MATRIX DERIVATION OF $\sum n^k$

Consider the formulae

$$\begin{aligned} 2^4 - 1^4 &= 1 + 4.1 + 6.1^2 + 4.1^3 \\ 3^4 - 2^4 &= 1 + 4.2 + 6.2^2 + 4.2^3 \\ 4^4 - 3^4 &= 1 + 4.3 + 6.3^2 + 4.3^3 \\ &\vdots \\ n^4 - (n-1)^4 &= 1 + 4.(n-1) + 6.(n-1)^2 + 4.(n-1)^3. \end{aligned}$$

By addition we see that

$$n^4 = n + 4S_1(n-1) + 3 + 6S_2(n-1) + 4S_3(n-1)$$

where

$$S_r(n) = \sum_{i=1}^{n} i^r$$

and similarly we get the general result

$$n^r = n + \binom{r}{1} S_1(n-1) + \binom{r}{r-2} S_2(n-1) + \cdots + \binom{r}{r-1} S_{r-1}(n-1).$$

The equations

$$\begin{aligned} n &= n \\ n^2 &= n + S_1(n-1) \\ n^3 &= n + 3S_1(n-1) + 3S_2(n-1) \\ n^4 &= n + 4S_1(n-1) + 6S_2(n-1) + 4S_3(n-1) \\ n^5 &= n + 5S_1(n-1) + 10S_2(n-1) + 10S_3(n-1) + 5S_4(n-1), \text{ etc.} \end{aligned}$$

can be written in matrix form as

$$\begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 0 & 0 & 0 & \dots \\ 1 & 4 & 6 & 4 & 0 & 0 & \dots \\ 1 & 5 & 10 & 10 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} n \\ S_1(n-1) \\ S_2(n-1) \\ S_3(n-1) \\ S_4(n-1) \\ \vdots \end{pmatrix}$$

Since this matrix of coefficients is triangular, it may be readily inverted to give

$$\begin{pmatrix} n \\ S_1(n-1) \\ S_2(n-1) \\ S_3(n-1) \\ S_4(n-1) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -1/2 & 1/2 & 0 & 0 & 0 & \\ 1/6 & -1/2 & 1/3 & 0 & 0 & \dots \\ 0 & 1/4 & -1/2 & 1/4 & 0 & \dots \\ -1/30 & 0 & 1/3 & -1/2 & 1/5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \vdots \end{pmatrix}.$$

To obtain the sums to n terms instead of $n-1$ we use $S_r(n) = S_r(n-1) + n^r$, which has the effect of adding $1s$ to the numbers immediately below the main diagonal of the matrix, turning each $-1/2$ into $1/2$. The final result is then

$$\begin{pmatrix} n \\ S_1(n) \\ S_2(n) \\ S_3(n) \\ S_4(n) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 1/2 & 0 & 0 & 0 & \dots \\ 1/6 & 1/2 & 1/3 & 0 & 0 & \dots \\ 0 & 1/4 & 1/2 & 1/4 & 0 & \dots \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \vdots \end{pmatrix}.$$

The numbers in the first column of this matrix are known as the “Bernoulli numbers”.

P.M.L.