + [y_i u_{i+h} + u_i y_{i+h}] + [u_i u_{i+h}].$$

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- The implication is that

$$\text{var}(\widetilde{K}(X_\delta)) = \text{[No Noise]} + \text{[Cross Terms]} + \text{[Pure Noise]}\ .$$

## ❑ How (in)efficient?

→ Suppose $U_t \sim iid\, N(0, \omega^2)$ and $\sigma_t^2 = \sigma^2$.

→ Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \sim N_n(0, \begin{pmatrix} \frac{\sigma^2}{n} + 2\omega^2 & \bullet & \bullet & \bullet \\ -\omega^2 & \frac{\sigma^2}{n} + 2\omega^2 & \bullet & \bullet \\ 0 & -\omega^2 & \frac{\sigma^2}{n} + 2\omega^2 & \bullet \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}),$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.17)*

- So we have that for $\omega^2 > 0$

$$\left\{ \begin{array}{c} n^{1/4}(\hat{\sigma}^2_{\mathrm{ML}} - \sigma^2) \\ n^{1/2}(\hat{\omega}^2_{\mathrm{ML}} - \omega^2) \end{array} \right\} \xrightarrow{L} N\left(0, \left( \begin{array}{cc} 8\omega\sigma^3 & 0 \\ 0 & 2\omega^4 \end{array} \right)\right).$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.18)*

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- The case with $\omega^2 = 0$ results in faster rates of convergence for $\hat{\sigma}^2_{\text{ML}}$:

$$n^{1/2}\left(\hat{\sigma}^2_{\text{ML}} - \sigma^2\right) \xrightarrow{L} N\left(0, 6\sigma^4\right).$$

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- When $\varpi^2 = 0$ and known a priori to be so:

$$n^{1/2}\left(\hat{\sigma}^2_{\mathrm{ML}} - \sigma^2\right) \xrightarrow{L} N\left(0, 2\sigma^4\right).$$

# 2. CLT for $\gamma(X_\delta)$

**Proposition**

Suppose $Y \in \mathcal{BSM}$, $U \in \mathcal{WN}$ and $Y \perp\!\!\!\perp U$, then as $\delta \downarrow 0$ for fixed $S$ and $h > 0$

$$\gamma_h(Y_\delta) - \gamma_{-h}(Y_\delta) = O_p(\delta),$$

$$\gamma_h(Y_\delta, U_\delta) - \gamma_{-h}(Y_\delta, U_\delta) = O_p(\delta),$$

$$\gamma_h(U_\delta)_t - \gamma_{-h}(U_\delta)_t = O_p(1).$$

**Theorem 1**

$$\delta^{-1/2}\begin{pmatrix} [Y_\delta]_t - \int_0^t \sigma_u^2 \mathrm{d}u \\ \gamma_1(Y_\delta) \\ \vdots \\ \gamma_H(Y_\delta) \end{pmatrix} \xrightarrow{L} MN\left(0, \begin{pmatrix} 2 & \bullet & \cdots & \bullet \\ 0 & 1 & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \times \int_0^t \sigma_u^4 \mathrm{d}u \right).$$

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The convergence is in law stably

$$\widetilde{\gamma}(Y_\delta, U_\delta)|Y \xrightarrow{L} MN\left(0, 2\omega^2[Y]B\right),$$

where $B$ is a known symmetric matrix.

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$$\mathrm{E}\left\{\widetilde{\gamma}(U_\delta)\right\} = 2\omega^2 n\,(1, -1, 0, 0, \ldots, 0)',$$

$$\mathrm{Cov}\left\{\widetilde{\gamma}(U_\delta)\right\} = 4\omega^4\left(nC + \widetilde{D}\right).$$

Here $C$ and $\widetilde{D}$ have known structures.

- Jacod (1994), Jacod & Protter (1998), Barndorff-Nielsen & Shephard (2002*a*)...

$$\sqrt{n}\left(\sum y_i^2 - IV\right) \xrightarrow{L} MN\left(0, 2\int_0^1 \sigma^4(s)ds\right).$$

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$$\sqrt{n}\left(\sum y_i^2 + 2\sum y_i y_{i-1} - IV\right) \xrightarrow{L} MN\left(0, 6\int_0^1 \sigma^4(s)ds\right).$$

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- More generally...

$$\sqrt{\frac{n}{H\int_0^1 k^2(x)dx}}\left(K_w(Y) - IV\right) \xrightarrow{L} MN\left(0, 4\int_0^1 \sigma^4(s)ds\right).$$

# 3. Behaviour of kernels

**Theorem 2** *(or putting it together)*

Write

$$w = (1, k(\tfrac{0}{H}), \ldots, k(\tfrac{H-1}{H}))'.$$

Then

$$\text{var}(\widetilde{K}(X) - IV) =$$

$$\frac{w'Aw}{n} 4IQ + w'Bw 8\omega^2 IV + n \times w'Cw 4\omega^4 + w'\widetilde{D}w 4\omega^4$$

**Theorem 2, cont.** Let $k(0) = 1$ and $k(1) = 0$ and set

$$\frac{1}{n} w' A w =$$

$$w' B w =$$

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**Theorem 2, cont.** Let $k(0) = 1$ and $k(1) = 0$ and set

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$$w' B w = -\frac{1}{H} \left\{ k'(0) + \int_0^1 k(x) k''(x) \mathrm{d}x \right\} + O(\tfrac{1}{H^2})$$

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w' B w &= -\frac{1}{H} \left\{ k'(0) + \int_0^1 k(x) k''(x) \mathrm{d}x \right\} + O(\tfrac{1}{H^2}) \\
n w' C w &= \frac{n}{H^2} \left\{ k'(0)^2 + k'(1)^2 \right\} \\
&\quad + \frac{n}{H^3} \left\{ k'''(0) + \int_0^1 k(x) k''''(x) \mathrm{d}x \right\} + O(\tfrac{n}{H^4}) \\
w' \widetilde{D} w &=
\end{aligned}
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**Theorem 2, cont.** Let $k(0) = 1$ and $k(1) = 0$ and set

$$\frac{1}{n} w' A w = \frac{H}{n} \int_0^1 k(x)^2 \mathrm{d}x + O(\tfrac{1}{n})$$

$$w' B w = -\frac{1}{H} \left\{ k'(0) + \int_0^1 k(x) k''(x) \mathrm{d}x \right\} + O(\tfrac{1}{H^2})$$

$$n w' C w = \frac{n}{H^2} \left\{ k'(0)^2 + k'(1)^2 \right\} + \frac{n}{H^3} \left\{ k'''(0) + \int_0^1 k(x) k''''(x) \mathrm{d}x \right\} + O(\tfrac{n}{H^4})$$

$$w' \widetilde{D} w = -\frac{1}{H} \left\{ k'(0) + \frac{1}{2} k'(0)^2 + \int_0^1 k(x) k''(x) \mathrm{d}x \right\} + O(\tfrac{1}{H^2})$$

- When

$$k'(0)^2 + k'(1)^2 \neq 0,$$

then

$$H = cn^{2/3},$$

is optimal, where

$$c = \left( 2 \frac{k'(0)^2 + k'(1)^2}{\int_0^1 k(x)^2 \mathrm{d}x} \frac{\omega^4}{\int_0^t \sigma_u^4 \mathrm{d}u} \right)^{1/3}.$$

and

$$n^{1/6} \left\{ \widetilde{K}(X) - \int_0^t \sigma_u^2 \mathrm{d}u \right\} \xrightarrow{L} MN \left( 0, 4c \int_0^1 k(x)^2 \mathrm{d}x \int_0^t \sigma_u^4 \mathrm{d}u + 4\omega^4 \frac{k'(0)^2 + k'(1)^2}{c^2} \right).$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.24)*

- The Bartlett kernel (which minimizes $w'Bw$) has

$$c = \left(2\frac{1+1}{1/3}\frac{\omega^4}{\int_0^t \sigma_u^4 \mathrm{d}u}\right)^{1/3} = \left(12\frac{\omega^4}{\int_0^t \sigma_u^4 \mathrm{d}u}\right)^{1/3}.$$

- It has the same asymptotic variance as TSRV

$$2(12)^{\frac{1}{3}}\frac{\omega^{4/3}}{\left(\int_0^t \sigma_u^4 \mathrm{d}u\right)^{2/3}}.$$

- When

$$k'(0)^2 + k'(1)^2 = 0,$$

then

$$H = cn^{1/2},$$

is optimal, and

$$n^{1/4}\left\{\widetilde{K}(X) - \int_0^t \sigma_u^2 \mathrm{d}u\right\} \xrightarrow{L} MN(0, *)\,.$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.26)*

- To simplify notation... define

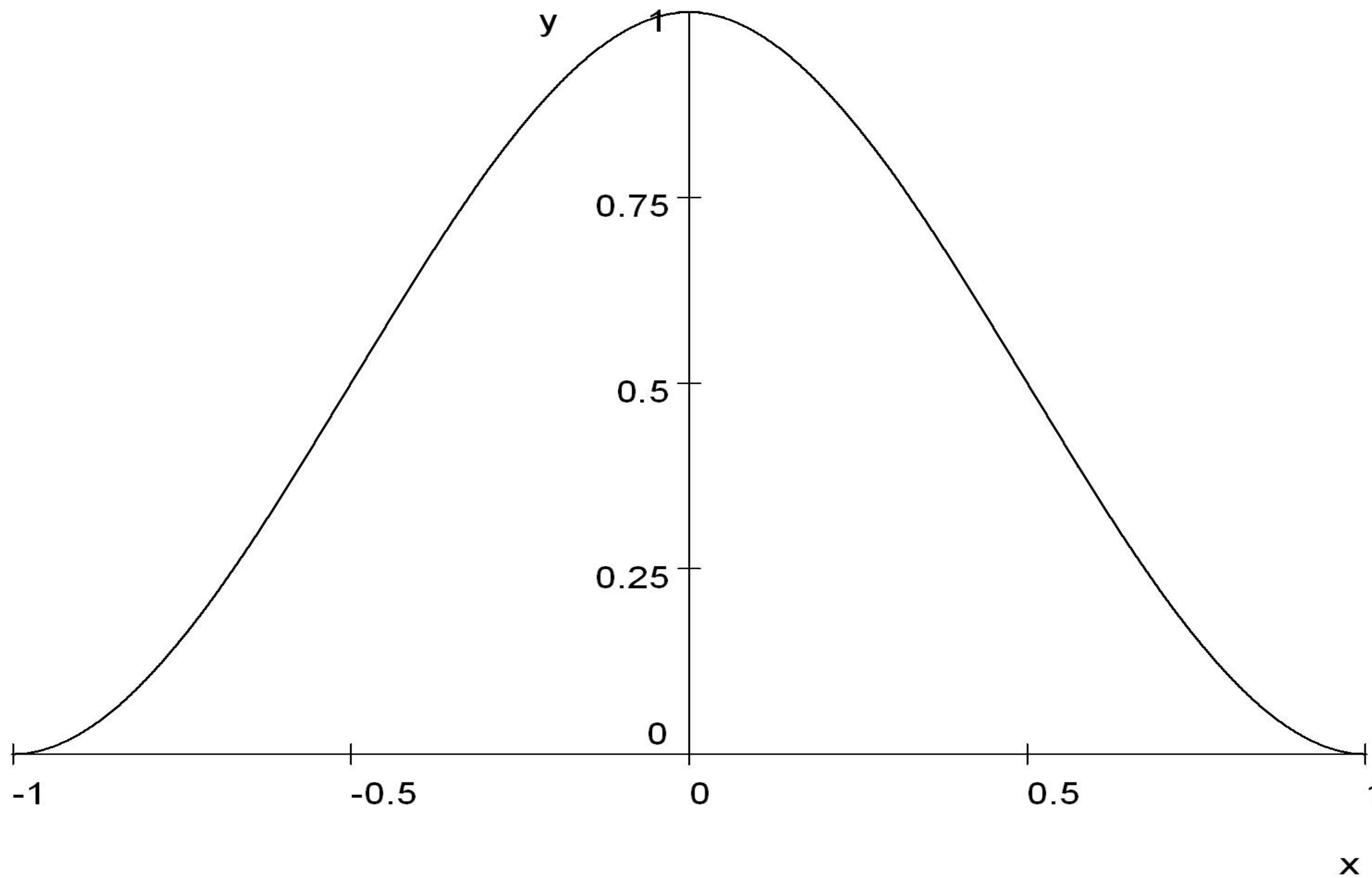
$$k_\bullet^{0,0} = \int_0^1 k(x)^2 \mathrm{d}x, \quad k_\bullet^{0,2} = \int_0^1 k(x)k''(x)\mathrm{d}x, \quad k_\bullet^{0,4} = \int_0^1 k(x)k''''(x)\mathrm{d}x.$$

| | $k(x)$ | $k_\bullet^{0,0}$ | $k_\bullet^{0,2}$ | $g$ |
|---|---|---|---|---|
| Cubic kernel | $1-3x^2+2x^3$ | 0.371 | −1.2 | 9.04 |
| 5-th order kernel | $1-10x^3+15x^4-6x^5$ | 0.391 | −1.42 | 10.2 |
| Parzen | | 0.269 | −1.5 | 8.54 |
| Tukey-Hanning | $\{1+\cos(\pi x)\}/2$ | 0.375 | −1.23 | 9.18 |
| | $\{1-\cos(\pi(1-x)^2)\}/2$ | 0.218 | −1.71 | 8.29 |

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.27)*

The cubic kernel (minimizes $w'Cw$) asymptotically equal to Multi Scale RV of Zhang (2004)

$$k(x) = 2|x|^3 - 3x^2 + 1$$



# 4. EFFECT OF STOCHASTICALLY SPACED DATA

- Measured prices at regularly spaced intervals of length $\delta$.

- Natural to measure returns in tick time.

- Extend the above theory to cover stochastically spaced data.

- When times independent of $Y$, done by Mykland & Zhang (2005) and Barndorff-Nielsen & Shephard (2005).

- But in financial econometrics spacing is endogenous. Need a new approach.

- $Y \in \mathcal{BSM}$ and assume we have measurements at times

$$t_j = T_{\delta j}, \qquad j = 1, 2, \ldots, n,$$

where $0 = t_0 < \ldots < t_n = T_1$ and

$$T_t = \int_0^t \tau_u^2 \mathrm{d}u,$$

where $\tau$ has strictly positive càdlàg sample paths and $T_0 = 0$.

- $Y \in \mathcal{BSM}$ and assume we have measurements at times

$$t_j = T_{\delta j}, \qquad j = 1, 2, \ldots, n,$$

where $0 = t_0 < \ldots < t_n = T_1$ and

$$T_t = \int_0^t \tau_u^2 \mathrm{d}u,$$

where $\tau$ has strictly positive càdlàg sample paths and $T_0 = 0$.

- New process $Z$ defined as $Z_t = Y_{T_t}$, so at the measurement times

$$Z_{\delta j} = Y_{T_{\delta j}}, \quad j = 1, 2, \ldots, n.$$

$Z$ made at equally spaced times = irregularly spaced data on $Y$.

- $Y \in \mathcal{BSM}$ and assume we have measurements at times

$$t_j = T_{\delta j}, \qquad j = 1, 2, \ldots, n,$$

where $0 = t_0 < \ldots < t_n = T_1$ and

$$T_t = \int_0^t \tau_u^2 \mathrm{d}u,$$

where $\tau$ has strictly positive càdlàg sample paths and $T_0 = 0$.

- New process $Z$ defined as $Z_t = Y_{T_t}$, so at the measurement times

$$Z_{\delta j} = Y_{T_{\delta j}}, \quad j = 1, 2, \ldots, n.$$

$Z$ made at equally spaced times = irregularly spaced data on $Y$.

- Now $$Z_t = \int_0^t a_{T_u} \tau_u \mathrm{d}u + \int_0^t \sigma_{T_u} \tau_u \mathrm{d}B_u,$$ where $B$ is Brownian motion.

- $Y \in \mathcal{BSM}$ and assume we have measurements at times

$$t_j = T_{\delta j}, \qquad j = 1, 2, \ldots, n,$$

where $0 = t_0 < \ldots < t_n = T_1$ and

$$T_t = \int_0^t \tau_u^2 \mathrm{d}u,$$

where $\tau$ has strictly positive càdlàg sample paths and $T_0 = 0$.

- New process $Z$ defined as $Z_t = Y_{T_t}$, so at the measurement times

$$Z_{\delta j} = Y_{T_{\delta j}}, \quad j = 1, 2, \ldots, n.$$

$Z$ made at equally spaced times = irregularly spaced data on $Y$.

- Now $$Z_t = \int_0^t a_{T_u} \tau_u \mathrm{d}u + \int_0^t \sigma_{T_u} \tau_u \mathrm{d}B_u,$$ where $B$ is Brownian motion.

Apply realised kernel on this.

# 5. DEPENDENT NOISE

- A higher-order kernel..

$$k'(0)^2 + k'(1)^2 = 0,$$

and the inefficient choice:

$$H = cn^{2/3},$$

eliminates cross- and pure-noise terms:

$$n^{1/6}\left\{\widetilde{K}(X) - \int_0^t \sigma_u^2 \mathrm{d}u\right\} \xrightarrow{L} MN\left(0, 4ck_\bullet^{0,0}\int_0^t \sigma_u^4 \mathrm{d}u\right),$$

# 6. Endogenous noise

- Key assumption: $Y \perp\!\!\!\perp U$.

- Relax by assuming

$$U_{\delta i} = \sum_{h=0}^{\overline{H}} \beta_h \left( Y_{\delta(i-h)} - Y_{\delta(i-1-h)} \right) + \overline{U}_{\delta i},$$

where $Y \perp\!\!\!\perp \overline{U}$ and $\overline{U} \in \mathcal{WN}$.

- Then

$$\gamma_h(Y_\delta, U_\delta) = \sum_{j=0}^{\overline{H}} \beta_j \gamma_{h+j}(Y_\delta) - \sum_{j=0}^{\overline{H}} \beta_j \gamma_{h+j+1}(Y_\delta) + \gamma_h(Y_\delta, \overline{U}_\delta).$$

✘ In particular

$$\gamma_h(Y_\delta, U_\delta) - \gamma_h(Y_\delta, \overline{U}_\delta) = \begin{cases} \beta_0[Y] + O_p(n^{-1/2}), & h = 0, \\ -\beta_0[Y] + O_p(n^{-1/2}), & h = -1, \\ O_p(n^{-1/2}), & |h| \neq 1. \end{cases}$$

✘ Hence flat-top kernels will be robust to this type of endogenous noise.

# 7. SIMULATION STUDY

- We simulate over the interval in time $[0, 1]$.
- We normalize one second to be $1/23,400$, so that the interval $[0, 1]$ contains 6.5 hours.
- In generating the observed price, we discretize $[0, 1]$ into a number $N = 23,400$ of intervals.
- When assessing the performance of the estimators under sparse sampling, we look at values of $n$ which are factors of $N$.

# 7.1. Stochastic Volatility

- We consider the follow SV model (see e.g. Huang & Tauchen (2005))

$$\mathrm{d}Y_t = \mu \mathrm{d}t + \sigma_t \mathrm{d}W_t,$$

$$\sigma_t = \exp\left(\beta_0 + \beta_1 \tau_t\right), \qquad \text{with} \quad \mathrm{corr}(\mathrm{d}W_t, \mathrm{d}B_t) = \rho$$

$$\mathrm{d}\tau_t = \alpha \tau_t \mathrm{d}t + \mathrm{d}B_t.$$

- We restart the volatility process each day at $\tau_0 \sim N(0, (-2\alpha)^{-1})$.
- For the noise we use

$$U_{j/N} \overset{i.i.d.}{\sim} N(0, \omega^2), \qquad j = 0, \ldots, N.$$

## 7.2. Implementation of realised kernel

- The asymptotic variance of the realised kernels we will use are

$$\text{Var}\left\{\widetilde{K}_H(X_\delta)\right\} \simeq \varpi(IV, IQ, \omega^2, n, H)$$

- Our simple rule-of-thumb is in the infeasible case

$$H^*_{simple} = 3.6867 \frac{\omega}{\sqrt{\int_0^1 \sigma_u^2 \mathrm{d}u}} \sqrt{n}$$

- The feasible version of this is

$$\widehat{H}^*_{simple} = 3.6867 \frac{\widetilde{\omega}_\delta}{\sqrt{[X_{\delta^*}]_1}} \sqrt{n},$$

where $[X_{\delta^*}]_1$ is a RV estimator based on low frequency data.

- Our focus is on the asymptotically pivotal t-statistic

$$T(IV, IQ, \omega^2, n, H) = \frac{\left(\widetilde{K}_H(X_\delta)_1 - IV\right)}{\sqrt{\varpi(IV, IQ, \omega^2, n, H)/\sqrt{n}}} \xrightarrow{L} N(0,1),$$

- We do this in two ways:

  (i) [infeasible case] when $\omega$ and $\sigma$ are known, imply $\widehat{H}$ and $\varpi(\bullet)$,

## ❑ Summary Statistics - Modified Tukey-Hanning - Infeasible case

$\omega^2$ = *0.001, number of reps. = 150,000*

| $n$ | $\overline{H}^*_{simple}$ | Mean | Stdv. | 0.5% | 2.5% | 97.5% | 99.5% |
|---|---|---|---|---|---|---|---|
| 195 | 4.54 | -0.001 | 0.999 | 0.12 | 1.40 | 96.61 | 98.94 |
| 390 | 6.21 | -0.000 | 1.001 | 0.18 | 1.70 | 96.75 | 99.09 |
| 780 | 8.58 | 0.000 | 0.999 | 0.22 | 1.87 | 96.93 | 99.19 |
| 1560 | 11.9 | 0.000 | 1.003 | 0.26 | 2.04 | 96.99 | 99.23 |
| 4680 | 20.3 | -0.000 | 1.007 | 0.36 | 2.21 | 97.01 | 99.27 |
| 5850 | 22.6 | 0.001 | 1.007 | 0.35 | 2.24 | 97.04 | 99.29 |
| 7800 | 26.0 | -0.001 | 1.010 | 0.33 | 2.28 | 97.06 | 99.24 |
| 11700 | 31.8 | -0.000 | 1.009 | 0.37 | 2.38 | 97.12 | 99.31 |
| 23400 | 44.8 | -0.001 | 1.017 | 0.43 | 2.37 | 97.06 | 99.29 |

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.38)*

(ii) [feasible case] estimating $\omega$ and $IV$ from the data, implying plugged in values of $\widehat{H}$ and $\varpi(\bullet)$.

- Two-step approach

1. choose H use a low frequency RV to estimate [Y]

2. then compute the estimator

3. then compute the standard error

- Use an unbiased kernel estimator of $\omega^2$

$$N_w(X_\delta) = \left\{ w_0 \widetilde{\gamma}_0(X_\delta) + \sum_{h=1}^{H} w_h \widetilde{\gamma}_h(X_\delta) \right\} / (-2n)$$

has $w = \left(\bar{w}', v'\right)'$, where $v$ is freely chosen and $\bar{w} = (0, 1)'$ and

$$\widehat{v}_h^{\omega} = (h+1) \frac{(H-h)(H-h+1)}{H(H+1)}.$$

- We use $H = c_\omega n^{1/3}$ with $c_\omega \simeq 2.8377\omega / \sqrt{\int_0^t \sigma_u^2 \mathrm{d}u}$.

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.40)*

- Estimating integrated quarticity is a tougher problem than estimating QV as the effect of noise is magnified up.

- Define the subsampled squared returns

$$x_{j,\cdot}^2 = \frac{1}{S+1}\sum_{s=0}^{S} x_{j,s/(S+1)}^2, \quad j = 1, 2, \ldots, n.$$

- This allows us to define a bipower variation estimator of integrated quarticity

$$\left\{X_\delta, \omega^2; S\right\}^{[2,2]} = \delta^{-1}\sum_{j=1}^{\lfloor t/\delta \rfloor}\left(x_{j,\cdot}^2 - 2\omega^2\right)\left(x_{j-2,\cdot}^2 - 2\omega^2\right).$$

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.41)*

- We can improve the finite sample performance of our estimator of integrated quarticity by using the inequality

$$\int_0^t \sigma_u^4 \mathrm{d}u \geq \frac{1}{t} \left( \int_0^t \sigma_u^2 \mathrm{d}u \right)^2 .$$

- Thus our preferred way of estimating integrated quarticity is

$$\widehat{IQ}_{\delta,S} = \max \left[ \frac{1}{t} \left( \widetilde{K}_v(X_\delta; S) \right)^2 , \{X_\delta, N_{\widehat{v}^\omega}(X_\delta); S\}^{[2,2]} \right] .$$

## ❑ Summary Statistics - Modified Tukey-Hanning - Feasible case

$\omega^2$ = *0.001, number of reps. = 150,000*

| $n$ | $\overline{H}^*_{simple}$ | Mean | Stdv. | 0.5% | 2.5% | 97.5% | 99.5% |
|---|---|---|---|---|---|---|---|
| 195 | 4.31 | -0.232 | 1.065 | 2.88 | 6.45 | 99.69 | 99.99 |
| 390 | 5.79 | -0.177 | 1.027 | 2.11 | 5.25 | 99.40 | 99.97 |
| 780 | 7.95 | -0.138 | 1.003 | 1.60 | 4.52 | 99.18 | 99.95 |
| 1560 | 11.0 | -0.112 | 0.995 | 1.29 | 4.08 | 98.93 | 99.91 |
| 4680 | 18.7 | -0.083 | 0.992 | 0.97 | 3.54 | 98.56 | 99.83 |
| 5850 | 20.8 | -0.077 | 0.991 | 0.94 | 3.47 | 98.53 | 99.82 |
| 7800 | 23.9 | -0.073 | 0.994 | 0.90 | 3.41 | 98.44 | 99.78 |
| 11700 | 29.2 | -0.065 | 0.993 | 0.83 | 3.34 | 98.38 | 99.77 |
| 23400 | 41.1 | -0.056 | 1.001 | 0.79 | 3.20 | 98.14 | 99.70 |

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.43)*

- An alternative to this is to use the delta method and base the asymptotic analysis on a log transformation

$$T_{\log} = \frac{\log\left\{\widetilde{K}_v(X_\delta)_1 + d\right\} - \log\left\{\int_0^1 \sigma_u^2 \mathrm{d}u + d\right\}}{\sqrt{\varpi} / \left\{\widetilde{K}_v(X_\delta)_1 + d\right\}} \xrightarrow{L} N(0,1).$$

- The presence of $d \geq 0$ allows for the possibility that $\widetilde{K}_v(X_\delta)_1$ may be truncated to be exactly zero.

- In our simulations we have taken $d = 0.2$.

## ❑ Summary Statistics - Modified Tukey-Hanning - Log version

$\omega^2$ = *0.001, number of reps. = 150,000*

| $n$ | $\overline{H}^*_{simple}$ | Mean | Stdv. | 0.5% | 2.5% | 97.5% | 99.5% |
|---|---|---|---|---|---|---|---|
| 195 | 4.31 | -0.153 | 1.004 | 1.49 | 4.36 | 98.89 | 99.87 |
| 390 | 5.79 | -0.118 | 0.992 | 1.16 | 3.76 | 98.68 | 99.81 |
| 780 | 7.95 | -0.093 | 0.982 | 0.92 | 3.34 | 98.54 | 99.78 |
| 1560 | 11.0 | -0.076 | 0.983 | 0.77 | 3.13 | 98.36 | 99.75 |
| 4680 | 18.7 | -0.057 | 0.986 | 0.68 | 2.90 | 98.08 | 99.67 |
| 5850 | 20.8 | -0.053 | 0.985 | 0.65 | 2.91 | 98.07 | 99.67 |
| 7800 | 23.9 | -0.051 | 0.990 | 0.63 | 2.86 | 98.01 | 99.63 |
| 11700 | 29.2 | -0.045 | 0.990 | 0.59 | 2.85 | 98.00 | 99.62 |
| 23400 | 41.1 | -0.039 | 0.999 | 0.61 | 2.79 | 97.78 | 99.59 |

# 8. Empirical study

- Inference procedure for the daily increments of $[Y]$ on trades of General Electric (GE) shares on the NYSE in 2004.

- Realised kernel implemented on returns recorded every $k$ trades, so on average these returns are recorded every 60 seconds.

- Compared to the feasible procedure of Barndorff-Nielsen & Shephard (2002*b*), which ignores the presence of market microstructure effects, based on returns calculated over 20 minutes within each day.

- General Electric shares are traded between $1,500$ and $6,000$ a day.

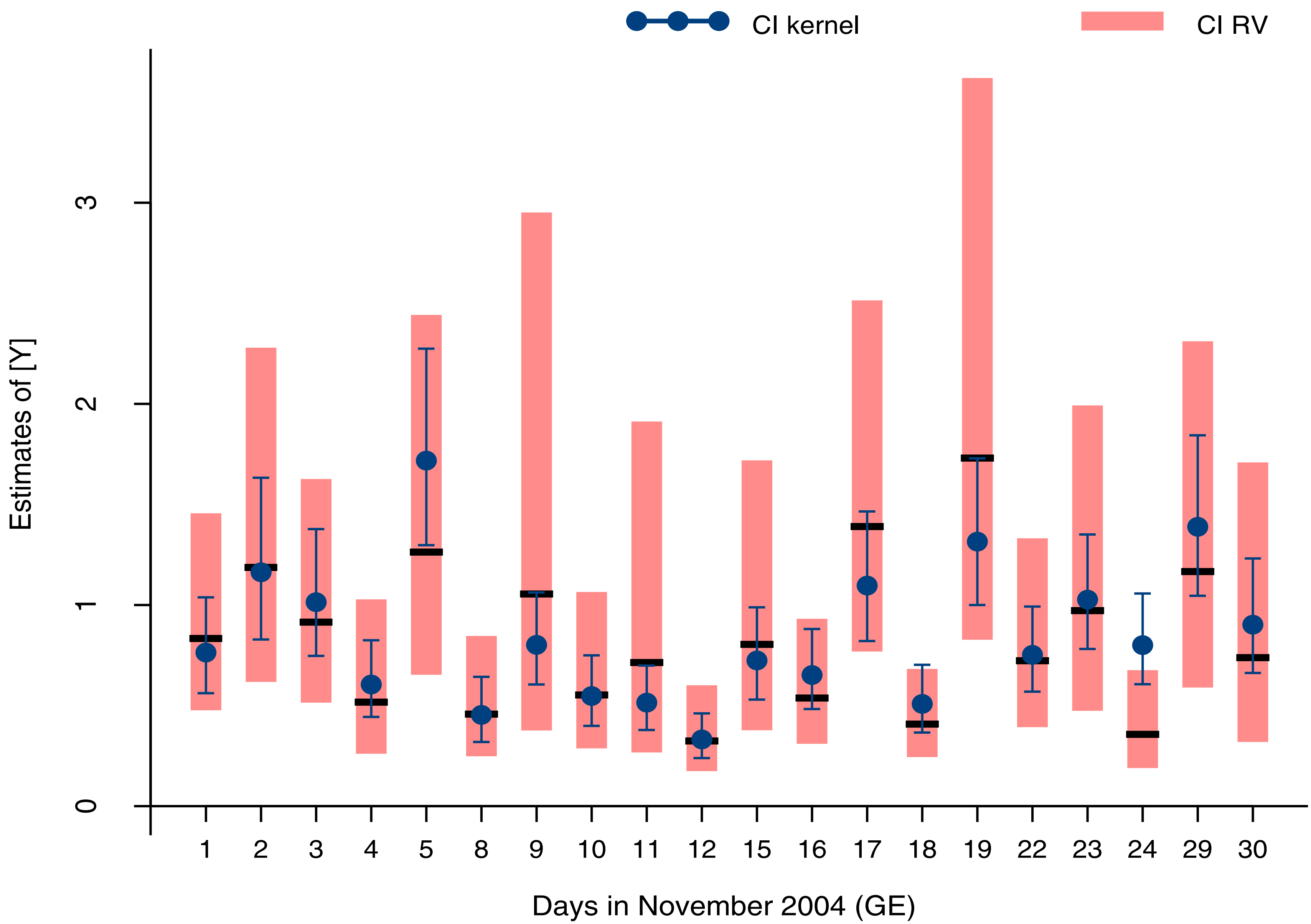
- Hansen & Lunde (2006*b*): 60 second intervals approx $Y \perp\!\!\!\perp U$ and $U$ is roughly $\mathcal{WN}$.

- In the 2004 sample period, we found $\omega^2$ to be very small.

❑ Inference for General Electric volatility; Five days in November 2004.

| | | *BNS Theory* | | | | *BNHLS Theory* | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Day | Trans | Lower | RV20m | Upper | $n$ | Lower | KV60s | Upper | $n$ | H | $\bar{\varpi}^2$ |
| 1 | 4631 | 0.48 | 0.83 | 1.46 | 20 | 0.56 | 0.76 | 1.04 | 357 | 4 | 0.0016 |
| 2 | 4974 | 0.62 | 1.19 | 2.28 | 20 | 0.83 | 1.16 | 1.63 | 356 | 4 | 0.0025 |
| 3 | 4918 | 0.51 | 0.92 | 1.63 | 20 | 0.75 | 1.01 | 1.38 | 352 | 4 | 0.0021 |
| 4 | 5493 | 0.26 | 0.52 | 1.03 | 20 | 0.44 | 0.60 | 0.82 | 344 | 4 | 0.0013 |
| 5 | 5504 | 0.65 | 1.26 | 2.44 | 20 | 1.30 | 1.72 | 2.27 | 344 | 3 | 0.0028 |

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.47)*




*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.48)*

## ❑ Summary statistics for alternative estimators, (GE) 2004.

| | Mean | Std. (HAC) | $\rho(\widehat{[Y]}, \tilde{K})$ | acf(1) | acf(2) | acf(5) | acf(10) |
|---|---|---|---|---|---|---|---|
| *Modified Tukey-Hanning kernel* ($H = cn^{1/2}$) | | | | | | | |
| $\tilde{K}_w^{\text{TH2}}(X_{\text{ap. 1 min}})$ | 0.962 | 0.568 (1.195) | 1.000 | 0.34 | 0.32 | 0.28 | 0.08 |
| *Alternative kernels* ($H = cn^{1/2}$) | | | | | | | |
| $\tilde{K}_w^{\text{PAR}}(X_{\text{ap. 1 min}})$ | 0.962 | 0.570 (1.197) | 1.000 | 0.34 | 0.32 | 0.27 | 0.08 |
| $\tilde{K}_w^{\text{CUB}}(X_{\text{ap. 1 min}})$ | 0.959 | 0.568 (1.192) | 1.000 | 0.34 | 0.32 | 0.27 | 0.08 |
| $\tilde{K}_w^{\text{5TH}}(X_{\text{ap. 1 min}})$ | 0.971 | 0.558 (1.186) | 0.999 | 0.35 | 0.32 | 0.28 | 0.08 |
| $\tilde{K}_w^{\text{8TH}}(X_{\text{ap. 1 min}})$ | 0.965 | 0.578 (1.212) | 0.995 | 0.34 | 0.32 | 0.27 | 0.09 |
| *Top-Flat Bartlett kernel* ($H = cn^{2/3}$) | | | | | | | |
| $\tilde{K}_w^{\text{BART}}(X_{\text{ap. 1 min}})$ | 0.963 | 0.562 (1.184) | 0.997 | 0.34 | 0.31 | 0.27 | 0.07 |
| *Simple RV* | | | | | | | |
| $[X_{\text{20 minutes}}]$ | 0.879 | 0.524 (1.008) | 0.793 | 0.28 | 0.24 | 0.26 | 0.06 |
| $[X_{\text{1 minutes}}]$ | 0.941 | 0.382 (0.919) | 0.878 | 0.44 | 0.40 | 0.38 | 0.11 |
| $[X_{\text{10 seconds}}]$ | 1.330 | 0.389 (1.142) | 0.801 | 0.60 | 0.56 | 0.51 | 0.32 |
| $[X_{\text{1 second}}]$ | 2.183 | 0.569 (1.828) | 0.739 | 0.69 | 0.66 | 0.57 | 0.48 |

*Barndorff-Nielsen, Hansen, Lunde & Shephard, 2006 (p.49)*

## 8.1. Speculative analysis

- Above does not use all of the available data efficiently, for the realised kernel is computed only on every 15 or so trades.

- **Reason:** empirical reality of the GE data matched the assumptions of our feasible central limit theory, allowing us to calculate daily confidence intervals.

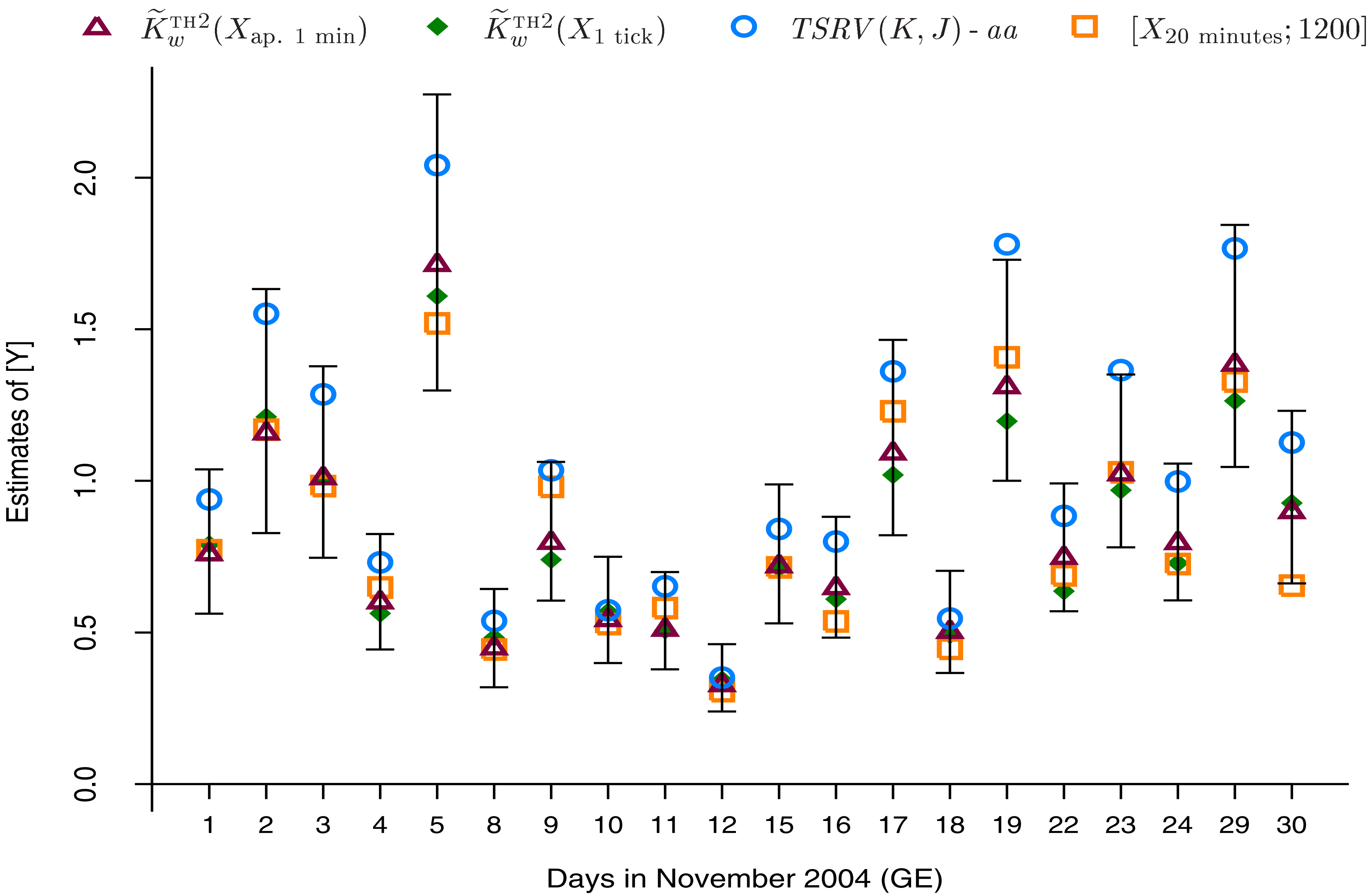
- Section 6 suggest our efficient realised kernel can do this, even though the white noise assumption and independence of $Y$ and $U$ are no long empirically well grounded assumptions.

- Inevitably then, the results in this subsection will be more speculative than those given in the previous analysis.

## Summary statistics for alternative estimators, (GE) 2004.

| | Mean | Std. (HAC) | $\rho(\widehat{[Y]}, \widetilde{K})$ | acf(1) | acf(2) | acf(5) | acf(10) |
|---|---|---|---|---|---|---|---|
| *Modified Tukey-Hanning kernel* $(H = cn^{1/2})$ | | | | | | | |
| $\widetilde{K}_w^{\text{TH2}}(X_{\text{ap. 1 min}})$ | 0.962 | 0.568 (1.195) | 1.000 | 0.34 | 0.32 | 0.28 | 0.08 |
| *Modified Tukey-Hanning kernel (inefficient rate* $H = cn^{2/3}$*)* | | | | | | | |
| $\widetilde{K}_w^{\text{TH2}}(X_{\text{1 tick}})$ | 0.945 | 0.521 (1.127) | 0.990 | 0.37 | 0.31 | 0.30 | 0.08 |
| *Alternative kernels (inefficient rate* $H = cn^{2/3}$*)* | | | | | | | |
| $\widetilde{K}_w^{\text{PAR}}(X_{\text{1 tick}})$ | 0.947 | 0.524 (1.133) | 0.990 | 0.37 | 0.31 | 0.30 | 0.08 |
| $\widetilde{K}_w^{\text{CUB}}(X_{\text{1 tick}})$ | 0.948 | 0.528 (1.142) | 0.991 | 0.37 | 0.32 | 0.30 | 0.08 |
| $\widetilde{K}_w^{\text{5TH}}(X_{\text{1 tick}})$ | 0.951 | 0.531 (1.148) | 0.989 | 0.37 | 0.31 | 0.30 | 0.08 |
| $\widetilde{K}_w^{\text{8TH}}(X_{\text{1 tick}})$ | 0.954 | 0.573 (1.207) | 0.998 | 0.34 | 0.31 | 0.27 | 0.09 |
| *AMZ (2006)* | | | | | | | |
| $TSRV(K, J)$ | 0.596 | 0.375 (0.807) | 0.985 | 0.36 | 0.35 | 0.27 | 0.09 |
| $TSRV(K, J)\text{-}aa$ | 1.206 | 0.757 (1.632) | 0.985 | 0.36 | 0.35 | 0.27 | 0.09 |

## ❑ Illustration.



# 9. Summary

- General treatment of kernel estimators.
- Feasible inference... CLT.
  - Estimator of IQ
- General Noise...
  - time-dependent...
  - endogenous!
- Effect of Stochastically Spaced Data.
- Monte Carlo and Emprical Analysis.

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