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THE PLANE SYMMETRY GROUPS: THEIR RECOGNITION AND NOTATION

DORIS SCHATTSCHNEIDER

Introduction. Groups of transformations which leave invariant a specified item are familiar objects of study for students and researchers alike. Finite groups of plane isometries which leave invariant a regular polygon are elementary examples: C_n, the cyclic group of order n, can be realized as the group of rotations leaving invariant a regular n-gon, and D_n, the dihedral group of order $2n$, can be realized as the group of all isometries (rotations and reflections) leaving invariant the same polygon. A very interesting collection of discrete groups of plane isometries which are natural extensions of these examples exists, but is lacking in most introductory algebra texts. These are the groups of plane isometries which leave invariant a design or pattern in the plane. If the pattern is finite, such a group is necessarily a subgroup of some dihedral group. If the pattern is repeated regularly in one or in two directions, translations and glide-reflections are additional possible isometries of the pattern, and so the group leaving such a design invariant will be an infinite discrete group. Designs which are invariant under all multiples of just one translation are frieze, or border ornaments, and their associated groups are commonly called "frieze groups." Patterns which are invariant under linear combinations of two linearly independent translations repeat at regular intervals in two directions, and hence their groups are often termed "wallpaper groups."

The interweaving of elementary aspects of Euclidean transformation geometry and group theory makes these groups excellent ones for study—but there are several non-mathematical bonuses which make their study especially appealing. To analyze a repeating design to see what makes it "work," and to create original designs using the power of the mathematical "laws" which govern these designs, is a strong non-mathematical motive for studying these groups. (Suddenly, the word "symmetry" has true dual meaning; both its artistic and mathematical connotations are seen as inseparable.) Rudiments of elementary crystallography are part of the theory as well—another bonus.

A very specific incentive to learn about these groups is the opportunity to study examples of the imaginative interlocking patterns by the Dutch artist M. C. Escher (1898–1972). His work is perhaps the most concrete testament to the power gained in understanding these groups. He struggled for several years to produce animate interlocking designs, with very primitive results. When he became aware that these types of designs were governed by groups of isometries, he studied the mathematical literature available. In examining Escher's notebooks, this author discovered that he copied in full the

Doris Schattschneider received her Ph.D. from Yale University in 1966 (in the area of algebraic groups). After teaching at Northwestern University and the University of Illinois at Chicago Circle, she came to Moravian College in 1968. Here, her interest in art led her to create a January Term course, "Tessellations, the Mathematical Art," in which much of the information for this MONTHLY article was developed. A deepened interest in the work of M. C. Escher was a natural outgrowth of this course. Recently, she has collaborated with a graphic artist to produce a book and a collection of unique geometric models, *M. C. Escher Kaleidocycles*, published by Ballantine Books.—*Editors*.

Escher Foundation, Haags Gemeentemuseum, The Hague.

Two periodic drawings by M. C. Escher contrast his early effort at repeating design with his later masterful skill. The pattern of lions, dated "1926 or 1927," was done before he developed a system which grew out of his study of mathematical articles and periodic designs on the Alhambra. The pattern of bugs is dated 1942, one year after Escher recorded his codified system in notebooks.

paper by G. Pólya **[18]** which outlines the important properties of each of the groups and includes a chart of illustrative designs. (This chart is also reproduced in **[12]**, page 78.) Escher records that this visual information was of more importance to him than the written text. Another rich source of visual information for Escher was found in the Moorish tile patterns of the Alhambra, in Granada, Spain. He visited this site and carefully recorded in sketchbooks many of these periodic geometric designs. The designs he produced after he digested this information (and ultimately worked out his own system) are amazingly intricate, even mind boggling, to the innocent viewer.

The literature available on the plane symmetry groups is scattered, and often incomplete. Good descriptions of the frieze groups do exist (**[1]**, **[4]**, **[5]**, **[6]**, **[11]**, **[20]**). However, gathering complete and coherent information from the various sources on the "wallpaper groups" can be frustrating, since terminology is not standard and several different notations for the groups are used. In addition, a frequent error occurs—the notations for two of the groups are interchanged in several sources.

This article attempts to provide in compact form information to correct these problems and, in addition, provide useful visual references for readers of the literature on the plane symmetry groups. The sources used are listed in the references. Many other books and articles contain information on this topic; the remarks that follow pertain to these as well.

Terminology. Classification of periodic patterns. Part of the difficulty in reading from various sources on the plane symmetry groups is the variation in terminology used by authors. Not only are different terms used to identify the same object, but sometimes the same terms are employed (in different sources) to identify different objects. In this section we define terms as used in this presentation and indicate some other common terminology. In using any source, the reader should be especially careful to determine the definition of terms used by the author.

A "*periodic*" or "*repeating*" *pattern* in the plane is a design having the following property: There exist a finite region and two linearly independent translations such that the set of all images of the region when acted on by the group generated by these translations produces the original design. In addition (although rarely stated explicitly) it is assumed that there is a translation vector of minimum length that maps the pattern onto itself. This excludes a pattern of stripes from being termed periodic.

The *translation group* of a periodic pattern is the set of all translations which map the pattern onto itself. A smallest region of the plane having the property that the set of its images under this translation group covers the plane is called a *unit* of the pattern. All units have the same area, but their outlines can have infinite variation. They are like tiles, all alike, which fill the plane without gaps or overlaps, and are laid in parallel rows. Some periodic designs incorporate part or all of the boundary of a unit as part of the design; others suppress this outline and one sees only a repeated figure against a blank background. For example, in the Escher lion design, four interlocked lions, each one facing in a different direction, form a unit of the pattern.

Every periodic pattern has naturally associated to it a *lattice* of points; choosing any point in the pattern, this lattice is the set of all images of that point when acted on by the translation group of the pattern. A *lattice unit* is a unit which is a parallelogram whose vertices are lattice points. The vectors which form the sides of a lattice unit generate the translation group of the pattern. (Crystallographers use the term *primitive cell* for a lattice unit; some authors use the term *unit cell*, or *cell*.)

In addition to translations, a periodic pattern may also be mapped onto itself by any of the other plane isometries: rotations, reflections or glide reflections. The *symmetry group* of the pattern is the set of all isometries which map the pattern onto itself. The classification of periodic patterns according to their symmetry groups is the two-dimensional counterpart of the system used by crystallographers to classify crystals. Hence, these groups are also termed the *two-dimensional crystallographic groups*.

The symmetry group of a periodic pattern necessarily maps a lattice associated to the pattern onto itself. Since centers of rotation of a pattern are mapped by translations to new centers of rotation (having the same order), only rotations of order 2, 3, 4, or 6 can occur as isometries of a periodic design. (This is often referred to as the crystallographic restriction.) If a pattern has no rotational symmetry, but reflections or glide reflections are in its symmetry group, then the lattice must have parallel rows of points at right angles to each other. These restrictions imply that there are five distinct types of lattice which can occur as the most general lattice possible for a plane symmetry group. For each lattice type there are conventionally chosen lattice units for purposes of classification. Chart 1 shows the five types of lattice, and for each a lattice unit.

LATTICES FOR PERIODIC PLANE PATTERNS

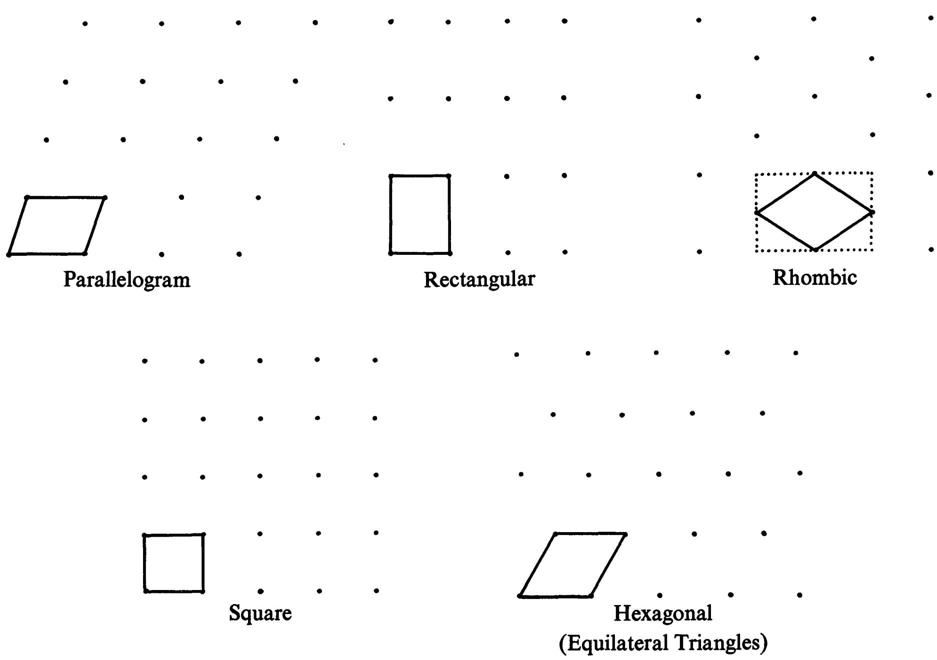


CHART 1. The lattice units outlined are those chosen by crystallographers for purposes of classification. The "centered cell," containing 2 units, is shown in dotted outline on the rhombic lattice.

LATTICE UNITS WITH SYMMETRIES OF PERIODIC PLANE PATTERNS

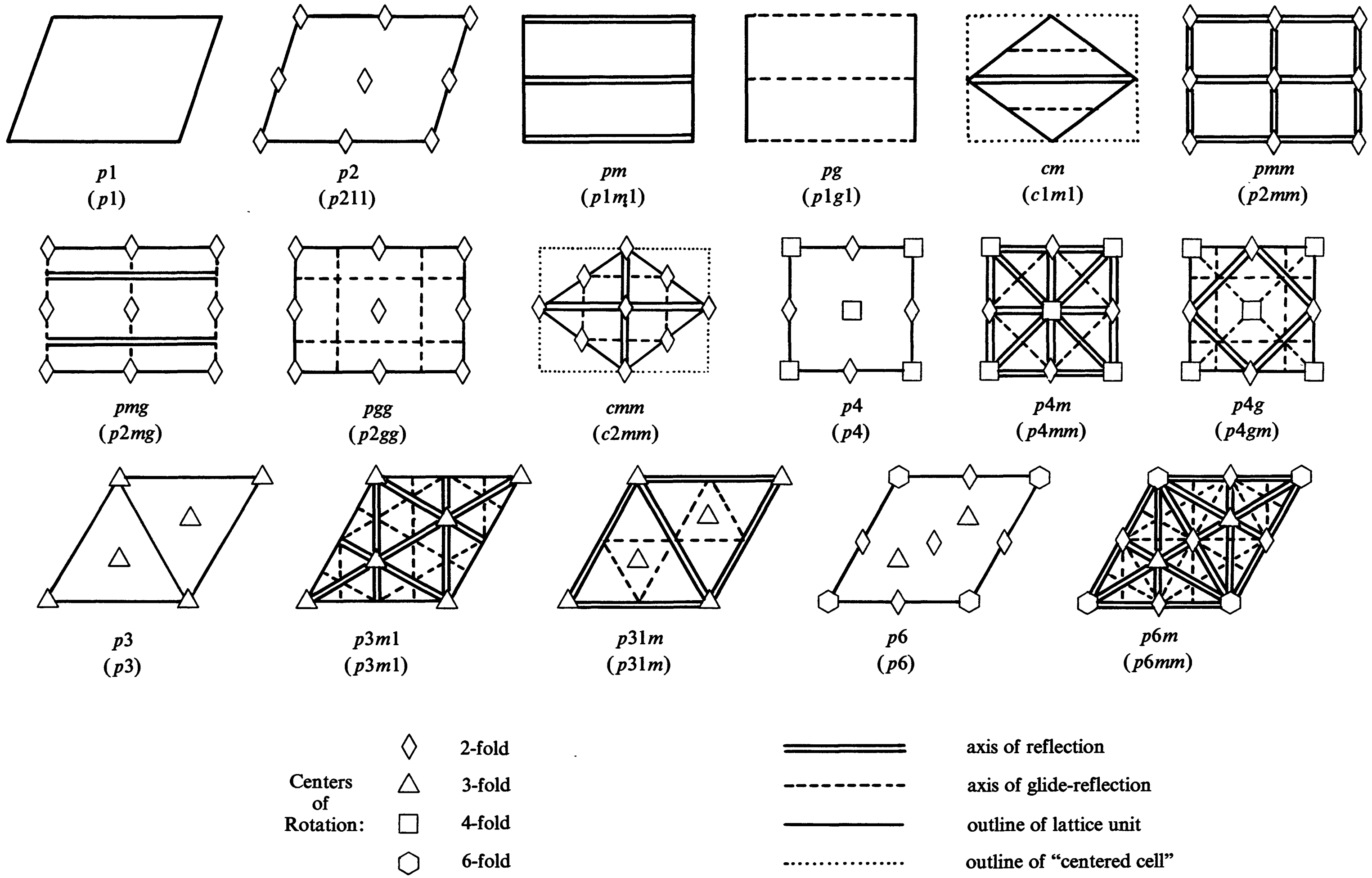


CHART 2. International notation identifies the seventeen two-dimensional crystallographic groups. The short form is given first, with the full notation in parentheses.

Arguing on the isometries possible for each of the five lattice types, it can be shown that there are seventeen distinct plane symmetry groups. In Chart 2, we show for each group a lattice unit and the placement of symmetry elements in the group relative to that lattice unit (i.e., centers of rotation, axes of reflection and glide reflection). This is an adaptation of the symbolism used in the International Tables for X-ray Crystallography [13]. Under each diagram is the crystallographic name or symbol for that group, both in the short and full form. Full explanation of this notation is found in [13] but is lacking in the mathematical literature, and so it might be helpful to include it here.

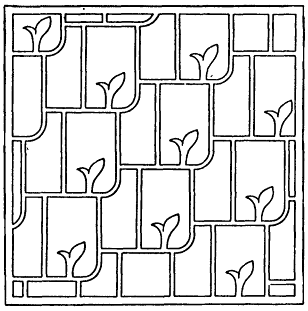
The crystallographic notation consists of four symbols which identify the conventionally chosen "cell," the highest order of rotation, and other fundamental symmetries. Usually a "primitive cell" (a lattice unit) is chosen with centers of highest order of rotation at the vertices. In two cases a "centered cell" is chosen so that reflection axes will be normal to one or both sides of the cell. The "x-axis" of the cell is the left edge of the cell (the vector directed downward). The interpretation of the full international symbol (read left to right) is as follows: (1) letter p or c denotes primitive or centered cell; (2) integer n denotes highest order of rotation; (3) symbol denotes a symmetry axis normal to the x-axis: m (mirror) indicates a reflection axis, g indicates no reflection, but a glide-reflection axis, 1 indicates no symmetry axis; (4) symbol denotes a symmetry axis at angle α to x-axis, with α dependent on n, the highest order of rotation: $\alpha = 180°$ for $n = 1$ or 2, $\alpha = 45°$ for $n = 4$, $\alpha = 60°$ for $n = 3$ or 6; the symbols m, g, 1 are interpreted as in (3). No symbols in the third and fourth position indicate that the group contains no reflections or glide-reflections. The many symmetry axes you see in the diagrams on Chart 2 result from the combination of translations or rotations with the symmetries indicated in the third and fourth position of the international symbol. Except in the case

Recognition Chart for Plane Periodic Patterns

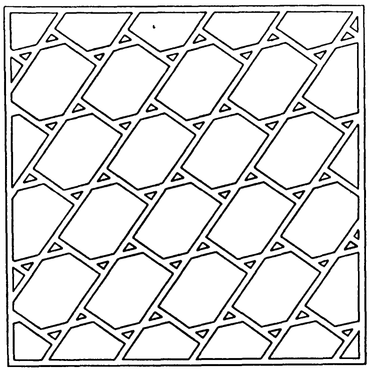
Type	Lattice	Highest Order of Rotation	Reflections	Non-Trivial Glide Reflections	Generating Region	Helpful Distinguishing Properties
$p1$	parallelogram	1	no	no	1 unit	
$p2$	parallelogram	2	no	no	1/2 unit	
pm	rectangular	1	yes	no	1/2 unit	
pg	rectangular	1	no	yes	1/2 unit	
cm	rhombic	1	yes	yes	1/2 unit	
pmm	rectangular	2	yes	no	1/4 unit	
pmg	rectangular	2	yes	yes	1/4 unit	parallel reflection axes
pgg	rectangular	2	no	yes	1/4 unit	
cmm	rhombic	2	yes	yes	1/4 unit	perpendicular reflection axes
$p4$	square	4	no	no	1/4 unit	
$p4m$	square	4	yes	yes	1/8 unit	4-fold centers on reflection axes
$p4g$	square	4	yes	yes	1/8 unit	4-fold centers not on reflection axes
$p3$	hexagonal	3	no	no	1/3 unit	
$p3m1$	hexagonal	3	yes	yes	1/6 unit	all 3-fold centers on reflection axes
$p31m$	hexagonal	3	yes	yes	1/6 unit	not all 3-fold centers on reflection axes
$p6$	hexagonal	6	no	no	1/6 unit	
$p6m$	hexagonal	6	yes	yes	1/12 unit	

Chart 3. A rotation through an angle of $360°/n$ is said to have order n. A glide-reflection is non-trivial if its component translation and reflection are not symmetries of the pattern.

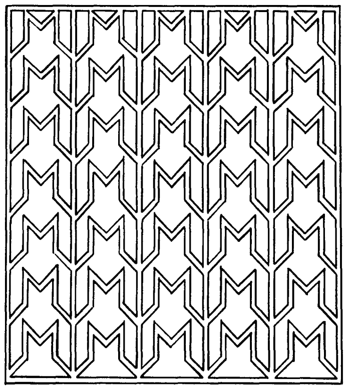
REPRESENTATIVE PATTERNS FOR



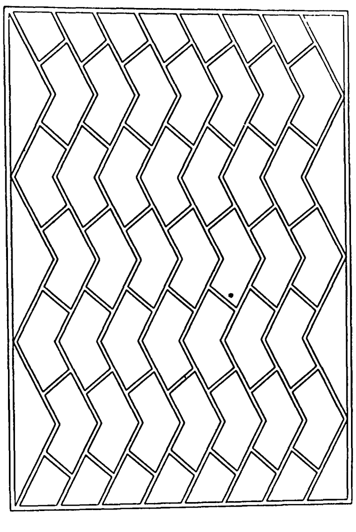
*p*1



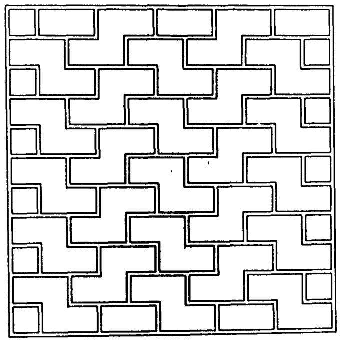
*p*2



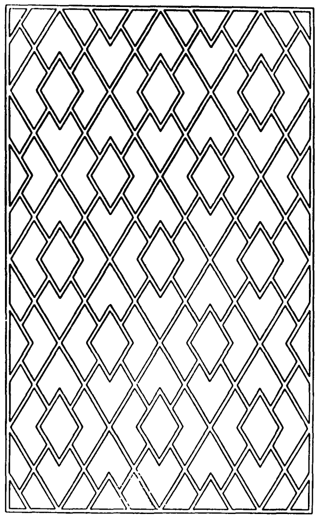
pm



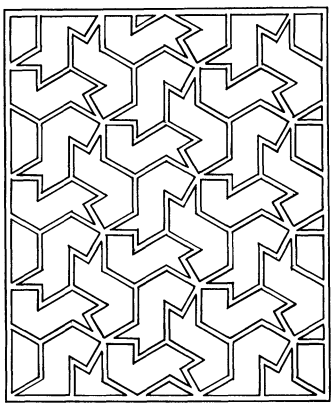
pmg



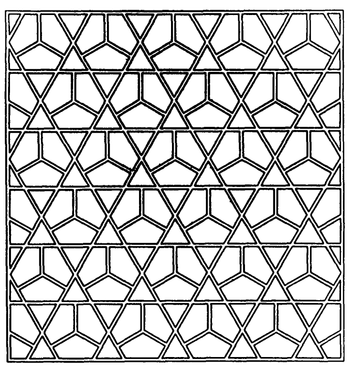
pgg



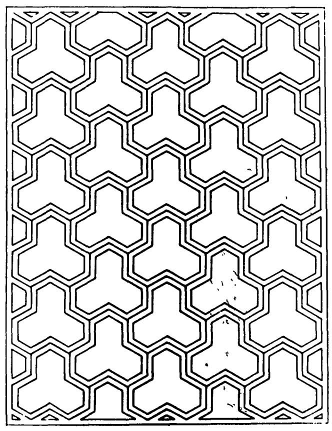
cmm



*p*3



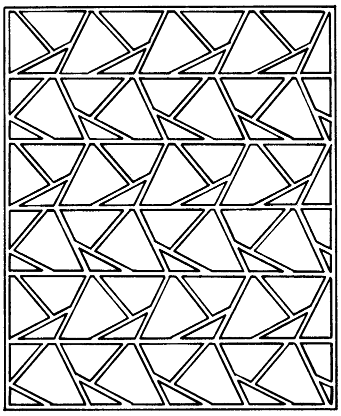
*p*3*m*1



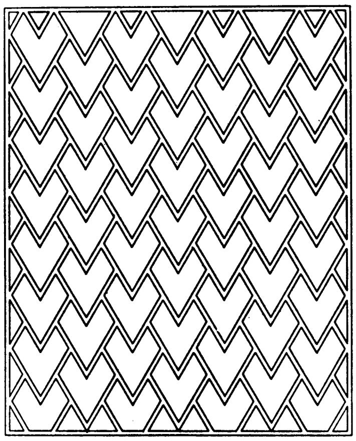
*p*31*m*

CHART 4. All designs except *pm*, *p*3, *pg* are found in **[10]**. The designs for *p*3 and *pg* are based on elements of Chinese lattice desi found in this book; the design for *pm* is based on a weaving pattern from the Sandwich Islands, found in **[14]**.

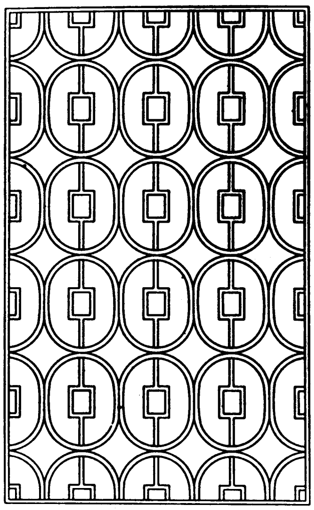
THE PLANE SYMMETRY GROUPS



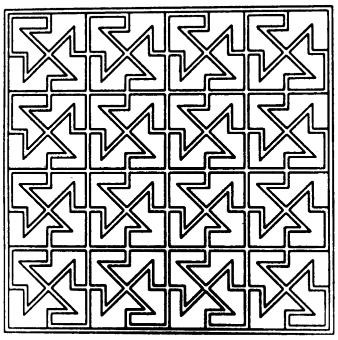
pg



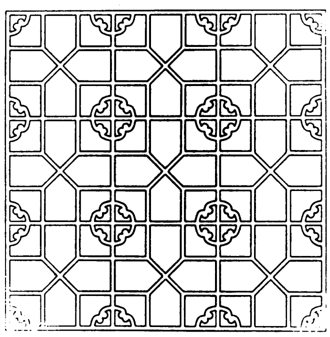
cm



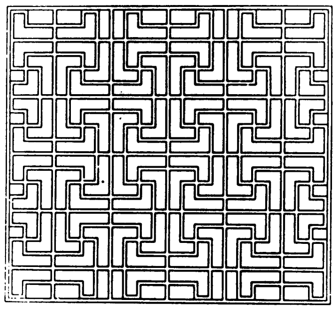
pmm



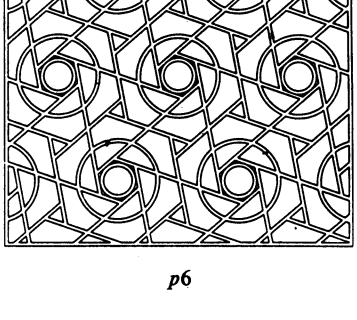
*p*4



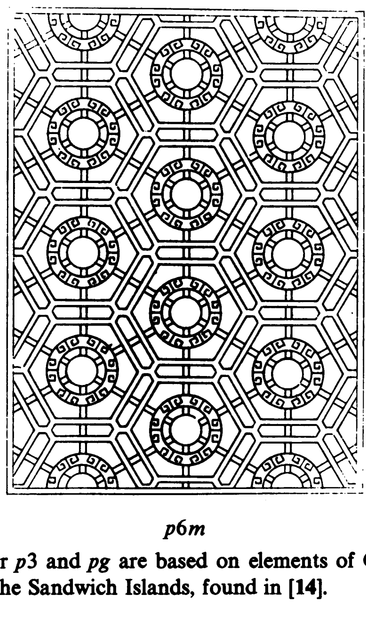
*p*4*m*



*p*4*g*



*p*6



*p*6*m*

ART 4. All designs except *pm*, *p*3, *pg* are found in **[10]**. The designs for *p*3 and *pg* are based on elements of Chinese lattice designs nd in this book; the design for *pm* is based on a weaving pattern from the Sandwich Islands, found in **[14]**.

of $p3m1$ and $p31m$, the four-place symbols can be shortened without loss of identification and the shortened form of notation is in most common usage.

Recognition and classification of periodic patterns can be fun; in fact once you begin looking for them, you become aware of how surrounded we are by these ornamental designs. (There is a slight warning to those who engage in this pastime. While staring at a stranger's printed dress or gazing intently at a carpet design, you may find yourself the object of curious stares!) To classify a periodic design as one of the seventeen types, it is not necessary to obtain all the information indicated on Chart 2. A check list for recognition of patterns is provided in Chart 3. This was the result of several attempts by the author and students to reduce to a minimum the information necessary to distinguish between designs. Using this you can classify any design as to its symmetry group. For example, the Escher design of lions has only 2-fold rotations (centered where the paws meet) and has glide reflections, but no reflections; hence it is type *pgg*. The reader should assign the correct pattern type to the design of bugs.

In addition to pattern books for wallpaper, tiles, floor coverings and fabrics, collections of decorative art also provide rich sources of patterns. The collection of M. C. Escher's designs **[15]** is a delightful source for analyzing patterns and contains commentary by C. H. MacGillavry, a crystallographer, aimed at helping the beginner discover the symmetries of the patterns. Three other widely differing collections currently in print appear in the bibliography: **[2]**, **[10]**, **[14]**. Journal articles **[8]**, **[9]**, **[19]** are also of interest. Museum collections often contain a wide variety of sources of repeating patterns which attest to the timelessness and universality of their use as decorative art. See the article "Mathematics and Islamic Art," by John Niman and Jane Norman, this MONTHLY, pp. 489–490.

We provide representative patterns in Chart 4 to test the reader's ability to recognize the various types of designs associated to the seventeen groups. For each of these patterns, an instructive exercise is to find a lattice unit of the type shown in Chart 2. (Hint: Begin by looking for a center of rotation of highest order for the pattern; next find axes of reflection or glide reflection.)

Group generators. Creation of periodic patterns. Finding generators for a group is a standard task. In the case of the plane symmetry groups, however, it has more than algebraic importance. Not only will a few isometries generate the symmetry group of a periodic design, but the same isometries, acting on a small portion of the design, will produce replicas of this region, and create the total plane-covering design. We call a *generating region* of a periodic pattern a smallest region of the plane whose images under the full symmetry group of the pattern cover the plane. (Crystallographers use the term *asymmetric unit* for a generating region; several mathematicians use the term *fundamental region* or *fundamental domain*.) The area of a generating region will always be a rational part of a unit, and, as with units, all generating regions of a pattern will have the same area, even though their outlines can vary greatly. In the Escher design of lions, one lion is a generating region. Often the term *motif* is used to denote the smallest portion of a design which generates the whole periodic design when acted on by the symmetry group of the design; in this usage the motif is a symbol which can be located within a generating region.

For algebraic (and geometric) analysis of these groups, a minimal set of generators is the most desirable choice. However, if we wish to use a set of generators to create a design by having it act on a generating region, a different choice of generators may be better suited to the task. In Chart 5 we show for each group two sets of generators and their location relative to a lattice unit containing a generating region of the pattern. The choice of a minimal set of generators is adapted from **[6**, Table 1] (other minimal sets of generators are indicated in **[11**, p. 40]). For each group, the second set of generators given includes the translation vectors which form the sides of the lattice unit. These generators may be preferred for producing a tile or printing a pattern by hand or computer. The rotations, reflections, or glide reflections shown fill out a unit with images of the generating region; then the translations repeat this unit to cover the plane. (A lattice unit is filled out for the first 12 types; a unit in the shape of a regular hexagon is filled out for the last 5 types.)

GENERATORS FOR THE PLANE SYMMETRY GROUPS

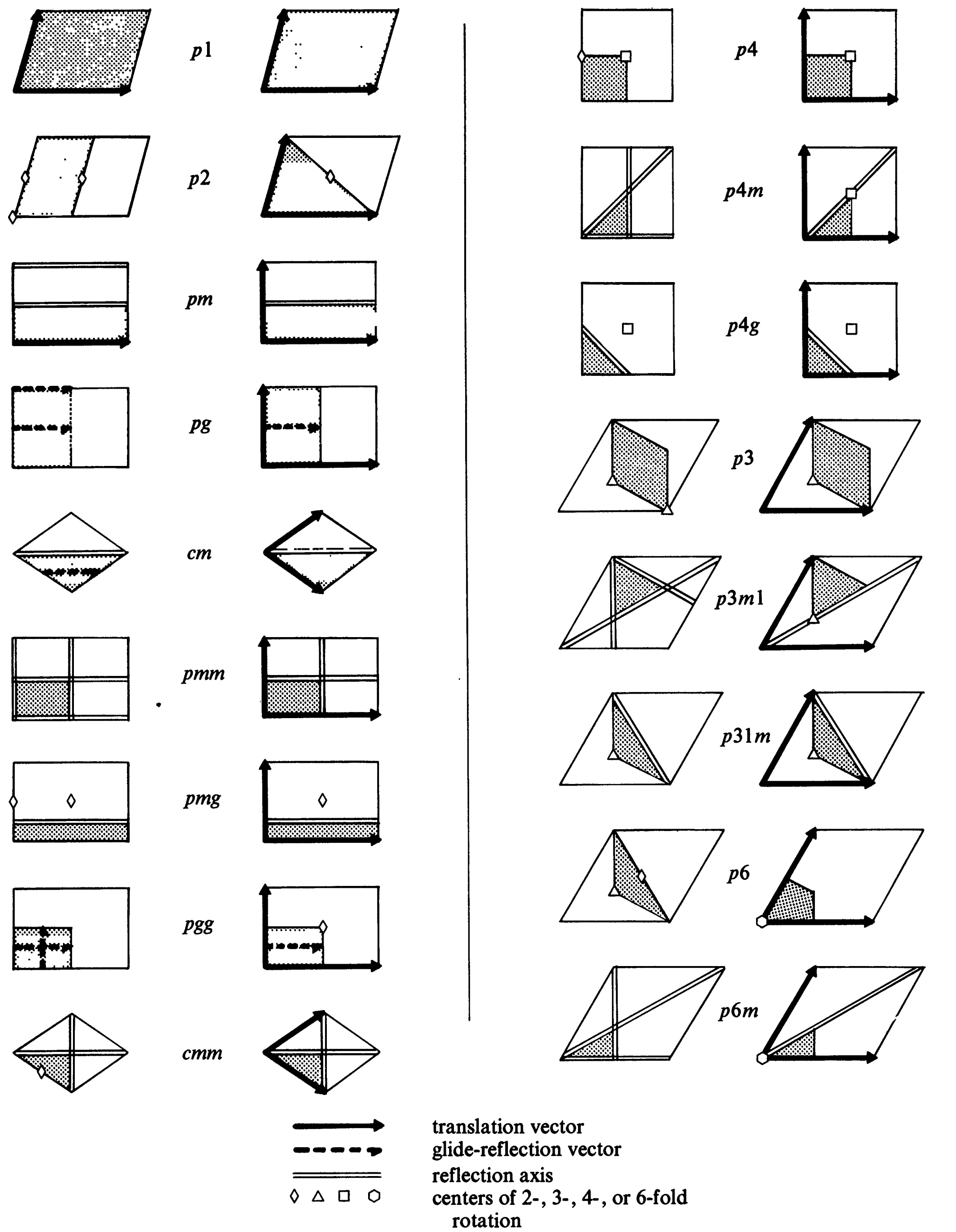


CHART 5. For each group, two sets of generators are indicated relative to a lattice unit containing a shaded generating region. A minimal set of generators is shown at the left, while a set of generators which includes the lattice unit translation vectors is shown at the right.

Since patterns of types $p3m1$ and $p31m$ are often confused, we demonstrate in Figure 1 how to use Chart 5 by creating a pattern of each type generated from the same motif. In each case, we begin with a single "hockey stick" motif, placed in a shaded generating region of each pattern type, and having its endpoints at centers of three-fold rotation. Each pattern is then produced by acting on the generating region by the isometries indicated on Chart 5, in the sequence shown.

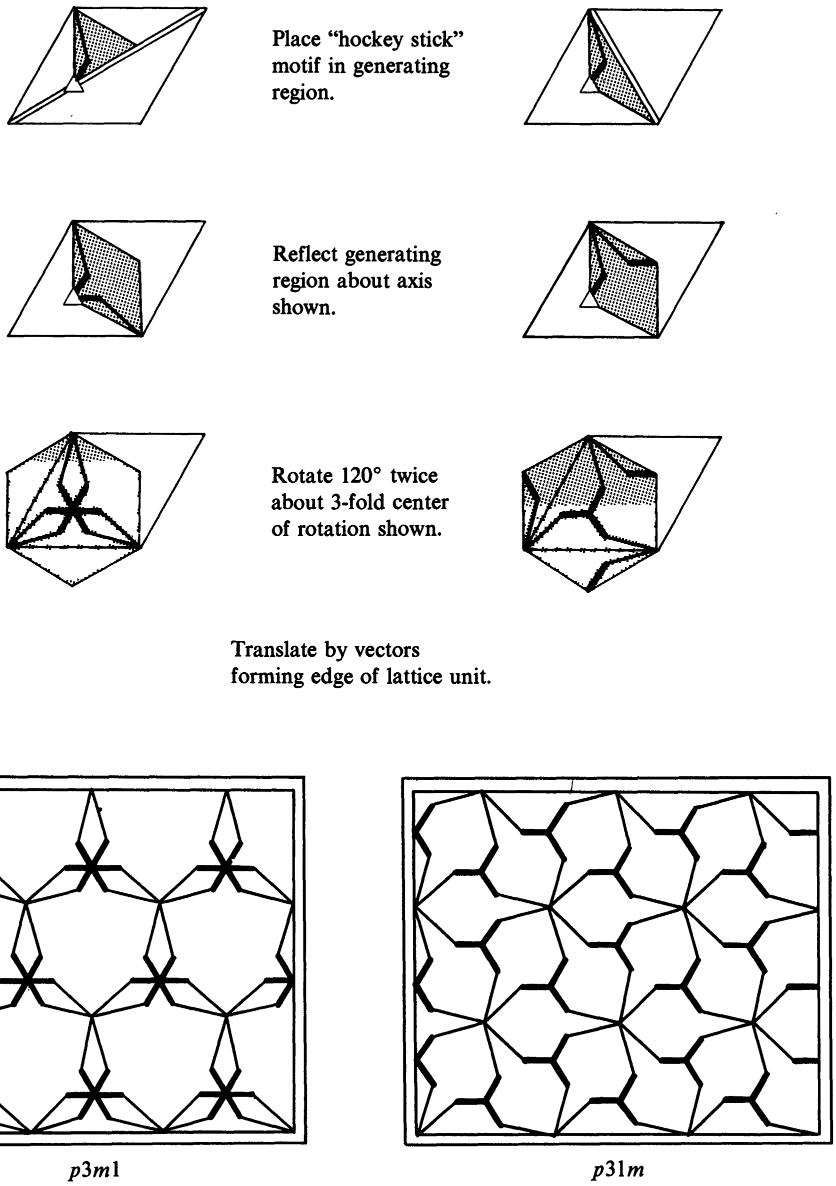


FIG. 1. Generating patterns of types $p3m1$ and $p31m$, beginning with the same motif. Note that in the completed $p31m$ pattern, a "natural" generating region is half of the arrow-shaped tile, while a "natural" unit is 3 of these interlocked tiles whose outline looks like a rotor.

Visually, the differences in the two patterns in Figure 1 are striking—there is no possibility that one pattern could be mistaken for the other. Charts which show each of the seventeen patterns which arise when the same motif (properly placed in a generating region) is acted on by each of the symmetry groups provide a clear visual demonstration of their differences. Such charts can be found in [4], [5], [11], [17], [20]. (Note the remarks in the next section concerning $p3m1$ and $p31m$.)

Notations for the groups. Interchange of p3m1 and p31m. The history of the classification of the two-dimensional crystallographic groups dates back to the late nineteenth century. Both [7] and [11] give a brief account of this history. An extensive discussion and comparison of the early literature is given in [3]. Many mathematicians have produced varying notations for the groups, and the variety of notations continues, even in books published within the last twenty years. Thus it is difficult to read the literature without some crossreference chart of notation. The notation adopted by the International Union of Crystallography in 1952 is in most widespread use, and this was used as the "norm" in preparing our crossreference table of notation, Chart 6.

COMPARISON OF NOTATION FOR THE PLANE SYMMETRY GROUPS

Internat'l (short)	Pólya; Guggenheimer	Niggli	Speiser	Fejes Tóth; Cadwell	Shubnikov-Koptsik	Wells Bell & Fletcher
$p1$	C_1	${C_1}^{\mathrm{I}}$	C_1, Abb. 17	W_1	$(b/a)1$	1
$p2$	C_2	${C_2}^{\mathrm{I}}$	C_2, Abb. 18	W_2	$(b/a):2$	2
pm	D_1kk	${C_s}^{\mathrm{I}}$	C_s^{I}, Abb. 19	W_1^2	$(b:a):m$	3
pg	D_1gg	${C_s}^{\mathrm{II}}$	C_s^{II}, Abb. 20	W_1^3	$(b:a):\tilde{b}$	4
cm	D_1kg	${C_s}^{\mathrm{III}}$	C_s^{III}, Abb. 21	W_1^1	$(a/a)/m$	8
pmm	D_2kkkk	${C_{2v}}^{\mathrm{I}}$	C_{2v}^{I}, Abb. 22	W_2^2	$(b:a):2\cdot m$	5
pmg	D_2kkgg	${C_{2v}}^{\mathrm{III}}$	C_{2v}^{III}, Abb. 24	W_2^3	$(b:a):m:\tilde{a}$	6
pgg	D_2gggg	${C_{2v}}^{\mathrm{II}}$	C_{2v}^{II}, Abb. 23	W_2^4	$(b:a):\tilde{b}:\tilde{a}$	7
cmm	D_2kgkg	${C_{2v}}^{\mathrm{IV}}$	C_{2v}^{IV}, Abb. 25	W_2^1	$(a/a):2\cdot m$	9
$p4$	C_4	${C_4}^{\mathrm{I}}$	C_4, Abb. 26	W_4	$(a:a):4$	10
$p4m$	D_4^*	${C_{4v}}^{\mathrm{I}}$	C_{4v}^{I}, Abb. 27	W_4^1	$(a:a):4\cdot m$	11
$p4g$	D_4°	${C_{4v}}^{\mathrm{II}}$	C_{4v}^{II}, Abb. 28	W_4^2	$(a:a):4\odot\tilde{a}$	12
$p3$	C_3	${C_3}^{\mathrm{I}}$	C_3, Abb. 29	W_3	$(a/a):3$	13
$p3m1$	D_3^*	${C_{3v}}^{\mathrm{I}}$	C_{3v}^{II}, Abb. 31	W_3^1	$(a/a):m\cdot 3$	15
$p31m$	D_3°	${C_{3v}}^{\mathrm{II}}$	C_{3v}^{I}, Abb. 30	W_3^2	$(a/a)\cdot m\cdot 3$	14
$p6$	C_6	${C_6}^{\mathrm{I}}$	C_6, Abb. 32	W_6	$(a/a):6$	16
$p6m$	D_6	${C_{6v}}^{\mathrm{I}}$	C_{6v}, Abb. 33	W_6^1	$(a/a):m\cdot 6$ (some alternatives exist)	17

CHART 6. Sources referred to in the table are listed in the References. The groups are listed in consecutive order as they appear in the International Tables of X-ray Crystallography, [13]. Note that Speiser interchanges the Niggli notations of ${C_{3v}}^{\mathrm{I}}$ and C_{3v}^{II} (figure numbers in the Speiser column are for the 2nd, 3rd, and 4th editions of his book).

In the preparation of this chart, it became apparent that the notation for the two groups $p3m1$ and $p31m$ was frequently interchanged in the literature, and so other crossreference charts could not be assumed to be accurate. The earliest occurrence of this interchange which was noted occurs in Speiser's book, [21]. He uses the notation of the paper by Niggli, [16], but interchanges Niggli's notation for these two groups. Since it is natural to assume these notations are the same, we include information from both sources on our chart. Other books which include this interchange of notation are: Bell and Fletcher [1], Budden, [4], Coxeter, [6], and Coxeter and Moser, [7]. It is quite likely that this notational error has been perpetuated in other works referring to these sources. (The crossreference Table 3 in [7] is correct if in the left column $p31m$ and $p3m1$ are interchanged.)

If the interpretation of the crystallographic notation explained earlier is understood, then it is always possible to determine the correct name for the symmetry group of a periodic design. This, together with the other information provided here, should enable the reader to make any necessary corrections of inaccurate identification in the literature.

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EULER'S FORMULA FOR *n*th DIFFERENCES OF POWERS

Dedicated to Professor L. Carlitz on his seventieth birthday.

H. W. GOULD

1. Introduction. Write down the sequence of fourth powers of the non-negative integers. Below these, write the first differences. Below these, write differences again. Repeat this process as long as you wish, and you obtain the following array of numbers:

H. W. Gould did his graduate work at the Universities of Virginia and North Carolina. Since 1958 he has been at West Virginia University, where he is now a professor. He has lectured and written extensively on his fields of interest, which include number theory, combinatorics, special functions, and history of mathematics. *Editors.*