Bonds and Cash Flow News:

Mysteries of Convexity

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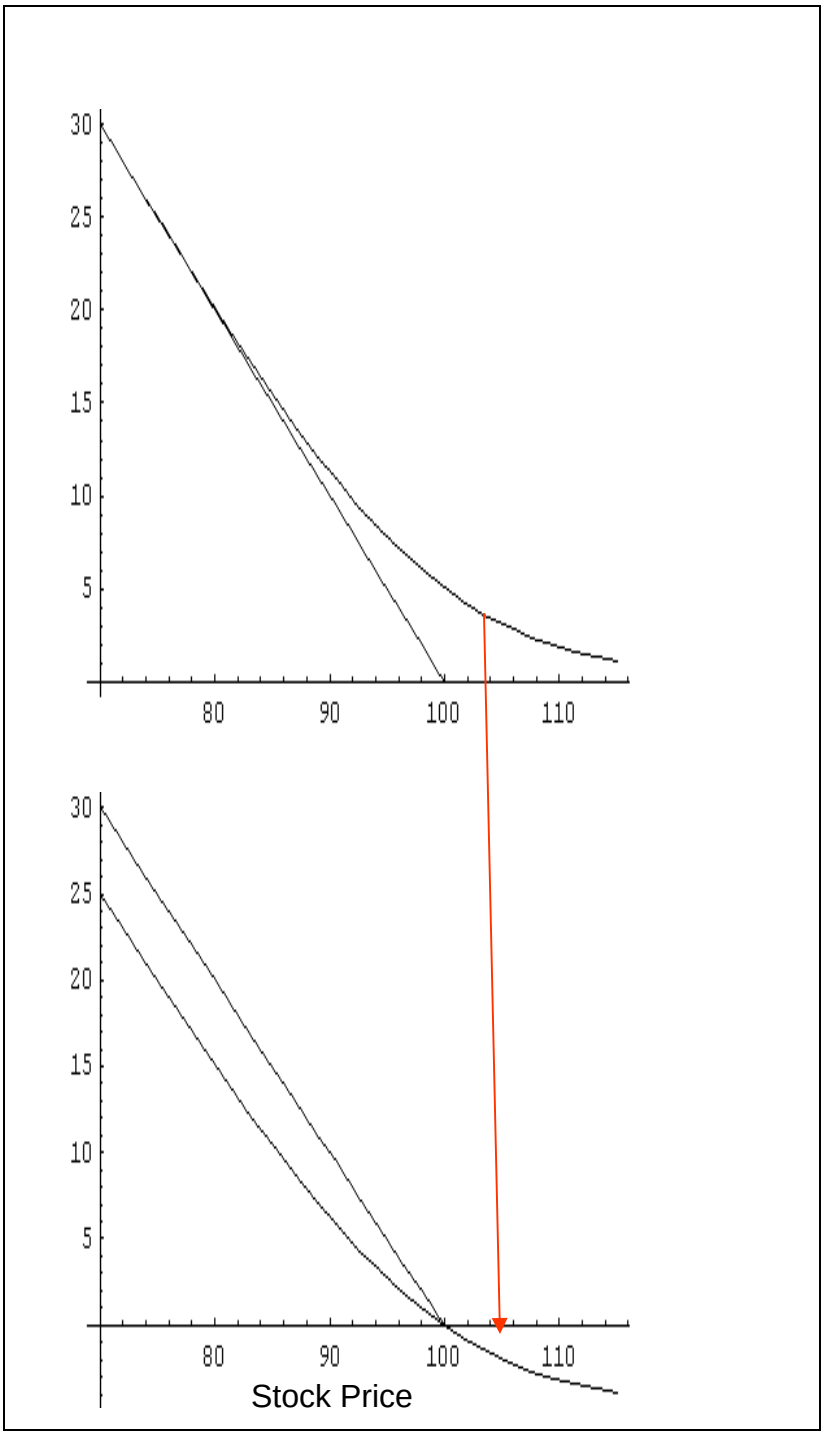
June 2-3, 2006

Introduction

- Suppose we have two assets whose prices depend on the same, univariate, risk factor $r_t \in [0, \infty)$, monotonically. Let these prices be denoted by $B(r_t)$ and $S(r_t)$ respectively.

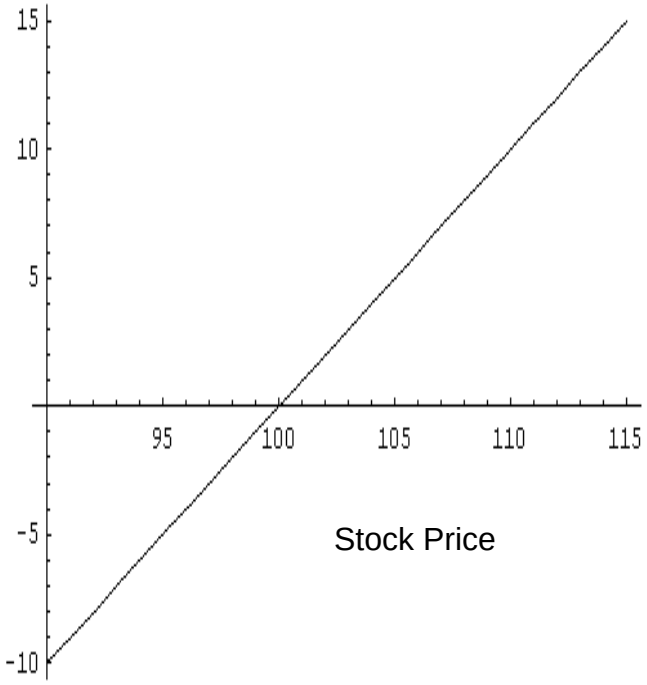
- Then, unless the convexity of the two assets are the same, one can always put together a portfolio, such that the portfolio *gains* regardless on whether r_t increases or decreases.

- In other words, the portfolio gains(loses) with volatility. Everything else being the same, the higher the volatility, the higher will be the returns to holding the more convex instrument.



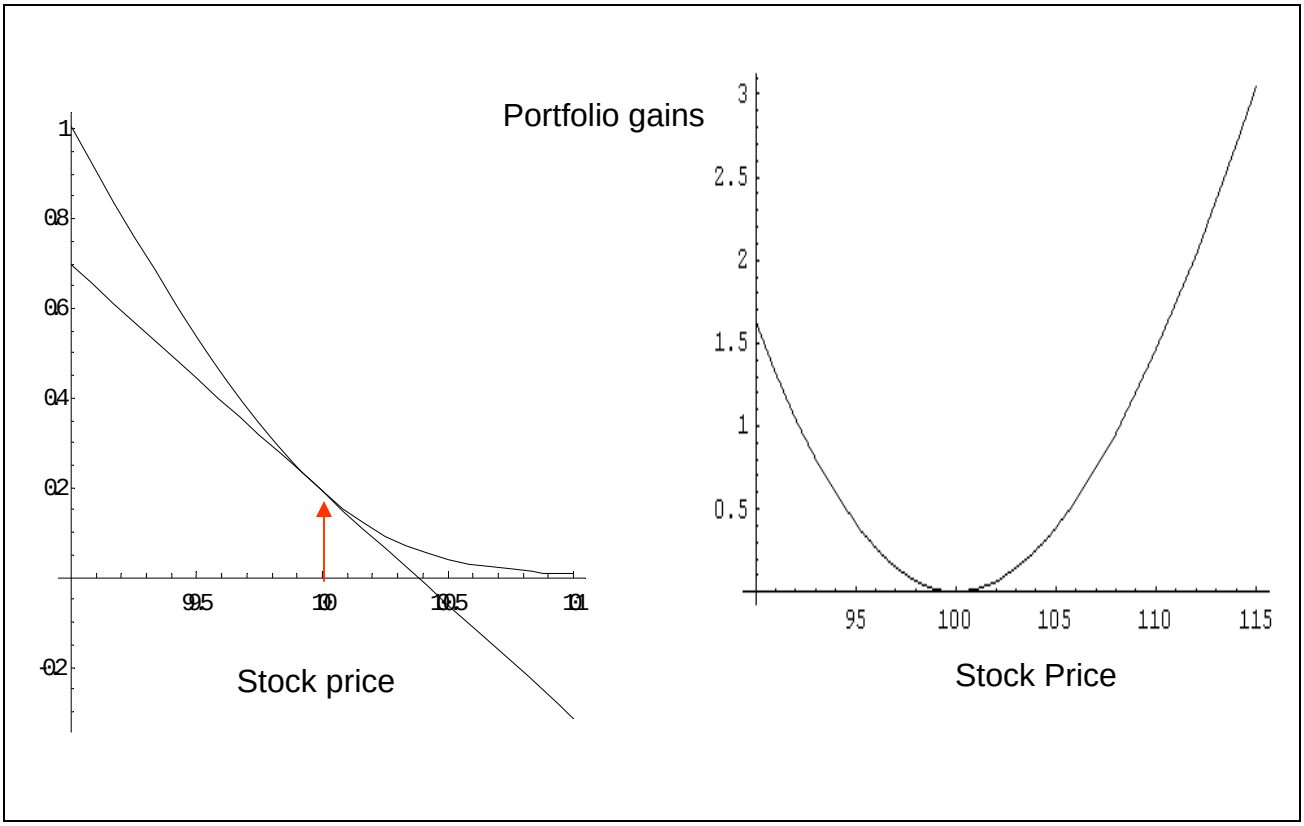
Consider a put option written on a stock

Borrow the premium and buy the put



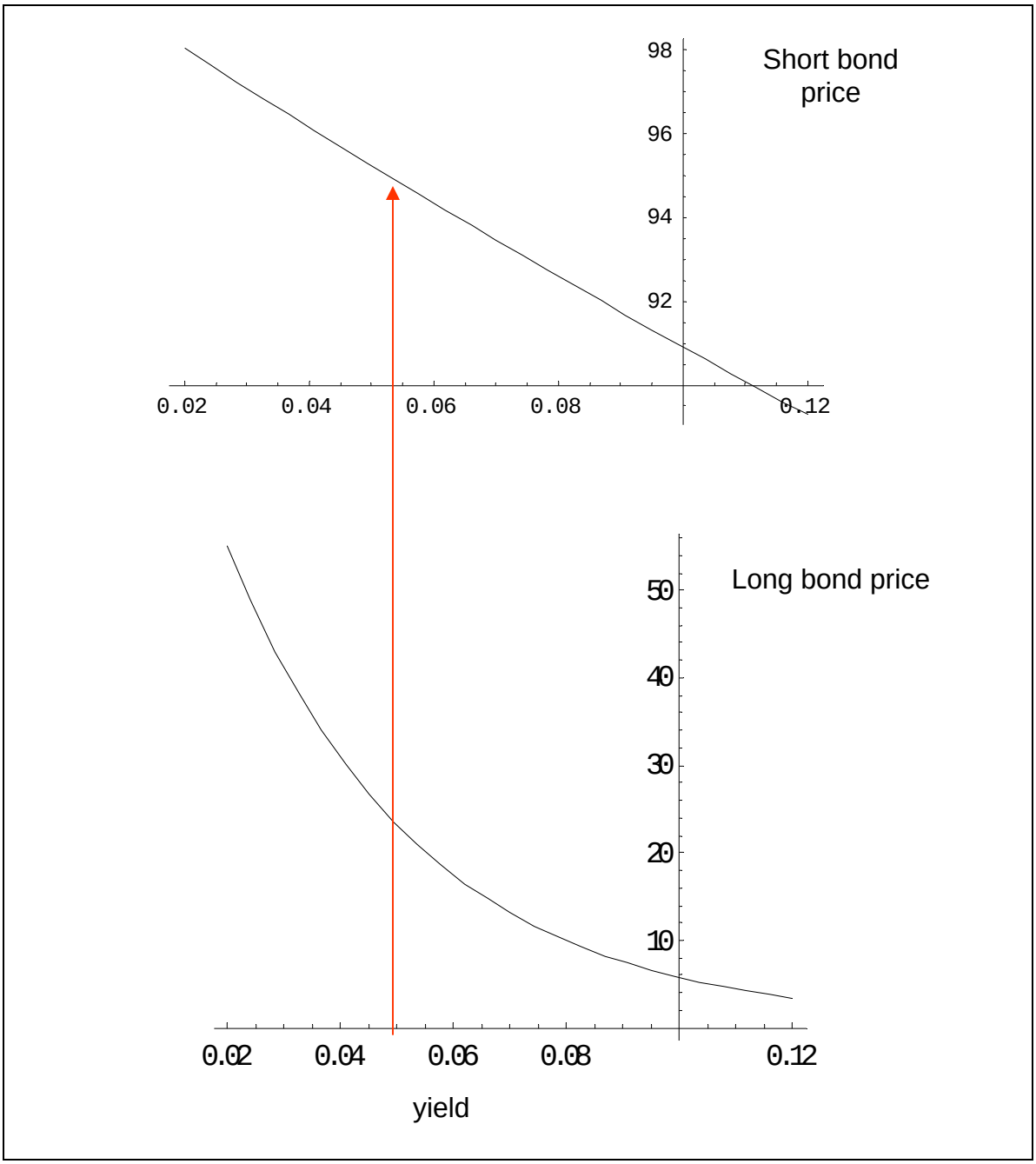
Go long in the futures market by *Delta* units of the stock.

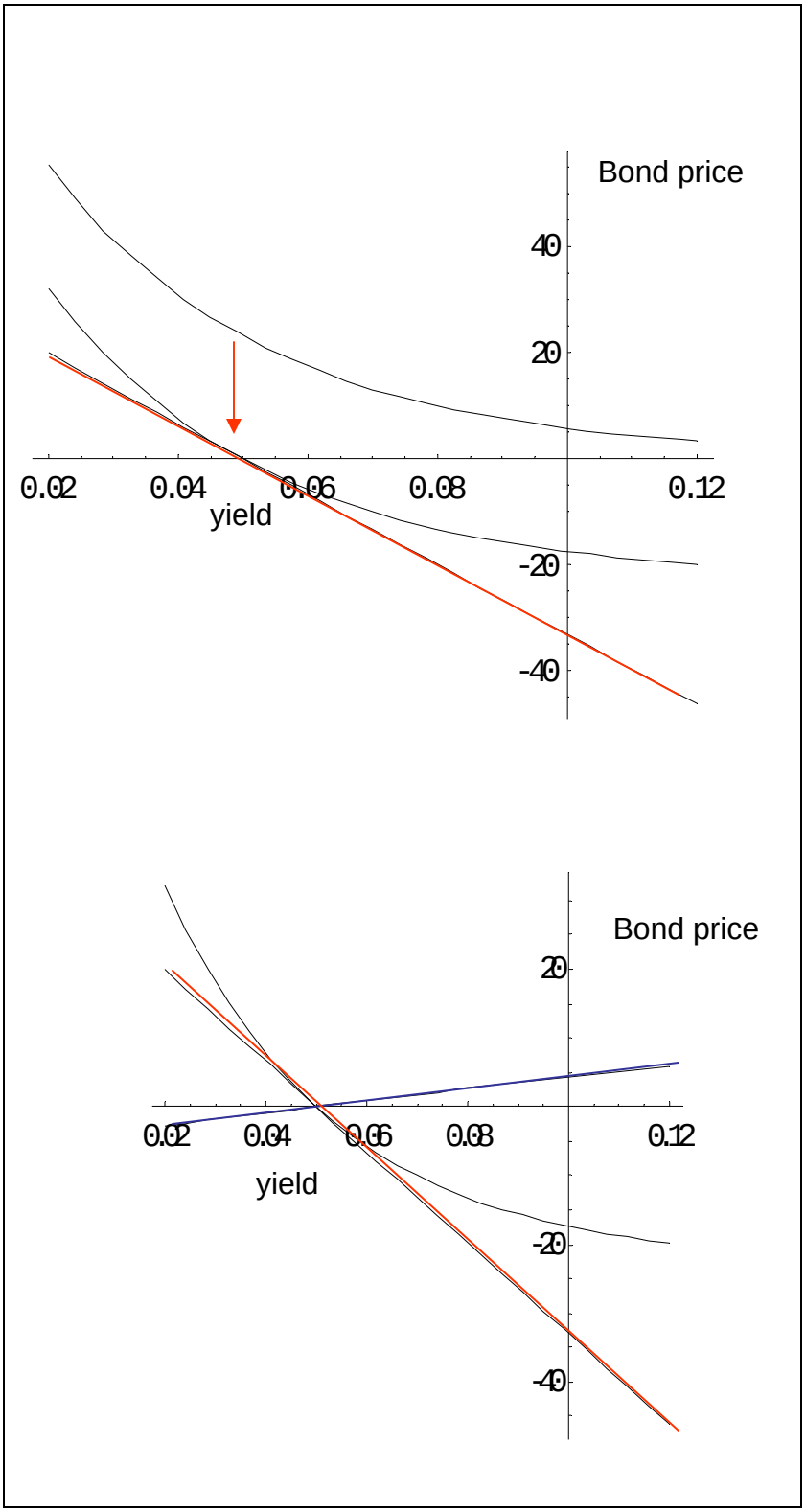
The final outcome is…



Now consider two bonds.

One long-dated, one short-dated

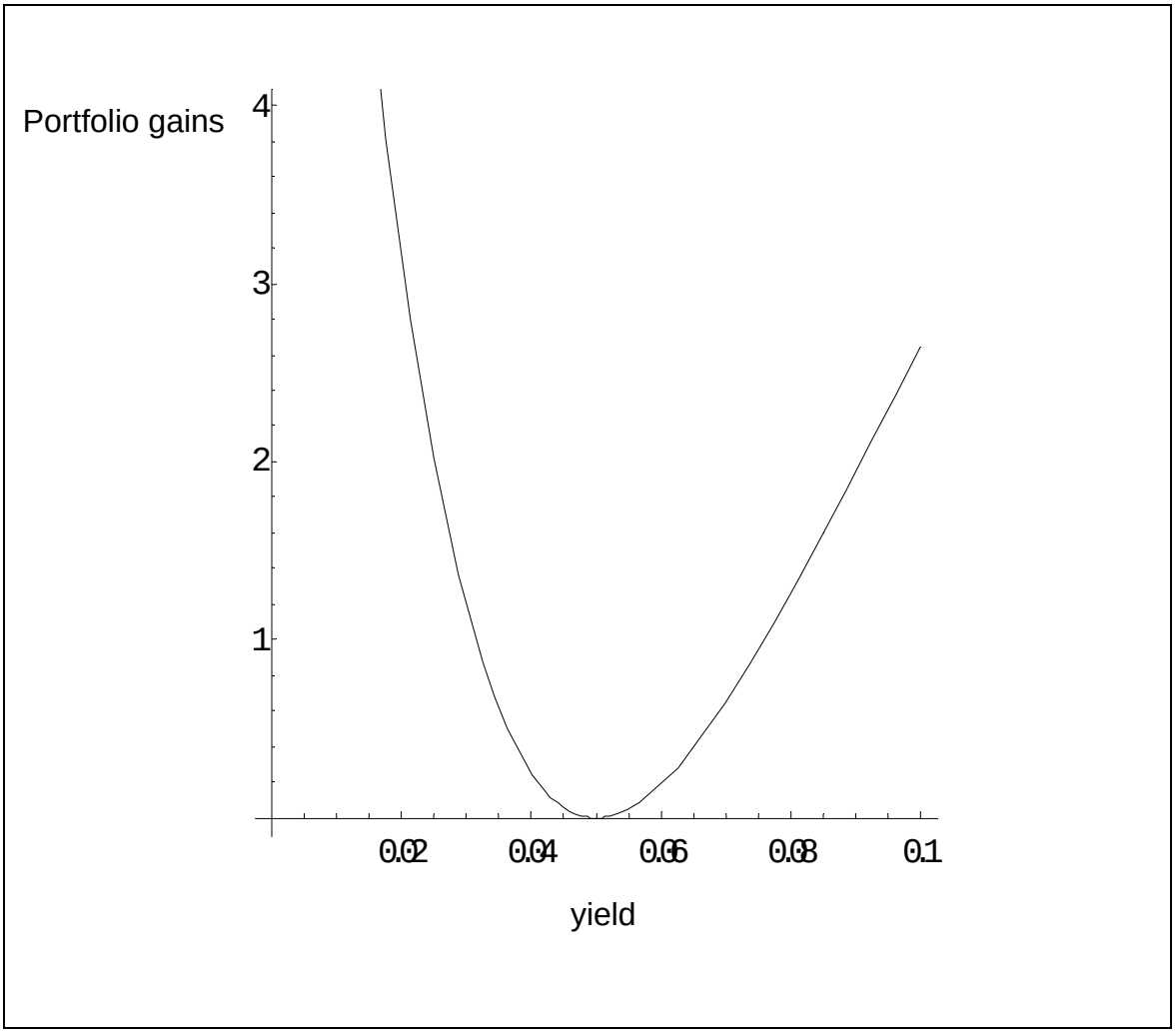




Buy the long-dated bond

Short Delta units of the short-dated bond

Convexity gains of a delta-hedged long bond



Convexity gains

$$C_t = E_t^P\left[\sum_{i=1}^{n} d(t,\ t_i)\,\Delta[\partial_y B(30,\ y)]\,\Delta y\right]$$

$$= \sum_{i=1}^{n} d(t,\ t_i)\,\Delta^2[\partial_y B(30,\ y)]\,E_t^P\left[\Delta y^2\right]$$

$$= \sum_{i=1}^{n} d(t,\ t_i)\,\Delta^2[\partial_y B(30,\ y)]\,\sigma^2(y,\ t_i)$$

Bond convexity trades

A "long" long bond, or swap, similar to a vanilla option, is a *both* convex instrument.

It co-exists with instruments, such as short bonds that are either linear or that have less convexity.

Better synthetic bond constructed using a barbell portfolio, with the same duration as a 5-year bond will have different convexity characteristics than its benchmark.

Hence a delta-hedged swap (bond) portfolio can be put together for long maturity swaps just as in the case of options to benefit from volatility-generated gains.

For simplicity assume that all yields depend on the same risk factor y_t.

Consider a 15-yr maturity default-free discount bond with price $B(t,T)$, at time t, where T is the maturity.

This price can be expressed using the corresponding time-t yield y_t^T:

$$B(t,T) = \frac{1}{(1+y_t^T)^T} \tag{1}$$

Define two other bonds, $B(t,s), B(t,L)$ where S is, say 1 year and L is 30 years.

We can calculate a synthetic $T = 15$-maturity bond using the barbell

$$S[t,T] = B(t,S) + hB(t,L)$$

where the hedge factor h is a proper hedge ratio obtained from:

$$\frac{\partial}{\partial r_t} S[t,T] = T$$

Delta-hedged bond portfolios

We form a delta-neutral portfolio that is independent of the level effects.

Buy one unit of the 15-year bond, sell h units of the synthetic.

Delta sensitivity will cancel for small y_t

The change in the value of this portfolio due to a small change Δy_t will be given by the gamma gains which depend essentially on the second derivative, B_{rr}.

As y_t vibrates, the position is adjusted by buying and (short) selling the less-convex asset using the partial derivative h_t at each readjustment.

Closed form Bond pricing formulas

Let the *risk-neutral dynamics* of the short rate r_t be given as,

$$dr_t = \mu(r_t, t)dt + \sigma r_t dW_t, \qquad (4)$$

where percentage volatility σ is constant.

Under different assumptions concerning short rate dynamics one can indeed solve the implied bond PDE and obtain closed form formulas.

One well known case is the case of Cox-Ingersoll-Ross(CIR) model.

In the CIR model the spot rate r_t is assumed to obey the slightly different mean-reverting SDE,

$$dr_t = \alpha(\kappa - r_t)dt + \sigma\sqrt{r_t}dW_t \qquad (5)$$

which is known as the square-root specification for interest rate volatility. Here the W_t will be a Wiener process under *real-world* probability.

The PDE that will correspond to this case will be given by:

$$(\alpha(\kappa - r) - \sigma^2 r\lambda)B_r + B_t + \frac{1}{2}B_{rr}\sigma^2 r - rB = 0 \quad (6)$$

with boundary condition

$$B(T,T) = 1. \quad (7)$$

Here the λ is known as the market price of interest-rate risk.

The closed-form bond pricing equation is given by:

$$B(t,T) = A(t,T)e^{-C(t,T)r_t} \quad (8)$$

where the functions $A(t,T), C(t,T)$ are given by

$$A(t,T) = \left(2\frac{\gamma\, e^{1/2\,(\alpha+\lambda+\gamma)(T-t)}}{(\alpha+\lambda+\gamma)\left(e^{\gamma\,(T-t)}-1\right)+2\,\gamma}\right)^{2\frac{\alpha\,\kappa}{\sigma^2}} \quad (9)$$

$$C(t,T) = 2\frac{e^{\gamma(T-t)} - 1}{(\alpha+\lambda+\gamma)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \tag{10}$$

and where the γ is given by:

$$\gamma = \sqrt{(\alpha+\lambda)^2 + 2\sigma^2}. \tag{11}$$

The λ, together with Bond volatility σ, will determine the *risk premia* in expected discount bond returns.

Figure 3 shows an example. Let the yields be fixed at 5% and the short rate volatility is given by 20%. The Figure shows that as we increase the volatility from say, 20% to 30%, the bond prices increase.

The Framework

We consider the issue within a "classical" framework.

Let $B(t,T)$ denote the price of a T-maturity default-free discount bond at time t.

$$B(t,T) = E_t^{\tilde{P}}[e^{-\int_t^T r_s ds} B(T,T)]$$

We assume that the following SDE describes

$$dB = \mu(B,t)Bdt + \sigma(Z_t,t)BdW_t,$$

Where the $\sigma(.)$ is an instantaneous volatility parameter that depends on a process Z_t to be specified below.

The W_t is a standard Wiener process under the (real world) probability P.

Here we let $Z_t = R_t$, so that there are no other exogenous variables.

The dynamics of the spot interest rate, r_t be given by:

$$dr_t = a(r_t, t)dt + b(Z_t, t)dW_t,$$

where the drift $a(r_t, t)$ and the diffusion $b(Z_t, t)$ parameters are assumed to be *known*.

The W_t is the same Wiener process defined above.

Bond price PDEs

A PDE is derived using Ito's expansion for $B(t,T)$.

$$dB(t,T) = B_r dr_t + B_t dt + \frac{1}{2} B_{rr} b(r_t,t)^2 dt.$$

Substituting for dr_t from

$$dr_t = a(r_t,t)dt + b(r_t,t)dW_t$$

$$dB(r_t,t) = \left(B_r a(r_t,t) + B_t + \frac{1}{2} B_{rr} b(r_t,t)^2 \right) dt + b(r_t,t) B_r dW_t,$$

This SDE must be identical to the original equation that drives the bond price dynamics. This means that we can equate the drift and diffusion coefficients.

$$b(r_t,t) B_r = \sigma(r_t,t) B,$$

$$\mu(B,t)B = B_r a(r_t,t) + B_t + \frac{1}{2}B_{rr}b(r_t,t)^2.$$

The Ito expansion of the Bond dynamics under risk-adjusted paths becomes:

$$B_r(a(r_t,t) - b(r_t,t)\lambda_t) + B_t + \frac{1}{2}B_{rr}b(r_t,t)^2 - r_t B = 0.$$

This is a PDE for the price of a default-free pure discount bond $B(t,T)$. The associated boundary condition is

$$B(T,T) = 1.$$

If one had an interest rate model with known drift $a(r_t,t)$ and diffusion coefficient $b(r_t,t)$, to use this PDE in practice, one would *still* need an estimate for the λ_t.

This is equivalent to using the drift from the spot-rate dynamics, written under the risk-neutral measure $\tilde{P}$.

$$dr_t = (a(r_t,t) - b(r_t,t)\lambda_t)dt + b(r_t,t)d\tilde{W}_t.$$

The PDE becomes

$$B_t + \frac{1}{2} B_{rr} b^2 r - rB = 0,$$

Rewrite this

$$B_t = -\frac{1}{2} B_{rr} b^2 r + rB = 0,$$

Discretize and reintroduce the time subscripts,

$$B(t+\delta, T) - B(t,T) = r_t B(t,T)\Delta - \frac{1}{2} B_{rr} \sigma_t^2 \Delta$$

$$B(t+\delta, T) = (1 + r_t \Delta) B(t,T) - \frac{1}{2} B_{rr} \sigma_t^2 \Delta$$

As time passes, the bond price increases by the accrued interest, but decreases by the (expected) convexity gain.

We can also look at it as

$$B(t,T) = E_t^P[\frac{1}{(1+r\Delta)}(B(t+\delta,T)+\frac{1}{2}B_{rr}\sigma_t^2\Delta)]$$

Now index the convexity parameter B_{rr} by t, since

$$\lim_{t\to T} B_{rr}^t = 0$$

Introducing the Expectation operator, under the real-world, or risk-adjusted probability, and solving the difference equation:

$$B(t,T) = E_t^P[\frac{1}{\prod_{i=0}^{n}(1+r_{t+i}\Delta)} + \frac{1}{2}\sum_{j=o}^{n}\frac{B_{rr}^{t+j}\sigma_{t+j}^2\Delta}{\prod_{i=0}^{j}(1+r_{t+i}\Delta)}]$$

Since $B(T,T) = 1$.

The cash flow news

This interpretation refers to the Corporate finance literature on Cash Flow news. (For example, Campbell(1999))

This literature assumes that bonds have no cash flow news in contrast to stocks. In case of stocks, dividends relate to cash flow news and lead to interesting testable models.

The paper argues that in case of bonds convexity may provide a link between cash flow news and bond prices.

Several hypotheses can be tested.

Below the term N_{cf} is the cash flow news component:

$$N_{cf} = E_t^P[\frac{1}{2}\sum_{j=o}^{n}\frac{B_{rr}^{t+j}\sigma_{t+j}^2\Delta}{\prod_{i=0}^{j}(1+r_{t+i}\Delta)}]$$

It can be written as:

$$N_{cf} = E_t^P[\frac{1}{2}\sum_{j=o}^{n}\frac{\gamma_{t+j}\sigma(r_t,t)^2\Delta}{D_{t+j}}]$$

Where: γ_{t+j} = Bond convexity

$\sigma(r_{t+j})^2$ = Annual yield variance

Δ = Hedging frequency

D_{t+j} = Discount rate

Estimation

Consider the empirical model that describes our preliminary results.

$$L_t = \beta_1 L_{t-1} + \epsilon_t$$

$$\sigma(\epsilon)_t^2 = \alpha_1 \sigma(\epsilon)_t^2 + \alpha_2 \epsilon_{t-1} + \alpha_3 |\epsilon_{t-1} - \epsilon_{t-2}| + v_t$$

$$C_t = \gamma_1 C_{t-1} + \gamma_2 \sigma(\epsilon)_{t-1} + \eta_t$$

$$\sigma(\eta)_t^2 = a_1 \sigma(\eta)_t^2 + a_2 \eta_{t-1} + a_3 |\eta_{t-1} - \eta_{t-2}| + n_t$$

Where L_t is the long swap rate, C_t is the curvature defined as the 2-15-30 butterfly.

Estimate the current variances and introduce these to the equations governing the level, curvature dynamics.

Results

Parameter	Value	Standard Error	T Statistic
C	0.0080321	0.0057087	1.407
AR(1)	0.99801	0.0010485	951.8071
K	-0.029725	0.024705	-1.2032
GARCH(1)	0.99472	0.0043672	227.7699
ARCH(1)	0.028554	0.011701	2.4403
Leverage(1)	-0.022906	0.0071239	-3.2154
DoF	5.392	0.50921	10.589

Results

Parameter	Value	Error	Statistic
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C	-0.016326	0.0052557	-3.1064
AR(1)	0.99293	0.002133	465.5167
Regress(1)	0.16042	0.058529	2.7408
Regress(2)	0.002045	0.03185	0.0642
Regress(3)	0.16661	0.10138	1.6434
K	-0.68904	0.18184	-3.7893
GARCH(1)	0.9114	0.023477	38.821
ARCH(1)	0.26011	0.037933	6.8569
Leverage(1)	0.043987	0.027023	1.6278
DoF	5.6925	0.60179	9.4592

Results

Parameter	Value	Standard Error	T Statistic
C	0.00052869	0.0025286	0.2091
AR(1)	0.99818	0.0016894	590.8459
K	-0.13277	0.057795	-2.2972
GARCH(1)	0.97888	0.031807	30.7752
ARCH(1)	0.21731	0.19357	1.1226
Leverage(1)	0.050805	0.024994	2.0327
DoF	5.0297	0.038778	129.7024

Results

Parameter	Value	Error	Statistic
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C	-0.008821	0.0025286	-3.4885
AR(1)	0.99484	0.0016894	588.871
Regress(1)	0.17682	0.057795	3.0594
Regress(2)	0.0044237	0.031807	0.1391
K	-0.7485	0.19357	-3.8668
GARCH(1)	0.90371	0.024994	36.1572
ARCH(1)	0.27345	0.038778	7.0517
Leverage(1)	0.03529	0.02772	1.2731
DoF	5.6814	0.60048	9.4614

Results

Parameter	Value	Standard Error	T Statistic
C	-0.0088828	0.0024564	-3.6162
AR(1)	0.99484	0.0016644	597.7214
Regress(1)	0.1807	0.046771	3.8636
K	-0.74825	0.19339	-3.8691
GARCH(1)	0.90374	0.02497	36.1929
ARCH(1)	0.27309	0.038781	7.0418
Leverage(1)	0.035414	0.027607	1.2828
DoF	5.6805	0.5987	9.4881