

Firm Heterogeneity and Credit Risk Diversification*

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March 2006

Abstract

This paper considers a simple model of credit risk and derives the limit distribution of losses under different assumptions regarding the structure of systematic and idiosyncratic risks and the nature of firm heterogeneity. It documents a rich and complex interaction between the underlying model parameters and the resulting loss distributions. The theoretical results indicate that neglecting heterogeneity in firm returns and/or default thresholds leads to *underestimation* of expected losses (EL), and its effect on portfolio risk is ambiguous. But once EL is controlled for, neglecting parameter heterogeneity leads to *overestimation* of risk. These results are verified empirically where it is shown that heterogeneity in the default threshold or unconditional probability of default, measured for instance by a credit rating, is of first order importance in affecting the shape of the loss distribution: including ratings heterogeneity alone results in a more than one-quarter drop in loss volatility and a more than one-half drop in 99.9% VaR, the level to which the risk weights of the New Basel Accord are calibrated.

JEL Classifications: C33, G13, G21.

Key Words: Risk management, correlated defaults, factor models, portfolio choice.

*We would like to thank Richard Cantor, Paul Embrechts, Rafael Repullo, Joshua Rosenberg, Jose Scheinkman, Zhenyu Wang, and especially David Lando, as well as participants at the 13th Annual Conference on Pacific Basin Finance, Economics, and Accounting at Rutgers University, June 2005, the BIS-ECB 4th Joint Central Bank Research Conference in Frankfurt, November 2005, the NBER conference on Risks of Financial Institutions, Cambridge, MA, November 2005, and seminar participants at the Newton Institute, University of Cambridge, UMass Amherst and the Federal Reserve Bank of New York for helpful comments and suggestions. We are also grateful to Yue Chen and Chris Metli for research assistance.

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1 Introduction

The importance of modeling correlated defaults has been recognized in the credit risk literature for some time. Early treatment can be traced to the single homogeneous factor model due to Vasicek (1987, 1991), which also forms the basis of New Basel Accord (BCBS, 2005) as outlined in detail by Gordy (2003). Extensions to multiple factors were proposed by Wilson (1997a,b) and Gupton, Finger and Bhatia (1997) in the form of the industry credit portfolio model CreditMetrics.¹ Practically all of these models are adaptations of Merton's (1974) options based approach, which develops a simple model of firm performance with a threshold value below which the firm defaults. Typically stochastic simulation is needed to obtain the portfolio loss distribution, though Vasicek and Gordy show that by imposing sufficient structure and homogeneity, closed form solutions for the loss distribution can be obtained in terms of an average portfolio default rate and an average asset correlation. Their results have shaped industry practice and, in the case of Gordy (2003), regulatory policy in the specific form of the regulatory capital formula in the New Basel Accord (BCBS, 2005, §272).

In this paper we build on the work of Vasicek and Gordy and consider the implications of relaxing the homogeneity assumptions that underlie these pioneering contributions. In particular, we examine the consequences of incorrectly neglecting the heterogeneity of return correlations and default thresholds across firms for the analysis of loss distribution. The default threshold captures a variety of firm characteristics such as balance sheet structure, including leverage, as well as intangibles like the quality of management. This heterogeneity can be random – firms, say, have on average the same factor loadings – and/or the differences could be systematic – mean factor loadings could differ across industries but are randomly distributed around the industry mean, across firms within an industry.

Our theoretical set-up is quite general and imposes few distributional and parametric restrictions. The theoretical results show a complex interaction between the sources of heterogeneity and the resulting loss distribution. We find that incorrectly neglecting heterogeneity results in *under*estimation of expected losses (EL), and its effect on portfolio risk is ambiguous. This is a new result and arises due to the nonlinear nature of the relationships that prevail between the return process, the default threshold and the resultant default (and hence loss) process. Differences in asset values and default thresholds across firms do not disappear by cross section averaging even if the differences across firms are random and the underlying portfolio is sufficiently large.

In comparing heterogeneous loss portfolios it is therefore important that appropriate adjustments are made so that the different portfolios all have the same EL's. This is only possible by allowing for systematic heterogeneity across firms, e.g. by grouping into industries, regions, uncon-

¹For a summary of this and other industry models, see Saunders and Allen (2002), and for detailed comparisons, see Koyluoglu and Hickman (1998) and Gordy (2000).

ditional distances to default (credit rating), or a combination of those. In that case we prove that neglected heterogeneity results in *overestimated* risk so that falsely imposing homogeneity can be quite costly.

The importance of these theoretical insights are investigated empirically using a portfolio of over 800 firms across U.S. and Japan. Return regressions subject to different degrees of parameter heterogeneity are estimated recursively using six ten-year rolling estimation windows, and for each estimation window the loss distribution is then simulated out-of-sample over a one-year period. The predictions made by theory are confirmed by the empirical results, and are found to be robust across the six years. We show that heterogeneity in the default threshold or unconditional probability of default (PD), measured for instance by a credit rating, is of first order importance in affecting the shape of the loss distribution: allowing for ratings heterogeneity alone results in a more than one-quarter drop in loss volatility (keeping EL's constant) and about a one-half drop in 99.9% VaR, the level to which the risk weights in the New Basel Accord are calibrated. Allowing for additional heterogeneity results in a further 10% drop in 99.9% VaR. This result has important policy implications as a PD estimate through a credit rating, whether internal or external, is the one parameter that is allowed to vary in the New Basel Accord.

This paper is the first, to our knowledge, which analyzes the impact of neglected heterogeneity on credit risk. We use a simple multifactor approach which is easily adapted to this task. Multifactor models have been used extensively in finance following Ross (1976) and Chamberlain and Rothschild (1983).² Their application to credit risk has been more recent. A notable example is its use in the CreditMetrics model as set out in Gupton, Finger and Bhatia (1997). Gordy (2000) and Schönbucher (2003, ch. 10) provide useful reviews.

A separate line of research has focused on correlated default intensities as in Schönbucher (1998), Duffie and Singleton (1999), Duffie and Gârleanu (2001), Collin-Dufresne, Goldstein and Hugonnier (2004), and Duffie, Saita and Wang (2005); with a review by Duffie (2005). There are also a host of other approaches, including correlated (but non-systematic) jumps-at-default (Driessen, 2005, Jarrow, Lando, and Yu, 2005), the contagion model of Davis and Lo (2001) as well as Giesecke and Weber's (2004) indirect dependence approach, where default correlation is introduced through local interaction of firms with their business partners as well as via global dependence on economic risk factors. The idea of generalizing default dependence beyond correlation using copulas is discussed in Li (2000), Embrechts, McNeil, and Straumann (2001), Schönbucher (2002) and Frey and McNeil (2003). Clearly, the issue of heterogeneity is relevant to all these approaches.

In short, the literature on modeling default dependence is growing rapidly along different paths, and there is as yet no consensus which approach is best. Our paper does not address that issue, but it does highlight, using a factor approach, the importance of accounting for heterogeneity. The issue of heterogeneity clearly also arises in the case of other approaches that focus on correlated

²Connor and Korajczyk (1995) provide an excellent survey.

default intensities or copulas; we leave that for others to explore. The factor structure considered here does allow us to explore two distinct channels of heterogeneity: one that is shared, namely factor sensitivities, and one which is specific to firms within a given grouping (e.g. credit rating), namely the default threshold or the distance to default.

Our results have bearing on risk and capital management as well as the pricing of credit assets. For example, in the case of a commercial bank, ignoring heterogeneity may result in underprovisioning for loan losses since EL is underestimated, and may result in overcapitalization against (bank) default since risk is overestimated. The risk assessment and pricing of complex credit asset such as collateralized debt obligations (CDOs) may be adversely affected since they are driven by the shape of the loss distribution which is then segmented into tranches.

The plan for the remainder of the paper is as follows: Section 2 introduces the basic model of firm value and default and considers the problem of correlated defaults. Section 3 derives the portfolio loss distribution under different heterogeneity assumptions, starting with the simple case of a homogeneous portfolio as introduced by Vasicek. Section 4 explores the impact of heterogeneity empirically using returns for firms in the U.S. and Japan across seven sectors and analyzes the resulting loss distributions by stochastic simulations. Section 5 provides some concluding remarks. A technical Appendix presents generalizations of some material in Sections 2 and 3.

2 Firm Value, Default and Default Dependence

Much of the research on credit risk modeling, including our own, is based on the option theoretic default model of Merton (1974). Merton recognized that a lender is effectively writing a put option on the assets of the borrowing firm; owners and owner-managers (i.e. shareholders) hold the call option. If the value of the firm falls below a certain threshold, the owners will put the firm to the debt-holders. Thus a firm is expected to default when the value of its assets falls below a threshold value determined by its liabilities.³

2.1 Firm Value and Default

Consider a firm i having asset value V_{it} at time t , and an outstanding stock of debt, D_{it} . Under the Merton model default occurs at the maturity date of the debt, $t + h$, if the firm's assets, $V_{i,t+h}$, are less than the face value of the debt at that time, $D_{i,t+h}$. The value of the firm at time t is the sum of debt and equity, namely

$$V_{it} = D_{it} + E_{it}, \text{ with } D_{it} > 0. \tag{1}$$

³An alternative to Merton's end of period approach are the first-passage models where default would occur the first time that firm value falls below a default boundary (or threshold) over the period, as in Zhou (2001).

Conditional on time t information, default will take place at time $t + h$ if $V_{i,t+h} \leq D_{i,t+h}$. In the Merton model debt is assumed to be fixed over the horizon h . For simplicity we set $h = 1$; extensions to multiple periods can be found in Pesaran, Schuermann, Treutler, and Weiner (2005), hereafter PSTW. Because default is costly and violations to the absolute priority rule in bankruptcy proceedings are common, in practice debtholders have an incentive to put the firm into receivership even before the equity value of the firm hits the zero value.⁴ Similarly, the bank might also have an incentive of forcing the firm to default once the firm's equity falls below a non-zero threshold.⁵ Importantly, default in a credit relationship is typically a weaker condition than outright bankruptcy. An obligor may meet the technical default condition, e.g. a missed coupon payment, without subsequently going into bankruptcy. As a result we shall assume that default takes place if

$$0 < E_{i,t+1} < C_{i,t+1}, \quad (2)$$

where $C_{i,t+1}$ is a positive default threshold which could vary over time and with the firm's characteristics (such as region or industry sector). Natural candidates that affect the default threshold include observable factors such as leverage, profitability, and firm age (most of which appear in models of firm default), as well as non-observable ones such as management quality.⁶

We are now in a position to consider the evolution of firm equity value which we assume follows a standard geometric random walk model:

$$\ln(E_{i,t+1}) = \ln(E_{it}) + \mu_i + \xi_{i,t+1}, \quad \xi_{i,t+1} \sim iidN(0, \sigma_{\xi_i}^2), \quad (3)$$

with a non-zero drift, μ_i , and idiosyncratic Gaussian innovations with a zero mean and firm-specific volatility, σ_{ξ_i} . Consequently, default occurs if

$$\ln(E_{i,t+1}) = \ln(E_{i,t}) + \mu_i + \xi_{i,t+1} < \ln(C_{i,t+1}), \quad (4)$$

or if the one-period change in equity value or return falls below some threshold defined by

$$\ln\left(\frac{E_{i,t+1}}{E_{it}}\right) < \ln\left(\frac{C_{i,t+1}}{E_{it}}\right) = \lambda_{i,t+1}. \quad (5)$$

Equation (5) tells us that the relative (rather than absolute) decline in firm value must be large enough over the period to result in default. Note that firm-specific information such as leverage and management quality, embedded in the default threshold C_i , carry over to λ_i . Thus for highly levered firms with poor management, the threshold is lower (in the sense of being more negative)

⁴See, for instance, Leland and Toft (1996) who develop a model where default is determined endogenously, rather than by the imposition of a positive net worth condition. More recently, Broadie, Chernov, and Sundaresan (2004) show that in the presence of APR default can be optimal when $E_{it} > 0$ even in the case of a single debt class.

⁵For a treatment of this scenario, see Garbade (2001).

⁶For models of bankruptcy and default at the firm level, see, for instance, Altman (1968), Lennox (1999), Shumway (2001), and Hillegeist, Keating, Cram and Lundstedt (2004).

than for well capitalized and well managed firms. The important issue of measuring λ_i empirically is taken up in Section 4.1.

Under the assumption of Gaussian innovations in (3), the probability that firm i defaults at the end of the period is given by

$$\pi_{i,t+1} = \Phi\left(\frac{\lambda_{i,t+1} - \mu_i}{\sigma_{\xi_i}}\right), \quad (6)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. In the theoretical discussions that follows we shall assume that the firm-specific default thresholds are given.

2.2 Cross Firm Default Dependence: Some Preliminaries

In the context of the Merton model, cross firm default dependence can be introduced by assuming that shocks to the value of a firm's equity, $\xi_{i,t+1}$, defined by (3), have the following multifactor structure:

$$\xi_{i,t+1} = \gamma_i' \mathbf{f}_{t+1} + \sigma_i \varepsilon_{i,t+1}, \quad \varepsilon_{i,t+1} \sim iid(0, 1) \quad (7)$$

where \mathbf{f}_{t+1} is an $m \times 1$ vector of common factors, γ_i is the associated vector of factor loadings, and $\varepsilon_{i,t+1}$ is the firm-specific idiosyncratic shock, assumed to be distributed independently across i . The common factors could be treated as either unobserved or observed through macroeconomic variables such as output growth, inflation, interest rates and exchange rates.⁷

In what follows we suppose the factors are unobserved, distributed independently of $\varepsilon_{i,t+1}$, and have constant variances.⁸ Thus, without loss of generality we assume that $\mathbf{f}_{t+1} \sim (\mathbf{0}, \mathbf{I}_m)$, where \mathbf{I}_m is an identity matrix of order m .

Using (7) in (3) we now have

$$\ln(E_{i,t+1}) - \ln(E_{it}) = r_{i,t+1} = \mu_i + \gamma_i' \mathbf{f}_{t+1} + \sigma_i \varepsilon_{i,t+1}. \quad (8)$$

Under the above assumptions

$$\sigma_{\xi_i}^2 = \gamma_i' \gamma_i + \sigma_i^2, \quad (9)$$

which decomposes the return variance into the part due the systematic risk factors, $\gamma_i' \gamma_i$, and the residual or idiosyncratic variance, σ_i^2 . The presence of the common factors also introduces a varying degree of asset return correlations across firms, which in turn leads to variation in cross firm default correlations for a given set of default thresholds, $\lambda_{i,t+1}$. The extent of default correlation depends on the size of the factor loadings, γ_i , the importance of the idiosyncratic shocks, σ_i , the values of

⁷For instance, PSTW provide an empirical implementation of this model by linking the (observable) factors, \mathbf{f}_{t+1} , to the variables in a global vector autoregressive model comprising around 80% of world output.

⁸The more general case where the factors may exhibit time varying volatility can be readily dealt with by allowing the factor loadings to vary over time, in line with market volatilities. But in this paper we shall not pursue this line of research, primarily because the focus of our empirical analysis is on quarterly and annual default risks, and over such horizons asset return volatility dynamics tend to be rather weak and of second order importance.

the default thresholds, $\lambda_{i,t+1}$, and the shape of the distribution assumed for $\varepsilon_{i,t+1}$, particularly its left tail properties. The correlation coefficient of returns of firms i and j is given by

$$\rho_{ij} = \frac{\boldsymbol{\delta}'_i \boldsymbol{\delta}_j}{(1 + \boldsymbol{\delta}'_i \boldsymbol{\delta}_i)^{1/2} (1 + \boldsymbol{\delta}'_j \boldsymbol{\delta}_j)^{1/2}}, \quad (10)$$

where $\boldsymbol{\delta}_i = \boldsymbol{\gamma}_i / \sigma_i$ is the standardized $m \times 1$ vector of factor loadings (systematic risk exposures) of firm i .

To derive the cross correlation of firm defaults, which we denote by $\rho_{ij,t+1}^*$, let $z_{i,t+1}$ be the default outcome for firm i over a single period such that

$$z_{i,t+1} = I(\lambda_{i,t+1} - r_{i,t+1}), \quad (11)$$

where $I(A)$ is an indicator function that takes the value of unity if $A \geq 0$, and zero otherwise. Then

$$\rho_{ij,t+1}^* = \frac{E(z_{i,t+1} z_{j,t+1}) - \pi_{i,t+1} \pi_{j,t+1}}{\sqrt{\pi_{i,t+1}(1 - \pi_{i,t+1})} \sqrt{\pi_{j,t+1}(1 - \pi_{j,t+1})}}, \quad (12)$$

where $\pi_{i,t+1} = E(z_{i,t+1})$ is firm i 's default probability over the period t to $t + 1$. For given values of the thresholds, $\lambda_{i,t+1}$, a relatively simple expression for $\rho_{ij,t+1}^*$ can be obtained if conditional on \mathbf{f}_{t+1} , $\varepsilon_{i,t+1}$ and $\varepsilon_{j,t+1}$ are cross sectionally independent, and \mathbf{f}_{t+1} and $\varepsilon_{i,t+1}$ have a joint Gaussian distribution. In this case, known as *conditionally independent double-Gaussian* model, we have

$$\pi_{i,t+1} = \Phi\left(\frac{\lambda_{i,t+1} - \mu_i}{\sqrt{\sigma_i^2 + \boldsymbol{\gamma}'_i \boldsymbol{\gamma}_i}}\right). \quad (13)$$

The argument of $\Phi(\cdot)$ in (13) is commonly referred to as “distance to default” (DD) such that

$$DD_{i,t+1} = \Phi^{-1}(\pi_{i,t+1}) = \frac{\lambda_{i,t+1} - \mu_i}{\sqrt{\sigma_i^2 + \boldsymbol{\gamma}'_i \boldsymbol{\gamma}_i}}. \quad (14)$$

For future reference note that under the double-Gaussian assumption $E(z_{i,t+1} z_{j,t+1})$ is given by

$$\begin{aligned} E(z_{i,t+1} z_{j,t+1}) &= E[I(\lambda_{i,t+1} - r_{i,t+1}) I(\lambda_{j,t+1} - r_{j,t+1})] \\ \Pr[r_{i,t+1} < \lambda_{i,t+1} \ \& \ r_{j,t+1} < \lambda_{j,t+1}] &= \Phi_2[\Phi^{-1}(\pi_{i,t+1}), \Phi^{-1}(\pi_{j,t+1}), \rho_{ij}], \end{aligned} \quad (15)$$

where $\Phi_2[\cdot]$ is the bivariate standard normal cumulative distribution function.

3 Losses in a Credit Portfolio

Consider now a credit portfolio composed of N different credit assets such as loans, each with exposures or weights w_{it} , at time t , for $i = 1, 2, \dots, N$, such that⁹

$$\sum_{i=1}^N w_{it} = 1, \quad \sum_{i=1}^N w_{it}^2 = O(N^{-1}), \quad w_{it} \geq 0. \quad (16)$$

⁹The assumption that N is time-invariant is made for simplicity and can be relaxed.

A sufficient condition for (16) to hold is given by $w_{it} = O(N^{-1})$, which is the standard granularity condition where no single exposure dominates the portfolio.¹⁰ For now, and without loss of generality, we impose that a defaulted asset has no recovery value.¹¹ Under this set-up the portfolio loss over the period t to $t + 1$ is given by

$$\ell_{N,t+1} = \sum_{i=1}^N w_{it} z_{i,t+1}. \quad (17)$$

The probability distribution function of $\ell_{N,t+1}$ can now be derived both conditional on an information set available at time t , \mathcal{I}_t , or unconditionally. The two types of distributions coincide when the factors, \mathbf{f}_{t+1} , are assumed to be serially independent, a case often maintained in the literature. However, this assumption precludes the use of any business cycle models in the analysis of credit risk. For the theoretical results we therefore consider a dynamic factor model and allow the factors to be serially correlated. In particular, we shall assume that \mathbf{f}_{t+1} follows a covariance stationary process, and \mathcal{I}_t contains at least \mathbf{f}_t and its lagged values, or their determinants when they are unobserved. This structure corresponds to the empirical application in PSTW which makes use of a global macroeconometric model, though later in this paper (Section 4) we impose serial independence on the factor process for expositional simplicity.

A simple example of a dynamic factor model is the Gaussian vector autoregressive specification

$$\mathbf{f}_{t+1} = \mathbf{\Lambda} \mathbf{f}_t + \boldsymbol{\eta}_{t+1}, \quad \boldsymbol{\eta}_{t+1} \mid \mathcal{I}_t \sim iidN(\mathbf{0}, \boldsymbol{\Omega}_{\eta\eta}), \quad (18)$$

where \mathcal{I}_t is the public information known at time t , and $\mathbf{\Lambda}$ is an $m \times m$ matrix of fixed coefficients with all its eigenvalues inside the unit circle such that

$$Var(\mathbf{f}_{t+1} \mid \mathcal{I}_t) = \sum_{s=0}^{\infty} \mathbf{\Lambda}^s \boldsymbol{\Omega}_{\eta\eta} \mathbf{\Lambda}'^s = \mathbf{I}_m. \quad (19)$$

The focus of our analysis will be on the limit distribution of $\ell_{N,t+1} \mid \mathcal{I}_t$, as $N \rightarrow \infty$. Not surprisingly, this limit distribution depends on the nature of the return process $\{r_{i,t+1}\}$ and the extent to which the returns are cross-sectionally correlated. Our theoretical discussion shall be in terms of the variance of the loss distribution, though occasionally we refer to the standard deviation or loss volatility, known as unexpected loss (UL) in the credit risk literature. In practice, one may also be interested in quantiles of the loss distributions, or VaR, and those can be easily obtained through stochastic simulations.

¹⁰Conditions (16) on the portfolio weights was in fact embodied in the initial proposal of the New Basel Accord in the form of the Granularity Adjustments which was designed to mitigate the effects of significant single-borrower concentrations on the credit loss distribution (BCBS, 2001, Ch.8). See also the discussion in Lucas, Klaassen, Sprei, and Straetmans (2001).

¹¹The case where default and recovery are correlated through common business cycle effects presents new technical difficulties and is addressed briefly in Appendix A of an earlier version of this paper, available at <http://fic.wharton.upenn.edu/fic/papers/05/p0505.html>.

3.1 Credit Risk under Firm Homogeneity

Vasicek (1987) was one the first to consider the limit distribution of $\ell_{N,t+1}$ using asset return equations with a factor structure. However, he focused on the perfectly homogeneous case with the same factor loadings, $\gamma_i = \gamma$, the same default thresholds, $\lambda_{i,t+1} = \lambda$, the same firm-specific volatilities, $\sigma_i = \sigma$, and zero unconditional returns, $\mu_i = 0$, for all i and t . Note that a multifactor model with homogeneous factor loadings is equivalent to a single factor model, so that under Vasicek's homogeneity assumptions we have

$$r_{i,t+1} = \gamma f_{t+1} + \sigma \varepsilon_{i,t+1}, \quad \begin{pmatrix} \varepsilon_{i,t+1} \\ f_{t+1} \end{pmatrix} | \mathcal{I}_t \sim iidN(\mathbf{0}, \mathbf{I}_2).$$

In this model the pair-wise asset return correlations, ρ_{ij} , is identical for all obligor pairs in the portfolio. Furthermore, since default depends on the sign of $\lambda - r_{i,t+1} = \lambda - (\gamma f_{t+1} + \sigma \varepsilon_{i,t+1})$, and not its magnitude, without loss of generality the normalization, $\sigma^2 + \gamma^2 = 1$ is often used in the literature, thus yielding $\gamma = \pm\sqrt{\rho}$, so that

$$r_{i,t+1} = \sqrt{\rho}f_{t+1} + \sqrt{1 - \rho}\varepsilon_{i,t+1}. \quad (20)$$

The remaining parameter, λ , is then calibrated to a pre-specified default probability, π , so that the distance to default and default thresholds are the same for all firms and can be easily estimated from historical default frequency of the portfolio using

$$\lambda = DD = \Phi^{-1}(\pi). \quad (21)$$

When default thresholds are allowed to vary across firms, identification issues arise which are discussed in Section 4.1.

In Vasicek's model the pair-wise correlation of firm defaults is given by (see (12) and (15))

$$\rho^*(\pi, \rho) = \frac{\Phi_2[\Phi^{-1}(\pi), \Phi^{-1}(\pi), \rho] - \pi^2}{\pi(1 - \pi)}. \quad (22)$$

For example, for the typical parameter values of $\pi = 0.01$, and $\rho = 0.30$, we have $\rho^* = 0.046$.¹² For future reference we also note that $\partial\rho^*(\pi, \rho)/\partial\rho > 0$ so long as $\rho > 0$. (See, for example, Zhou (2001, p. 562)).

Under the Vasicek model portfolio loss variance depends on π and ρ^* :

$$Var(\ell_{N,t+1} | \mathcal{I}_t) = \pi(1 - \pi) \left\{ \rho^* + (1 - \rho^*) \sum_{j=1}^N w_{jt}^2 \right\}. \quad (23)$$

¹²Determinants of ρ^* in the case where the errors have Student- t distribution with the same degree of freedom is discussed in an earlier version of this paper, available at <http://fic.wharton.upenn.edu/fic/papers/05/p0505.html>.

Under the granularity condition, (16), for N sufficiently large the second term in brackets becomes negligible. Hence, in the limit

$$\lim_{N \rightarrow \infty} \text{Var}(\ell_{N,t+1} | \mathcal{I}_t) = \pi(1 - \pi)\rho^* = \Phi_2[\Phi^{-1}(\pi), \Phi^{-1}(\pi), \rho] - \pi^2. \quad (24)$$

The larger the default correlation, ρ^* , the larger will be the portfolio loss variance. For a finite value of N , loss variance is minimized by adopting an equal weighted portfolio, with $w_{jt} = 1/N$. For sufficiently large N , only the granularity condition (16) matters, and nothing can be gained by further optimization with respect to the weights, w_{jt} .

More broadly, Vasicek's credit loss limit distribution is fully determined by two parameters, namely the unconditional default probability, π , and the pair-wise return correlation coefficient, ρ (see Appendix A for further detail). The former fixes the expected loss of the portfolio, while the latter controls the shape of the loss distribution. In effect one parameter, ρ , controls all aspects relating to the shape of the loss distribution: its volatility, skewness and kurtosis.

3.2 Credit Risk with Firm Heterogeneity

Building on Vasicek's work we now consider models that allow for firm heterogeneity across a number of relevant parameters. In this section we provide some analytical derivations and show how the theoretical work of Vasicek can be generalized. An empirical evaluation of the importance of allowing for firm heterogeneity in credit risk analysis is discussed in Section 4.

Under the heterogeneous multifactor return process (8), the portfolio loss, $\ell_{N,t+1}$, can be written as

$$\ell_{N,t+1} = \sum_{i=1}^N w_{it} I(a_{i,t+1} - \boldsymbol{\delta}'_i \mathbf{f}_{t+1} - \varepsilon_{i,t+1}), \quad (25)$$

where

$$a_{i,t+1} = \frac{\lambda_{i,t+1} - \mu_i}{\sigma_i}, \quad \boldsymbol{\delta}_i = \frac{\boldsymbol{\gamma}_i}{\sigma_i}. \quad (26)$$

In addition to allowing for parameter heterogeneity, we also relax the assumption that conditional on \mathcal{I}_t the common factors, \mathbf{f}_{t+1} , and the idiosyncratic shocks, $\varepsilon_{i,t+1}$, are normally distributed with zero means. Accordingly we assume that

$$\begin{aligned} \varepsilon_{i,t+1} &| \mathcal{I}_t \sim iid(0, 1), \text{ for all } i \text{ and } t, \\ \mathbf{f}_{t+1} &| \mathcal{I}_t \sim iid(\boldsymbol{\mu}_{f_t}, \mathbf{I}_m), \text{ for all } t, \end{aligned}$$

where under the dynamic factor model (18), $\boldsymbol{\mu}_{f_t} = \Lambda \mathbf{f}_t$. Allowing $\boldsymbol{\mu}_{f_t}$ to be time-varying enables us to explicitly consider the possible effects of business cycle variations on the loss distribution. In the credit risk literature $\boldsymbol{\mu}_{f_t}$ is usually set to zero. For future use we shall denote the \mathcal{I}_t -conditional probability density and the cumulative distribution functions of $\varepsilon_{i,t+1}$ and \mathbf{f}_{t+1} , by $f_\varepsilon(\cdot)$ and $F_\varepsilon(\cdot)$, and $f_{\mathbf{f}}(\cdot)$ and $F_{\mathbf{f}}(\cdot)$, respectively.

To deal with parameter heterogeneity across firms we abstract from time variations in the default thresholds (namely set $a_{i,t+1} = a_i$) and adopt the following random coefficient model:

$$\boldsymbol{\theta}_i = \boldsymbol{\theta} + \mathbf{v}_i, \quad \mathbf{v}_i \sim iid(\mathbf{0}, \Omega_{vv}), \quad \text{for } i = 1, 2, \dots, N, \quad (27)$$

where

$$\boldsymbol{\theta}_i = (a_i, \boldsymbol{\delta}'_i)', \quad \boldsymbol{\theta} = (a, \boldsymbol{\delta}')', \quad \mathbf{v}_i = (v_{ia}, \mathbf{v}'_{i\boldsymbol{\delta}})', \quad (28)$$

and

$$\Omega_{vv} = \begin{pmatrix} \omega_{aa} & \omega_{a\boldsymbol{\delta}} \\ \omega_{\boldsymbol{\delta}a} & \Omega_{\boldsymbol{\delta}\boldsymbol{\delta}} \end{pmatrix}, \quad (29)$$

is a positive semi-definite symmetric matrix, and \mathbf{v}_i 's are distributed independently of $(\varepsilon_{j,t+1}, \mathbf{f}_{t+1})$ for all i, j and t .

Allowing for such parameter heterogeneity may be desirable when firms have different sensitivities to the systematic risk factors \mathbf{f}_{t+1} , and those sensitivities or factor loadings are known only up to their distributional properties described in (27). A practical example might be assessing the credit risk for a portfolio of borrowers which are privately held, i.e. not publicly traded. This is typically the case for much of middle market and most of small business lending. For such firms it would be very difficult or even impossible to obtain individual estimates of $\boldsymbol{\theta}_i$, and an average estimate based on $\boldsymbol{\theta}$ and Ω_{vv} may need to be used. See also Section 4.6.

The heterogeneity described in (27) to (29) states that firm differences are purely random. However, firms could in addition exhibit systematic parameter differences, say by industry and/or region, so that parameter means and covariances are also industry and/or region specific. This generalization is taken up in Section 3.4.2.

3.3 Limits to Unexpected Loss under Parameter Heterogeneity

The extent to which credit losses are diversifiable can be investigated using a number of different measures. Before exploring the entire loss distribution, for reasons of analytical tractability we focus here on loss variance, $Var(\ell_{N,t+1} | \mathcal{I}_t)$, or its square root, unexpected loss, and note that in general

$$Var(\ell_{N,t+1} | \mathcal{I}_t) = E_f[Var(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t)] + Var_f[E(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t)]. \quad (30)$$

Because of the dependence of the default indicators, $z_{i,t+1}$, across i , through the common factors \mathbf{f}_{t+1} , unexpected loss remains even with a portfolio of infinitely many exposures. The problem of correlated defaults can be dealt with by first conditioning the analysis on the source of cross-dependence (namely \mathbf{f}_{t+1}) and noting that conditional on \mathbf{f}_{t+1} the default indicators, $z_{i,t+1} = I(a_i - \boldsymbol{\delta}'_i \mathbf{f}_{t+1} - \varepsilon_{i,t+1})$, $i = 1, 2, \dots, N$, are independently distributed.

The conditional variance of $z_{i,t+1}$ is bounded since

$$\text{Var}(z_{i,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) = E(z_{i,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) - [E(z_{i,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t)]^2 \leq \frac{1}{4}. \quad (31)$$

Then by the conditional independence of the $z_{i,t+1}$ we have

$$\text{Var}(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) = \sum_{i=1}^N w_{it}^2 \text{Var}(z_{i,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) \leq \frac{1}{4} \left(\sum_{i=1}^N w_{it}^2 \right). \quad (32)$$

Hence, under (16)

$$E[\text{Var}(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t)] \leq \frac{1}{4} \left(\sum_{i=1}^N w_{it}^2 \right) \rightarrow 0, \text{ as } N \rightarrow \infty, \quad (33)$$

and in the limit the loss variance, $\text{Var}(\ell_{N,t+1} | \mathcal{I}_t)$, is dominated by the second term in (30). Namely, we have

$$\lim_{N \rightarrow \infty} \text{Var}(\ell_{N,t+1} | \mathcal{I}_t) = \lim_{N \rightarrow \infty} \{ \text{Var}[E(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t)] \}, \quad (34)$$

which is similar to Proposition 2 in Gordy (2003). This result clearly shows that when the portfolio weights satisfy the granularity condition, (16), the limit behavior of the unexpected loss does not depend on the weights w_{it} . Furthermore, this result holds irrespective of whether a_i and δ_i are homogeneous or vary randomly across i .

Under the random coefficient model, (27), asymptotic loss variance, given by (34), can be obtained by integrating out the heterogeneous effects of a_i and δ_i . First note that $\ell_{N,t+1} = \sum_{i=1}^N w_{it} I(a_i - \delta_i' \mathbf{f}_{t+1} - \varepsilon_{i,t+1})$, which under (27) can be written as

$$\ell_{N,t+1} = \sum_{i=1}^N w_{it} I(a - \delta' \mathbf{f}_{t+1} - \zeta_{i,t+1}), \quad (35)$$

where

$$\zeta_{i,t+1} = \varepsilon_{i,t+1} - \mathbf{v}'_i \mathbf{g}_{t+1} \quad (36)$$

captures all innovations, and $\mathbf{g}_{t+1} = (1, -\mathbf{f}'_{t+1})'$. Conditional on \mathbf{f}_{t+1} , $\zeta_{i,t+1}$ is distributed independently across i with zero mean and variance

$$\omega_{t+1}^2 = 1 + \mathbf{g}'_{t+1} \boldsymbol{\Omega}_{vv} \mathbf{g}_{t+1}, \quad (37)$$

where $\mathbf{g}'_{t+1} \boldsymbol{\Omega}_{vv} \mathbf{g}_{t+1}$ is the variance contribution arising from the random coefficients model (i.e. explicitly due to parameter heterogeneity). The expected loss conditional on \mathbf{f}_{t+1} is given by

$$\begin{aligned} E(\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) &= \sum_{i=1}^N w_{it} \Pr(\zeta_{i,t+1} \leq a - \delta' \mathbf{f}_{t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t) \\ &= \sum_{i=1}^N w_{it} F_{\mathcal{Z}} \left(\frac{\boldsymbol{\theta}' \mathbf{g}_{t+1}}{\omega_{t+1}} \right), \end{aligned}$$

and since $\sum_{i=1}^N w_{it} = 1$, then

$$E(\ell_{N,t+1} \mid \mathbf{f}_{t+1}, \mathcal{I}_t) = F_{\varkappa} \left(\frac{\boldsymbol{\theta}' \mathbf{g}_{t+1}}{\omega_{t+1}} \right), \quad (38)$$

where $F_{\varkappa}(\cdot)$ is the cumulative distribution function of the standardized composite innovations

$$\varkappa_{i,t+1} = \frac{\zeta_{i,t+1}}{\omega_{t+1}} \mid \mathbf{f}_{t+1}, \mathcal{I}_t \sim iid(0, 1). \quad (39)$$

The loss distribution (38) describes the general case of parameter heterogeneity, and evaluation such as computing EL and VaR, may be done using stochastic simulation by taking independent draws from any given distribution of $\varkappa_{i,t+1}$. In some cases we are able to make predictions analytically, e.g. when heterogeneity is limited to mean returns and/or default thresholds, or to the factor loadings. Those cases are taken up in Section 3.4.

In the limit, therefore, using (34) we have

$$\lim_{N \rightarrow \infty} Var(\ell_{N,t+1} \mid \mathcal{I}_t) = Var \left[F_{\varkappa} \left(\frac{\boldsymbol{\theta}' \mathbf{g}_{t+1}}{\omega_{t+1}} \right) \mid \mathcal{I}_t \right], \quad (40)$$

which does not depend on the exposure weights, w_{it} . This result represents a generalization of the limit variance obtained for the homogeneous case, given above by (24).

The implication for credit risk management is clear: changing the exposure weights that satisfy the granularity condition (16) will have no risk diversification impact so long as all firms in the portfolio have the same risk factor loading distribution. To achieve systematic diversification one needs different firm types, e.g. along industry or country lines, and we treat this case below in Section 3.4.1.

3.4 Impact of Neglecting Heterogeneity

Parameter heterogeneity can significantly affect the shape of the loss distribution as well as expected and unexpected losses. This is most easily illustrated with a single factor model. Multifactor generalizations are given in Appendix A. As before, portfolio losses are (replacing w_{it} with w_i to simplify the notation)

$$\ell_{N,t+1} = \sum_{i=1}^N w_i I(a - \delta f_{t+1} - \zeta_{i,t+1}),$$

where $a = (\lambda - \mu)/\sigma$, and $\delta = \gamma/\sigma$; and $\zeta_{i,t+1} = \varepsilon_{i,t+1} - v_{ia} + v_{i\delta} f_{t+1}$, is a composite innovation comprising. Recall that in the absence of heterogeneity, δ and a can be written in terms of the return correlation, ρ , and default probability, π , namely

$$\delta = \sqrt{\frac{\rho}{1-\rho}}, \text{ for } \rho > 0, \quad (41)$$

and

$$a = \frac{\Phi^{-1}(\pi)}{\sqrt{1-\rho}} < 0 \text{ for } \pi < 1/2, \quad (42)$$

which yields the following useful relationship between a , δ and π :

$$a = \sqrt{1 + \delta^2} \Phi^{-1}(\pi). \quad (43)$$

Therefore, for a given value of $\pi < 1/2$, a and δ are negatively related and can not vary freely of one another.

Under the conditionally independent normal assumption,

$$\begin{pmatrix} \varepsilon_{i,t+1} \\ v_{ia} \\ v_{i\delta} \end{pmatrix} | f_t \sim iidN \left(\mathbf{0}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{aa} & \omega_{a\delta} \\ 0 & \omega_{a\delta} & \omega_{\delta\delta} \end{pmatrix} \right),$$

then $\zeta_{i,t+1} | f_t \sim iidN(0, 1 + \omega_{aa} + \omega_{\delta\delta} f_t^2 - 2\omega_{a\delta} f_t)$, and hence as $N \rightarrow \infty$ the loss distribution can be simulated using¹³

$$x(f) = \Phi \left(\frac{a - \delta f}{\sqrt{1 + \omega_{aa} + \omega_{\delta\delta} f^2 - 2\omega_{a\delta} f}} \right), \quad (44)$$

for random draws of $f \sim N(0,1)$. Note that the asymptotic loss distribution is given by the distribution of x (the fraction of the portfolio lost) over $(0, 1]$. Equation (44) is a key expression which we use below to analyze the impact of heterogeneity (or its neglect), manifested through non-zero values of ω_{aa} , $\omega_{\delta\delta}$, and $\omega_{a\delta}$, on the loss distribution, especially its tail.

3.4.1 Heterogeneity of the Mean Returns and/or Default Thresholds

Consider first the case where the standardized factor loading is the same for all firms, namely $\delta_i = \delta$, $\forall i$, but allow for differences in a_i . This also imposes $\sigma_i^2 = \sigma^2$, $\forall i$, and implies the same pair-wise return correlation, ρ , across all firms. As a result, any variation in a_i is due to cross firm variation in $\lambda_i - \mu_i$, the difference between the default threshold and the mean return. It is unlikely that one would see differences in firm thresholds, perhaps due to management quality, but not in expected returns, so that variation in λ_i will likely be accompanied by variation in μ_i .

With that in mind, portfolio losses are

$$x = \Phi \left(\tilde{a} - \tilde{\delta} f \right),$$

where

$$\tilde{a} = \frac{a}{\sqrt{1 + \omega_{aa}}}, \quad \tilde{\delta} = \frac{\delta}{\sqrt{1 + \omega_{aa}}}. \quad (45)$$

¹³Here to simplify the exposition we have denoted the limit of $\ell_{N,t+1}$ by x , and have abstracted from the subscript t since f_t is serially uncorrelated.

In this case the CDF of x would have the same form as Vasicek's loss distribution, namely¹⁴

$$\begin{aligned} F_\ell(x) &= \Phi\left(\frac{\Phi^{-1}(x) - \tilde{a}}{\tilde{\delta}}\right), \\ &= \Phi\left(\frac{\sqrt{(1 + \omega_{aa})(1 - \rho)}\Phi^{-1}(x) - \Phi^{-1}(\pi)}{\sqrt{\rho}}\right), \end{aligned} \quad (46)$$

where the second expression makes use of (42), (41) and (45). This clearly reduces to the CDF of the Vasicek's model for $\omega_{aa} = 0$; see (A.4) in Appendix A.1.

It is also easily seen that in this case EL for the heterogeneous portfolio, denoted $\tilde{\pi}$, is given by

$$\tilde{\pi} = E(x) = \Phi\left(\frac{\tilde{a}}{\sqrt{1 + \tilde{\delta}^2}}\right) = \Phi\left(\frac{a}{\sqrt{1 + \omega_{aa} + \delta^2}}\right) = \Phi\left(\frac{\Phi^{-1}(\pi)}{\sqrt{1 + (1 - \rho)\omega_{aa}}}\right), \quad (47)$$

which differs from the EL of the homogeneous portfolio. Since we are interested in values of $\pi < 1/2$ for which $\Phi^{-1}(\pi) < 0$, we have

$$\frac{\partial \tilde{\pi}}{\partial \omega_{aa}} = \phi\left(\frac{\Phi^{-1}(\pi)}{\sqrt{1 + (1 - \rho)\omega_{aa}}}\right) \frac{-\Phi^{-1}(\pi)(1 - \rho)}{2(1 + (1 - \rho)\omega_{aa})^{3/2}} \geq 0, \text{ for } \pi < 1/2,$$

and it readily follows that $\tilde{\pi} \geq \pi$, meaning EL is *underestimated* when $\omega_{aa} > 0$ and this source of heterogeneity is neglected.

To derive the impact of ω_{aa} on unexpected loss, first without holding EL fixed, note that the pair-wise correlation of asset returns in this case is given by

$$\tilde{\rho} = \frac{\delta^2}{1 + \delta^2 + \omega_{aa}} = \frac{\rho}{1 + \omega_{aa}(1 - \rho)},$$

and using the results in Section 3.1 for N sufficiently large we have

$$Var(x) = \tilde{\pi}(1 - \tilde{\pi})\rho^*(\tilde{\pi}, \tilde{\rho}),$$

where

$$\rho^*(\tilde{\pi}, \tilde{\rho}) = \frac{\Phi_2[\Phi^{-1}(\tilde{\pi}), \Phi^{-1}(\tilde{\pi}), \tilde{\rho}] - \tilde{\pi}^2}{\tilde{\pi}(1 - \tilde{\pi})}, \quad (48)$$

and, as before in (15), $\Phi_2[\cdot]$ is the bivariate standard normal cumulative distribution function.

Thus, if we allow EL to vary, the effect ω_{aa} on loss variance is

$$\begin{aligned} \frac{\partial Var(x)}{\partial \omega_{aa}} &= \frac{\partial Var(x)}{\partial \tilde{\pi}} \times \frac{\partial \tilde{\pi}}{\partial \omega_{aa}} + \frac{\partial Var(x)}{\partial \tilde{\rho}} \times \frac{\partial \tilde{\rho}}{\partial \omega_{aa}} \\ &= \left[(1 - 2\tilde{\pi})\rho^*(\tilde{\pi}, \tilde{\rho}) + \tilde{\pi}(1 - \tilde{\pi}) \left(\frac{\partial \rho^*(\tilde{\pi}, \tilde{\rho})}{\partial \tilde{\pi}} \right) \right] \frac{\partial \tilde{\pi}}{\partial \omega_{aa}} + \tilde{\pi}(1 - \tilde{\pi}) \left(\frac{\partial \rho^*(\tilde{\pi}, \tilde{\rho})}{\partial \tilde{\rho}} \right) \frac{\partial \tilde{\rho}}{\partial \omega_{aa}}. \end{aligned}$$

The first term of this derivative is positive so long as $0 \leq \tilde{\pi} < 1/2$ and $\rho > 0$ (and hence $\rho^*(\tilde{\pi}, \tilde{\rho}) > 0$).¹⁵ However, the second term is negative since $\partial \tilde{\rho} / \partial \omega_{aa} < 0$. Thus the net effect of heterogeneity in mean returns and/or default thresholds on portfolio loss variance is ambiguous.

¹⁴See Appendix A.

¹⁵Note that $\partial \rho^* / \partial \tilde{\pi} > 0$.

3.4.2 Systematic and Random Heterogeneity

Now suppose we wish to consider the effect of a non-zero ω_{aa} on the loss variance *holding EL fixed*. It is easy to see that such a scenario is not possible if a_i 's are taken to be random draws from the same distribution. To control the EL we need to introduce an additional *systematic* source of heterogeneity. One possible approach would be to introduce firm types where for each type a_i 's are draws from different distributions or from the same distribution but with different parameters. As an illustration, suppose the loan portfolio contains two types of firms, \mathcal{H} and \mathcal{L} , with the portfolio weights $w_{\mathcal{H}}$ and $w_{\mathcal{L}}$, and the unconditional default probabilities, $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{L}}$, respectively, such that $0 < \pi_{\mathcal{L}} < \pi_{\mathcal{H}} < 1/2$. The differences in the default probabilities across the two types of firms could be due to differences in leverage or management quality, summarized in a credit rating. The portfolio loss in this case is given by

$$\ell_{N,t+1} = \sum_{i=1}^{N_{\mathcal{H}}} w_{i\mathcal{H}} I(a_{i\mathcal{H}} - \delta f_{t+1} - \varepsilon_{i\mathcal{H},t+1}) + \sum_{i=1}^{N_{\mathcal{L}}} w_{i\mathcal{L}} I(a_{i\mathcal{L}} - \delta f_{t+1} - \varepsilon_{i\mathcal{L},t+1}), \quad (49)$$

where $N = N_{\mathcal{H}} + N_{\mathcal{L}}$, $w_{k,N_k} = \sum_{i=1}^{N_k} w_{ik}$, $w_{\mathcal{H},N_{\mathcal{H}}} + w_{\mathcal{L},N_{\mathcal{L}}} = 1$, $I(\cdot)$ is the indicator function as in (11),

$$a_{i\mathcal{H}} = a_{\mathcal{H}} + v_{i\mathcal{H}a}, \quad a_{i\mathcal{L}} = a_{\mathcal{L}} + v_{i\mathcal{L}a}, \quad (50)$$

with $f \sim N(0, 1)$, $\varepsilon_{ik,t+1} \sim N(0, 1)$ and $v_{ika} \sim N(0, \omega_{aa})$, for $k = \mathcal{H}, \mathcal{L}$. It is also assumed that $\varepsilon_{ik,t+1}$ and v_{ika} are cross-independently distributed across all i and k .

Assuming that the granularity condition (16) holds for each firm type, then as $N_{\mathcal{H}}, N_{\mathcal{L}} \rightarrow \infty$, we have

$$x|f = w_{\mathcal{H}} \Phi(\tilde{a}_{\mathcal{H}} - \tilde{\delta}f) + w_{\mathcal{L}} \Phi(\tilde{a}_{\mathcal{L}} - \tilde{\delta}f), \quad (51)$$

where $w_k = \lim_{N_k \rightarrow \infty} w_{k,N_k}$, $\tilde{a}_k = (1 + \omega_{aa})^{-1/2} a_k$, for $k = \mathcal{H}, \mathcal{L}$, $w_{\mathcal{H}} + w_{\mathcal{L}} = 1$, and as before $\tilde{\delta} = (1 + \omega_{aa})^{-1/2} \delta$. Since $f \sim N(0, 1)$, it is now easily seen that

$$E(x) = \tilde{\pi} = w_{\mathcal{H}} \pi_{\mathcal{H}} + w_{\mathcal{L}} \pi_{\mathcal{L}},$$

where

$$\pi_k = \Phi\left(\frac{a_k}{\sqrt{1 + \omega_{aa} + \delta^2}}\right) = \Phi\left(\frac{a_k \sqrt{1 - \rho}}{\sqrt{1 + (1 - \rho)\omega_{aa}}}\right), \quad \text{for } k = \mathcal{H}, \mathcal{L},$$

and hence

$$a_k = \frac{\sqrt{1 + (1 - \rho)\omega_{aa}} \Phi^{-1}(\pi_k)}{\sqrt{1 - \rho}}, \quad \text{for } k = \mathcal{H}, \mathcal{L}.$$

To ensure the same expected losses under the homogeneous and heterogeneous cases we must have

$$\pi = w_{\mathcal{H}} \pi_{\mathcal{H}} + w_{\mathcal{L}} \pi_{\mathcal{L}}, \quad (52)$$

and this can be achieved, for given values of $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{L}}$, by an appropriate choice of the portfolio weights on the types \mathcal{L} and \mathcal{H} (note that the granularity condition implies that changing the weights within type has no effect), so long as $\pi_{\mathcal{H}} \neq \pi_{\mathcal{L}}$, and $0 < \pi_k < 1$, for $k = \mathcal{H}, \mathcal{L}$.¹⁶

Using (49), and recalling the result in (15), we now have

$$V(x) = w_{\mathcal{H}}^2 [F(\pi_{\mathcal{H}}, \pi_{\mathcal{H}}, \tilde{\rho}) - \pi_{\mathcal{H}}^2] + w_{\mathcal{L}}^2 [F(\pi_{\mathcal{L}}, \pi_{\mathcal{L}}, \tilde{\rho}) - \pi_{\mathcal{L}}^2] + 2w_{\mathcal{H}}w_{\mathcal{L}} [F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}) - \pi_{\mathcal{H}}\pi_{\mathcal{L}}],$$

where

$$F(\pi_i, \pi_j, \tilde{\rho}) = \Phi_2 [\Phi^{-1}(\pi_i), \Phi^{-1}(\pi_j), \tilde{\rho}].$$

Hence, under (52) the variance of the heterogeneous portfolio reduces to

$$V_{\text{het}}(x) = w_{\mathcal{H}}^2 F(\pi_{\mathcal{H}}, \pi_{\mathcal{H}}, \tilde{\rho}) + w_{\mathcal{L}}^2 F(\pi_{\mathcal{L}}, \pi_{\mathcal{L}}, \tilde{\rho}) + 2w_{\mathcal{H}}w_{\mathcal{L}} F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}) - \tilde{\pi}^2. \quad (53)$$

The following theorem now summarizes the above results and establishes the conditions under which $V_{\text{het}}(x) < V_{\text{hom}}(x)$, where $V_{\text{hom}}(x)$ is the variance of the associated homogeneous portfolio given by $F(\pi, \pi, \rho) - \pi^2$. Importantly, if this variance reducing heterogeneity is ignored, risk will be *overestimated*.

Theorem 1: Consider the loss portfolio given by (49), composed of heterogeneous obligors that differ by types, $k = \mathcal{H}$ and \mathcal{L} , and by firm characteristics, a_{ik} , within each type. Denote the number of obligors of each type by $N_{\mathcal{H}}$ and $N_{\mathcal{L}}$, and suppose that $N_{\mathcal{H}}$ and $N_{\mathcal{L}} \rightarrow \infty$, so that for large N_k default probabilities by types are given by

$$\pi_k = \Phi \left(\frac{a_k \sqrt{1 - \rho}}{\sqrt{1 + (1 - \rho) \omega_{aa}}} \right), \text{ for } k = \mathcal{H} \text{ and } \mathcal{L},$$

where $\rho = \delta^2 / (1 + \delta^2)$, ω_{aa} measures the degree of within-type (random) heterogeneity, and $a_{\mathcal{H}} - a_{\mathcal{L}}$ captures the extent of across-type (systematic) heterogeneity. Under $f \sim N(0, 1)$, $\varepsilon_{ik,t+1} \sim N(0, 1)$ and $v_{ika} \sim N(0, \omega_{aa})$, for $k = \mathcal{H}, \mathcal{L}$, and assuming that $\varepsilon_{ik,t+1}$ and v_{ika} are distributed independently across all i and k , we have

(a): The limiting loss variance under heterogeneity is

$$\lim_{N_{\mathcal{H}}, N_{\mathcal{L}} \rightarrow \infty} V(\ell_{N,t+1}) = V_{\text{het}}(x) = w_{\mathcal{H}}^2 F(\pi_{\mathcal{H}}, \pi_{\mathcal{H}}, \tilde{\rho}) + w_{\mathcal{L}}^2 F(\pi_{\mathcal{L}}, \pi_{\mathcal{L}}, \tilde{\rho}) + 2w_{\mathcal{H}}w_{\mathcal{L}} F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}) - \tilde{\pi}^2,$$

where $w_k = \lim_{N_k \rightarrow \infty} \sum_{i=1}^{N_k} w_{ik} > 0$, for $k = \mathcal{H}, \mathcal{L}$, $w_{\mathcal{H}} + w_{\mathcal{L}} = 1$,

$$\tilde{\rho} = \frac{\rho}{1 + \omega_{aa}(1 - \rho)},$$

and

$$\tilde{\pi} = w_{\mathcal{H}}\pi_{\mathcal{H}} + w_{\mathcal{L}}\pi_{\mathcal{L}}.$$

¹⁶Note that the possibility of $\pi_{\mathcal{H}} = \pi_{\mathcal{L}}$ is ruled out only if $a_{\mathcal{H}} \neq a_{\mathcal{L}}$, which requires $a_{i\mathcal{H}}$ and $a_{i\mathcal{L}}$ to be draws from distributions with different means.

(b): The loss variance of the associated homogeneous alternative (obtained by setting $a_{\mathcal{H}} = a_{\mathcal{L}} = a$ and $\omega_{aa} = 0$) is given by

$$V_{\text{hom}}(x) = F(\pi, \pi, \rho) - \pi^2,$$

where $\pi = \Phi(a\sqrt{1-\rho})$.

(c): Supposing that $w_{\mathcal{H}}$ (or $w_{\mathcal{L}}$) is set such that $\tilde{\pi} = \pi$, then for $\rho > 0$, $\omega_{aa} > 0$, and $a_{\mathcal{H}} \neq a_{\mathcal{L}}$, we have

$$V_{\text{hom}}(x) > V_{\text{het}}(x). \quad (54)$$

The first two parts of the theorem are already established. To prove the third part, note that since $\rho > \tilde{\rho}$, and $\partial F(\pi, \pi, \rho)/\partial \rho > 0$ we have¹⁷

$$F(\pi, \pi, \rho) \geq F(\pi, \pi, \tilde{\rho}).$$

Therefore, to establish (54) it is sufficient to show that under $\pi = w_{\mathcal{H}}\pi_{\mathcal{H}} + w_{\mathcal{L}}\pi_{\mathcal{L}}$,

$$F(\pi, \pi, \tilde{\rho}) > w_{\mathcal{H}}^2 F(\pi_{\mathcal{H}}, \pi_{\mathcal{H}}, \tilde{\rho}) + w_{\mathcal{L}}^2 F(\pi_{\mathcal{L}}, \pi_{\mathcal{L}}, \tilde{\rho}) + 2w_{\mathcal{H}}w_{\mathcal{L}}F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}). \quad (55)$$

Consider now $F(x, y, \tilde{\rho})$ and note that $\partial^2 F(x, y, \tilde{\rho})/\partial x^2 < 0$,¹⁸ and hence for given values of y and $\tilde{\rho}$, $F(x, y, \tilde{\rho})$ is concave in x and we have

$$F(\pi, \pi, \tilde{\rho}) = F(w_{\mathcal{H}}\pi_{\mathcal{H}} + w_{\mathcal{L}}\pi_{\mathcal{L}}, \pi, \tilde{\rho}) > w_{\mathcal{H}}F(\pi_{\mathcal{H}}, \pi, \tilde{\rho}) + w_{\mathcal{L}}F(\pi_{\mathcal{L}}, \pi, \tilde{\rho}). \quad (56)$$

Similarly, $\partial^2 F(x, y, \tilde{\rho})/\partial y^2 < 0$, and

$$\begin{aligned} F(\pi_{\mathcal{H}}, \pi, \tilde{\rho}) &> w_{\mathcal{H}}F(\pi_{\mathcal{H}}, \pi_{\mathcal{H}}, \tilde{\rho}) + w_{\mathcal{L}}F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}), \\ F(\pi_{\mathcal{L}}, \pi, \tilde{\rho}) &> w_{\mathcal{H}}F(\pi_{\mathcal{L}}, \pi_{\mathcal{H}}, \tilde{\rho}) + w_{\mathcal{L}}F(\pi_{\mathcal{L}}, \pi_{\mathcal{L}}, \tilde{\rho}). \end{aligned}$$

Using these results in (56), and noting that by symmetry $F(\pi_{\mathcal{H}}, \pi_{\mathcal{L}}, \tilde{\rho}) = F(\pi_{\mathcal{L}}, \pi_{\mathcal{H}}, \tilde{\rho})$, then (55) is readily established as required.

The above theorem is easily extended to portfolios containing more than two types of firms. Moreover, as the distance between $\pi_{\mathcal{L}}$ and $\pi_{\mathcal{H}}$ widens, the difference between the risks of the two portfolio types increases, suggesting that efficient credit portfolios should follow a “barbell” strategy combining exposures to very high quality credit with very low quality credits, so long as $\pi_k < 1/2$ for $k = \mathcal{H}, \mathcal{L}$. As a result ignoring this type of heterogeneity would result in *overestimation* of risk when holding EL fixed.

¹⁷Recall that $V_{\text{hom}}(x) = F(\pi, \pi, \rho) - \pi^2 = \pi(1-\pi)\rho^*$, where ρ^* is the default correlation given by (22), and as noted in Zhou (2001, p. 562), $\partial \rho^*/\partial \rho > 0$ for $\rho > 0$.

¹⁸A proof is provided in Appendix B.

3.4.3 Full Parameter Heterogeneity

The impact of allowing for full parameter heterogeneity in the multifactor case is discussed in Appendix A.3. For the single factor case, the analysis of allowing for non-zero values of ω_{aa} , $\omega_{\delta\delta}$, and $\omega_{a\delta}$ is easily carried out using (44) through random draws $f^{(r)} \sim iidN(0, 1)$ for $r = 1, 2, \dots, R$. Given these simulated values one can readily compute UL, VaR, and other distributional characteristics as desired.

4 An Empirical Application: The Impact of Neglected Heterogeneity

In this section we consider different types of heterogeneity across firms and illustrate their effects on the resulting loss distribution by simulating losses for credit portfolios comprised of public firms from the U.S. and Japan. Specifically, we want to see if the predictions based on the random coefficient model, as set out in Section 3.4, hold empirically. We are also interested in understanding which source of heterogeneity is the most important in affecting the shape of the loss distribution: the firm return process and associated factor loadings or the default threshold through information on distance to default or a credit rating.

4.1 Heterogeneity in Default Thresholds: Specification and Identification

We begin with a brief discussion of the specification and identification of the default thresholds. The probability of default for the i^{th} firm is given by (6), which we reproduce here for convenience:

$$\pi_{i,t+1} = \Phi\left(\frac{\lambda_{i,t+1} - \mu_i}{\sigma_{\xi_i}}\right).$$

This provides a functional relationship between a firm's equity returns (as characterized by μ_i and σ_{ξ_i}), its default threshold, $\lambda_{i,t+1}$, and the default probability, $\pi_{i,t+1}$. In the case of publicly traded companies, μ_i and σ_{ξ_i} can be consistently estimated from market returns based on historical data using either rolling or expanding observation windows. In general, however, $\lambda_{i,t+1}$ and $\pi_{i,t+1}$ can not be directly observed. One possibility would be to use balance sheet and other accounting data to estimate $\lambda_{i,t+1}$. This approach is taken up by Duffie, Saita and Wang (2005), to cite an academic example, and KMV as an industry example. But as argued in PSTW, the accounting information is likely to be noisy and might not be all that reliable due to information asymmetries and agency problems between managers, share-, and debtholders.¹⁹ Moreover, in a multi-country setting, the accounting based route presents additional challenges such as different accounting standards and bankruptcy rules that exist across countries. In addition to accounting data, other

¹⁹With this in mind, Duffie and Lando (2001) allow for the possibility of imperfect information about the firm's assets and default threshold in the context of a first-passage model.

firm characteristics, such as firm age and perhaps size, as well as management quality could also be important in the determination of default thresholds that are quite difficult to observe. In view of these measurement problems, PSTW propose an alternative estimation approach where firm-specific default thresholds are obtained using firm-specific credit ratings and historical default frequencies. These credit ratings could be either external, e.g. supplied by a rating agency, or internal from a bank's rating unit.

Broadly two identification schemes are possible, and they imply in turn assumptions about the unconditional distance to default, DD . One approach would be to impose the same default threshold for all firms of a given rating. Alternatively one could impose the same DD for all firms of a given rating, meaning

$$DD_{i,t+1} = \frac{\lambda_{i,t+1} - \bar{\mu}_i}{\bar{\sigma}_{\xi_i}} = DD_{\mathcal{R},t+1}, \quad (57)$$

for all firms i with rating \mathcal{R} , where $\bar{\mu}_i$ and $\bar{\sigma}_{\xi_i}$ are the *unconditional* estimates of μ_i and σ_{ξ_i} obtained using observations on firm-specific returns up to the end of period t . In this case the default threshold is different for every firm and can be computed using

$$\hat{\lambda}_{i,t+1} = DD_{\mathcal{R},t+1} \bar{\sigma}_{\xi_i} + \bar{\mu}_i, \text{ for } i \in \mathcal{R}_t, \quad (58)$$

where

$$DD_{\mathcal{R},t+1} = \Phi^{-1}(\hat{\pi}_{\mathcal{R},t+1}), \quad (59)$$

$\Phi^{-1}(\cdot)$ is the inverse of the cumulative distribution function of the standard normal, and $\hat{\pi}_{\mathcal{R},t+1}$ is the observed default frequency of \mathcal{R} -rated firms.²⁰ This approach is analogous to the systematic heterogeneity by types discussed in the theory Section 3.4.2. Once again the estimated default thresholds, $\hat{\lambda}_{i,t+1}$, will be finite so long as $\hat{\pi}_{\mathcal{R},t+1} \neq \{0, 1\}$.

The identification conditions can be summed up as follows: condition (57) imposes the same unconditional probability of default for each \mathcal{R} -rated firm, whereas the alternative strategy simply imposes that this needs to hold on average across \mathcal{R} -rated firms in the portfolio. Of the two, the assumption of the same distance-to-default seems more in line with the way credit ratings are established by the main rating agencies. First, the idea that firms with similar distances-to-default have similar probabilities of default is central to structural models of default. For instance, KMV makes use of a one-to-one mapping from DD s to EDFs (expected default frequencies). Second, rating agencies attempt to group firms according to their (unconditional) probability of default (subject possibly to some adjustments for differences in their expected loss given defaults), and in a structural model this is equivalent to grouping firms according to distance-to-default. In our empirical analysis we shall focus on the threshold estimates given by (58).²¹

²⁰Note that $\Phi^{-1}(\pi_{i,t+1}) < 0$ for $\pi_{i,t+1} < \frac{1}{2}$. In practice $\pi_{i,t+1}$ tends to be quite small.

²¹More detail as well as results using the same-threshold (λ) identifying assumption are given in Pesaran, Schuermann and Treutler (2005).

4.2 Data and Portfolio Construction

We form credit portfolios at the end of each year from 1997 to 2002 and then simulate portfolio losses for the following year. Parameters are estimated recursively using 10-year (40-quarter) rolling windows. The simulations are out-of-sample in that the models, fitted over a ten-year sample, are used to simulate losses for the subsequent 11th year. This recursive procedure allows us to explore the robustness of any results to possible time variation in the underlying parameters.

The loss simulations require an estimate of the unconditional probability of default for each firm. These may be obtained at the level of the credit rating, \mathcal{R} , assigned to the firm by rating agencies such as Moody's, S&P or Fitch. In keeping with our overall empirical strategy, we estimate probabilities of default recursively for each grade using 10-year rolling windows of all firm rating histories from S&P. These probabilities are estimated using the time-homogeneous Markov or parametric duration estimator discussed in Lando and Skødeberg (2002) and Jafry and Schuermann (2004). We impose a minimum annual probability of default (PD) of 0.001% or 0.1 basis points. Our estimated PD s for both AAA and AA fall below this minimum.

In order to be selected for inclusion in our portfolios, a firm needs a credit rating as well as 10 years of consecutive quarterly equity returns that match the rolling estimation window. In case both ratings are available the S&P rating is chosen.²² For the first sample or cohort (which ends in 1997) we have 211 Japanese firms and 628 U.S. firms, a portfolio of 839 firms in total. At the end of the following year the portfolio is rebalanced, *retaining* surviving firms and *augmenting* the portfolio with new firms that have a rating at the end of that year, i.e. 1998, and also have 40 consecutive quarters of returns. All returns are computed in U.S. dollars (USD).²³ Our analysis and conclusions are clearly conditional on the population of publicly traded firms with sufficiently long track records and need not extend to firms that are not publicly traded or have relatively short histories.

To make the portfolio exposures representative of the rated universe in each country, we reweight the portfolio exposures (in USD) by rating in the following manner. Suppose that each obligor begins with \$100 of exposure. If 10% of all rated Japanese firms have a BB rating, but 15.6% of the Japanese firms in our portfolio are BB -rated, then each of these firms is given $\frac{10.0}{15.6} \times \$100 = \$63$ of exposure. In the U.S. the difference in the ratings distribution across the two agencies is modest, but not so in Japan where Moody's rates more than twice as many firms as S&P. To address this

²²Our decision rule is driven by the use of S&P ratings histories to compute the default probabilities $\pi_{\mathcal{R}}$.

²³Our source of return data for U.S. firms is CRSP while for Japanese firms it is Datastream.

issue we take the average of the two agencies' ratings distributions by rating for each country.²⁴

The portfolio composition is adjusted annually, starting with 1998, to reflect defaults, upgrades and downgrades which may have occurred during the year. Since credit migration matrices even at annual frequencies are diagonally dominant, with average staying probabilities exceeding 90% for investment grades, annual portfolio rebalancing seems a reasonable compromise between accuracy and computational burden; the alternative would be quarterly rebalancing. We also update the ratings distribution each year to allow for compositional changes in the universe of rated firms. For example, at the end of 1997 \mathcal{B} -rated firms made up only 1.54% of all rated Japanese firms, but by year-end 2002 this proportion had risen to 6.75%. In Table 1 we show the average ratings distribution for each country for 1997 and 2002. It becomes clear that there has been a systematic deterioration in average credit quality over this period. In addition, estimated probabilities of default for non-investment grade ratings, and for \mathcal{CCC} in particular, have risen noticeably over this period. As a result, the weighted average annual probability of default, $\hat{\pi}$, has increased from 1.23% for the year-end 1997 portfolio to 3.26% for the year-end 2002 portfolio.

Using two-digit SIC codes we group firms into seven broad sectors to ensure a sufficient number of firms per sector. The sectors and percentage of firms by sector by country at year-end 1997 are summarized in Table 2.

4.3 Model Specifications

To explore the role of geographic and industry or sectoral heterogeneity we introduce two new indices into the notation of the previous sections. Specifically, denote $r_{ijc,t+1}$ to be the return of firm i in sector j in country c over the quarter t to $t + 1$. The application will explore two countries (Japan and U.S.) and seven sectors/industries in each country. Following the multi-factor return model given by (8), we employ the following return regressions adapted to our empirical applications:

$$r_{ijc,t+1} = \alpha_{ijc} + \beta'_{ijc} \mathbf{f}_{t+1} + u_{ijc,t+1}, \quad (60)$$

where $\mathbf{f}_{t+1} \sim (\boldsymbol{\mu}_f, \boldsymbol{\Sigma}_f)$, $\boldsymbol{\mu}_f$ is an $m \times 1$ vector of constants, and $\boldsymbol{\Sigma}_f$ is the covariance matrix of the common factors, also assumed fixed. In terms of the return parameters of (3) and (8), the expected return can be written as

$$\mu_{ijc,t+1} = \alpha_{ijc} + \beta'_{ijc} \boldsymbol{\mu}_f, \quad (61)$$

²⁴The precise exposure allocation is as follows. Denote FV_{ic} to be the (face value) exposure to firm i in country c . The portfolio total nominal face value is \$1bn. Then

$$FV_{ic} = \$1bn \cdot w_c \cdot \left(\frac{1}{N_c} \right) \cdot \theta(\mathcal{R})_c \text{ for } i \in \mathcal{R},$$

where w_c is the share of the total portfolio for country c (75% for the U.S., 25% for Japan), N_c is the number of firms in country c , $\theta(\mathcal{R})_c$ is the rating representation adjustment. Note that $\theta(\mathcal{R})_c$ will vary across time to reflect compositional changes in the rated universe of firms in country c .

and the unexpected component as

$$\xi_{ijc,t+1} = \beta'_{ijc}(\mathbf{f}_{t+1} - \boldsymbol{\mu}_f) + u_{ijc,t+1}. \quad (62)$$

The total return variance is given by

$$\sigma_{\xi_{ijc}}^2 = \beta'_{ijc} \boldsymbol{\Sigma}_f \beta_{ijc} + \sigma_{ijc}^2, \quad (63)$$

where σ_{ijc}^2 is the variance of the idiosyncratic component, $u_{ijc,t+1}$. Note that (60) is not a forecasting equation. Moreover, since the factors are assumed to be serially uncorrelated here, there is no meaningful distinctions between conditional and unconditional returns and hence loss distributions. This is in contrast to the observable factor model presented in PSTW where the (global) factor structure is modeled as a vector autoregressive error correcting mechanism, thereby explicitly introducing serial correlation in the returns, making multi-period forecasts possible, conditional on the factor values at the end of the sample period.

Following a standard approach in the finance literature, we model firm returns using an unobserved components or factor approach, either single or multiple, with increasing degrees of heterogeneity. One obvious source of heterogeneity is geography or country. As we have two countries, we estimate each model specification first by pooling the U.S. and Japanese firms (referred to as the ‘‘pooled model’’ specification) and then by estimating two separate models for each of the two countries (the ‘‘modeled separately’’ specification).

The empirical exercise involves a number of variations on the basic firm return equation given by (60) using market-cap weighted market returns for each country $\bar{r}_{c,t+1}$ as proxies for two of the possible m common factors. Sector returns for a country c , $\bar{r}_{jc,t+1}$, are computed in a similar fashion, namely using the market-cap weighted average of firm returns in that sector.²⁵ The ‘‘global’’ market return index, \bar{r}_{t+1} , is made up of just the two countries U.S. and Japan and is simply the weighted sum of the two individual country returns,

$$\bar{r}_{t+1} = w_{US} \bar{r}_{US,t+1} + (1 - w_{US}) \bar{r}_{JP,t+1}, \quad (64)$$

where w_{US} measures the relative size of the U.S. economy. We estimate w_{US} by taking the average U.S. share of PPP-denominated GDP over 1997-2002, and obtain $w_{US} = 0.75$. To obtain the global sector return $\bar{r}_{j,t+1}$ for a particular sector j we proceed similarly to (64) and define

$$\bar{r}_{j,t+1} = w_{US} \bar{r}_{US,j,t+1} + (1 - w_{US}) \bar{r}_{JP,j,t+1}. \quad (65)$$

The simplest model is the fully homogeneous return specification analogous to the one assumed by Vasicek:

$$r_{ijc,t+1} = \alpha_c + \beta_c \bar{r}_{c,t+1} + u_{ijc,t+1}, \quad (66)$$

²⁵The weights for period $t + 1$ are based on the average of the market capitalization (in USD) at end of periods t and $t + 1$.

with $u_{ijc,t+1} \sim iidN(0, \sigma_c^2)$. For the pooled model, $\sigma_c^2 = \sigma^2$, $\alpha_c = \alpha$, $\beta_c = \beta$ and $\bar{r}_{c,t+1} = \bar{r}_{t+1}$ as in (64) for $c = US$ and JP .

The second model tests empirically the predictions made by theory in Section 3.4 by introducing heterogeneity in default thresholds by rating (Model II). The third model specification allows for full parameter heterogeneity where firm alphas, factor loadings and error variances are allowed to vary across firms. In the fourth specification we add an industry or sector factor so that each firm's returns is regressed on $\bar{r}_{c,t+1}$ as well as on $\bar{r}_{cj,t+1}$. To be clear, for the pooled model, $\bar{r}_{c,t+1} = \bar{r}_{t+1}$ as in (64), and $\bar{r}_{cj,t+1} = \bar{r}_{j,t+1}$ as in (65), for $c = US$ and JP .

In the loss simulations we must impose conditional independence, but if we have failed to capture this dependence in the return model specifications, we will subsequently underestimate risk. With that in mind, the fifth and final model specification is the principal components (PC) model. We selected \hat{m} , the number of factors, using the IC_1 and IC_2 selection criteria proposed in Bai and Ng (2002), with the maximum number of factors set to 5. Both criteria yielded the same result. In the application, two factors were selected for the U.S., three for Japan and three for the pooled model. The procedure was conducted for the 1997 cohort of firms, using the prior ten years of quarterly data. For tractability the number of factors was kept fixed for the subsequent cohort of firms, though the actual factors were, of course, re-estimated. Table 3 summarizes the five model specifications that we consider.

4.4 Return Regressions: Recursive Estimates

The return regression parameters, estimated recursively using a 10-year rolling window, are summarized in Table 4. We focus our discussion on the average pair-wise correlation of returns and the average pair-wise correlation of residuals as they map naturally into our loss modeling framework. The average pair-wise correlation of residuals is of particular interest since it gives an indication of how close a particular model is to conditional independence.

Starting with the results in Panel A of Table 4, we note that the in-sample average pair-wise correlation of quarterly returns for the first ten years (1988-1997) is 0.1933 for the U.S. firms as compared to the much higher figure of 0.6011 for the Japanese firms. The average pair-wise correlation for the pooled sample is very close to the U.S.-only sample at 0.1937.²⁶ The factor models generally do a good job of accounting for the cross-section correlation of returns, at least in-sample. Considering first the U.S. and Japan pooled results, the average pair-wise correlation of residuals for the whole portfolio is around 0.022 for the Vasicek and single factor CAPM models. Adding an industry factor reduces that residual correlation to 0.015, and the PCA model by construction

²⁶For 1988-1997, the average pairwise correlation of USD-denominated returns for Japanese firms of 0.60 is slightly higher than the average correlation of Yen-denominated returns of 0.55 due to the common currency adjustment. However, local currency returns for Japanese firms in our sample are still noticeably more correlated than those for U.S. firms. This pattern holds for the later periods as well.

leaves almost no cross-section residual correlation. To be sure, there is no guarantee that this will hold out-of-sample. In-sample goodness of fit across models as measured by \bar{R}^2 (not reported in the table) range from 0.135 for the Vasicek to 0.229 for the sector CAPM to 0.339 for the PCA model.

Staying with the pooled model, notice the high degree of residual correlation that remains for the Japanese firms, ranging from around 42% (Vasicek) to 39% (CAPM). The reason is simple: the “global” market weighted return, \bar{r}_{t+1} , is dominated by U.S. firms. The overall portfolio average is low since residuals from U.S. and Japanese firm regressions are relatively uncorrelated and in some cases even negatively correlated.

Estimating the models separately for each country helps, and this is seen clearly in the last three columns of Table 4, Panel A. While the overall average pair-wise correlation of residuals is quite similar at around 1.5% to 2%, for Japanese firms it is reduced dramatically, from a range of 39% to 42% under the pooled specification to a range of 2% to 6% when estimated separately. Similarly for U.S. firms, the average pair-wise correlation is reduced from a range of 7% to 9.5% in the pooled approach to a range of 2% to 3.5% when estimated separately. Clearly geographic heterogeneity plays an important role.

The results reported in Table 4 also show the high degree of variability in the coefficient estimates that exists across firms. This is illustrated by Figure 1 where the empirical densities (smoothed histograms)²⁷ of the firm betas based on the one factor or CAPM model (Model III) are displayed separately for the two countries. The estimates of the Japanese betas are more tightly distributed around their mean than are U.S. betas. We see a similar pattern with the firm alphas. These results are line with the theoretical results in Section 3.4 in that the parameters across the two countries can be viewed as draws from two different distributions; they are systematically different.

Panels B through F in Table 4 show the recursive results using a 10-year rolling window for the next five ten-year periods. We note that average pair-wise cross-sectional correlations of firm returns remain at around 20% through 1999 (though they show a steady decline for Japanese firms), but starting with the cohort of 1991-2000 (Panel D), the average correlation for the portfolio drops to 0.139. The sudden and substantial market reversals in the U.S. in March 2000 and the subsequent market declines probably play a strong role in explaining these results.²⁸ Similar patterns are also

²⁷Estimated betas are from Model III – CAPM where U.S. & Japan are modeled separately. All densities are estimated with an Epanechnikov kernel using Silverman’s (1986) optimal bandwidth.

²⁸Throughout the analysis we have been assuming time invariant volatilities. While it is well known that high frequency (daily, weekly) firm returns exhibit volatility clustering, this effect tends to vanish as the data frequency declines due to temporal aggregation effects. Nonetheless, we conducted standard diagnostic tests for ARCH effects on all firm return regressions in the case of Model III and calculated the percentage of firm-specific return regressions in which the ARCH effects are significant at the 5% level. For most periods the percentage of firms with significant ARCH effects fell between 5 and 10%; the detailed results are available upon request from the authors. Overall the

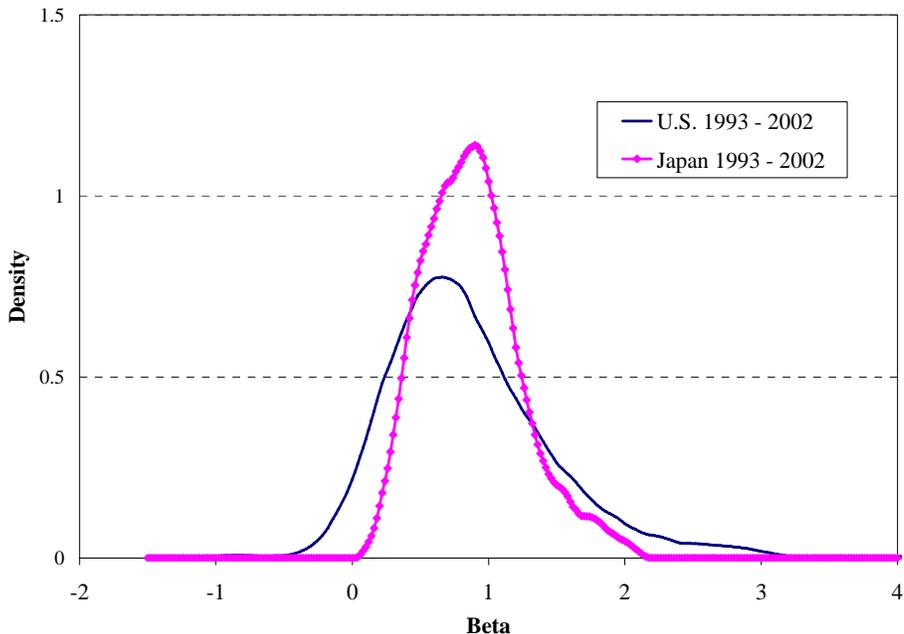


Figure 1: Kernel density estimates of estimated betas, Model III (CAPM), modeled separately (by country)

observed across the different models over the successive periods.

A further source of systematic heterogeneity across firms is the unconditional default probability or distance to default, captured for instance by the differences in their credit rating. It is reasonable to expect that the return processes of firms with a relatively high credit rating should on average exhibit a relatively low error variances and vice versa. This is indeed the case when we compare the average estimates of $\sigma_{ijc}^2 = \text{Var}(u_{ijc,t+1})$ across ratings. Table 5 shows averages of $\hat{\sigma}_{ijc}$ for Models I (Vasicek) and II (Vasicek + Rating) where the estimates do not vary across firms, and for the CAPM where they do, based on the final 10-year cohort, 1993 – 2002 (the same cohort on which Figure 1 is based). Similar results are obtained for other sample periods. The firm beta, $\hat{\beta}$, for Models I & II (recall this is a pooled estimate) is 0.824, and the firm error volatility, $\hat{\sigma}$, is 0.191. The next three rows show the average estimates by credit rating for Model III. Taking the last row first, $\bar{\sigma}_{\mathcal{R}}$ increases monotonically as we descend the credit spectrum, from 0.115 for \mathcal{AAA} and \mathcal{AA} firms, to 0.275 for \mathcal{B} -rated and 0.282 for \mathcal{CCC} -rated firms. No such clear pattern can be seen for firm betas, $\bar{\beta}_{\mathcal{R}}$. Thus credit ratings seem to sort firms by firm-specific risk but not by firm beta or factor loading, and in this way it might be reasonable to consider credit rating as being able to distinguish systematic differences in unconditional distance to default.

evidence is not sufficiently overwhelming to motivate ARCH modeling across all firms.

4.5 Impact of Heterogeneity on Credit Losses

We are now ready to generate loss distributions using the return and distance to default parameter estimates from Section 4.4. We begin by calibrating the analytical (asymptotic with $N \rightarrow \infty$) results presented in Section 3.4 assuming the parameter estimates are random draws from Gaussian processes. We then simulate the corresponding loss distributions using firm-specific estimates in the context of our finite-sized portfolio. In both approaches we maintain the double-Gaussian assumption applied to the common factors, \mathbf{f}_t , and the idiosyncratic shocks, $\varepsilon_{ijc,t+1}$. This allows us to compare the analytical and simulated approaches, and will help shed light on the validity of the Gaussian random coefficient model for the analysis of loss distributions. As argued in Section 4.6 this is an important consideration in the application of our proposed heterogeneous loss distribution to portfolios where reliable firm-specific estimates of β and σ can not be obtained due to lack of adequate return histories, or because some of the firms in the portfolio might not be publicly traded companies.

4.5.1 Calibrating the Asymptotic Loss Distributions

For the homogeneous case the asymptotic results are given in Appendix A.1. In what follows we focus on unexpected losses given by the square root of (23), and various quantiles or VaRs. It turns out that our two-country portfolio is quite close to an asymptotically diversified portfolio. For example, the portfolio for the first 10-year cohort has 839 firms, or an effective number of 638 equal-sized exposures.²⁹ For this portfolio the simulated UL is 1.47%, only 7bp above the asymptotic result, and similarly for the two quantiles 99.0% (within 19bp) and 99.9% VaR (within 23bp).

For the heterogeneous case, we calibrate the asymptotic loss distribution by restricting our numerical analysis to the single (world) factor version of the pooled country models. The return process in this case, using the notations introduced in Section 3.4, is defined by

$$r_{i,t+1} = \mu_i + \gamma_i f_{t+1} + \sigma_i \varepsilon_{i,t+1}, \quad \varepsilon_{i,t+1} \sim iidN(0, 1), \quad f_{t+1} \sim N(0, 1).$$

Given the idiosyncratic nature of the firm-specific shocks, $\varepsilon_{i,t+1}$, the common factor can be consistently estimated using the market return, denoted by \bar{r}_{t+1} (see Pesaran, 2006). This yields the familiar CAPM specification:

$$r_{i,t+1} = \alpha_i + \beta_i \bar{r}_{t+1} + \sigma_i \varepsilon_{i,t+1}, \quad \varepsilon_{i,t+1} \sim iidN(0, 1), \quad \bar{r}_{t+1} \sim N(\bar{r}, \sigma_{\bar{r}}^2),$$

with the following relationships between the parameters of the two specifications

$$\mu_i = \alpha_i + \beta_i \bar{r}, \quad \text{and} \quad \gamma_i = \beta_i \sigma_{\bar{r}}.$$

²⁹If N is the number of obligors in our portfolio, each with exposure weight w_i which is randomly assigned, then $N^* = \left(\sum_{i=1}^N w_i^2\right)^{-1}$ is the equivalent number of equally weighted exposures.

Thus firm i 's default probability is

$$\pi_{i,t+1} = \Pr(r_{i,t+1} \leq \lambda_{i,t+1}) = \Phi\left(\frac{\lambda_{i,t+1} - \mu_i}{\sigma_{\xi_i}}\right),$$

where $\sigma_{\xi_i}^2 = \beta_i^2 \sigma_{\bar{r}}^2 + \sigma_i^2$.

Using the identification strategy of same distance-to-default by rating \mathcal{R} implies the following rating-specific default thresholds:

$$\lambda_{i\mathcal{R}} = \sigma_{\xi_i} \Phi^{-1}(\pi_{\mathcal{R}}) + \mu_i. \quad (67)$$

A firm of type \mathcal{R} defaults if $\mu_{i\mathcal{R}} + \gamma_{i\mathcal{R}} f_{t+1} + \sigma_{i\mathcal{R}} \varepsilon_{i,t+1} \leq \lambda_{i\mathcal{R}}$, or if $\delta_{i\mathcal{R}} f_{t+1} + \varepsilon_{i,t+1} \leq a_{i\mathcal{R}}$, where

$$\delta_{i\mathcal{R}} = \frac{\gamma_{i\mathcal{R}}}{\sigma_{i\mathcal{R}}} = \frac{\beta_{i\mathcal{R}} \sigma_{\bar{r}}}{\sigma_{i\mathcal{R}}}, \quad (68)$$

and

$$a_{i\mathcal{R}} = \sqrt{1 + \delta_{i\mathcal{R}}^2} \Phi^{-1}(\pi_{\mathcal{R}}). \quad (69)$$

The reduced form parameters $\delta_{i\mathcal{R}}$ and $a_{i\mathcal{R}}$ can now be estimated using the estimates of β_i and σ_i from the CAPM regressions (categorized by credit rating at the end of the sample), the default probability estimates by rating, $\hat{\pi}_{\mathcal{R}}$, given for 2002 in the last column of Table 1, and the unconditional mean and variance of the market return, \bar{r} and $\hat{\sigma}_{\bar{r}}^2$. The parameters that enter the asymptotic loss distribution can then be computed as sample moments by rating which we denote by $\hat{a}_{\mathcal{R}}, \hat{\delta}_{\mathcal{R}}, \hat{\omega}_{a_{\mathcal{R}} a_{\mathcal{R}}}, \hat{\omega}_{\delta_{\mathcal{R}} \delta_{\mathcal{R}}}$ and $\hat{\omega}_{a_{\mathcal{R}} \delta_{\mathcal{R}}}$, for $\mathcal{R} = 1, 2, \dots, K$, where K is the number of credit rating categories (in our application 7), $\hat{\delta}_{i\mathcal{R}} = \hat{\beta}_{i\mathcal{R}} \hat{\sigma}_{\bar{r}} / \hat{\sigma}_{i\mathcal{R}}$, $\hat{a}_{i\mathcal{R}} = \sqrt{1 + \hat{\delta}_{i\mathcal{R}}^2} \Phi^{-1}(\hat{\pi}_{\mathcal{R}})$, $N_{\mathcal{R}}$ is the number of firms in the rating category \mathcal{R} at the end of the sample (where the loss distribution is to be simulated). Note also that $\sum_{\mathcal{R}=1}^K N_{\mathcal{R}} = N$.

Using the above parameter estimates the asymptotic losses can be simulated as

$$x^{(r)} = \sum_{\mathcal{R}=1}^K w_{\mathcal{R}} \Phi\left(\frac{\hat{a}_{\mathcal{R}} - \hat{\delta}_{\mathcal{R}} f^{(r)}}{\sqrt{1 + \hat{\omega}_{a_{\mathcal{R}} a_{\mathcal{R}}} + \hat{\omega}_{\delta_{\mathcal{R}} \delta_{\mathcal{R}}} f^{(r)2} - 2\hat{\omega}_{a_{\mathcal{R}} \delta_{\mathcal{R}}} f^{(r)}}}\right), \quad (70)$$

where $w_{\mathcal{R}}$ is the weight of \mathcal{R} -rated firms in the portfolio ($\sum_{\mathcal{R}=1}^K w_{\mathcal{R}} = 1$), and $f^{(r)}$, $r = 1, 2, \dots, R$ are random draws from $N(0, 1)$.³⁰ This formulation automatically sets the expected loss to be equal when introducing credit rating information since portfolio loss is just the weighted average of loss by rating.

Table 6 shows the relevant parameter estimates needed to generate losses in (70), by rating, for the last 10-year sample window, 1993 – 2002. The estimate $\hat{a}_{\mathcal{R}}$ is just distance to default scaled by $\sqrt{1 + \hat{\delta}_{i\mathcal{R}}^2}$. Since the average standardized factor loading, $\hat{\delta}_{\mathcal{R}}$, varies little across rating, the increase in $\hat{a}_{\mathcal{R}}$ as we descend the rating spectrum is driven by the increase in unconditional probability of default, $\hat{\pi}_{\mathcal{R}}$.

³⁰Clearly, draws from other distributions can also be considered.

A similar effect is driving the declining within-rating parameter variance, $\hat{\omega}_{a_{\mathcal{R}}a_{\mathcal{R}}}$. When $\hat{\pi}_{\mathcal{R}}$ is high (e.g. for low ratings such as \mathcal{B} and \mathcal{CCC}), $\Phi^{-1}(\hat{\pi}_{\mathcal{R}})$ is close to zero and the within-type variance of $\hat{a}_{i\mathcal{R}}, \hat{\omega}_{a_{\mathcal{R}}a_{\mathcal{R}}}$, shrinks. Note that this within-type variation is assumed to be purely random. We do not expect any systematic pattern with regard to factor loadings $\hat{\delta}_{\mathcal{R}}$ nor their within-rating dispersion, $\hat{\omega}_{\delta_{\mathcal{R}}\delta_{\mathcal{R}}}$. Finally, the correlation coefficient between the two sets of parameter estimates, denoted by $\hat{\rho}_{a_{\mathcal{R}}\delta_{\mathcal{R}}}$, is always negative, as expected from (69).

In Table 7 we report results using (70) for the last 10-year window (ending in 2002, thus estimating the loss distribution for 2003), for $R = 1,000,000$, for the three specifications which are single factor models, namely homogeneous Vasicek (Model I), Vasicek plus rating (Model II), and CAPM (Model III). For easy comparison we hold EL fixed across models. For each model and each year, we show EL, UL, and two commonly reported quantiles (VaR), 99.0% and 99.9%.

We see clearly that allowing for parameter heterogeneity reduces risk, whether measured by UL or VaR. Taking for instance the first simulation year, 1998, we see that allowing for only heterogeneity in the a -parameter through rating-specific distance to default, UL drops by about one-fifth, from 1.40% to 0.82%, and 99.9% VaR drops by a quarter from 11.87% to 6.16%, as expected from the theoretical results in Section 3.4.1. Allowing in addition for factor loading heterogeneity results in a further reduction of more than one-third in UL to 0.52%, and of nearly one-sixth in 99.9% VaR to 5.30%.

4.5.2 Simulating Finite Sample Credit Losses under Heterogeneity

In this section we simulate the loss distribution for our finite sample portfolio using firm-specific parameter estimates. This exercise allows us to assess how well the estimates based on analytical results (that apart from the rating information do not use any other firm-specific information) predict tail properties of the loss distribution of the finite sample loan portfolio under consideration.

We simulate firm returns out-of-sample using (60), assuming that the systematic and idiosyncratic components are serially uncorrelated and independently distributed, thus imposing conditional independence. The loss distributions for the different model specifications are then simulated using appropriate default thresholds, and assuming, for simplicity, no recovery in the event of default. All simulations are based on 200,000 replications.

Before embarking on a detailed model-by-model, year-by-year discussion of the loss simulation results, it is helpful to consider Figure 2 to gain an overview of the results. In Figure 2 we show the loss densities for 2003 across the five different specifications (separate country models holding EL fixed). It is immediately apparent that the models are grouped into two sets. While the models differ in several ways, the main distinction between the two groups is the use of credit ratings. The more skewed density with the mode closer to the vertical axis is generated by the fully homogeneous Model I which does not make use of credit rating information while the others do. Indeed Model II

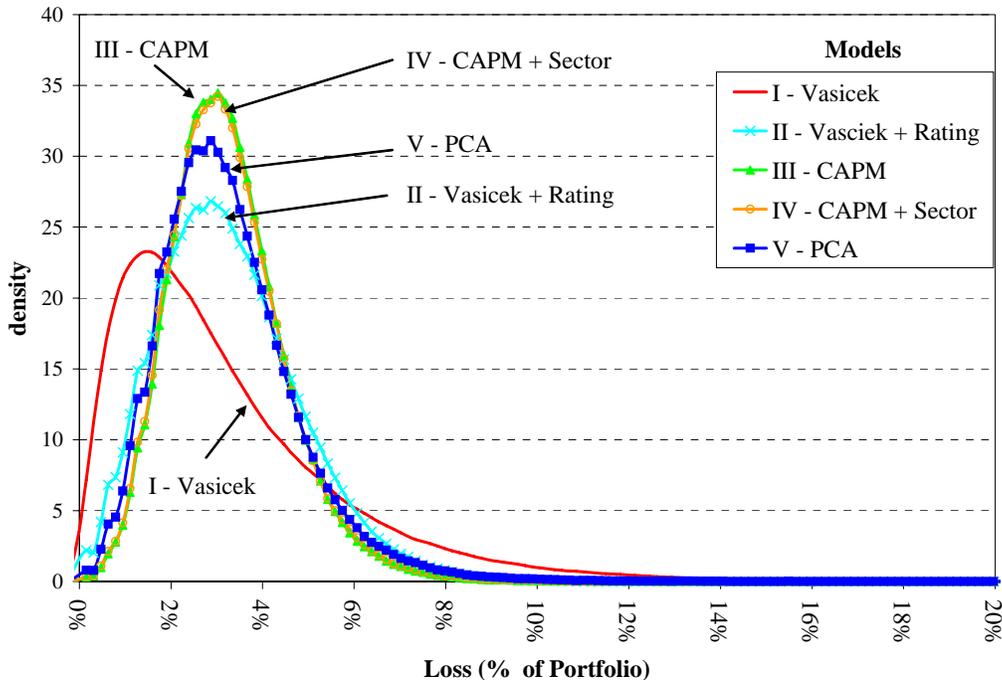


Figure 2: Simulated finite-sample loss densities, holding EL fixed, for alternative models (U.S. and Japan modeled separately, 2003)

adds only this source of information. Whatever other sources of heterogeneity may be important, an estimate of the unconditional probability of, or distance to, default, as provided by a credit rating, clearly has a significant influence on the overall shape of the loss density.

We turn now to Table 8 where we report the loss simulations results for each of the six rolling windows. We proceed by discussing in some detail the loss simulations for the first period and then draw comparisons across years. In addition to the model number and name in Table 8, we provide the EL calibration level.

For each year we report the first four simulated moments of the loss distribution (note that the first moment or expected loss is the same across all models by construction) as well as two quantiles (VaR), 99.0% and 99.9%. We also calculated expected shortfall; the results are qualitatively no different, and so we report here only the VaR results. The first set of columns is for the pooled specification while the second set is for the country specific models, analogous to the presentation of the in-sample regression results in Table 4. Broadly speaking, risk, measured either by UL or VaR, declines as model heterogeneity increases, and thus ignoring it would result in overestimation of risk.

The finite sample simulations confirm rather precisely the predictions made by the asymptotic theory. The fully homogeneous model of Vasicek (Model I) generates the most extreme losses and

has the largest unexpected losses. Adding ratings information (Model II) results in a significant reduction in risk (while controlling for expected losses). UL drops by more than one-quarter from 1.47% to 1.07% while 99.9% VaR is reduced by nearly one-half from 12.05% to 6.72%. Credit ratings seem to capture relevant firm-specific information, and this is useful even though the information is grouped together into just a few (seven) rating categories. Models III and IV allow for heterogeneous slopes (factor loadings) and firm-specific error variances, with Model IV also adding an industry return factor. UL falls from 1.07% in Model II to 0.86% in Model III, while 99.9% VaR declines from 6.72% to 5.56%. Thus, the ranking across these three models in the finite sample simulations are *exactly* as predicted by theory using the calibrations in Section 4.5.1.

Adding an industry factor in Model IV results in a very small increase in risk from Model III. However, the distributions are extremely similar, and the small differences are likely due to estimation noise: UL is nearly the same, 0.86% vs. 0.88%, as are VaR levels, e.g. 5.56% vs. 5.58% at the 99.9% VaR level. Finally, the principal components Model V generates UL results that are similar to Model II, which is Model I with ratings information, namely 1.08% vs. 1.07%. VaR, however, is higher. For instance, 99.9% VaR is 7.69%, compared to 6.72% for Model II. In this way Model V also generates tail losses which are higher than Models III and IV.

This upturn in risk may appear counter-intuitive – adding heterogeneity results not in risk reduction but in an increase. However, it is important to keep in mind that the out-of-sample loss simulations are performed under the maintained assumption of *conditional independence*. Recall from Table 4 that only Model V has an (in-sample) average pair-wise cross-sectional correlation of residuals which is effectively zero. All other models have some remaining correlation. Put differently, while Model V is conditionally independent on an in-sample basis, it seems that the others are not. So long as on an out-of-sample basis Model V is still closer to conditional independence than the others, and there is currently no way of verifying this, the other models will generate risk forecasts which are biased downward, meaning that risk would be underestimated since return correlations are underestimated. Measuring and evaluating out-of-sample conditional dependence is an important topic which requires further investigation.³¹

Turning now to the set of columns labeled “U.S. & Japan Modeled Separately” we see that allowing for geographic heterogeneity in the simple Vasicek case reduces risk. For the Vasicek model this amounts to doubling the number of parameters as there is one set for each country. UL drops by about 14% from 1.47% to 1.29%, skewness and kurtosis both decline, as does VaR. For instance, 99.9% VaR declines by nearly 20% from 12.05% to 10.14%.

In general, however, pooled and separate country models generate similar loss distributions.

³¹The absence of conditional independence empirically, especially on an out-of-sample basis, we think has been neglected in the literature. Das, Duffie, Kapadia, and Saita (2005) also find significant remaining correlation, or default clustering, even after accounting for observable factors. They propose an unobserved factor approach they call “frailty” to absorb the remaining dependence, though this approach makes out-of-sample forecasting challenging.

Modeling the U.S. and Japan separately usually results in lower risk for Model II. Once firm-specific factor loadings are allowed for, as in Models III and IV, VaR is actually slightly higher for the separate country model, although the second through fourth moments are the same. For instance, in the case of the basic CAPM model, Model III, 99.9% VaR increases from 5.56% to 5.75%. A similar pattern can be seen when adding an industry factor in Model IV. Model V, however, shows risk reduction by allowing for country heterogeneity. Moreover, the results for the last period shown in Panel F, Table 8, indicates that even for the simple Vasicek model, Model I, pooling need not always increase risk: 99.9% VaR is 17.47% for the pooled model but 17.81% for the country-specific model. Indeed for this last period, allowing the parameters to vary by country increases VaR for all models save the last one, Model V. Some of these results could be due to parameter instability and the associated estimation errors. Nevertheless, the model ranking is robust to comparisons over time, i.e. comparing Panels A through F. Overall it seems that allowing for country-specific factor loadings is more important than requiring different country models to have their own specific factors.

Returning to Figure 2, the loss densities are clearly very different. The Vasicek model has only three parameters per country $(\alpha_c, \beta_c, \sigma_c)$, and once credit rating information is included in Model II, the distributional shape changes dramatically. Indeed Model II yields a loss distribution which is remarkably similar to those generated by the fully heterogeneous model specification. Credit ratings seem indeed to be a useful and informative summary statistic for firm-level unconditional credit risk.

Moving down the panels in Table 8 we notice that the portfolio is getting riskier over time; expected loss rises every year. If we compare value-at-risk, say at the 99.9%, for a model, say the one-factor CAPM model (Model III) applied to the two countries separately, we see that VaR increases from 5.75% in 1998 to 7.99% in 2000 to 9.30% in 2003.

On the whole, what is clear from the discussion is that there is a rich and complex interaction between the underlying model parameters and the resulting loss distributions.

4.6 Heterogeneity and the New Basel Capital Accord

The theoretical and empirical results indicate that neglecting firm heterogeneity can have significant impact on risk, and that a dominant source of heterogeneity is the difference in the unconditional probability of default across firms as summarized, for instance, in a credit rating. Credit ratings can be provided by an external rating agency or can be generated by the bank itself using internal rating models and tools. Indeed the New Basel Capital Accord allows banks, under Pillar 1 (prescribed minimum capital), to assign ratings to their obligors but does not allow them to compute their own factor loadings or return correlations. Those are fixed by the policy makers in the form of the risk weight function which assigns capital to credit exposures. It thus appears that the Basel Committee

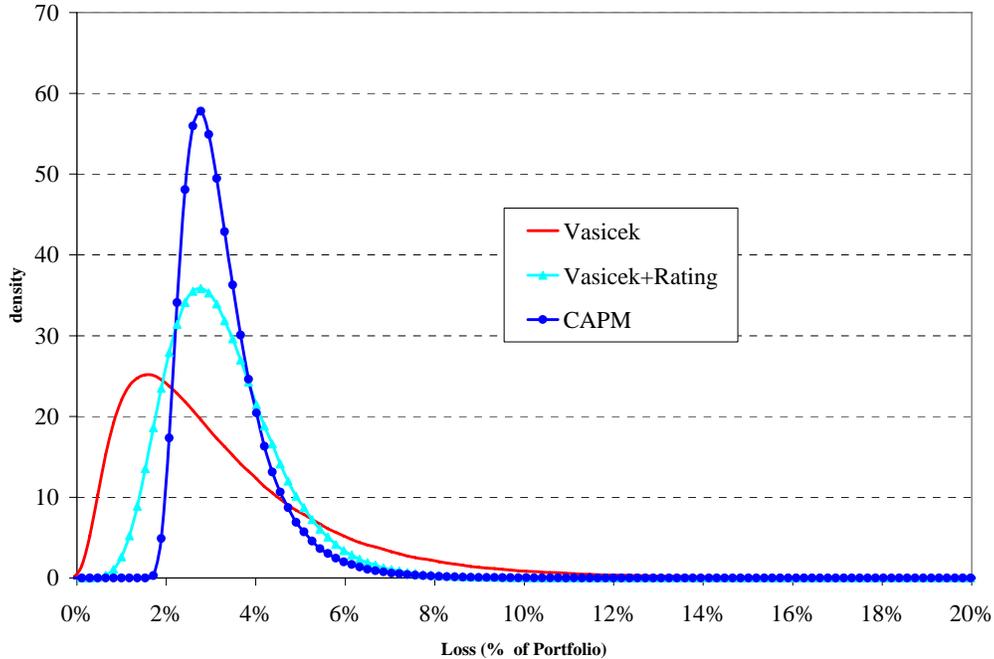


Figure 3: Calibrated (asymptotic) loss densities under random parameter framework (same EL, for 2003)

has, perhaps unwittingly, allowed banks to model the most important source of heterogeneity.

Nonetheless, full-blown economic capital models, which are limited only by available data, will play an important role in the New Basel Accord under Pillar 2 (supervisory discretion). For obligors that are publicly traded, the full simulation approach discussed in Section 4.5.2 would be such an example. However, the bulk of a bank’s lending portfolio is to firms which are privately held, and so parameters such as factor loadings cannot be estimated at the firm level. It is still possible to estimate credit ratings using firm financials (these are reported to banks as part of the lending relationship). In this case the theoretical results based on the random coefficient model could be very helpful as the parameter means and their covariances could be estimated using data from publicly traded firms. Moreover, additional (systematic) heterogeneity with respect to country or industry sector could be accommodated in this way. Simply put, our theoretical results provide an easy way of incorporating fairly rich parameter heterogeneity for a portfolio of credit exposures with limited data needed for parameter estimation.

As an illustration, in Figure 3 we show the calibrated analytical loss densities introduced in Section 4.5.1 above using the estimates based on the last 10-year window (ending in 2002, thus estimating the loss distribution for 2003); see also Table 7. The chart shows the densities for the three models (I: Vasicek; II: Vasicek plus rating; III: CAPM) while keeping EL fixed, and therefore

the chart is comparable to the densities in Figure 2 which make use of the individual parameter estimates rather than their moments. Even with this limited information and the assumption of multivariate normality of the parameters made by the random coefficient framework, the (theory based) densities in Figure 3 are remarkably close to those (small sample based) in Figure 2. The simple homogeneous Vasicek specification generates the most skewed and fat-tailed distribution. Allowing for ratings (Model II) has a significant effect on the shape of the loss density, as does adding factor loading heterogeneity (Model III). The difference in VaR between the theoretical and simulated loss distributions is quite small. For the simulation year used in Figures 2 and 3 (2003), the difference in 99.9% VaR is 2% for Model I, 6% for Model II and 4% for Model III. Similar differences are obtained for the other five years (not reported).

In this way our normal random parameter framework can be readily applied to cases where only ratings are available and other sources of heterogeneity are ignored (Model II), or when sensitivities or factor loadings are known only up to their distributional properties (Model III).

5 Concluding Remarks

In this paper we have considered a simple model of credit risk and derived the limit distribution of losses under different distributional assumptions regarding the structure of systematic and idiosyncratic risks and the nature of firm heterogeneity. The analytical and simulation results point to some interesting conclusions. Theory indicates that under the maintained assumption of conditional independence, meaning that all cross-firm dependence is captured by the systematic risk factors, if the firm parameters are heterogeneous but come from a common distribution, asymptotically (when the number of exposures, N , is sufficiently large) there is no scope for further risk reduction through active credit portfolio management. However, if firms are systematically different in that their parameters come from different distributions, as could be the case for firms from different sectors or countries, then further risk reduction is possible, even asymptotically, by changing the portfolio weights across types. In either case, neglecting parameter heterogeneity can lead to *underestimation* of expected losses. Then once expected losses are controlled for, neglecting parameter heterogeneity can lead to *overestimation* of risk, whether measured by unexpected loss or value-at-risk. Effectively the loss distribution is more skewed and fat-tailed when heterogeneity is ignored.

In light of these observations a natural question is: which sources of heterogeneity are most important from the perspective of portfolio losses? Here the answer seems clear: allowing for differences in the default threshold or unconditional probability of default (PD), measured for instance by a credit rating, is of first order importance in affecting the shape of the loss distribution. Including ratings heterogeneity alone results in a drop in loss volatility of more than one-quarter, and a drop of nearly one-half in 99.9% VaR, the VaR-level to which the New Basel Accord is calibrated.

For policy makers and risk managers alike, this is good news. After all, an obligor PD, in the form of a rating, whether internal or external to the bank, is one of the key parameters in the New Basel Accord which is allowed to vary. Indeed, early U.S. supervisory guidance indicates that banks must group their obligors into at least seven (non-default) grades, each with a unique PD (FRB, 2003, p.201). Our results suggest that possibly finer differentiation or grouping along these lines may be fruitful, so long as it is properly done.³²

When considering the return specification, firm fixed effects do not seem very important. However, allowing for flexible factor sensitivities does appear to be important, especially for capturing cross-firm dependence. If the maintained assumption of conditional independence is violated, i.e. if there remains cross-sectional dependence in the residuals from the return regressions, then risk will be *underestimated*. Thus proper specification of the return model is key by allowing for heterogeneous factor loadings and the possible addition of industry return factors.

By contrast, the differences in pooled versus country-specific results suggest that further subdividing the firm return specification and error variances by country matters less. Thus allowing for country-specific factor loadings is important, but requiring country-specific factors is less so.

³²See, for instance, Hanson and Schuermann (2005) for a discussion on PD estimation and their grouping into ratings.

A Appendix A: Limit Behavior of Credit Loss Distribution

A.1 Loss Densities under Homogeneous Parameters

In order to show how our approach relates to that of Vasicek, here we consider the homogeneous parameter case but do not require f_{t+1} and $\varepsilon_{i,t+1}$ to have Gaussian distributions. Since in the homogeneous case the multifactor model is equivalent to a single factor model, we consider scalar values for δ_i and f_{t+1} and denote them by δ and f_{t+1} , respectively. In this case we note that conditional on f_{t+1} , the random variables $z_{i,t+1}$ are identically and independently distributed as well as being integrable. (Recall that $|w_i z_{i,t+1}| \leq 1$ for all i and t .) Hence, conditional on f_{t+1} and as $N \rightarrow \infty$, we have

$$\ell_{N,t+1} \mid f_{t+1}, \mathcal{I}_t \xrightarrow{a.s.} F_\varepsilon(a - \delta f_{t+1}).$$

In the limit the probability density function of $\ell_{N,t+1} \mid I_t$ can be obtained from the probability density functions of f_{t+1} and $\varepsilon_{i,t+1}$, which we denote here by $f_f(\cdot)$ and $f_\varepsilon(\cdot)$, respectively. It will be helpful to write the loss density $f_\ell(\cdot)$ in terms of the systematic risk factor density $f_f(\cdot)$ and the standardized idiosyncratic shock density $f_\varepsilon(\cdot)$.

Therefore, conditional on I_t and denoting the limit of $\ell_{N,t+1}$ as $N \rightarrow \infty$, by ℓ_{t+1} we have (with probability 1)

$$\ell_{t+1} = F_\varepsilon(a - \delta f_{t+1}). \quad (\text{A.1})$$

Now making use of standard results on transformation of probability densities, for $\delta \neq 0$ we have

$$f_\ell(\ell_{t+1} \mid \mathcal{I}_t) = \left| \frac{\partial F_\varepsilon(a - \delta f_{t+1})}{\partial f_{t+1}} \right|^{-1} f_f(f_{t+1} \mid \mathcal{I}_t),$$

where f_{t+1} is given in terms of ℓ_{t+1} , via (A.1), namely

$$f_{t+1} = \frac{a - F_\varepsilon^{-1}(\ell_{t+1})}{\delta},$$

and $|\partial F_\varepsilon(a - \delta f_{t+1}) / \partial f_{t+1}|$ is the Jacobian of the transformation which is given by

$$\frac{\partial F_\varepsilon(a - \delta f_{t+1})}{\partial f_{t+1}} = -\delta f_\varepsilon(a - \delta f_{t+1}) = -\delta f_\varepsilon[F_\varepsilon^{-1}(\ell_{t+1})].$$

Hence

$$f_\ell(\ell_{t+1} \mid \mathcal{I}_t) = \frac{f_f\left(\frac{a - F_\varepsilon^{-1}(\ell_{t+1})}{\delta} \mid \mathcal{I}_t\right)}{|\delta| f_\varepsilon[F_\varepsilon^{-1}(\ell_{t+1})]}, \text{ for } 0 < \ell_{t+1} \leq 1. \quad (\text{A.2})$$

A.2 Relation to Vasicek's Loss Distribution

The above results provide a simple generalization of Vasicek's one-factor loss density distribution, derived in Vasicek (1991, 2002) and Gordy (2000), given by

$$f_\ell(x \mid \mathcal{I}_t) = \sqrt{\frac{1-\rho}{\rho}} \left\{ \frac{\phi\left[\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(\pi)}{\sqrt{\rho}}\right]}{\phi[\Phi^{-1}(x)]} \right\}, \text{ for } 0 < x \leq 1, \rho \neq 0, \quad (\text{A.3})$$

where x denotes the fraction of the portfolio lost to defaults. The corresponding loss distribution is

$$F_\ell(x | \mathcal{I}_t) = \Phi \left[\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(\pi)}{\sqrt{\rho}} \right]. \quad (\text{A.4})$$

The density (A.2) reduces to (A.3) when $\mu_{ft} = 0$, and assuming that the innovations, f_{t+1} and $\varepsilon_{i,t+1}$ are both Gaussian. In this case

$$\begin{aligned} f_f(f_{t+1} | \mathcal{I}_t) &= \phi(f_{t+1}), \\ f_\varepsilon(\varepsilon_{i,t+1} | \mathcal{I}_t) &= \phi(\varepsilon_{i,t+1}), \quad F_\varepsilon(\cdot) = \Phi(\cdot), \end{aligned}$$

and

$$f_\ell(x | \mathcal{I}_t) = \frac{1}{|\delta|} \left\{ \frac{\phi \left[\frac{a - \Phi^{-1}(x)}{\delta} \right]}{\phi[\Phi^{-1}(x)]} \right\}, \text{ for } 0 < x \leq 1, \quad |\delta| \neq 0 \quad (\text{A.5})$$

where we have used x for ℓ_{t+1} . Furthermore, in the homogeneous case

$$\delta = \sqrt{\frac{\rho}{1-\rho}}, \text{ for } \rho > 0, \quad (\text{A.6})$$

and

$$\pi = \Phi \left(\frac{a}{\sqrt{1+\delta^2}} \right). \quad (\text{A.7})$$

Hence

$$a = \frac{\Phi^{-1}(\pi)}{\sqrt{1-\rho}}. \quad (\text{A.8})$$

Using (A.6) and (A.8) in (A.5) now yields Vasicek's loss density given by (A.3) (note that $\phi(x) = \phi(-x)$).

Under the double-Gaussian assumption, the distribution of $\delta f_{t+1} + \varepsilon_{t+1}$ (conditional on I_t) is also Gaussian with mean $\delta \mu_{ft}$ and variance $1 + \delta^2$. Therefore,

$$E(\ell_{N,t+1} | \mathcal{I}_t) = \Phi \left(\frac{a - \delta \mu_{ft}}{\sqrt{1+\delta^2}} \right).$$

Using (A.8) and (A.6) the conditional mean loss can therefore be written as

$$E(\ell_{N,t+1} | \mathcal{I}_t) = \Phi \left[\Phi^{-1}(\pi) - \sqrt{\rho} \mu_{ft} \right], \quad (\text{A.9})$$

and reduces to π only when $\mu_{ft} = 0$. It is also interesting to note that under $\mu_{ft} \neq 0$, Vasicek's loss density and distributions become

$$f_\ell(x | \mathcal{I}_t) = \sqrt{\frac{1-\rho}{\rho}} \left\{ \frac{\phi \left[\frac{\sqrt{1-\rho}}{\rho} \Phi^{-1}(x) - \sqrt{\frac{1}{\rho}} \Phi^{-1}(\pi) + \mu_{ft} \right]}{\phi[\Phi^{-1}(x)]} \right\}, \text{ for } 0 < x \leq 1, \quad \rho > 0. \quad (\text{A.10})$$

For $\rho > 0$, the cumulative distribution function associated with this density is given by

$$F_\ell(x | \mathcal{I}_t) = \Phi \left(\sqrt{\frac{1-\rho}{\rho}} \Phi^{-1}(x) - \sqrt{\frac{1}{\rho}} \Phi^{-1}(\pi) + \mu_{ft} \right). \quad (\text{A.11})$$

Also

$$\frac{\partial F_\ell(x | \mathcal{I}_t)}{\partial \mu_{ft}} = \phi \left(\sqrt{\frac{1-\rho}{\rho}} \Phi^{-1}(x) - \sqrt{\frac{1}{\rho}} \Phi^{-1}(\pi) + \mu_{ft} \right) > 0,$$

which shows that good news (a rise in μ_{ft}) reduces the probability of losses above a given thresholds, i.e. reduces value-at-risk, as to be expected.

A.3 Loss Densities under Full Parameter Heterogeneity

The impact on the loss distribution is considerably more complicated when we consider heterogeneity across the full set of parameters, including for instance the factor loadings. In this case, $\ell_{N,t+1}$, is given by (35): $\ell_{N,t+1} = \sum_{i=1}^N w_i I(a - \boldsymbol{\delta}' \mathbf{f}_{t+1} - \zeta_{i,t+1})$. Since conditional on f_{t+1} , the composite errors, $\zeta_{i,t+1} = \varepsilon_{i,t+1} - v_{ia} + v'_{i\delta} f_{t+1}$, are independently distributed across i , then

$$\ell_{N,t+1} | \mathbf{f}_{t+1}, \mathcal{I}_t \xrightarrow{a.s.} F_{\varkappa} \left(\frac{\boldsymbol{\theta}' \mathbf{g}_{t+1}}{\omega_{t+1}} \right),$$

where *a.s.* denotes almost sure convergence, and as before $F_{\varkappa}(\cdot)$ denotes the cumulative distribution function of the standardized composite errors, $\varkappa_{i,t+1}$, defined by (39), $\mathbf{g}_{t+1} = (1, -\mathbf{f}'_{t+1})'$ and ω_{t+1} is given by (37). Once again the limiting distribution of credit loss depends on the conditional densities of $\zeta_{i,t+1}$ and f_{t+1} . For example, if $(\varepsilon_{i,t+1}, v_{ia}, \mathbf{v}'_{i\delta})$ follows a multivariate Gaussian distribution, then $\varkappa_{i,t+1} | f_{t+1}, I_t \sim iidN(0, 1)$.

The probability density of the fraction of the portfolio lost, x , over the range (0,1), can be derived from the (conditional) joint probability density function assumed for the factors, f , by application of standard change-of-variable techniques to the non-linear transformation

$$x = F_{\varkappa} \left(\frac{a - \boldsymbol{\delta}' \mathbf{f}}{\sqrt{1 + \omega_{aa} - 2\boldsymbol{\omega}'_{\alpha\delta} \mathbf{f} + \mathbf{f}' \Omega_{\delta\delta} \mathbf{f}}} \right). \quad (\text{A.12})$$

For a general m factor set up analytical derivations are quite complicated and will not be attempted here. Instead, we consider the relatively simple case of a single factor model, where f is a scalar, f . Suppose $f = \psi(x)$ satisfies the transformation, (A.12), and note that

$$f_\ell(x | \mathcal{I}_t) = |\psi'(x)| f_f [\psi(x) - \mu_{ft}], \text{ for } 0 < x \leq 1,$$

where $|\psi'(x)| = |x'(f)|^{-1}$. In other words, $\psi(x)$ is that value of the systematic factor f which generated loss of x . In the double-Gaussian case, for example, we have

$$\begin{aligned} x'(f) &= \left(\frac{f(\delta\omega_{a\delta} - a\omega_{\delta\delta}) + a\omega_{a\delta} - \delta(1 + \omega_{aa})}{(1 + \omega_{aa} - 2\omega_{a\delta}f + \omega_{\delta\delta}f^2)^{3/2}} \right) \\ &\quad \times \phi \left(\frac{a - \delta f}{\sqrt{1 + \omega_{aa} - 2\omega_{a\delta}f + \omega_{\delta\delta}f^2}} \right). \end{aligned}$$

Hence

$$|\psi'(x)| = \left(\frac{1}{\phi[\Phi^{-1}(x)]} \right) \left| \frac{[1 + \omega_{aa} - 2\omega_{a\delta}\psi(x) + \omega_{\delta\delta}\psi^2(x)]^{3/2}}{\psi(x)(\delta\omega_{a\delta} - a\omega_{\delta\delta}) + a\omega_{a\delta} - \delta(1 + \omega_{aa})} \right|.$$

and for $0 < x \leq 1$ we have

$$f_\ell(x | \mathcal{I}_t) = \left| \frac{[1 + \omega_{aa} - 2\omega_{a\delta}\psi(x) + \omega_{\delta\delta}\psi^2(x)]^{3/2}}{\psi(x)(\delta\omega_{a\delta} - a\omega_{\delta\delta}) + a\omega_{a\delta} - \delta(1 + \omega_{aa})} \right| \left\{ \frac{\phi[\psi(x) - \mu_{ft}]}{\phi[\Phi^{-1}(x)]} \right\}, \quad (\text{A.13})$$

This limiting loss distribution does not depend on the individual values of the portfolio weights, w_i , $i = 1, 2, \dots, N$, so long as the granularity conditions in (16) are satisfied.

B Appendix B

Proposition 1 *Let $F(x_1, y_1, \rho) = \Phi_2(\Phi^{-1}(x_1), \Phi^{-1}(y_1), \rho)$. Then for $\rho > 0$, $\partial^2 F(x_1, y_1, \rho)/\partial^2 x_1 < 0$ and $\partial^2 F(x_1, y_1, \rho)/\partial^2 y_1 < 0$.*

Proof: By the symmetry of $F(x_1, y_1, \rho) = \Phi_2(\Phi^{-1}(x_1), \Phi^{-1}(y_1), \rho)$ in x_1 and y_1 , it suffices to show that $\partial^2 F(x_1, y_1, \rho)/\partial x_1^2 < 0$. Let $G(x) = \Phi^{-1}(x)$ and note that

$$G'(x) = \Phi^{-1'}(x) = \frac{1}{\phi(\Phi^{-1}(x))} \text{ and } G''(x) = \Phi^{-1''}(x) = \frac{\Phi^{-1}(x)}{\phi(\Phi^{-1}(x))^2}.$$

We have

$$\begin{aligned} F(x_1, y_1, \rho) &= \Phi_2(G(x_1), G(y_1), \rho), \\ &= \int_{-\infty}^{G(x_1)} \int_{-\infty}^{G(y_1)} \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xp + y^2}{2(1-\rho^2)}\right) dy dx, \\ &= \int_{-\infty}^{G(x_1)} \int_{-\infty}^{G(y_1)} \frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{y - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x) dy dx. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial F(x_1, y_1, \rho)}{\partial x_1} &= \int_{-\infty}^{G(y_1)} \frac{G'(x_1)}{\sqrt{1-\rho^2}} \phi\left(\frac{y - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) \phi(G(x_1)) dy, \\ &= \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{G(y_1)} \phi\left(\frac{y - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) dy > 0, \end{aligned}$$

where the second line follows from noting that

$$G'(x_1)\phi(G(x_1)) = \phi(\Phi^{-1}(x_1))/\phi(\Phi^{-1}(x_1)) = 1.$$

Thus, the second partial derivative is given by

$$\frac{\partial^2 F(x_1, y_1, \rho)}{\partial x_1^2} = \rho \frac{G'(x_1)}{1-\rho^2} \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{G(y_1)} [y - \rho G(x_1)] \phi\left(\frac{y - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) dy$$

The integral in this term can be evaluated using standard results on the expectation of truncated normal variables, noting that $y|x_1 \sim N(\rho G(x_1), 1 - \rho^2)$, we have

$$\begin{aligned} \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{G(y_1)} y \phi\left(\frac{y - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) dy &= E[y|y \leq G(y_1)] \Pr(y \leq G(y_1)) \\ &= \Phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) \left[\rho G(x_1) - \sqrt{1-\rho^2} \frac{\phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right)}{\Phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right)} \right]. \end{aligned}$$

We also have

$$-\rho G(x_1) \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{G(y_1)} \phi\left(\frac{y - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) dy = -\rho G(x_1) \Phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right)$$

Combining these terms we have

$$= \rho \frac{G'(x_1)}{1-\rho^2} \left[\Phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) \rho G(x_1) - \sqrt{1-\rho^2} \phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) - \rho G(x_1) \Phi\left(\frac{G(y_1) - \rho G(x_1)}{\sqrt{1-\rho^2}}\right) \right].$$

Thus, we conclude that for $\rho > 0$,

$$\frac{\partial^2 F(x_1, y_1, \rho)}{\partial x_1^2} = -\frac{\rho}{\sqrt{1-\rho^2}} \left(\frac{1}{\phi(\Phi^{-1}(x_1))} \right) \phi\left(\frac{\Phi^{-1}(y_1) - \rho \Phi^{-1}(x_1)}{\sqrt{1-\rho^2}}\right) \leq 0,$$

as desired.

References

- [1] Altman, Edward I. (1968). “Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy.” *Journal of Finance* 20, 589-609.
- [2] Bai, Jushan and Serena Ng (2002). “Determining the Number of Factors in Approximate Factor Models.” *Econometrica*, 70, 191-221.
- [3] Basel Committee on Banking Supervision (2001). “The Internal Ratings Based Approach.” Basel Committee on Banking Supervision, May; <<http://www.bis.org/publ/bcbsca.htm>>.
- [4] Basel Committee on Banking Supervision (2005). “International Convergence of Capital Measurement and Capital Standards: A Revised Framework.” <<http://www.bis.org/publ/bcbs118.htm>>, November.
- [5] Black, Fischer, and John C. Cox (1976). “Valuing Corporate Securities: Some Effects of Bond Indenture Provisions.” *Journal of Finance* 31, 351-367.
- [6] Broadie, Mark, Mikhail Chernov and Suresh Sundaresan (2004). “Optimal Corporate Securities Values in the Presence of Chapter 7 and Chapter 11.” mimeo, Graduate School of Business, Columbia University.
- [7] Chamberlain, Gary and Michael Rothschild (1983). “Arbitrage, Factor Structure and Mean-Variance Analysis on Large Asset Markets.” *Econometrica* 51, 1281-1304.
- [8] Collin-Dufresne, Pierre, Robert S. Goldstein, and Julien Hugonnier (2004). “A General Formula for Pricing Defaultable Claims.” *Econometrica* 72, 1377–1407.
- [9] Connor, Gregory and Robert A. Korajczyk (1995). “The Arbitrage Pricing Theory and Multifactor Models of Asset Returns.” ch. 4 in Robert A. Jarrow, Vojislav Maksimovic, and W.T. Ziemba (eds.), *Handbooks in Operations Research and Management Science*, Vol. 9, Amsterdam, NL: Elsevier.
- [10] Das, Sajiv. R., Darrell Duffie, Nikunj Kapadia, and Leandro Saita (2005), “Common Failings: How Corporate Defaults are Correlated.” forthcoming, *Journal of Finance*.
- [11] Davis, Mark and Violet Lo (2001). “Infectious Defaults.” *Quantitative Finance* 1, 382-387.
- [12] Driessen, Joost (2005). “Is default event risk priced in corporate bonds?” *Review of Financial Studies* 18, 165–195.
- [13] Duffie, Darrell (2005). “Credit Risk Modeling with Affine Processes.” *Journal of Banking & Finance* 29, 2751–2802.
- [14] Duffie, Darrell and Nicolae Gârleanu (2001). “Risk and Valuation of Collateralized Debt Obligations.” *Financial Analyst Journal*, January/February, 41-59.

- [15] Duffie, Darrell and David Lando (2001). “Term Structures of Credit Spreads with Incomplete Accounting Information.” *Econometrica* 69, 633-664.
- [16] Duffie, Darrell, Leandro Saita, and Ke Wang (2005). “Multi-Period Corporate Failure Prediction with Stochastic Covariates.” Working Paper, Stanford Graduate School of Business.
- [17] Duffie, Darrell and Kenneth J. Singleton (1999). “Simulating Correlated Defaults.” Working Paper, Stanford Graduate School of Business.
- [18] Duffie, Darrell and Kenneth J. Singleton (2003). *Credit Risk: Pricing, Measurement and Management*. Princeton, NJ: Princeton University Press.
- [19] Embrechts, Paul, Alexander J. McNeil and D. Straumann (2001). “Correlation and Dependency in Risk Management: Properties and Pitfalls.” pp.176-223 in M.A.H. Dempster and H.K Moffat (eds.) *Risk Management: Value at Risk and Beyond*, Cambridge, UK: Cambridge University Press.
- [20] Federal Reserve Board, 2003, “Supervisory Guidance on Internal Ratings-Based Systems for Corporate Credit,” Attachment 2 in <http://www.federalreserve.gov/boarddocs/meetings/2003/20030711/attachment.pdf>.
- [21] Fernholz, Robert, Ioannis Karatzas and Constantinos Kardaras (2003). “Diversity and Relative Arbitrage in Financial Markets.” Department of Mathematics and Statistics, Columbia University.
- [22] Frey, Rüdiger and Alexander J. McNeil (2003). “Dependent Defaults in Models of Portfolio Credit Risk.” *Journal of Risk* 6, 59-92.
- [23] Garbade, Kenneth (2001). *Pricing Corporate Securities as Contingent Claims*, Cambridge, MA: MIT Press.
- [24] Giesecke, Kai and Stefan Weber (2004). “Cyclical Correlations, Credit Contagion and Portfolio Losses.” *Journal of Banking & Finance* 28, 3009-3036.
- [25] Gordy, Michael B. (2000). “A Comparative Anatomy of Credit Risk Models.” *Journal of Banking & Finance* 24, 119-149.
- [26] Gordy, Michael B. (2003). “A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules.” *Journal of Financial Intermediation* 12, 199-232.
- [27] Gupton, Gregory M., Christopher C. Finger and Mickey Bhatia (1997). CreditMetricsTM – Technical Document, this version: April 2, 1997. J.P. Morgan, New York.
- [28] Hanson, Samuel G. and Til Schuermann (2005). “Confidence Intervals for Probabilities of Default.” forthcoming, *Journal of Banking & Finance*.

- [29] Hillegeist, Stephen A., Elizabeth K. Keating, Donald P. Cram and Kyle G. Lundsted (2004). "Assessing the Probability of Bankruptcy." *Review of Accounting Studies* 9, 5-34.
- [30] Jafry, Yusuf and Til Schuermann (2004). "Measurement, Estimation and Comparison of Credit Migration Matrices." *Journal of Banking & Finance* 28, 2603-2639.
- [31] Jarrow, Robert A., David Lando, and Fan Yu (2005). "Default Risk and Diversification: Theory and Empirical Implications." *Mathematical Finance* 15, 1-26.
- [32] Koyluoglu, H. Ugur, and Andrew Hickman (1998). "Reconcilable Differences." *Risk* 11, October, 56-64
- [33] Lando, David (2004). *Credit Risk Modeling: Theory and Applications*. Princeton, NJ: Princeton University Press.
- [34] Lando, David and Torben Skødeberg (2002). "Analyzing Ratings Transitions and Rating Drift with Continuous Observations." *Journal of Banking & Finance* 26, 423-444.
- [35] Leland, Hayne E. and Klaus Bjerre Toft (1996). "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads." *Journal of Finance* 51, 987-1019.
- [36] Lennox, Clive (1999). "Identifying Failing Companies: A Re-Evaluation of the Logit, Probit and DA Approaches." *Journal of Economics and Business* 51, 347-364.
- [37] Li, David X. (2000). "Default Correlation: A Copula Approach." *Journal of Fixed Income*, March, 43-54.
- [38] Lucas, André, Pieter Klaasen, Peter Spreij, and Stefan Straetmans (2001). "An Analytic Approach to Credit Risk of Large Corporate Bond and Loan Portfolios." *Journal of Banking & Finance* 25, 1635-1664.
- [39] Merton, Robert C. (1974). "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance* 29, 449-470.
- [40] Pesaran, M.Hashem (2006). "Estimation and Inference in Large Heterogeneous Panels with a Multifactor Error Structure." forthcoming, *Econometrica*.
- [41] Pesaran, M.Hashem, Til Schuermann, and Björn-Jakob Treutler (2005). "Global Business Cycles and Credit Risk." forthcoming in Mark Carey and René Stulz (eds.), *Risks of Financial Institutions*, Chicago, IL: University of Chicago Press. Available as NBER Working Paper w11493.
- [42] Pesaran, M. Hashem, Til Schuermann, Björn-Jakob Treutler, and Scott M. Weiner (2005). "Macroeconomic Dynamics and Credit Risk: A Global Perspective." forthcoming, *Journal of Money, Credit and Banking*.

- [43] Ross, Stephen A. (1976). "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory* 13, 341-360.
- [44] Saunders, Anthony and Linda Allen (2002). *Credit Risk Measurement- New Approaches to Value at Risk and Other Paradigms*. 2nd Ed., New York: John Wiley & Sons.
- [45] Schönbucher, Philipp J. (1998). "The Term Structure of Defaultable Bond Prices." *Review of Derivatives Research* 2, 161-192.
- [46] Schönbucher, Philipp J. (2002). "Taken to the Limit: Simple and Not-so-Simple Loan Loss Distributions." Working Paper, Department of Statistics, Bonn Univ.
- [47] Schönbucher, Philipp J. (2003), *Credit Derivatives Pricing Models: Models, Pricing and Implementation*, New York, NY: John Wiley & Sons.
- [48] Shumway, Tyler (2001). "Forecasting Bankruptcy more Accurately: A Simple Hazard Model." *Journal of Business* 74, 101-124.
- [49] Silverman, Bernard W. (1986). *Density Estimation for Statistics and Data Analysis*. New York, NY: Chapman & Hall.
- [50] Vasicek, Oldrich (1987). "Probability of Loss on Loan Portfolio." KMV Corp.; available at www.kmv.com.
- [51] Vasicek, Oldrich (1991). "Limiting Loan Loss Distribution." KMV Corp.; available at www.kmv.com.
- [52] Vasicek, Oldrich (2002). "Loan Portfolio Value." *Risk* 15, 160-162.
- [53] Wilson, Thomas (1997a). "Portfolio Credit Risk, Part I." *Risk* 10, September, 111-117.
- [54] Wilson, Thomas (1997b). "Portfolio Credit Risk, Part II." *Risk* 10, October, 56-61.
- [55] Zhou, Chuncheng (2001). "An Analysis of Default Correlations and Multiple Defaults." *Review of Financial Studies* 14, 555-576.

Table 1
Ratings Distributions and Probabilities of Default

Credit Rating	1997				2002			
	Ratings Distribution (%)				Ratings Distribution (%)			
	Japan	U.S.	Both	$\hat{\pi}_R(\%)$	Japan	U.S.	Both	$\hat{\pi}_R(\%)$
AAA	4.8	2.86	3.35	0.001	0.75	2.37	1.97	0.001
AA	22.62	10.81	13.76	0.001	16.03	9.25	10.95	0.001
A	37.93	25.61	28.69	0.005	29.38	21.49	23.46	0.006
BBB	23.16	22.33	22.54	0.064	34.89	25.67	27.97	0.106
BB	9.94	16.3	14.71	0.481	11.85	15.72	14.75	0.63
B	1.54	19.79	15.22	3.343	6.75	19.62	16.4	5.429
CCC	0	2.31	1.73	36.49	0.35	5.89	4.5	49.78
Portfolio	1.23				3.26			

Note: The table presents the distribution of firms by rating for U.S. and Japan as of year-end 1997 and 2002. These distributions are calculated by taking the average of the distribution for Moody's and the distribution for Standard and Poor's. We construct our portfolios so that the exposure weights for each country are consistent with that country's rating distribution. The column label "Both" represents the exposure weights by rating for the combined portfolio (75% U.S. / 25% Japan). The final column, $\hat{\pi}_R(\%)$, contains the estimated annual probabilities of default (PD) that are used in the simulation exercises. These PDs are estimated using the time-homogeneous Markov or parametric duration estimator discussed in Lando and Skødeberg (2002) and Jafry and Schuermann (2004). A minimum annual PD of 0.001% or 0.1 basis points is imposed.

Table 2
Industry Breakdowns by Country

Industry	% of Firms at	
	Year-End 1997	
	U.S.	Japan
Agriculture, Mining & Construction	5.3	8.5
Communication, Electric & Gas	16.7	6.2
Durable Manufacturing	22.1	34.1
Finance, Insurance & Real Estate	23.1	14.7
Non-durable Manufacturing	18.2	24.6
Service	4.8	6.6
Wholesale & Retail Trade	9.9	5.2
Total	100.0	100.0

Note: The table presents the distribution of firms by industry group for both the U.S. and Japan sub-portfolios as of year-end 1997.

Table 3
Specifications of Return Equations for
Separate Country Models

Model	Return Specification
I Vasicek	$r_{ijc,t+1} = \alpha_c + \beta_c \bar{r}_{c,t+1} + u_{ijc,t+1}$
II Vasicek + Rating	$r_{ijc,t+1} = \alpha_c + \beta_c \bar{r}_{c,t+1} + u_{ijc,t+1}$
III CAPM	$r_{ijc,t+1} = \alpha_{ijc} + \beta_{ijc} \bar{r}_{c,t+1} + u_{ijc,t+1}$
IV CAPM + Sector	$r_{ijc,t+1} = \alpha_{ijc} + \beta_{1,ijc} \bar{r}_{c,t+1} + \beta_{2,ijc} \bar{r}_{cj,t+1} + u_{ijc,t+1}$
V PCA	$r_{ijc,t+1} = \alpha_{ijc} + \beta'_{ijc} \mathbf{f}_{c,t+1} + u_{ijc,t+1}$

Note: $r_{ijc,t+1}$ denotes the return of firm i in sector j in country c over the quarter t to $t+1$; $c = \text{US, JP}$. In Models I through IV, $\bar{r}_{c,t+1}$ denotes the market-cap weighted return in country c over the quarter t to $t+1$ and $\bar{r}_{cj,t+1}$ is the market-cap weighted return for sector j in country c over the same period. For the “pooled” country models we drop the c subscript on $\bar{r}_{c,t+1}, \bar{r}_{cj,t+1}$, and $f_{c,t+1}$ so that the common factors are global factors as opposed to country-specific factors. For Model I there is a single default threshold for all firms. For Models II to V there is one default threshold per rating for the “pooled” specification, and one threshold per rating per country for the “modeled separately” specification.

Table 4
Average Pair-wise Correlation of Returns and
In-sample Residuals Based on Ten-Year Rolling Windows

	Sample Window	Average Pair-wise Correlation of Returns			Model Specifications	Average Pair-wise Correlation of Residuals					
		US&JP	US	JP		US & Japan Pooled			US & Japan Modeled Separately		
						US&JP	US	JP	US&JP	US	JP
Panel A	1988-1997	0.1937	0.1933	0.6011	I Vasicek	0.0222	0.0951	0.4217	0.0239	0.0365	0.0607
	# of firms	839	628	211	III CAPM	0.0218	0.0797	0.3868	0.0242	0.0374	0.0639
					IV CAPM + Sector	0.0147	0.0711	0.3869	0.0142	0.0221	0.0227
					V PCA	-0.0001	0.0016	0.0037	0.0000	0.0005	-0.0045
Panel B	1989-1998	0.2150	0.2114	0.5913	I Vasicek	0.0235	0.0940	0.3793	0.0249	0.0440	0.0633
	# of firms	854	633	221	III CAPM	0.0237	0.0825	0.3553	0.0255	0.0456	0.0666
					IV CAPM + Sector	0.0177	0.0773	0.3582	0.0193	0.0324	0.0258
					V PCA	0.0003	0.0011	0.0028	0.0000	0.0004	-0.0041
Panel C	1990-1999	0.2097	0.2237	0.5666	I Vasicek	0.0533	0.1282	0.3917	0.0627	0.0731	0.1391
	# of firms	842	613	229	III CAPM	0.0568	0.1246	0.3807	0.0661	0.0778	0.1473
					IV CAPM + Sector	0.0419	0.1168	0.3817	0.0417	0.0621	0.0390
					V PCA	0.0000	0.0005	0.0010	-0.0003	-0.0001	-0.0036
Panel D	1991-2000	0.1391	0.1691	0.4638	I Vasicek	0.0533	0.1207	0.3427	0.0633	0.0783	0.1359
	# of firms	816	588	228	III CAPM	0.0559	0.1206	0.3442	0.0649	0.0821	0.1428
					IV CAPM + Sector	0.0379	0.1039	0.3470	0.0392	0.0615	0.0427
					V PCA	0.0000	0.0018	0.0030	0.0002	0.0008	-0.0041
Panel E	1992-2001	0.1309	0.1633	0.4411	I Vasicek	0.0482	0.1132	0.3282	0.0578	0.0740	0.1277
	# of firms	811	585	226	III CAPM	0.0501	0.1135	0.3324	0.0583	0.0772	0.1339
					IV CAPM + Sector	0.0378	0.1077	0.3361	0.0381	0.0624	0.0421
					V PCA	-0.0007	0.0001	0.0009	-0.0004	-0.0003	-0.0042
Panel F	1993-2002	0.1545	0.1999	0.4191	I Vasicek	0.0549	0.1098	0.3332	0.0628	0.0772	0.1254
	# of firms	818	600	218	III CAPM	0.0569	0.1157	0.3488	0.0641	0.0811	0.1324
					IV CAPM + Sector	0.0439	0.1099	0.3543	0.0426	0.0658	0.0391
					V PCA	-0.0008	-0.0006	0.0001	-0.0007	-0.0009	-0.0043

Note: This table presents the results of recursive estimation of return equations using quarterly return data. All estimation results are calculated using a 40-quarter rolling window. The results for the “pooled” country models are given in the first set of columns and the “separate” country models in the second set of columns. Portfolio determination and sample construction are discussed in Section 4.2. Specification of the return models is discussed in Section 4.3 (see Table 3 for further detail). The data source for returns of U.S. firms is CRSP. The source for Japanese firms is Datastream. Yen-denominated Japanese returns are converted to USD-denominated returns by subtracting the percentage change in the Yen/USD exchange rate. .

Table 5
Parameter Estimates by Credit Ratings based on Return Regressions
1993-2002

Model	Parameter	<i>AAA/AA</i>	<i>A</i>	<i>BBB</i>	<i>BB</i>	<i>B</i>	<i>CCC</i>
Vasicek /	$\bar{\beta}$	0.824	0.824	0.824	0.824	0.824	0.824
Vasicek + Rating	$\bar{\sigma}$	0.191	0.191	0.191	0.191	0.191	0.191
CAPM	$\bar{\beta}_{\mathcal{R}}$	0.719	0.795	0.738	0.933	1.095	0.596
	$\bar{\sigma}_{\mathcal{R}}$	0.115	0.135	0.158	0.217	0.275	0.282

Note: Averages of estimated parameters from US & JP pooled return regressions from Models I and III; see Table 4. Note that the return specification for Model II is the same as for Model I. $\bar{\beta}$ and $\bar{\sigma}$ are the pooled estimates of return factor loading (“beta”) and firm error variance, respectively. In the CAPM model these parameters are estimated separately for each firm, so that $\bar{\beta}_{\mathcal{R}}$ and $\bar{\sigma}_{\mathcal{R}}$ are averages by rating.

Table 6
Parameter Means Used in Asymptotic Loss Distribution
“Pooled Country Model” 1993-2002

Parameter	<i>AAA/AA</i>	<i>A</i>	<i>BBB</i>	<i>BB</i>	<i>B</i>	<i>CCC</i>
$\hat{\alpha}_{\mathcal{R}}$	-4.860	-4.293	-3.306	-2.674	-1.709	-0.006
$\hat{\delta}_{\mathcal{R}}$	0.476	0.421	0.341	0.326	0.291	0.195
$\omega_{a_{\mathcal{R}}a_{\mathcal{R}}}$	0.452	0.217	0.057	0.039	0.020	<0.001
$\omega_{\delta_{\mathcal{R}}\delta_{\mathcal{R}}}$	0.097	0.078	0.047	0.049	0.058	0.035
$\rho_{a_{\mathcal{R}}\delta_{\mathcal{R}}}$	-0.963	-0.956	-0.954	-0.929	-0.911	-0.819

Note: Averages (by rating) for reduced form parameters for CAPM model from Table 5. *AAA* and *AA* are grouped since their default probabilities are both assigned the minimum of 0.01bp. For details on how the parameters are computed, please see Section 4.5.1.

Table 7
Simulated Losses Using Analytical (Asymptotic) Results of
Random Parameters Framework

Sample	Simulation Year	Model	EL	UL	VaR	
					99.0%	99.9%
1988-1997	1998	I. Vasicek	1.23%	1.40%	6.82%	11.87%
		II. Vasicek+Rating	1.23%	0.82%	4.11%	6.16%
		III. CAPM	1.23%	0.52%	3.22%	5.30%
1989-1998	1999	I. Vasicek	1.60%	1.89%	9.17%	15.91%
		II. Vasicek+Rating	1.60%	1.08%	5.38%	8.07%
		III. CAPM	1.60%	0.75%	4.61%	7.62%
1990-1999	2000	I. Vasicek	2.10%	2.14%	10.37%	17.01%
		II. Vasicek+Rating	2.10%	1.18%	6.05%	8.60%
		III. CAPM	2.10%	0.82%	5.28%	8.04%
1991-2000	2001	I. Vasicek	2.28%	1.65%	8.10%	12.07%
		II. Vasicek+Rating	2.28%	0.91%	5.06%	6.58%
		III. CAPM	2.28%	0.89%	5.31%	7.37%
1992-2001	2002	I. Vasicek	2.74%	1.88%	9.22%	13.46%
		II. Vasicek+Rating	2.74%	1.00%	5.71%	7.25%
		III. CAPM	2.74%	0.85%	5.53%	7.39%
1993-2002	2003	I. Vasicek	3.26%	2.38%	11.61%	17.11%
		II. Vasicek+Rating	3.26%	1.23%	6.94%	8.88%
		III. CAPM	3.26%	0.95%	6.54%	8.84%

Note: Expected loss is held fixed across models for easy comparison. Parameters are given in Table 6. Loss distributions are simulated using equation (70), as discussed in Section 4.5.1, with 1,000,000 replications.

Table 8
Out-of-Sample Simulated Annual Losses Based on 10-Year Rolling Return Regressions

200,000 replications

Simulation	Using	Model Specifications	US & Japan Pooled						US & Japan Modeled Separately					
			Year	Sample	UL	Skew.	Kurt.	Value-at-Risk		UL	Skew.	Kurt.	Value-at-Risk	
								99.0%	99.9%				99.0%	99.9%
Panel A	1998	1988-1997	I	Vasicek	1.47%	3.1	19.5	7.00%	12.05%	1.29%	2.6	15.5	6.10%	10.14%
			II	Vasicek + Rating	1.07%	1.3	5.8	4.57%	6.72%	1.01%	1.1	5.1	4.30%	6.17%
			III	CAPM	0.86%	1.0	5.9	3.73%	5.56%	0.88%	1.0	5.7	3.82%	5.75%
			IV	CAPM + Sector	0.88%	1.0	6.1	3.81%	5.58%	0.90%	1.1	5.9	3.89%	5.81%
			V	PCA	1.08%	1.6	8.8	4.82%	7.69%	1.07%	1.5	7.8	4.74%	7.30%
Panel B	1999	1989-1998	I	Vasicek	1.94%	3.0	18.1	9.30%	15.91%	1.73%	2.8	17.9	8.25%	14.04%
			II	Vasicek + Rating	1.28%	1.4	6.7	5.80%	8.52%	1.21%	1.4	6.7	5.50%	7.95%
			III	CAPM	0.95%	1.3	8.2	4.53%	6.93%	0.98%	1.4	8.7	4.67%	7.06%
			IV	CAPM + Sector	0.99%	1.2	7.4	4.64%	7.06%	1.04%	1.5	8.9	4.88%	7.67%
			V	PCA	1.20%	1.7	9.3	5.69%	8.88%	1.20%	1.8	10.4	5.72%	8.95%
Panel C	2000	1990-1999	I	Vasicek	2.22%	2.6	14.7	10.65%	17.34%	2.06%	2.4	12.7	9.78%	15.87%
			II	Vasicek + Rating	1.43%	1.2	5.5	6.56%	9.18%	1.39%	1.1	5.3	6.37%	8.91%
			III	CAPM	1.05%	1.0	6.3	5.23%	7.54%	1.12%	1.1	6.1	5.51%	7.99%
			IV	CAPM + Sector	1.09%	1.0	5.7	5.33%	7.56%	1.19%	1.1	6.2	5.74%	8.25%
			V	PCA	1.39%	1.5	7.7	6.68%	10.16%	1.40%	1.6	8.4	6.77%	10.46%

Table 8 (continued)
Out-of-Sample Simulated Annual Losses Based on 10-Year Rolling Return Regressions

200,000 replications

Simulation	Using	Year	Sample	Model Specifications	US & Japan Pooled					US & Japan Modeled Separately				
					UL	Skew.	Kurt.	Value-at-Risk		UL	Skew.	Kurt.	Value-at-Risk	
								99.0%	99.9%				99.0%	99.9%
Panel D	2001	1991-2000	$\hat{\pi} = 2.28\%$	I Vasicek	1.75%	1.8	8.6	8.39%	12.72%	1.72%	1.7	8.2	8.20%	12.04%
				II Vasicek + Rating	1.22%	0.7	3.8	5.71%	7.38%	1.23%	0.7	3.9	5.71%	7.40%
				III CAPM	1.13%	0.8	4.4	5.54%	7.46%	1.20%	0.8	4.5	5.76%	7.69%
				IV CAPM + Sector	1.18%	0.7	4.2	5.64%	7.57%	1.27%	0.8	4.4	5.96%	8.06%
				V PCA	1.44%	1.1	5.4	6.74%	9.49%	1.44%	1.1	5.5	6.75%	9.66%
Panel E	2002	1992-2001	$\hat{\pi} = 2.74\%$	I Vasicek	1.98%	1.6	7.3	9.48%	13.71%	1.96%	1.6	7.5	9.45%	13.65%
				II Vasicek + Rating	1.34%	0.5	3.4	6.32%	7.91%	1.35%	0.6	3.5	6.39%	8.07%
				III CAPM	1.16%	0.5	3.7	5.87%	7.55%	1.22%	0.6	3.9	6.08%	7.89%
				IV CAPM + Sector	1.19%	0.6	3.8	5.98%	7.73%	1.26%	0.7	4.1	6.25%	8.23%
				V PCA	1.42%	0.9	4.8	6.95%	9.43%	1.44%	1.0	5.2	7.03%	9.86%
Panel F	2003	1993-2002	$\hat{\pi} = 3.26\%$	I Vasicek	2.48%	1.8	8.0	11.92%	17.47%	2.56%	1.8	8.2	12.20%	17.81%
				II Vasicek + Rating	1.51%	0.6	3.7	7.46%	9.46%	1.56%	0.7	3.9	7.65%	9.86%
				III CAPM	1.27%	0.8	4.7	6.98%	9.21%	1.27%	0.8	5.0	7.00%	9.30%
				IV CAPM + Sector	1.28%	0.8	4.7	7.01%	9.20%	1.30%	0.9	5.2	7.15%	9.60%
				V PCA	1.51%	1.1	5.9	7.96%	11.15%	1.49%	1.2	6.2	7.91%	11.11%

Note: This table presents results for simulated out-of-sample annual loss distributions (4-quarter ahead loss distributions). The table presents simulation results for the “pooled” country models in the first set of columns and the “separate” country models in the second set of columns. Model specifications, including the return regressions and determination of default thresholds, are discussed in Section 4.3 (see Table 3 for more detail on the model specifications). Simulation are carried out using 200,000 replications. For each year all models are calibrated to have the same expect loss given by $\hat{\pi}$. For each simulation, the table reports the standard deviation of losses (denoted Unexpected Losses - UL), the 3rd and 4th moments of the loss distributions, as well as the 99.0% and 99.9% quantiles of the distribution (denoted Value-at-Risk).